<table>
<thead>
<tr>
<th>Title</th>
<th>Spinal hypersurfaces in complex hyperbolic space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Kamiya, Shigeyasu</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1994), 882: 33-40</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1994-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/84247">http://hdl.handle.net/2433/84247</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
Spinal hypersurfaces in complex hyperbolic space

Shigeyasu Kamiya
神谷 茂保 (岡山理大)

The purpose of this article is to introduce spinal hypersurfaces in $H^n_C$, which are studied and developed by Goldman and Mostow.

In Section 1 we state the properties of elements of $U(1, n; C)$. Section 2 is devoted to discussing spinal hypersurfaces on which results are due to Goldman and Mostow (cf. [3], [5]). In Section 3 we show Phillips' theorem on the Dirichlet polyhedra (cf. [6], [7]).

0. First we recall definitions and notation. Let $C$ be the field of complex numbers. Let $V = V^{1,n}(C) \ (n \geq 1)$ denote the vector space $C^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(Z, W) = -\overline{Z_0}W_0 + \sum_{k=1}^{n} \overline{Z_k}W_k,$$

for $Z = (Z_0, Z_1, ..., Z_n), W = (W_0, W_1, ..., W_n) \in V$.

An automorphism $g$ of $V$, that is a linear bijection such that $\Phi(g(Z), g(W)) = \Phi(Z, W)$ for $Z, W \in V$, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1, n; C)$.

Let $V_0 = \{ Z \in V \mid \Phi(Z, Z) = 0 \}$ and $V_- = \{ Z \in V \mid \Phi(Z, Z) < 0 \}$. It is clear that $V_0$ and $V_-$ are invariant under $U(1, n; C)$.

Set $V^* = V_- \cup V_0 - \{ 0 \}$. Let $\pi : V* \rightarrow \pi(V^*)$ be the projection map defined by

$$\pi(Z_0, ..., Z_n) = (\frac{Z_1}{Z_0}, ..., \frac{Z_n}{Z_0}) = (z_1, ..., z_n).$$

Set $H^n_C = \pi(V_-)$.

An element $g$ of $U(1, n; C)$ operates in $\pi(V^*)$, leaving $\overline{H^n_C}$ (the closure of $H^n_C$ in $\pi(V^*)$) invariant. Since $H^n_C$ is identified with the complex unit ball

$$B^n = \{ z \in C^n \mid \|z\|^2 = \sum_{k=1}^{n} |z_k|^2 < 1 \},$$

we can regard a unitary transformation as a transformation operating on $B^n$. Hence discrete subgroups of $U(1, n; C)$ is the generalizations of Fuchsian groups.

We can introduce the Bergman metric in $B^n$. This hyperbolic distance $d(z, w)$ for $z, w \in B^n$ is expressed by the use of the Hermitian form $\Phi$ as follows.

$$d(z, w) = 2 \cosh^{-1} \frac{|\Phi(Z, W)|}{[\Phi(Z, Z)\Phi(W, W)]^{1/2}},$$

where $Z \in \pi^{-1}(z), W \in \pi^{-1}(w)$. This does not depend on the choice of $Z, W$. 
To discuss some properties of unitary transformations, it may be more convenient to use another matrix representation for \( U(1, n; \mathbb{C}) \). By changing the basis of \( V \), we introduce the group \( \tilde{U}(1, n; \mathbb{C}) \) as follows. Let

\[
D = \begin{pmatrix}
1/\sqrt{2} & 1/\sqrt{2} & 0 \\
-1/\sqrt{2} & 1/\sqrt{2} & 0 \\
0 & 0 & I_{n-1}
\end{pmatrix}.
\]

Define \( \tilde{U}(1, n; \mathbb{C}) \) by \( D^{-1}U(1, n; \mathbb{C})D \). We see that \( \tilde{U}(1, n; \mathbb{C}) \) is the automorphism group of the Hermitian form

\[
\tilde{\Phi}(Z, W) = -(\overline{Z_0}W_1 + \overline{Z_1}W_0) + \sum_{k=2}^{n} \overline{Z_k}W_k.
\]

We can regard \( D^{-1} \) as a mapping of complex unit ball \( B^n \) to

\[
\overline{H^n} = \{ z \in \mathbb{C}^n | \ Re(z_1) > \frac{1}{2} \sum_{k=2}^{n} |z_k|^2 \},
\]

which is called the Siegel domain.

1. The nontrivial elements fall into three general conjugacy types, depending on the number and location of their fixed points. Since each element acts on the closure of \( B^n \), the Brouwer fixed point theorem implies that it has a fixed point. Let \( g \neq id \). We call \( g \) elliptic if it has a fixed point in \( B^n \) and \( g \) parabolic if it has exactly one fixed point and this lies on the boundary. An element \( g \) will be called loxodromic if it has exactly two fixed points and they lie on the boundary. If \( g \) is conjugate to an element (different from the identity) in the identity component \( U_0(1,1; \mathbb{R}) \), it will be called hyperbolic. Hyperbolic elements are special kinds of loxodromic elements.

Now we state properties of each kind of elements. Let

\[
U(1; \mathbb{C}) \times U(n; \mathbb{C}) = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & A \end{pmatrix} \mid |\alpha| = 1, AA^* = I_n \right\}.
\]

Proposition 1.

1) Let \( g \) be an elliptic element. Then:

1) \( g \) is conjugate to an element of \( U(1; \mathbb{C}) \times U(n; \mathbb{C}) \).

2) \( g \) is semisimple with eigenvalues of absolute value 1.

2) Let \( g \) be a loxodromic element. Then:

1) \( g \) is semisimple with \( n - 1 \) eigenvalues of absolute value 1.
2) \( g \) leaves the geodesic joining the two fixed points, invariant. This is called the axis of \( g \) and denoted by \( A_g \).

3) \( g \) moves every point \( z \) in \( A_g \) the same distance \( T(g) = d(z, g(z)) \). This \( T(g) \) is called the translation length of \( g \).

4) \( T(g) = \min_{z \in B^n} d(z, g(z)) \).

3) Let \( g \) be a parabolic element. Then:

1) \( g \) is not semisimple.

2) All absolute values of eigenvalues are 1.

If a parabolic element \( g \) is unipotent (that is, all eigenvalues are 1), then \( g \) is called strictly parabolic. A standard form of strictly parabolic element of \( \tilde{U}(1, n; \mathbb{C}) \) is as follows.

\[
\tilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \bar{a}^T \\ 0 & 0 & I_{n-1} \end{pmatrix},
\]

where \( \text{Re}(s) = \frac{1}{2} \|a\|^2 \).

In particular, a conjugate element to

\[
\begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}
\]

\((s \neq 0, \text{Re}(s) = 0)\) is called translation. We note that strictly parabolic elements are not necessarily conjugate to translations, because their minimal polynomials are different.

2. Before defining a spinal hypersurface we recall the following important proposition.

Proposition 2 ([2, Proposition 2.5.1]). A totally geodesic submanifold in \( H^n_C \) is equivalent to \( H^n_C \) or \( H^m_R \) \((m \leq n)\) under \( U(1, n; \mathbb{C}) \).

Given two points \( z_1, z_2 \in H^n_C \), the equidistant surface \( E\{z_1, z_2\} \) of \( z_1, z_2 \) is by definition

\[
E\{z_1, z_2\} = \{ z \in H^n_C \mid d(z, z_1) = d(z, z_2) \}.
\]

We call this \( E\{z_1, z_2\} \) a spinal hypersurface of \( \{z_1, z_2\} \). By Proposition 2, a spinal hypersurface is not a totally geodesic submanifold in \( H^n_C \), but a real analytic hypersurface in \( H^n_C \) diffeomorphic to \( \mathbb{R}^{2n-1} \). Two points \( z_1, z_2 \) in \( H^n_C \) determine a unique complex geodesic \( \Sigma \). Set \( \sigma\{z_1, z_2\} = E\{z_1, z_2\} \cap \Sigma \), which is called a spine of \( E \). It follows that the spine \( \sigma\{z_1, z_2\} \) is a (real) geodesic in \( \Sigma \).

Theorem 3.

\[
E = \Pi^\Sigma_1(\sigma) = \bigcup_{s \in \sigma} \Pi^\Sigma_1(s),
\]
where $\Pi_{\Sigma} : H_{C}^{n} \rightarrow \Sigma$ is orthogonal projection onto $\Sigma$.

For our proof we need a lemma.

Lemma 4 ([3, Lemma III.2.3.1]). Let $L \subset H_{C}^{n}$ be a complex linear subspace with orthogonal projection $\Pi$. Then for all $u \in H_{C}^{n}$ and $s \in L$, then the geodesic for $\Pi(u)$ to $u$ and to $s$ are orthogonal and span a totally real 2-space. Furthermore

$$\cosh \left( \frac{d(u,s)}{2} \right) = \cosh \left( \frac{d(u,\Pi(u))}{2} \right) \cosh \left( \frac{d(\Pi(u),s)}{2} \right).$$

Proof of Theorem 3. It follows from Lemma 4 that for $i = 1, 2$

$$\cosh \left( \frac{d(z,z_{i})}{2} \right) = \cosh \left( \frac{d(z,\Pi_{\Sigma}(z))}{2} \right) \cosh \left( \frac{d(\Pi_{\Sigma}(z),z_{i})}{2} \right).$$

It is seen that $z \in E\{z_{1}, z_{2}\} \iff d(z,z_{1}) = d(z,z_{2})$

$$\iff d(z_{1},\Pi_{\Sigma}(z)) = d(z_{2},\Pi_{\Sigma}(z))$$

$$\iff \Pi_{\Sigma}(z) \in \sigma\{z_{1}, z_{2}\}.$$ 

The complex hyperplanes $\Pi_{\Sigma}^{-1}(s)$ for $s \in \sigma$ are called the slices of $E$. The slice of $E$ is a maximal holomorphic submanifold of $E$.

Theorem 5. There is a bijective correspondence between spinal hypersurfaces in $H_{C}^{n}$ and (real) geodesics in $H_{C}^{n}$.

Proof. Let $E\{z_{1}, z_{2}\}$ be a spinal hypersurface. Then there is a unique complex geodesic $\Sigma$ which is orthogonal to slices of $E$. By setting $\sigma = E\{z_{1}, z_{2}\} \cap \Sigma$ we have a unique geodesic $\sigma$ corresponding to $E\{z_{1}, z_{2}\}$. Conversely, if a geodesic $\sigma$ is given, then there is a unique complex geodesic $\Sigma$ and a reflection $P_{\sigma}$ of $\Sigma$ with $P_{\sigma}(\sigma) = \sigma$. For any point $z_{1}$ set $z_{2} = P_{\sigma}(z_{1})$. Then it is easy to show that $E\{z_{1}, z_{2}\} = \Pi_{\Sigma}^{-1}(\sigma)$.

Two end points in $\partial H_{C}^{n}$ of the spine $\sigma$ are called vertices of $E$.

Next we shall characterize slices of spinal hypersurfaces. Let $S$ be a complex hypersurface and let $i_{S}$ be an inversion of $S$.

Theorem 6. Let $u_{1}, u_{2}$ be two points in $\partial H_{C}^{n}$. Then the complex hypersurface $S$ is a slice of $E$ with ends points $u_{1}, u_{2}$ if and only if $i_{S}$ interchanges $u_{1}$ and $u_{2}$.

Let $z_{1}, z_{2}$ be two points in $H_{C}^{n}$. Then the inversion $i_{S}$ interchanges $z_{1}$ and $z_{2}$ if and only if $S$ is the slice of $E\{z_{1}, z_{2}\}$.

Theorem 7. Let $H_{1}, H_{2}$ be two ultraparallel complex hyperplanes in $H_{C}^{n}$. Then there exists a unique spinal hypersurface $E$ having $H_{1}, H_{2}$ as slices.
Proof. Let $U_1, U_2 \in \mathbb{C}^{n+1}$ be polar vectors to complex hyperplanes $H_1$ and $H_2$, respectively. Set
$$U_{\pm} = U_1 - (\Phi(U_1, U_2) \pm \sqrt{\Phi(U_1, U_2)^2 - 1})U_2.$$ Let $E$ be a spinal hypersurface with the vertices $\pi(U_1), \pi(U_2)$. It is not difficult to show that $E$ is a spinal hypersurface with slices $H_1$ and $H_2$.

Definition 8. Two spinal hypersurfaces $E_1, E_2$ in $H^n_C$ are coequidistant if $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, where $\Sigma_i$ denotes the complex spine of $E_i$ \ $(i = 1, 2)$. We say that $E_1$ and $E_2$ are covertical if $\Sigma_1$ and $\Sigma_2$ are parallel. When $E_1$ and $E_2$ have a common slice, these are said to be cotranchal.

Theorem 9. If two spinal hypersurfaces $E_1, E_2$ are coequidistant or covertical, then they are not cotranchal.

We prepare a lemma.

Lemma 10 ([1, Theorem 7.16.2]). Let $\theta_1, \theta_2, \ldots, \theta_n$ be any ordered $n$-tuple with $0 < \theta_j < \pi$, $j = 1, 2, \ldots, n$. Then there exists a polygon $P$ with interior angles $\theta_1, \theta_2, \ldots, \theta_n$ occurring in this order around $\partial P$, if and only if $\theta_1 + \theta_2 + \ldots + \theta_n < (n - 2)\pi$.

Proof of Proposition 9. As both $\Sigma_1$ and $\Sigma_2$ are orthogonal to $S$, the triangle formed by $\Sigma_1, \Sigma_2, S$ has two right angles. This contradicts Lemma 10.

Theorem 11. If two spinal hypersurfaces $E_1$ and $E_2$ contain a common point $x$, then either $S_1$ and $S_2$ transversely intersect at $x$ or there exists a unique common slice $S$ including $x$.

Proof. The tangent spaces $T_xE_1$ and $T_xE_2$ at $x$ are real hyperplanes in the tangent space $T_xH^n_C$ of $H^n_C$ at $x$. Suppose that $E_1$ and $E_2$ do not transversely meet at $x$. Then $T_xE_1 = T_xE_2$. Noting that $T_xS_1$ and $T_xS_2$ are maximal complex submanifolds of $T_xE_1$ and $T_xE_2$, respectively, we see that $T_xS_1 = T_xS_2$. Since $S_1$ and $S_2$ are totally geodesic and $S_1 \cap S_2$ contains $x$, $S_1 = S_2$.

Corollary 12. If two spinal hypersurfaces $E_1$ and $E_2$ are coequidistant or covertical, then they meet transversely.

Proof. Suppose that $E_1$ and $E_2$ do not meet transversely. Theorem 11 implies that there exists a common slice $S$, which is orthogonal to complex spines $\Sigma_1$ and $\Sigma_2$. It follows from Lemma 10 that $\Sigma_1$ and $\Sigma_2$ are ultraparallel. This is a contradiction.

Theorem 13. If two spinal hypersurfaces $E_1$ and $E_2$ have two common slices, then $E_1 = E_2$.

Proof. Let $\sigma_i$ denote the spine of $E_i$ $(i = 1, 2)$ and let $l_1$ and $l_2$ be distinct geodesics orthogonal to both $\sigma_1$ and $\sigma_2$. Then the quadrilateral formed by $l_1, \sigma_1, l_2, \sigma_2$ has four right angles. Lemma 10 implies that $E_1 = E_2$. 
Let $q_1, q_2$ be points in $\partial H^n_C$ and let $c$ be a complex hyperplane. Let $Q_1, Q_2$ be null vectors representing $q_1, q_2$ and $C$ a positive vector polar to $c$. Then the complex number

$$\eta(q_1, q_2; c) = \frac{\Phi(Q_1, C)\Phi(C, Q_2)}{\Phi(Q_1, Q_2)\Phi(C, C)}$$

is independent of the choices of representative vectors. It is easy to prove that $\eta(q_1, q_2; c)$ is $U(1, n; \mathbb{C})$-invariant.

Using the invariant $\eta(q_1, q_2; c)$, we can quantitatively discuss the intersection of a spinal hypersurface with a complex geodesic $c$.

**Theorem 14.** Let $E$ be a spinal hypersurface with vertices $q_1, q_2$. Let $c$ be a complex hypersurface. Assume that $c$ is not a slice of $E$ and $q_1, q_2 \notin \partial c$. Then

$$E \cap c \neq \emptyset \iff Im(\eta)^2 + 2Re(\eta) < 1.$$ 

**Theorem 15.** The number of components of $E_1 \cap E_2$ is at most 2.

**Proof.** Let $E_1$ and $E_2$ be spinal hypersurfaces with vertices $q_1^+, q_1^-$ and $q_2^+, q_2^-$, respectively. Take $Q_2^+ \in \pi^{-1}(q_2^+), Q_2^- \in \pi^{-1}(q_2^-)$ such that $\Phi(Q_2^-, Q_2^+) = 2$. Then vectors polar to the slices of $E_2$ are given by

$$Q_2(t) = \frac{1}{2}(tQ_2^+ + t^{-1}Q_2^-)$$

for $0 < t < \infty$, and these vectors satisfy $\Phi(Q_2(t), Q_2(t)) = 1$. Set

$$\eta(t) = \eta(q_1^-, q_1^+; q_2(t)) = \frac{\Phi(Q_2^-, Q_2(t))\Phi(Q_2(t), Q_1^+)}{\Phi(Q_2(t), Q_2(t))\Phi(Q_1^-, Q_1^+)}.$$ 

The connected components of $E_1 \cap E_2$ correspond to the connected components of the set of all $t > 0$ such that $\eta(t) \in D$, where $D = \{\eta|Im(\eta)^2 + 2Re(\eta) < 1\}$. It follows that $\eta(t) \in D$ if and only if $t^{-4}f(t) < 0$, where $t^{-4}f(t^2) = 2Re(\eta(t)) + Im(\eta(t))^2 - 1$ and $f(s)$ is a quartic polynomial with positive leading term. Thus $\eta^{-1}(D) = \{t > 0|f(t^2) < 0\}$ has at most two components.

**Theorem 16.** If two spinal hypersurfaces $E_1$ and $E_2$ are coequidistant or covertical, then $E_1 \cap E_2$ is connected.

3. Let $w$ be a point in $H^n_C$. We may assume that $w$ is not a fixed point of any element $g$ except the identity. Let $H_g(w) = \{z \in H^n_C \mid d(z, w) < d(z, g(w))\}$. It is easy to see that $H_g(w) = \{z \in H^n_C \mid d(z, w) < d(z, g^{-1}(w))\}$.

**Definition 17.** The Dirichlet polyhedron $D(w)$ for $G$ with the center $w$ is defined by

$$D(w) = \bigcap_{g \in G-\{id\}} H_g(w).$$
where $H_g(w) = \{z \in H_C^n \mid d(z, w) < d(z, g(w))\}$.

It follows from Proposition 2 that $D(w)$ is not necessarily convex. In the same manner as in [1] we have

Proposition 18.

1. The Dirichlet polyhedron $D(w)$ is locally finite.
2. The Dirichlet polyhedron $D(w)$ is star-shaped about $w$.

Theorem 19. Let $w$ be a point in $H_C^n$. Let $G = \langle g \rangle$ be a cyclic group generated by $g$, where $g$ is strictly parabolic or hyperbolic. Then the Dirichlet polyhedron $D(w)$ for $G$ with the center $w$ has exactly two disjoint faces.

Proof. Set $H_k = H(w, g^k) = \{z \in H_C^n \mid d(z, w) < d(z, g^k(w))\}$. It follows that

$$z \in H_k \iff d(z, w) < d(z, g^k(w)) \iff \cosh^2 \left( \frac{d(z, w)}{2} \right) < \cosh^2 \left( \frac{d(z, g^k(w))}{2} \right).$$

Let $f_z(k) = \cosh^2 \left( \frac{d(z, g^k(w))}{2} \right)$. We regard $f_z(k)$ as a function of $k$. Therefore $H_k = \{z \in H_C^n \mid f_z(0) < f_z(k)\}$. We can complete our proof by using the following two lemmas.

Lemma 20. If $f_z(k)$ is a convex function with respect to $k$ for any $z \in H_C^n$, then $D(w) = H_1 \cap H_{-1}$ and $\partial H_1 \cap \partial H_{-1} = \emptyset$.

Proof. If $z \in H_1$, then $f_z(0) < f_z(1)$. Let $k > 1$. Since $f_z(k)$ is a convex function with respect to $k$,

$$kf_z(1) < (k - 1)f_z(0) + 1f_z(k) = kf_z(0) - f_z(0) + f_z(k).$$

Hence $f_z(k) < k\{f_z(0) - f_z(1)\} + f_z(k)$. Noting that $f_z(0) - f_z(1) < 0$, we see that $z \in H_k$, that is $H_1 \subset H_k$. Similarly we have $H_1 \subset H_k$ for $k < -1$. Thus

$$D(w) = \bigcap_{k=-\infty,k \neq 0}^{\infty} H_k = H_1 \cap H_{-1}$$

Next we shall show that $\overline{H}_1 \cap \overline{H}_{-1} = \emptyset$. Suppose that $z \in \overline{H}_1 \cap \overline{H}_{-1}$. Then $f_z(0) \geq f_z(1)$ and $f_z(0) \geq f_z(-1)$. Therefore we have

$$0 < \{f_z(1) - f_z(0)\} + \{f_z(-1) - f_z(0)\}.$$ 

This is a contradiction. Thus $\overline{H}_1 \cap \overline{H}_{-1} = \emptyset$.

Lemma 21. If $g$ is strictly parabolic or hyperbolic, then $f_z(k)$ is a convex function with respect to $k$ for any $z \in H_C^n$.
References


Department of Mechanical Engineering
Okayama University of Science
1-1 Ridai-cho Okayama 700 JAPAN