数理解析研究所講究録
第 882 巻 1994 年 33-40

Spinal hypersurfaces in complex hyperbolic space

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The purpose of this article is to introduce spinal hypersuraces in $H_{\mathbf{C}}^n$, which are studied and developed by Goldman and Mostow.

In Section 1 we state the properties of elements of $U(1,n;\mathbf{C})$. Section 2 is devoted to discussing spinal hypersurfaces on which results are due to Goldman and Mostow (cf. [3], [5]). In Section 3 we show Phillips' theorem on the Dirichlet polyhedra (cf. [6], [7]).

0. First we recall definitions and notation. Let $\mathbf{C}$ be the field of complex numbers. Let $V=V^{1,n}(\mathbf{C})$ $(n\geq 1)$ denote the vector space $\mathbf{C}^{n+1}$, together with the unitary structure defined by the Hermitian form

$$\Phi(Z,W)=-\overline{Z_0}W_0+\sum_{k=1}^{n}\overline{Z}_kW_k,$$

for $Z=(Z_0,Z_1,\ldots,Z_n), W=(W_0,W_1,\ldots,W_n)$ in V.

An automorphism g of V, that is a linear bijection such that $\Phi(g(Z),g(W))=\Phi(Z,W)$ for Z,W in V, will be called a unitary transformation. We denote the group of all unitary transformations by $U(1,n;\mathbf{C})$.

Let $V_0=\{Z\in V|\ \Phi(Z,Z)=0\}$ and $V_-=\{Z\in V|\ \Phi(Z,Z)<0\}$. It is clear that V_0 and V_- are invariant under $U(1,n;\mathbf{C})$.

Set $V^*=V_-\cup V_0-\{0\}$. Let $\pi:V^*\longrightarrow\pi(V^*)$ be the projection map defined by

$$\pi(Z_0,\ldots,Z_n)=(\frac{Z_1}{Z_0},\ldots,\frac{Z_n}{Z_0})=(z_1,\ldots,z_n).$$

Set $H_{\mathbf{C}}^n=\pi(V_-)$.

An element g of $U(1,n;\mathbf{C})$ operates in $\pi(V^*)$, leaving $\overline{H}_{\mathbf{C}}^n$ (the closure of $H_{\mathbf{C}}^n$ in $\pi(V^*)$) invariant. Since $H_{\mathbf{C}}^n$ is identified with the complex unit ball

$$B^n=\{z\in\mathbf{C}^n|\ \|z\|^2=\sum_{k=1}^{n}|z_k|^2<1\},$$

we can regard a unitary transformation as a transformation operating on B^n. Hence discrete subgroups of $U(1,n;\mathbf{C})$ is the generalizations of Fuchsian groups.

We can introduce the Bergman metric in B^n. This hyperbolic distance $d(z,w)$ for $z,w\in B^n$ is expressed by the use of the Hermitian form Φ as follows.

$$d(z,w)=2\cosh^{-1}\frac{|\Phi(Z,W)|}{[\Phi(Z,Z)\Phi(W,W)]^{1/2}},$$

where $Z\in\pi^{-1}(z), W\in\pi^{-1}(w)$. This does not depend on the choice of Z,W.

To discuss some properties of unitary transformations, it may be more convenient to use another matrix representation for $U(1,n;\mathbf{C})$. By changing the basis of V, we introduce the group $\tilde{U}(1,n;\mathbf{C})$ as follows. Let

$$D = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}.$$

Define $\tilde{U}(1,n;\mathbf{C})$ by $D^{-1}U(1,n;\mathbf{C})D$. We see that $\tilde{U}(1,n;\mathbf{C})$ is the automorphism group of the Hermitian form

$$\tilde{\Phi}(Z,W) = -(\overline{Z_0}W_1 + \overline{Z_1}W_0) + \sum_{k=2}^{n} \overline{Z_k}W_k.$$

We can regard D^{-1} as a mapping of complex unit ball B^n to

$$\widetilde{H^n} = \{z \in \mathbf{C}^n \mid \ Re(z_1) > \frac{1}{2}\sum_{k=2}^{n} |z_k|^2\},$$

which is called the Siegel domain.

1. The nontrivial elements fall into three general conjugacy types, depending on the number and location of their fixed points. Since each element acts on the closure of B^n, the Brouwer fixed point theorem implies that it has a fixed point. Let $g \neq id$. We call g elliptic if it has a fixed point in B^n and g parabolic if it has exactly one fixed point and this lies on the boundary. An element g will be called loxodromic if it has exactly two fixed points and they lie on the boundary. If g is conjugate to an element (different from the identity) in the identity component $U_0(1,1;\mathbf{R})$, it will be called hyperbolic. Hyperbolic elements are special kinds of loxodromic elements.

Now we state properties of each kind of elements. Let

$$U(1;\mathbf{C}) \times U(n;\mathbf{C}) = \left\{\begin{pmatrix} \alpha & 0 \\ 0 & A \end{pmatrix} \mid \ |\alpha| = 1, AA^* = I_n\right\}.$$

Proposition 1.

(1) Let g be an elliptic element. Then:

1) g is conjugate to an element of $U(1;\mathbf{C}) \times U(n;\mathbf{C})$.

2) g is semisimple with eigenvalues of absolute value 1.

(2) Let g be a loxodromic element. Then:

1) g is semisimple with $n-1$ eigenvalues of absolute value 1.

2) g leaves the geodesic joining the two fixed points, invariant. This is called the axis of g and denoted by A_g.

3) g moves every point z in A_g the same distance $T(g) = d(z, g(z))$. This $T(g)$ is called the translation length of g.

4) $T(g) = \min_{z \in B^n} d(z, g(z))$.

(3) Let g be a parabolic element. Then:

1) g is not semisimple.

2) All absolute values of eigenvalues are 1.

If a parabolic element g is unipotent (that is, all eigenvalues are 1), then g is called strictly parabolic. A standard form of strictly parabolic element of $\tilde{U}(1, n; \mathbf{C})$ is as follows.

$$\tilde{g} = \begin{pmatrix} 1 & 0 & 0 \\ s & 1 & \overline{a}^T \\ a & 0 & I_{n-1} \end{pmatrix},$$

where $Re(s) = \frac{1}{2}\|a\|^2$.

In particular, a conjugate element to

$$\begin{pmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & I_{n-1} \end{pmatrix}$$

$(s \neq 0, Re(s) = 0)$ is called translation. We note that strictly parabolic elements are not necessarily conjugate to translations, because their minimal polynomials are different.

2. Before defining a spinal hypersurface we recall the following important proposition.

Proposition 2 ([2, Proposition 2.5.1]). *A totally geodesic submanifold in $H^n_{\mathbf{C}}$ is equivalent to $H^m_{\mathbf{C}}$ or $H^m_{\mathbf{R}}$ $(m \leq n)$ under $U(1, n; \mathbf{C})$.*

Given two points $z_1, z_2 \in H^n_{\mathbf{C}}$, the equidistant surface $E\{z_1, z_2\}$ of z_1, z_2 is by definition

$$E\{z_1, z_2\} = \{z \in H^n_{\mathbf{C}} |\ d(z, z_1) = d(z, z_2)\}.$$

We call this $E\{z_1, z_2\}$ a spinal hypersurface of $\{z_1, z_2\}$. By Proposition 2, a spinal hypersurface is not a totally geodesic submanifold in $H^n_{\mathbf{C}}$, but a real analytic hypersurface in $H^n_{\mathbf{C}}$ diffeomorphic to $\mathbf{R}^{2n-1}$. Two points z_1, z_2 in $H^n_{\mathbf{C}}$ determine a unique complex geodesic Σ. Set $\sigma\{z_1, z_2\} = E\{z_1, z_2\} \cap \Sigma$, which is called a spine of E. It follows that the spine $\sigma\{z_1, z_2\}$ is a (real) geodesic in Σ.

Theorem 3.

$$E = \Pi_\Sigma^{-1}(\sigma) = \bigcup_{s \in \sigma} \Pi_\Sigma^{-1}(s),$$

where $\Pi_\Sigma : H^n_{\mathbf{C}} \to \Sigma$ *is orthogonal projection onto* Σ.

For our proof we need a lemma.

Lemma 4 ([3, Lemma III.2.3.1]). *Let* $L \subset H^n_{\mathbf{C}}$ *be a complex linear subspace with orthogonal projection* Π. *Then for all* $u \in H^n_{\mathbf{C}}$ *and* $s \in L$, *then the geodesic for* $\Pi(u)$ *to* u *and to* s *are orthogonal and span a totally real 2-space. Futhermore*

$$\cosh\left(\frac{d(u,s)}{2}\right) = \cosh\left(\frac{d(u,\Pi(u))}{2}\right)\cosh\left(\frac{d(\Pi(u),s)}{2}\right).$$

Proof of Theorem 3. It follows from Lemma 4 that for $i = 1, 2$

$$\cosh\left(\frac{d(z,z_i)}{2}\right) = \cosh\left(\frac{d(z,\Pi_\Sigma(z))}{2}\right)\cosh\left(\frac{d(\Pi_\Sigma(z),z_i)}{2}\right).$$

It is seen that

$$\begin{aligned} z \in E\{z_1,z_2\} &\Longleftrightarrow d(z,z_1) = d(z,z_2) \\ &\Longleftrightarrow d(z_1,\Pi_\Sigma(z)) = d(z_2,\Pi_\Sigma(z)) \\ &\Longleftrightarrow \Pi_\Sigma(z) \in \sigma\{z_1,z_2\}. \end{aligned}$$

The complex hyperplanes $\Pi_\Sigma^{-1}(s)$ for $s \in \sigma$ are called the slices of E. The slice of E is a maximal holomorphic submanifold of E.

Theorem 5. *There is a bijective correspondence between spinal hypersurfaces in* $H^n_{\mathbf{C}}$ *and (real) geodesics in* $H^n_{\mathbf{C}}$.

Proof. Let $E\{z_1,z_2\}$ be a spinal hypersurface. Then there is a unique complex geodesic Σ which is orthogonal to slices of E. By setting $\sigma = E\{z_1,z_2\} \cap \Sigma$ we have a unique geodesic σ corresponding to $E\{z_1,z_2\}$. Conversely, if a geodesic σ is given, then there is a unique complex geodesic Σ and a reflection P_σ of Σ with $P_\sigma(\sigma) = \sigma$. For any point z_1 set $z_2 = P_\sigma(z_1)$. Then it is easy to show that $E\{z_1,z_2\} = \Pi_\Sigma^{-1}(\sigma)$.

Two end points in $\partial H^n_{\mathbf{C}}$ of the spine σ are called vertices of E.
Next we shall chracterize slices of spinal hypersurfaces. Let S be a complex hypersurface and let ι_S be an inversion of S.

Theorem 6. *Let* u_1, u_2 *be two points in* $\partial H^n_{\mathbf{C}}$. *Then the complex hypersurface* S *is a slice of* E *with ends points* u_1, u_2 *if and only if* ι_S *interchanges* u_1 *and* u_2.
Let z_1, z_2 *be two points in* $H^n_{\mathbf{C}}$. *Then the inversion* ι_S *interchanges* z_1 *and* z_2 *if and only if* S *is the slice of* $E\{z_1,z_2\}$.

Theorem 7. *Let* H_1, H_2 *be two ultraparallel complex hyperplanes in* $H^n_{\mathbf{C}}$. *Then there exists a unique spinal hypersurface* E *having* H_1, H_2 *as slices.*

Proof. Let $U_1, U_2 \in \mathbf{C}^{n+1}$ be polar vectors to complex hyperplanes H_1 and H_2, respectively. Set

$$U_{\pm} = U_1 - (\Phi(U_1, U_2) \pm \sqrt{\Phi(U_1, U_2)^2 - 1})U_2.$$

Let E be a spinal hypersurface with the vertices $\pi(U_1), \pi(U_2)$. It is not difficult to show that E is a spinal hypersurface with slices H_1 and H_2.

Definition 8. Two spinal hypersurfaces E_1, E_2 in $H^n_{\mathbf{C}}$ are coequidistant if $\Sigma_1 \cap \Sigma_2 \neq \emptyset$, where Σ_i denotes the complex spine of E_i $(i = 1, 2)$. We say that E_1 and E_2 are covertical if Σ_1 and Σ_2 are parallel. When E_1 and E_2 have a common slice, these are said to be cotranchal.

Theorem 9. *If two spinal hypersurfaces E_1, E_2 are coequidistant or covertical, then they are not cotranchal.*

We prepare a lemma.

Lemma 10 ([1, Theorem 7.16.2]). *Let $\theta_1, \theta_2, ..., \theta_n$ be any ordered n-tuple with $0 \leq \theta_j < \pi$, j=1, 2, ..., n. Then there exists a polygon P with interior angles $\theta_1, \theta_2, ..., \theta_n$ occurring in this order around ∂P, if and only if $\theta_1 + \theta_2 + ... + \theta_n < (n-2)\pi$.*

Proof of Proposition 9. As both Σ_1 and Σ_2 are orthogonal to S, the triangle formed by Σ_1, Σ_2, S has two right angles. This contradicts Lemma 10.

Theorem 11. *If two spinal hypersurfaces E_1 and E_2 contain a common point x, then either S_1 and S_2 transversely intersect at x or there exists a unique common slice S including x.*

Proof. The tangent spaces T_xE_1 and T_xE_2 at x are real hyperplanes in the tangent space $T_xH^n_{\mathbf{C}}$ of $H^n_{\mathbf{C}}$ at x. Suppose that E_1 and E_2 do not transversely meet at x. Then $T_xE_1 = T_xE_2$. Noting that T_xS_1 and T_xS_2 are maximal complex submanifolds of T_xE_1 and T_xE_2, respectively, we see that $T_xS_1 = T_xS_2$. Since S_1 and S_2 are totally geodesic and $S_1 \cap S_2$ contains x, $S_1 = S_2$.

Corollary 12. *If two spinal hypersurfaces E_1 and E_2 are coequidistant or covertical, then they meet transversely.*

Proof. Suppose that E_1 and E_2 do not meet transversely. Theorem11 implies that there exists a common slice S, which is orthogonal to complex spines Σ_1 and Σ_2. It follows from Lemma 10 that Σ_1 and Σ_2 are ultraparallel. This is a contradiction.

Theorem 13. *If two spinal hypersurfaces E_1 and E_2 have two common slices, then $E_1 = E_2$.*

Proof. Let σ_i denote the spine of E_i $(i = 1, 2)$ and let l_1 and l_2 be distinct geodesics orthogonal to both σ_1 and σ_2. Then the quadrilateral formed by $l_1, \sigma_1, l_2, \sigma_2$ has four right angles. Lemma 10 implies that $E_1 = E_2$.

Let q_1, q_2 be points in $\partial H^n_{\mathbf{C}}$ and let c be a complex hyperplane. Let Q_1, Q_2 be null vectors representing q_1, q_2 and C a positive vector polar to c. Then the complex number

$$\eta(q_1, q_2; c) = \frac{\Phi(Q_1, C)\Phi(C, Q_2)}{\Phi(Q_1, Q_2)\Phi(C, C)},$$

is independent of the choices of representative vectors. It is easy to prove that $\eta(q_1, q_2; c)$ is $U(1, n; \mathbf{C})$-invariant.

Using the invariant $\eta(q_1, q_2; c)$, we can quantitatively discuss the intersection of a spinal hypersurface with a complex geodesic c.

Theorem 14. *Let E be a spinal hypersurface with vertices q_1, q_2. Let c be a complex hypersurface. Assume that c is not a slice of E and $q_1, q_2 \notin \partial c$. Then*

$$E \cap c \neq \emptyset \Longleftrightarrow Im(\eta)^2 + 2Re(\eta) < 1.$$

Theorem 15. *The number of components of $E_1 \cap E_2$ is at most 2.*

Proof. Let E_1 and E_2 be spinal hypersurfaces with vertices q_1^+, q_1^- and q_2^+, q_2^-, respectively. Take $Q_2^+ \in \pi^{-1}(q_2^+), Q_2^- \in \pi^{-1}(q_2^-)$ such that $\Phi(Q_2^-, Q_2^+) = 2$. Then vectors polar to the slices of E_2 are given by

$$Q_2(t) = \frac{1}{2}(tQ_2^+ + t^{-1}Q_2^-)$$

for $0 < t < \infty$, and these vectors satisfy $\Phi(Q_2(t), Q_2(t)) = 1$. Set

$$\eta(t) = \eta(q_1^-, q_1^+; q_2(t)) = \frac{\Phi(Q_1^-, Q_2(t))\Phi(Q_2(t), Q_1^+)}{\Phi(Q_2(t), Q_2(t))\Phi(Q_1^-, Q_1^+)}.$$

The connected components of $E_1 \cap E_2$ correspond to the connected components of the set of all $t > 0$ such that $\eta(t) \in D$, where $D = \{\eta | Im(\eta)^2 + 2Re(\eta) < 1\}$. It follows that $\eta(t) \in D$ if and only if $t^{-4}f(t) < 0$, where $t^{-4}f(t^2) = 2Re(\eta(t)) + Im(\eta(t))^2 - 1$ and $f(s)$ is a quartic polynomial with positive leading term. Thus $\eta^{-1}(D) = \{t > 0 | f(t^2) < 0\}$ has at most two components.

Theorem 16. *If two spinal hypersurfaces E_1 and E_2 are coequidistant or covertical, then $E_1 \cap E_2$ is connected.*

3. Let w be a point in $H^n_{\mathbf{C}}$. We may assume that w is not a fixed point of any element g except the identity. Let $H_g(w) = \{z \in H^n_{\mathbf{C}} \mid d(z, w) < d(z, g(w))\}$. It is easy to see that $H_g(w) = \{z \in H^n_{\mathbf{C}} \mid d(z, w) < d(z, g^{-1}(w))\}$.

Definition 17. The Dirichlet polyhedron $D(w)$ for G with the center w is defined by

$$D(w) = \bigcap_{g \in G - \{id\}} H_g(w),$$

where $H_g(w) = \{z \in H^n_{\mathbf{C}} |\ d(z,w) < d(z, g(w))\}$.

It follows from Proposition 2 that $D(w)$ is not necessarily convex. In the same manner as in [1] we have

Proposition 18.
(1) The Dirichlet polyhedron $D(w)$ is locally finite.
(2) The Dirichlet polyhedron $D(w)$ is star-shaped about w.

Theorem 19. *Let w be a point in $H^n_{\mathbf{C}}$. Let $G = < g >$ be a cyclic group generated by g, where g is strictly parabolic or hyperbolic. Then the Dirichlet polyhedron $D(w)$ for G with the center w has exactly two disjoint faces.*

Proof. Set $H_k = H(w, g^k) = \{z \in H^n_{\mathbf{C}} \mid d(z,w) < d(z, g^k(w))\}$. It follows that

$$\begin{aligned} z \in H_k &\iff d(z,w) < d(z, g^k(w)) \\ &\iff \cosh^2\left(\frac{d(z,w)}{2}\right) < \cosh^2\left(\frac{d(z,g^k(w))}{2}\right). \end{aligned}$$

Let $f_z(k) = \cosh^2\left(\frac{d(z,g^k(w))}{2}\right)$. We regard $f_z(k)$ as a function of k. Therefore $H_k = \{z \in H^n_{\mathbf{C}} \mid f_z(0) < f_z(k)\}$. We can complete our proof by using the following two lemmas.

Lemma 20. *If $f_z(k)$ is a convex function with respect to k for any $z \in H^n_{\mathbf{C}}$, then $D(w) = H_1 \cap H_{-1}$ and $\partial H_1 \cap \partial H_{-1} = \emptyset$.*

Proof. If $z \in H_1$, then $f_z(0) < f_z(1)$. Let $k > 1$. Since $f_z(k)$ is a convex function with respect to k,

$$kf_z(1) < (k-1)f_z(0) + 1f_z(k) = kf_z(0) - f_z(0) + f_z(k).$$

Hence $f_z(0) < k\{f_z(0) - f_z(1)\} + f_z(k)$. Noting that $f_z(0) - f_z(1) < 0$, we see that $z \in H_k$, that is $H_1 \subset H_k$. Similarily we have $H_1 \subset H_k$ for $k < -1$. Thus

$$D(w) = \bigcap_{k=-\infty, k\neq 0}^{\infty} H_k = H_1 \cap H_{-1}$$

Next we shall show that $\overline{H}_1 \cap \overline{H}_{-1} = \emptyset$. Suppose that $z \in \overline{H}_1 \cap \overline{H}_{-1}$. Then $f_z(0) \geq f_z(1)$ and $f_z(0) \geq f_z(-1)$. Therefore we have

$$0 < \{f_z(1) - f_z(0)\} + \{f_z(-1) - f_z(0)\}.$$

This is a contradiction. Thus $\overline{H}_1 \cap \overline{H}_{-1} = \emptyset$.

Lemma 21. *If g is strictly parabolic or hyperbolic, then $f_z(k)$ is a convex function with respect to k for any $z \in H^n_{\mathbf{C}}$.*

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