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ON REDUCIBLE FINITE
SUBGROUPS OF MAPPING CLASS GROUPS OF SURFACES

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Introduction

Let $\Sigma_g$ be the closed connected orientable surface of genus $g \geq 2$. By an automorphism of $\Sigma_g$, we mean an element of the mapping class group $\mathcal{M}_g$ which is the group of the isotopy classes of orientation preserving diffeomorphisms. We recall some definitions mainly from [T]. A periodic automorphism is the one which is of finite order in $\mathcal{M}_g$. A non-empty 1-submanifold is said to be essential if it is compact, and its no two components are homotopic and no components are null-homotopic. A reducible automorphism is the one which fixes the isotopy class of some essential 1-submanifold of $\Sigma_g$.

In §1, we describe the relation between order and reducibility of periodic automorphisms. The result shows that the order of a periodic automorphism determine its reducibility unless $g$ is even and the order is $2g+2$. This exception occurs because there is a periodic diffeomorphism $\Sigma_g \to \Sigma_g$ of order $4g+2$ with a fixed point for any $g \geq 1$. The proof is based on the geometric characterization of irreducible finite subgroup of $\Sigma_g$ by Gilman, and cyclicity condition for 2-orbifolds by Harvey. Details of this section can be found in [Ka].

In §2, via Nielsen realization theorem [N, Ke], we consider decompositions of any finite subgroup of $\mathcal{M}_g$ along oriented essential 1-submanifolds, and describe the quotient orbifold types appearing in “irreducible” decompositions after capping off 2-disks to obtain closed orbifolds.
Notation. We denote by $\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ the 2-dimensional orbifold whose underlying surface is $\Sigma_\gamma$ and whose singular locus consists of $n$ cone points with singular indices $m_1, m_2, \cdots, m_n$, respectively. We also write $S^2(m_1, \cdots, m_n)$ when $\gamma = 0$.

1. Reducibility and orders of periodic automorphisms

This section is devoted to prove the following.

**Theorem 1.1.** Let $f \in \mathcal{M}_g$ be a periodic automorphism of order $N$. Then, the followings hold:

(I) if $f$ is irreducible, then $N \geq 2g + 1$,

(II) if $f$ is reducible, then $N \leq 2g + 2$ and $N \neq 2g + 1$;

furthermore, if the genus $g$ is odd, then $N \leq 2g$.

All the inequalities are best possible. That is to say, there certainly exists a periodic automorphism of $\Sigma_g$ having as order the value of the right-hand term of each inequality, with required reducibility. On the other hand, $\Sigma_g$ has always a periodic and irreducible automorphism of order $2g + 2$.

**Proof of inequalities.**

Given a periodic automorphism $f \in \mathcal{M}_g$ of order $N$, by Nielsen realization theorem, it can be represented by a periodic diffeomorphism $f : \Sigma_g \to \Sigma_g$ of the same order $N$. We denote by $O_f$ the quotient orbifold of $\Sigma_g$ by the cyclic action generated by $f$. Then $f$ is irreducible if and only if $O_f$ is of the form $S^2(m_1, m_2, m_3)$ where $m_1, m_2, m_3 \geq 2$ for any (and then necessarily all) Nielsen realization $f$ [Gi].

Then, the inequality of (i) directly follows from the Riemann-Hurwitz formula for the canonical projection $\pi : \Sigma_g \to O_f(= S^2(m_1, m_2, m_3))$ since each $m_i \leq N$.

To obtain the rest of the inequalities in (ii), instead of estimating order $N$ while the genus $g$ fixed, we obtain the minimum genus $g_{\min}(N)$ of surfaces which admit a periodic and reducible automorphism of a fixed order $N$. Depending on the form of prime decomposition of $N$, it is described as follows:

**Theorem 1.2.** Let $N$ be an integer $\geq 2$ with prime decomposition $p_1^{r_1} \cdots p_k^{r_k}$ where each $p_i$ is prime, each $r_i \geq 1$, and $p_1 < p_2 < \cdots < p_k$. Then, the minimum genus
$g_{\min}(N)$ of surfaces which admit a periodic and reducible automorphism of order $N$ is given by

(i) $g_{\min}(N) = \max \left\{ 2, (p_1 - 1) \frac{N}{p_1} \right\}$, if $r_1 > 1$ or $N$ is prime,

(ii) $g_{\min}(N) = N - \frac{1}{2} \left( \frac{N}{p_1} + \frac{N}{p_2} + \frac{N}{p_3} - 1 \right)$, if $N = p_1 p_2 p_3$ and $p_3 \leq \frac{p_1 p_2 - 2p_1 + 1}{p_2 - p_1}$,

(iii) $g_{\min}(N) = (p_1 - 1) \left( \frac{N}{p_1} - 1 \right)$, otherwise.

Now, we see that the rest of the inequalities follow from Theorem 1.2. Let $N$ be the order of any periodic and reducible automorphism of $\Sigma_g$. Then, by definition, it holds that $g_{\min}(N) \leq g$. According to the form of the prime decomposition of $N$, replacing the left-hand side by the term given by Theorem 1.2, we obtain $N \leq 2g + 2$. Next, we can see that $g_{\min}(2g + 1) > g$ and therefore $N$ cannot be $2g + 1$.

Suppose now $g$ is odd. Then we can also see $g_{\min}(2g + 2) > g$, which implies that $N$ cannot be $2g + 2$, and therefore $N \leq 2g$.

A sketchy proof of Theorem 1.2 is given in the end of this section.

**Examples.**

Now, we describe examples of periodic automorphisms which should assure the best possibility of each inequality. It is known that an orbifold $\Sigma_\gamma(m_1, m_2, \cdots, m_n)$ is an $N$-cyclic quotient of some compact surface if and only if it satisfies the following conditions [H]:

(i) $lcm(m_1, \cdots, \hat{m_i}, \cdots, m_n) = lcm(m_1, \cdots, m_n)$ where $m_i$ denotes the omission of $m_i$. ($i = 1, 2, \cdots, n$);

(ii) $lcm(m_1, \cdots, m_n)$ divides $N$, and if $\gamma = 0$, $lcm(m_1, \cdots, m_n) = N$;

(iii) $n \neq 1$;

(iv) if $lcm(m_1, \cdots, m_n)$ is even, then the number of $m_i$'s divisible by the maximum power of 2 dividing $lcm(m_1, \cdots, m_n)$ is even.

We call such an orbifold $N$-cyclic. Note that the genus of $N$-cyclically covering surface of a given $N$-cyclic orbifold is determined uniquely by the Riemann-Hurwitz...
formula. Now, it is easy to see that the following three orbifolds give examples of periodic and reducible automorphisms of $\Sigma_g$ which show that equality holds for each inequality of Theorem 1.1, respectively: $S^2(2g + 1, 2g + 1, 2g + 1); S^2(2, 2, g + 1, g + 1)$ $(g: \text{even}); S^2(2, 2g, 2g).$

Also, the orbifold $S^2(g + 1, 2g + 2, 2g + 2)$ gives an example of periodic and irreducible automorphism of $\Sigma_g$ of order $2g + 2$. This complete the proof of Theorem 1.1.

**Proof of Theorem 1.2.**

For an $N$-cyclic orbifold $\Sigma_\gamma(m_1, \ldots, m_n)$, the genus of the $N$-cyclic covering surface $g$ is given by

\[(*) \quad g = 1 + N(\gamma - 1) + \frac{1}{2}N\sum_{i=1}^{n}(1 - \frac{1}{m_i})\]

Therefore, $g_{\min}(N)$ is the minimum value of (*) where $\Sigma_\gamma(m_1, \ldots, m_n)$ varies all the orbifolds which are not of the type $S^2(m_1, m_2, m_3)$, satisfying Harvey's cyclicity conditions (i)-(iv).

So far as $\gamma = 0$ and $n = 4$, the minimum of (*) corresponds to the maximum of $1/m_1 + 1/m_2 + 1/m_3 + 1/m_4$ where $lcm(m_2, m_3, m_4) = lcm(m_1, m_3, m_4) = lcm(m_1, m_2, m_4) = lcm(m_1, m_2, m_3) = N$. By dividing into several subcases carefully, the calculation of this maximum is reduced to the calculation of the maximum of $1/x + 1/y + 1/z$ where $lcm(x, y) = lcm(y, z) = lcm(z, x) = \text{given positive integer}$. The latter maximum was given by Harvey [H]. The result of calculation gives the value expected for $g_{\min}(N)$.

If $\gamma \neq 0$ or $n \neq 4$, it can be checked that the value of (*) does not exceed the minimum for the case $\gamma = 0$ and $n = 4$ so far as $\gamma$ and $m_i$'s satisfy (i)-(iv). Therefore, $g_{\min}(N)$ is not less than the expected value.

The following three $N$-cyclic orbifolds realize the minimum genus according to the form of prime decomposition of $N$: $S^2(p_1, p_1, N, N); S^2(p_1, p_2, p_3) (N = p_1 p_2 p_3); S^2(p_1, p_1, N/p_1, N/p_1) (r_1 = 1, k \geq 2)$. This completes the proof of Theorem 1.2.
2. Irreducible decomposition

Let \( \mathcal{E} \) be the set of the isotopy classes of oriented essential 1-submanifolds of \( \Sigma_g \). Transformation of 1-submanifolds by diffeomorphisms naturally induces an action of \( \mathcal{M}_g \) on \( \mathcal{E} \). Let \( \mathfrak{G} \) be a finite subgroup of \( \mathcal{M}_g \). We denote by \( \mathcal{E}_{\mathfrak{G}} \) the subset of \( \mathcal{E} \) consisting of the elements fixed by every \( g \in \mathfrak{G} \). If \( G \subset \text{Diff}^+ \Sigma_g \) is any Niesen realization of \( \mathfrak{G} \), it is easy to see that any \( \mathcal{E} \in \mathcal{E}_{\mathfrak{G}} \) has a representative \( \mathcal{E} \subset \Sigma_g \) such that \( G(\mathcal{E}) = \mathcal{E} \). Then, the action of \( G \) on \( \Sigma_g \) decomposes into the pair of:

1. the permutation of the connected components of \( \Sigma_g \setminus \mathcal{E} \);
2. actions on each connected component of \( \Sigma_g \setminus \mathcal{E} \) of its stabilizer.

Note that any \( \mathcal{E} \in \mathcal{E}_{\mathfrak{G}} \) is contained in a maximal element of \( \mathcal{E}_{\mathfrak{G}} \) according to the inclusion order since the number of the connected components of an essential 1-submanifold is at most \( 3g - 3 \). Among the decompositions as above, it might be natural to call a decomposition corresponding to a maximal element of \( \mathcal{E}_{\mathfrak{G}} \) an irreducible decomposition of \( G \).

In this section, we describe the orbifolds appearing as the quotient of connected component of \( \Sigma_g \setminus \mathcal{E} \) by its stabilizer after capping off 2-disks to the boundary of the component.

Now, we set the notation. We fix \( G \) and \( \mathcal{E} \) as above. We denote by \( S_i \) a connected component of \( \Sigma_g \setminus \mathcal{E} \). We take a completion \( M'_i \) of \( S_i \) as follows. Let \( \tilde{S}_i \) be the universal covering of \( S_i \) embedded in \( \Sigma_g \) via a lift of the inclusion \( S_i \to \Sigma_g \). Then \( \pi_1(S_i) \) acts on the closure \( \tilde{S}_i \). We set \( M'_i \) as the quotient \( \tilde{S}_i / \pi_1(S_i) \). Next, for each boundary component of \( M'_i \), we cap off 2-disk identifying it with the cone of the boundary component, and obtain a closed surface \( M_i \). Then, the stabilizer \( G_i \) of \( S_i \) naturally acts on \( M_i \). We denote the quotient orbifold \( M_i / G_i \) by \( O_i \).

**Theorem 2.1.** Let \( \mathcal{E} \subset \Sigma_g \) be an oriented essential 1-submanifold which is invariant under the \( G \)-action. If its representing class \([\mathcal{E}]\) is maximal in \( \mathcal{E}_{\mathfrak{G}} \), then the corresponding quotient orbifold \( O_i = M_i / G_i \) for any connected component \( S_i \) of \( \Sigma_g \setminus \mathcal{E} \) is described as follows:

1. If \( G_i \) is a trivial group, then \( O_i \) is isomorphic to the 2-sphere \( S^2 \).
(ii) If $G_i$ is not trivial, then the orbifold isomorphism class of $O_i$ is one of the followings according to the genus $g_i$ of $M_i$.

(a) $g_i \geq 2$: $S^2(2,2,2,2), S^2(2,2,2,m) (m \geq 3), S^2(m_1,m_2,m_3) (m_1, m_2, m_3 \geq 2, \text{ and } \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1)$;
(b) $g_i = 1$: $S^2(2,2,2,2), S^2(3,3,3), S^2(2,4,4), S^2(2,3,6)$;
(c) $g_i = 0$: $S^2(2,3,3), S^2(2,3,4), S^2(2,3,5), S^2(2,2,m), S^2(m,m) (m \geq 2)$.

Moreover, any orbifold type above certainly appears in some irreducible decomposition for some $g \geq 2$.

The theorem follows from the next two lemmas.

**Lemma 2.2.** There exists an oriented essential 1-submanifold $\overline{E}_0$ of $M_i$ invariant under the $G_i$-action so that $\overline{E}_0 \subset \tilde{M}_i$.

**Lemma 2.3.** Let $\overline{E}_0 \subset S_i$ be another $G_i$-invariant oriented essential 1-submanifold of $\Sigma_g$. Suppose that $\overline{E}_0 \cup \overline{E}$ also form an essential 1-submanifold of $\Sigma_g$. Then, $G(\overline{E}_0) \cup \overline{E}$ is a $G$-invariant oriented essential 1-submanifold of $\Sigma_g$.

**References**


