数理解析研究所講究録
第 882 巻 1994 年 26-32

Margulis Decomposition and Translation Lengths of Discrete Möbius Groups

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1 Introduction

For any integer $n \geq 2$, let R^n denote the n-dimensional Euclidean space and $\overline{R^n} = R^n \cup \{\infty\}$ its one-point compactification. Any point $x \in R^n$ is represented as $x = (x_1, \ldots, x_n)$ and when matrices act on x, x is treated as a column vector. The subspace $H^n = \{x \in R^n | x_n > 0\}$ of R^n with metric $\rho(\ ,\ )$ induced by the line element $ds^2 = |dx|^2/dx_n^2$ is a model of the n -dimensional hyperbolic space and we call H^n the n-dimensional upper half-space.

A Möbius transformation of $\overline{R^n}$ is a finite product of reflections in $(n-1)$-dimensional spheres or hypersurfaces. A group of Möbius transformation of $\overline{R^n}$ is denoted by $M(\overline{R^n})$ and call the (full) Möbius group. Möbius transformtions are classified by their conjugacy class in $M(\overline{R^n})$. The canonical forms are as follows. An element in $M(\overline{R^n})$ is said to be loxodromic if it is conjugate to a transformation of the form

$$\gamma(x) = \lambda T x$$

where $\lambda > 0, \lambda \neq 1$, and $T \in O(n)$, the group of $n \times n$ -orthogonal matrices, and parabolic if it is conjugate to the transformation of the form

$$\gamma(x) = Tx + a$$

where $T \in O(n), a \in R^n$ and $Ta = a \neq 0$. A non-trivial element is said to be elliptic if it is neither loxodromic nor parabolic.

For $\gamma \in M(\overline{R^n})$ we denote the Jacobian matrix of γ at $x \in R^n$ by $\gamma'(x)$. Then chain rule implies that $\gamma'(x) = \nu U x$ with $\nu > 0, U \in O(n)$. We call the positive number ν the linear magnification of γ at x and denote by $|\gamma'(x)|$. If $\gamma \in M(\overline{R^n})$ does not fix ∞, the set $I(\gamma) = \{x \in R^n |\ |\gamma'(x)| = 1\}$ becomes an $(n-1)$-sphere centered at $\gamma^{-1}(\infty)$. We call $I(\gamma)$ the isometric sphere of γ. The action of γ on $\overline{R^n}$ is the composition of an inversion in $I(\gamma)$, followed by a Euclidean isometry. For $x \in \overline{R^n}$ denote x^* by the image of the reflection of x in the unit sphere centered at the origin. Let $\gamma \in M(\overline{R^n})$ be an arbitrary element which does not fix ∞. Then γ can be represented uniquely in the form

$$\gamma(x) = \lambda T(x-a)^* + b$$

where $\lambda > 0, T \in 0(n)$ and $a, b \in R^n$. In this expression $\lambda^{1/2}$ is the radius of $I(\gamma)$ and $a = \gamma^{-1}(\infty)$ (resp. $b = \gamma(\infty)$) is the center of $I(\gamma)$ (resp. $I(\gamma^{-1})$). If $\gamma \in M(\overline{R^n})$ fixes ∞, then γ can be written as a similarity in the form

$$\gamma(x) = \lambda T x + a$$

where $\lambda > 0, T \in O(n)$ and $a \in R^n$.

Let denote by $M(H^n)$ the subgroup of $M(\overline{R^n})$ consisting of elements which keep H^n invariant. Then $M(H^n)$ is the full group of hyperbolic isometries of H^n. For any subgroup Γ of $M(H^n)$, Γ is discrete if and only if Γ acts discontinuously on H^n. Also Γ acts on $\partial H^n = \overline{R^{n-1}}$ as a group of conformal automorphisms. For a discrete subgroup Γ of $M(H^n)$, the region of discontinuity $\Omega(\Gamma)$ of Γ is the subset of $\overline{R^{n-1}}$ on which Γ acts discontinuously. The limit set $\Lambda(\Gamma)$ of Γ is the complement of $\Omega(\Gamma)$ in $\overline{R^{n-1}}$. A discrete subgroup Γ of $M(H^n)$ whose limit set consists of at most two points is called elementary. If Γ is not elementary, $\Lambda(\Gamma)$ is a perfect, uncountable set.

Let γ be a loxodromic transformation. Then γ has exactly two fixed points on $\overline{R^{n-1}}$. The geodesic A_γ joining these two points is called the axis of γ. The axis A_γ is kept invariant under the action of γ. For a loxodromic transformation $\gamma \in M(H^n)$ we set

$$l_\gamma = \inf_{x \in H^n} \rho(x, \gamma(x)).$$

We know that l_γ is positive and attained at any point of A_γ. This constant l_γ is called the translation length of γ. We denote $L(\Gamma)$ by the set of translation lengths of all loxodromic transformations of Γ.

For a discrete subgroup Γ of $M(H^n)$, let E_Γ be the set of all geodesics in H^n whose end points belong to $\Lambda(\Gamma)$. The convex hull $Hull(\Lambda(\Gamma))$ is the intersection of all hyperbolically convex sets in H^n which contain E_Γ. Let $N_\Gamma = H^n/\Gamma$ be a quotient orbifold for Γ and $M_\Gamma = (H^n \cup \Omega(\Gamma))/\Gamma$ its closure. The quotient $C_\Gamma = Hull(\Lambda(\Gamma))/\Gamma$ is a subset of N_Γ and is called the Nielsen convex core for Γ.

2 The Margulis decomposition for quotient orbifolds

For a discrete subgroup Γ of $M(H^n)$, let $\tilde{\Gamma}$ be the subset of Γ consisting of all elements of infinite orders. For $\epsilon > 0$ and $x \in H^n$, we define

$$I_\epsilon(x) = \{\gamma \in \tilde{\Gamma} \mid \rho(x, \gamma(x)) < \epsilon\}$$

and

$$\Gamma_\epsilon(x) = \langle \Gamma \cap I_\epsilon(x) \rangle.$$

For our argument, the following result is essential (see [1], [3]).

Proposition 1.(Margulis lemma) *For each n, there exists a positive number $\epsilon(n)$ such that for any discrete subgroup Γ of $M(H^n)$, $x \in H^n$ and $\epsilon \leq \epsilon(n)$, $\Gamma_\epsilon(x)$ is a finite extension of an abelian group.*

We call $\epsilon(n)$ the Margulis constant in dimension n.

For any $\epsilon \in (0, \epsilon(n)]$ and a discrete subgroup Γ of $M(H^n)$, we write

$$R_\epsilon(\Gamma) = \{x \in H^n \mid \rho(x, \gamma(x)) < \epsilon \text{ for some } \gamma \in \tilde{\Gamma}\}.$$

We can easily see thet $R_\epsilon(\Gamma)$ is a Γ-invariant set of H^n. The quotient $R_\epsilon(\Gamma)/\Gamma \subset N_\Gamma$ is called the thin part of N_Γ and is denoted by $N_{(0,\epsilon)}$. The complement of $N_{(0,\epsilon)}$ in N_Γ is denoted by $N_{[\epsilon,\infty)}$ and is called the thick part of N_Γ. The decomposition

$$N_\Gamma = N_{(0,\epsilon)} \cup N_{[\epsilon,\infty)}$$

is called the Margulis decomposition for N_Γ.

A discrete subgroup Γ of $M(H^n)$ is said to be geometrically finite if there exists $\epsilon \in (0, \epsilon(n)]$ so that $C_\Gamma \cap N_{[\epsilon,\infty)}$ is compact.

Let Γ' be a subgroup of Γ. A set $X \subset H^n$ is precisely invariant under Γ' in Γ if $\gamma(X) = X$ for any $\gamma \in \Gamma$ and $\gamma(X) \cap X = \emptyset$ for any $\gamma \in \Gamma - \Gamma'$. Let $\Lambda_P(\Gamma)$ denote the set of parabolic fixed points of Γ. For $p \in \Lambda_P(\Gamma)$, we write $\Gamma_p = \{\gamma \in \Gamma | \gamma(p) = p\}$ and call the stabilizer of p.

The following is an immediate consequence of Margulis lemma.

Proposition 2. ([2], [3]) *Let Γ be a discrete subgroup of $M(H^n)$. Then there exists a constant $\epsilon \in (0, \epsilon(n)]$ so that the following holds:*

(1) *For any $p \in \Lambda_P(\Gamma)$ there exists an open region T_p in H^n which contains a component of $R_\epsilon(\Gamma)$ so that T_p is precisely invariant under Γ_p in Γ.*

(2) *For any distinct points $p, q \in \Lambda_P(\Gamma), T_p$ and T_q are mutually disjoint to each other.*

We say that $T = \bigcup_{p \in \Lambda_P(\Gamma)} T_p$ is a strictly invariant system of parabolic neighborhoods for Γ.

A parabolic fixed point p of Γ is called a bounded parabolic fixed point if there exists a compact subset of $\overline{R^{n-1}} - \{p\}$ whose translates by Γ_p cover $\Lambda(\Gamma) - \{p\}$. We say that a limit point y of Γ is a conical limit point of Γ if for some geodesic ray I in H^n ending at y, there is a compact set K in H^n so that $\{\gamma \in \Gamma | \gamma(I) \cap K \neq \emptyset\}$ is an infinite set.

The following is well known.

PROPOSITION 3. ([3], [4]) *Let* Γ *be a discrete subgroup of* $M(H^n)$. *Then the following statements are equivalent.*

(1) Γ *is geometrically finite.*

(2) $\Lambda(\Gamma)$ *consists of conical limit points or bounded parabolic fixed points.*

(3) *There exist* $p_1, \ldots, p_r \in \Lambda_P(\Gamma)$ *with respective horoball neighborhoods* $B_1, \ldots, B_r$ *such that the set* $B = \bigcup_{\gamma \in \Gamma} \gamma(B_1 \cup \ldots \cup B_r)$ *forms a strictly invariant system of parabolic neighborhoods for* Γ *and* $(Hull(\Lambda(\Gamma)) - B)/\Gamma$ *is compact.*

3 Translation lengths of discrete Möbius groups

Let Γ be a discrete subgroup of $M(H^n)$ and $\epsilon \in (0, \epsilon(n)]$, be chosen. We define

$$N_{\epsilon,1} = (R_\epsilon(\Gamma) \cap T)/\Gamma,$$

$$N_{\epsilon,2} = N_{(0,\epsilon)} - N_{\epsilon,1}$$

and call $N_{\epsilon,1}$ (resp. $N_{\epsilon,2}$) the parabolic part (resp. the non-parabolic part) of $N_{(0,\epsilon)}$. If Γ is a discrete subgroup of $M(H^3)$ consisting of orientation-preserving transformations (i,e Γ is a Kleinian group), then each component of $N_{(0,\epsilon)}$ is homeomorphic to either $\{D - \{0\}\} \times S^1, \{D - \{0\}\} \times (0,1)$ or $D \times S^1$, where D is a unit disk.

To investigate the structure of $N_{(0,\epsilon)}$, we consider $L(\Gamma)$, the set of translation lengths of loxodromic elements of Γ. First we deal with the geometrically finite case.

LEMMA 4. *Let* Γ *be a geometrically finite subgroup of* $M(H^n)$. *Then* $L(\Gamma)$ *is a discrete subset of* $[0, \infty)$.

PROOF. Assume the contrary. Then there exist a sequence $\{\gamma_m\}$ of distinct loxodromic elements of Γ and a constant $\alpha \geq 0$ such that $l_m \to \alpha \ \ (m \to \infty)$, where l_m is a translation length of γ_m.

Let denote by D_a a Dirichlet region for Γ centered at $a \in H^n$, with $\Gamma_a = \{id\}$. For any m, choose a point $x_m \in A_m$, the axis of γ_m. Then, for every m, there exists $g_m \in \Gamma$ such that $g_m(x_m) = y_m \in cl(D_a) \cap H^n$, where $cl(D_a)$ is the closure of D_a.

Suppose that $\{y_m\}$ has an accumulation point $y_0 \in cl(D_a) \cap H^n$. Then there exist a subsequence of $\{\gamma_m\}$ (use the same notation) and $\delta > 0$ so that $\{x \in H^n | \rho(y_0, x) < \delta\} \cap \tilde{A}_m \neq \emptyset$ for every m, where $\tilde{A}_m$ is the axis of $g_m \circ \gamma_m \circ g_m^{-1}$. It follows that there exists a positive integer m_0 with $(g_m \circ \gamma_m \circ g_m^{-1})(y_0) \in \{x \in H^n | \rho(y_0, x) < \delta + 2\alpha\} \subset H^n$ for $m \geq m_0$. Then there exist a subsequence of $\{\gamma_m\}$ (again use the same notation) and a point $y \in \{x \in H^n | \rho(y_0, x) \leq \delta + 2\alpha\}$ such that $(g_m \circ \gamma_m \circ g_m^{-1})(y_0) \to y \ \ (m \to \infty)$. This means $y \in H^n \cap \Lambda(\Gamma) \neq \emptyset$. It is a contradiction. So there exist a subsequence of $\{y_m\}$ (use the same notation) and a point $p \in \partial D_a \cap \overline{R^{n-1}}$ such that $y_m \to p \ \ (m \to \infty)$.

It is well known that conical limit points can not be contained in the boundary of any Dirichlet region. Since Γ is geometrically finite, we conclude that p is a bounded parabolic fixed point and there exists a horoball neighborhood B_p which is precisely invariant under Γ_p in Γ.

Note that $y_m \in \tilde{A}_m$ and the translation length is invariant under the conjugation in $M(H^n)$. So there exists a positive integer m_1 such that $\{x \in |\rho(x, y_{m_1}) < 2\alpha\} \subset B_p$. Hence we deduce that $(g_{m_1} \circ \gamma_{m_1} \circ g_{m_1}^{-1})(y_{m_1}) \in (g_{m_1} \circ \gamma_{m_1} \circ g_{m_1}^{-1})(B_p) \cap B_p \neq \emptyset$. It contradicts the fact that B_p is precisely invariant under Γ_p in Γ. Therefore we establish this lemma.

q.e.d.

If Γ is geometrically finite, then Lemma 4 yields that the number $l_\Gamma = \min L(\Gamma)$ is positive. Hence we have the following :

THEOREM 5. *Let Γ be a geometrically finite subgroup of $M(H^n)$. Then the non-parabolic part $N_{\epsilon,2}$ of $N_{(0,\epsilon)}$ is empty for any $\epsilon \in (0, \min(l_\Gamma, \epsilon(n)))$.*

PROOF. Choose a positive number with $\epsilon \in (0, \min(l_\Gamma, \epsilon(n)))$. Take an arbitrary point $x \in R_\epsilon(\Gamma)$. Then, from the definition of $R_\epsilon(\Gamma)$, there exists $\gamma \in \tilde{\Gamma}$ such that $\rho(x, \gamma(x)) < \epsilon$.

If γ is loxodromic, then $\rho(x, \gamma(x)) \geq l(\gamma) \geq l_\Gamma > \epsilon$ and it is a contradiction. So γ is parabolic and we have $x \in R_\epsilon(\Gamma) \cap T$. It implies $N_{\epsilon,1} = N_{(0,\epsilon)}$ and $N_{\epsilon,2} = \emptyset$.

q.e.d.

Next we consider the general case. The following lemma is essential for our discussion.

LEMMA 6. *For any $\alpha \geq 0$ there exist a non-elementary, discrete subgroup Γ of $M(H^n)$ and a sequence $\{\gamma_m\}$ of loxodromic elements of Γ such that $l_m \searrow \alpha \ \ (m \to \infty)$.*

PROOF. Let a sequence $\{r_m\}$ of positive numbers, with $r_m \searrow e^\alpha \ \ (m \to \infty)$, be given. We take hemispheres $\sigma, \sigma_1, \sigma_2, \ldots$ in H^n as the following:

$$\sigma = \{x \in H^n \mid \ |x| = 1\},$$

$$\sigma_m = \{x \in H^n \mid \ |x| = r_m\} \ \ (m = 1, 2, \ldots).$$

For each m we define a Möbius transformation g_m as $g_m = r_m x$. It can be easily seen that g_m is loxodromic, $g_m(\sigma) = \sigma_m$ and λ_m, the translation length of g_m, is equal to $\log r_m$.

Let $\{p_m\}$ be a sequence of points in $\overline{R^{n-1}}(= \partial H^n)$ with $r_{m+1} < |p_m| < r_m$ $(m = 1, 2, \ldots)$. We can take a sequence $\{R_m\}$ of positive numbers which satisfy

$$r_{m+1} + R_m < |p_m| < r_m - R_m \quad (m = 1, 2, \ldots).$$

Here we set

$$\Sigma_m = \{x \in H^n | \; |x - p_m| = R_m\}.$$

Then $\{\Sigma_m\}$ is a sequence of hemispheres in H^n which are mutually disjoint to each other. Let denote by ψ_m the reflection in Σ_m and set $\psi_m(\sigma) = S_m, \psi_m(\sigma_m) = S'_m$ $(m = 1, 2, \ldots)$. We can easily see that $S_m, S'_m \subset Int(\Sigma_m)$ and $Int(S_m) \cap Int(S'_m) = \emptyset$ $(m = 1, 2, \ldots)$.

We put $\gamma_m = \psi_m \circ g_m \circ \psi_m^{-1}$. Then we have that γ_m is loxodromic and the translation length of γ_m is equal to $\log r_m$. Let Γ be the group generated by $\gamma_1, \gamma_2, \ldots$. We show that Γ is a non-elementary, free, discrete subgroup of $M(H^n)$. Since Γ contains two loxodromic transformations which do not have common fixed points, Γ is a non-elementary group. Let γ be an element of Γ which is represented as a reduced word $\gamma = \gamma_{m_k} \circ \cdots \circ \gamma_{m_1}, \gamma_{m_i} \in \{\gamma_1^{\pm 1}, \gamma_2^{\pm 1}, \ldots\}$ $(i = 1, \ldots, k)$. Note that hemispheres $S_1, S'_1, S_2, S'_2, \ldots$ are mutually disjoint to each other. Take a point $x_0 = (x_1, \ldots, x_n) \in H^n$ with x_n sufficiently large. We may suppose that $B(x_0, \delta) = \{x \in H^n | \rho(x, x_0) < \delta\} \subset \bigcap_{i=1}^{\infty}(Ext(S_i) \cup Ext(S'_i))$. We can easily see $\gamma_{m_1}(B(x_0, \delta)) \subset Int(S_l)$ or $Int(S'_l)$ for some $l = 1, 2, \ldots$. and $\gamma_{m_1}(B(x_0, \delta)) \cap B(x_0, \delta) = \emptyset$. Repeat this procedure. Then we obtain $\gamma(B(x_0, \delta)) \subset Int(S_j)$ or $Int(S'_j)$ for some $j = 1, 2, \ldots$. It follows that $\gamma(B(x_0, \delta)) \cap B(x_0, \delta) = \emptyset$ and $\gamma \neq id$. Hence we have that Γ is free and discrete. Furthermore $\{\gamma_m\}$ is the sequence of loxodromic elements and $l_m = \log r_m \searrow \alpha$ $(m \to \infty)$. It completes the proof of this lemma.

q.e.d.

By using Lemma 6, we have the following result immediately.

THEOREM7. *For any positive integer $n \geq 2$ there exists a non-elementary, discrete subgroup Γ of $M(H^n)$ such that $N_{\epsilon,2} \neq \emptyset$ for any $\epsilon > 0$.*

Next we apply Lemma 6 to geometrically finite groups. Let $\epsilon \in (0, \epsilon(n)]$ be sufficiently small. Then, by using Lemma 6, we can take loxodromic transformations $\gamma_1, \ldots, \gamma_r$, such that $l_k < \epsilon$ $(k = 1, 2, \ldots, r)$ and $\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle$ is a non-elementary, geometrically finite subgroup of $M(H^n)$. Hence we have the following:

THEOREM 8. *For any positive integer* $n \geq 2$ *and any* $\epsilon > 0$, *there exists a geometrically finite subgroup* Γ *of* $M(H^n)$ *such that* $N_{\epsilon,2} \neq \emptyset$.

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