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A lower bound for the volume of Dirichlet fundamental polyhedrons

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1. Introduction. This paper is concerned with a lower bound for the volume of Dirichlet fundamental polyhedrons for Kleinian groups.

Let $H^3=\{(x,y,t)\in R^3;t>0\}$ with metric $d(\cdot,\cdot)$ induced by the line element $ds^2=(dx^2+dy^2+dt^2)/t^2$. Let f be an element of PSL(2,C) and identifies with a Möbius transformation of $\hat{C}=C\cup\{\infty\}$ onto itself,such as

$$f(z)=\frac{az+b}{cz+d}, \quad ad-bc=1.$$

Its action on the Riemann sphere $\hat{C}$ can be naturally extended to H^3.

Next for each f and g in PSL(2,C), we let [f,g] denote the commutator $fgf^{-1}g^{-1}$. We define the two complex numbers

(1) $\beta(f) = tr^2(f)-4, \quad \gamma(f,g) = tr([f,g])-2$,

for the two generator subgroup ⟨f,g⟩.

The following inequality [4] gives an important necessary condition for a two generator group ⟨f,g⟩ to be nonelementary and discrete.

Proposition 1([J1]). If ⟨f,g⟩ is nonelementary and discrete, then

(2) $|\gamma(f,g)|+|\beta(f)|\geqq 1$ and $|\gamma(f,g)|+|\beta(g)|\geqq 1$.

The above inequality is called Jφrgensen's inequality.

Proposition 2([R]). *If* $\langle f,g\rangle$ *is a nonelementary discrete Fuchsian subgroup of* $PSL(2,R)$, *then*

(3) $$|\gamma(f,g)|\geqq 2-2\cos(\pi/7).$$

On the other hand ,if $\langle f,g\rangle$ is not Fuchsian,it is known that there is a nonelementary discrete group $\langle f,g\rangle$ with an arbitrary small number $|\gamma(f,g)|$ ([J2]). But if we have some restrictive conditions for elements of generators, one can obtain the lower bounds for $|\gamma(f,g)|$. If we have a restriction $\beta(f)=\beta(g)$ for two-generator group $\langle f,g\rangle$. Jørgensen proved in [J3] that

(4) $$|\gamma(f,g)|\geqq 1/8.$$

The first purpose of this paper is to show the existence of a collar from the consideration of a lower bound of $|\gamma(f,g)|$. This gives an improvement of a previous paper [2].

2 Preliminary. We collect some elementary results.

Lemma 1. *If* f *and* g *are in* $PSL(2,C)$ *with* $\gamma(f,g)=\gamma$ *and* $\beta(f)=\beta$, *then*

(5) $$\gamma(f,gfg^{-1})=\gamma(\gamma-\beta) \text{ and } \beta([f,g])=\gamma(\gamma+4).$$

Lemma 2. *Let* G *be an elementary discrete subgroup of* $PSL(2,C)$. *If* $f,g\in G$ *with* $\gamma(f,g)\neq 0$, $\beta(f)=\beta(g)\neq -4$, *then*

(6) $$|\gamma(f,g)|\geqq 2-2\cos(\pi/5)\ (=0.3819\cdots).$$

It is easily seen that $\gamma(f,g)\neq 0$ if and only if $\mathrm{fix}(f)\cap \mathrm{fix}(g)=\phi$ where $\mathrm{fix}(f)$ denotes the fixed point set in $\hat{C}$. Now we need to show the followings.

Lemma 3. If $\langle f,g\rangle$ is discrete with $\gamma(f,g)=\beta(f)\neq 0$, then either f is elliptic of order 2,3,4, or 6 or g is elliptic of order 2.

3. A lower bound for the commutator. We show here a lower bound for the commutator of two generator groups.

Next we show Theorem 1 which is applicable to the collar lemma.

Theorem 1. Let $\langle f,g\rangle$ be a discrete subgroup of PSL(2,C) with $\gamma(f,g)\neq 0$, $\beta(g)\neq -4$ and

(7) $$0 < |\beta(f)| < 2\{2\cos(2\pi/7)-2\cos(\pi/7)+1\},\text{ then}$$

$$|\gamma(f,g)| > 2-2\cos(\pi/7).$$

Proof. Put $\gamma=\gamma(f,g)$ and $\beta=\beta(f)$. Suppose $|\gamma(f,g)|\leqq 2-2\cos(\pi/7)$. If $\gamma=\beta$, then Lemma 3 yields $|\gamma|=|\beta|\geqq 1$ or $\beta(g)=-4$ which contradict the assumption of Theorem 1. Therefore $\gamma\neq\beta$ and we have that $\langle [f,g],f[f,g]f^{-1}\rangle$ is nonelementary discrete subgroup of $\langle f,g\rangle$ by Lemma 2 with

$0<|\gamma([f,g],f[f,g]f^{-1})|=|\gamma^2(\gamma-\beta)(\beta+4)|<0.3$ and $|\beta([f,g])|=|\gamma(\gamma+4)|<1$.

We have the following result from Jφrgensen's inequality (2) that

(8) $$1\leqq|\gamma^2(\gamma-\beta)(\beta+4)|+|\gamma(\gamma+4)| .$$

Specially $|\beta|>|\gamma|$, if not we have

$1 \leqq |\gamma^2(\gamma-\beta)(\beta+4)|+|\gamma(\gamma+4)| \leqq (2|\gamma^3|+|\gamma|)(|\gamma|+4)<1.$

Set $S=\{z;|z|\leqq 2-2\cos(\pi/7)\}$. Let R be the union of the convex hulls of $S\cup\{-4\}$ and $S\cup\{\beta\}$. The function $u(z)=|z+4|+|z-\beta|$ is subharmonic in $D=\mathrm{int}(R)$ and hence there exists a point ξ in ∂D such that $|\gamma+4|+|\gamma-\beta| \leqq u(\xi)$. Put $0\leqq\theta=|\arg\beta|\leqq\pi$, then we have follows by estimating $u(\xi)$ from the above by the length of the component of $\partial D\setminus\{-4,\beta\}$ which contains ξ, that is

(9) $|\gamma+4|+|\gamma-\beta| \leqq r_1 + r_2 + |\gamma|\theta,$

where $r_1=(4^2-|\gamma|^2)^{1/2}+|\gamma|\sin^{-1}(|\gamma|/4)$ and $r_2=(|\beta|^2-|\gamma|^2)^{1/2}+|\gamma|\sin^{-1}(|\gamma|/|\beta|)$. Set $r = r_1 + r_2$ where $4.91<r<4.918$. Then (12) is reformed by

(10) $1\leqq|\beta+4|(\theta-1)|\gamma|^3+\{|\beta+4|(r-4)+1\}|\gamma|^2+4|\gamma|$.

Our goal is to estimate that the right hand side of above inequality is less than 1 and this contradicts the asuumption $|\gamma|\leqq 2-2\cos(\pi/7)=d$.

Now $|\beta+4|^2=|\beta|^2+4^2+8|\beta|\cos\theta=a+b\cos\theta$ where $16.7\leqq a\leqq 16.8$ and $7.1\leqq b\leqq 7.2$. Set $F(x)=(a+b\cos\theta)^{1/2}(\theta-1)x^3+\{(a+b\cos\theta)^{1/2}(r-4)+1\}x^2+4x$ where $0\leqq x\leqq d$. The derivative $F'(x)$ is a increasing function with respect to x, then $F(x)\leqq F(d)$ where $d=2-2\cos(\pi/7)$. Let $f(\theta)=F(d)$ and therefore we have $f'(\theta)=(a+b\cos\theta)^{-1/2}(d^2/2)g(\theta)$ where

$$g(\theta)=2(a+b\cos\theta)d-b\{(\theta-1)d+(r-4)\}\sin\theta.$$

We divide θ into 13 cases;

(I) If $0\leqq\theta\leqq4\pi/9$, then $g(\theta)>0$ and $f(\theta)\leqq f(4\pi/9)\leqq0.997789<1$.

(II) If $4\pi/9\leqq\theta\leqq5\pi/11$, then we have

$$f(\theta)\leqq\{16.8+7.2\cos(4\pi/9)\}^{1/2}(5\pi/11-1)d^3+[\{16.8+7.2\cos(4\pi/9)\}^{1/2}(r-4)+1]d^2+4d<0.99884<1.$$

(III) If $5\pi/11\leqq\theta\leqq8\pi/17$, similarly we have $f(\theta)\leqq0.999444<1$.

(IV) If $8\pi/17\leqq\theta\leqq9\pi/19$, similarly we hold $f(\theta)\leqq0.998055<1$.

(V) If $9\pi/19\leqq\theta\leqq\pi/2$, similarly $f(\theta)\leqq0.99785<1$.

(VI) If $10\pi/21\leqq\theta\leqq11\pi/23$, then $f(\theta)\leqq0.99785 < 1$

(VII) If $11\pi/23\leqq\theta\leqq\pi/2$, also we have $f(\theta)\leqq0.999901$

(VIII) If $\pi/2\leqq\theta\leqq5\pi/9$, then $g(\theta)<-0.64$ and $f(\theta)\leqq f(\pi/2)<0.9974<1$.

(IX) If $5\pi/9\leqq\theta\leqq2\pi/3$, also we have $g(\theta)<-0.47$ and $f(\theta)<1$

(X) If $2\pi/3\leqq\theta\leqq11\pi/15$, we have $f(\theta)\leqq0.99936 < 1$.

(XI) If $11\pi/15\leqq\theta\leqq7\pi/9$, then we have $f(\theta)\leqq0.995193<1$.

(XII) If $7\pi/9\leqq\theta\leqq8\pi/9$, also we have $f(\theta)\leqq0.9995<1$.

(XIII) If $8\pi/9\leqq\theta\leqq\pi$, also $f(\theta)\leqq0.9986<1$.

This completes the proof.

Theorem 2([G&M2]). <u>Let</u> $\langle f,g\rangle$ <u>be a discrete subgroup of</u> PSL(2,C) <u>with</u>

$\gamma(f,g)\neq0$, $\beta(f)=\beta(g)\neq-4$ <u>then</u>

(11) $$|\gamma(f,g)|\geqq 0.193.$$

The above constant is not sharp. The following theorem is sharp.

Theorem 3([G&M2]). Let $\langle f,g\rangle$ be a discrete subgroup of PSL(2,C) with

$\gamma(f,g)\neq 0$, $\beta(f)=\beta(g)\neq -4$ and

(12) $$\min\{|\beta(f)|,|\beta(fg)|,|\beta(fg^{-1})|\}\geqq 2\{\cos(2\pi/7)+2\cos(\pi/7)-1\},$$ then

$$|\gamma(f,g)|\geqq 2-2\cos(\pi/7).$$

The following two theorems are also proved in [F2].

Theorem 4([F2]). Let $\langle f,g\rangle$ be a nonelementary discrete group with

$\beta(g)\neq -4$ and $0<|\beta(f)|<2\{\cos(2\pi/7)+2\cos(\pi/7)-1\}$, then

(13) $$|\gamma(f,g)|\geqq 2-2\cos(\pi/7) \text{ or } |\gamma(f,g)-\beta(f)|>1.$$

Theorem 5([F2]). Let $\langle f,g\rangle$ be a nonelementary discrete group with

$\beta(g)\neq -4$ and $0<|\beta(f)|<2\{\cos(2\pi/7)+2\cos(\pi/7)-1\}$, then

(14) $$\max\{|\gamma(f,g)|,|\gamma(f,gfg^{-1})|\}\geqq 2-2\cos(\pi/7).$$

4. The collar lemma. Let G be a discrete subgroup of PSL(2,C) acting on the upper half space H^3. If $f\in G\backslash\{id.\}$ is not a parabolic element, then we denote A_f the geodesic in H^3 joining the fixed points of f on $\hat{C}$ the boundary of H^3 in R^3. For a positive number k, we define a tubular neighborhood about A_f as

$$N_k(f)=\{x\in H^3;\ d(x,A_f)\leqq k\},$$

where d is the hyperbolic metric. Let G_f be the subgroup of G which leaves A_f invariant. We call $N_k(f)$ a collar for f in G, if $g(N_k(f))\cap N_k(f)=\phi$ for all $g\in G\setminus G_f$ and $g(N_k(f))=N_k(f)$ for all $g\in G_f$. The number k is called the width of the collar $N_k(f)$.

Following [F1], we introduce the notion of complex distance between two geodesics in H^3 and also state the cosine rule. Denote a directed geodesic L by the ordered pair of its endpoints;so $L=(a,b)$ for its endpoints $a,b\in C$, $a\neq b$. The complex distance $t=\delta(L_1,L_2)\in C$ between two directed geodesics $L_1=(a_1,b_1)$ and $L_2=(a_2,b_2)$ is defined as follows:$|\mathrm{Re}(t)|\geqq 0$ is the hyperbolic distance between the geodesics and $\mathrm{Im}(t)$ is the angle made by the geodesics along their common perpendicular and is determined modulo 2π unless $\mathrm{Re}(t)\neq 0$, in which case $\pm\mathrm{Im}(t)$ is determined modulo 2π. We can compute the complex distance by the formula([F1]),

(15) $$\cosh^2(t/2)=(a_1,a_2,b_2,b_1).$$

The right hand side of this equality denotes the cross ratio of these four points. Therefore, for any $f\in PSL(2,C)$, we see $\delta(L_1,L_2)=\delta(f(L_1),f(L_2))$.

Let $f\in PSL(2,C)$ be non-parabolic and let A_f be directed geodesics in the hyperbolic space joining the fixed points of f. If L is a perpendicular to A_f then the complex distance t between L and $f(L)$ is called the complex translation length of f. In this case, we have

(16) $$tr^2(f)=4\cosh^2(t/2),$$

which makes sense even if f is not loxodromic.

For the geodesics L_0, L_1, L_2, put $\omega=\delta(L_1,L_2)$, $t_1=\delta(L_0,L_1)$, $t_2=\delta(L_0,L_2)$ and denote by α the complex distance from the perpendicular between L_0 and L_1 to the perpendicular between L_0, L_2. Then we have the so-called cosine rule:

(17) $$\cosh(\omega)=\cosh(t_1)\cosh(t_2)-\cosh(\alpha)\sinh(t_1)\sinh(t_2).$$

Let ω be the complex distance between A_f and $A_{gfg^{-1}}$. Then we can normalize f and gfg^{-1} as follows:

$$f=\begin{bmatrix}\cosh(t/2) & \sinh(t/2)\\ \sinh(t/2) & \cosh(t/2)\end{bmatrix},\quad gfg^{-1}=\begin{bmatrix}\cosh(t/2) & \exp(\omega)\sinh(t/2)\\ \exp(-\omega)\sinh(t/2) & \cosh(t/2)\end{bmatrix}$$

We have $\gamma=tr(fgf^{-1}g^{-1})-2=-(1-\cosh(t))(1-\cosh(\omega))$. Recall the cosine rule (17) and take $L_2=A_f$, $L_0=A_g$ and $L_1=A_{gfg-1}=g(A_f)$. It is easy to show that $\mu=\delta(A_g,A_f)=\delta(A_g,A_{gfg-1})$. Thus we have $\cosh(\omega)=\cosh^2(\mu)-\cosh(t')\sinh^2(\mu)$ where t' is a complex translation length of g. Therfore we have the following lemma([F1],[K]).

Lemma 4. Let f and g be non-parabolic elements in PSL(2,C) and let μ be the complex distance between A_f and A_g. If $\gamma=\mathrm{tr}(fgf^{-1}g^{-1})-2\neq 0$, then

(18) $$4\gamma=\beta(f)\beta(g)\sinh^2(\mu).$$

Making use of Lemma 4 and Theorem 1, we will show the following so-called collar lemma.

Theorem 6. Let G be a nonelementary discrete subgroup of PSL(2,C). Let f be an element of G\{id.} with $0<|\beta(f)|=2s<2\{2\cos(2\pi/7)-2\cos(\pi/7)+1\}=2c$, then there exist a collar $N_{k(s)}(f)$ with the width

(19) $$\sinh^2 k(s)=(c/s-1)/2.$$

And further let f and g be in G and suppose that f and g generate a nonelementary discrete group. If $0<|\beta(f)|=2s<2c$ and $0<|\beta(g)|=2s'<2c$, then the collars $N_{k(s)}(f)$ for f and $N_{k(s')}(g)$ for g are disjoint, where k is the function defined by (19).

Proof. Let f be an element of G\{id.} with $0<|\beta(f)|<2c$ and $g\in G\setminus G_f$. Suppose f is elliptic. The condition $|\beta(f)|<1$ implies that the order of f is not less than 7. Then $\langle f,gfg^{-1}\rangle$ is not elementary discrete subgroup of G. If f is not elliptic, then we see that $\mu\neq 0$ where μ is the complex distance between A_f and $A_{gfg^{-1}}$ for $g\in G\setminus G_f$. Thus we conclude that $\langle f,gfg^{-1}\rangle$ is non-elementary discrete group and we have from (2) , $|\beta(f)|+|\gamma(f,gfg^{-1})|\geqq 1$.

Therefore we have $\gamma(f,gfg^{-1})\neq 0$ by the assumption of Theorem 7. Thus $\langle f,gfg^{-1}\rangle$ is discrete with $\gamma(f,gfg^{-1})\neq 0$, $\beta(f)=\beta(gfg^{-1})\neq -4$, then we have $|\gamma(f,gfg^{-1})|\geqq 2-2\cos(\pi/7)=c^2$ for any $g\in G\setminus G_f$ by Theorem 1. It is already known that $4\gamma(f,gfg^{-1})=\beta^2(f)\sinh^2(\mu)$ from (18). By the simple calculation we have $c=|\beta(f)||\sinh(\mu)|\leqq|\beta(f)|\{2\sinh^2(\mathrm{Re}\,\mu/2)+1\}$. This completes the first part of theorem.

Next we prove the last part of theorem. Let μ be the complex distance between A_f and A_g. Then (18) and (19) imply

$$\begin{aligned}|\sinh^2(\mu)|&=4|\gamma(f,g)|/(|\beta(f)||\beta(g)|)\\ &\geqq c^2/(|\beta(f)||\beta(g)|)\\ &=(2\sinh^2k(s)+1)(2\sinh^2k(s')+1)\\ &=(\cosh^2k(s)+\sinh^2k(s))(\cosh^2k(s')+\sinh^2k(s'))\\ &\geqq\{\cosh k(s)\ \cosh k(s')+\sinh k(s)\ \sinh k(s')\}^2\\ &=\cosh^2(k(s)+k(s')),\end{aligned}$$

where k is the function defined by (19). From $\cosh^2\mathrm{Re}\,\mu\geqq|\sinh^2\mu|$, we have $\mathrm{Re}\,\mu\geqq k(s)+k(s')$, which proves the last part of the Theorem. We complete the proof.

Remark. The function $k(s)$ defined on the above is decresing function with respect to s and $k(s)\to\infty$ as $s\to\infty$.

Let f be a loxodromic element in PSL(2,C) with the multiplier $\exp(2\alpha(f)+2i\theta)$. The translation length $2\alpha(f)$ of f is also defined by $\inf\{d(\zeta,f(\zeta));\zeta\in H^3\}$. Next lemma is given by Zagier([Me2]).

Lemma 5. Let $x_1,x_2\in R$ and $0<x_1<\pi\sqrt{3}$, then there exist a positive integer n such that

$$\cosh nx_1 - \cos nx_2 \leqq \cosh(\sqrt{4\pi x_1/\sqrt{3}}) - 1. \tag{20}$$

If the multiplier of f is given by $\exp(2\alpha(f)+2i\theta)$, then

$$|\beta(f)| = 2(\cosh 2\alpha(f)-\cos 2\theta) \tag{21}$$

$$=4(\sinh^2\alpha(f)+\sin^2\theta).$$

If f is a loxodromic, then the axes of f and $f^n(n\neq 0)$ are same. By a simple computation and lemma 5 have $|\beta(f^n)|=2(\cosh 2n\alpha(f)-\cos 2n\theta)\leqq 2\{\cosh(\sqrt{8\pi\alpha(f)/\sqrt{3}}) - 1\}$ for some positive integer n. We restate Theorem 6, setting $2\alpha(f)=L$, $\cosh(\sqrt{4\pi L/\sqrt{3}}) - 1=s<c_1=2\cos(2\pi/7)-2\cos(\pi/7)+1\,(<0.445)$ and $\frac{\sqrt{3}}{4\pi}[\log(1+c_1+\sqrt{c_1^2+2c_1})]^2=c_2\,(\fallingdotseq 0.114519)$, then

Theorem 7. Let g be a non trivial closed geodesic with the length $L(g)<c_2$ in any complete hyperbolic 3-manifold M, then there exists a tubular neighbourhood N(g) around g in M. Let r be the the hyperbolic width of N(g). Then, the hyperbolic volume of N(g) is $\pi\cdot L(g)\cdot\sinh^2 r$ which is a decresing function of L(g).

Remark. If $L(g) \leqq 0.10857$, $r \geqq 0.17198$ and $\pi \cdot L(g) \cdot \sinh^2 r \geqq 0.01018$.

5. A lower bound of the volume of $V(H^3/\Gamma)$. Let $q(z_1,z_2)$ be a chordal distance between $z_1, z_2 \in \hat{C}$, that is $q(z_1,z_2)=2|z_1-z_2|(1+|z_1|^2)^{-1/2}(1+|z_2|^2)^{-1/2}$. We introduce two different norms which measure the distance from f in mobius transformation groups to id. The first of the two norms for f is given interms of the matrix,

(22) $\qquad m(f) = ||f-f^{-1}||$

where for any matrix A in SL(2,C) we let $||A||$ denote its euclidean norm $||A||^2=\mathrm{tr}(AA^*)$ and A^* its Hermitian transpose. The second is defined by

(23) $\qquad \rho(f) = d(f(j),j)$

where j is the point (0,0,1) in H^3.

If f is in $M(\hat{C})$ with $\mathrm{fix}(f)=\{z_1,z_2\}$, then $|\beta(f)|=\dfrac{1}{2}\dfrac{q(z_1,z_2)^2}{8-q(z_1,z_2)^2}m(f)^2$, $2\cosh(\rho(f))=||f||^2$ and simple computation leads $4||f||^2=m(f)^2+2|\mathrm{tr}^2(f)|$.

If f is in $M(\hat{C})\backslash\{id.\}$ with $\mathrm{fix}(f)=\{z_1,z_2\}$ and multiplier $\exp(2\alpha(f)+2i\theta)$, then,

(24) $$\sinh^2(\rho(f)/2)=\frac{4-q^2}{8-q^2}\,\frac{m(f)^2}{8}+\sinh^2\alpha(f)\ ,$$

(25) $$\sinh^2(\rho(f)/2)=\frac{4}{8-q^2}\,\frac{m(f)^2}{8}-\sin^2\theta\ ,$$

where $q=q(z_1,z_2)$.

Let N denote the set of positive integers and for each ρ in $[0,\infty)$ set

(26) $$s(\alpha)=\sup_{\theta}\left(\inf_{N}(\sinh^2 k\alpha+\sin^2 k\theta)\right).$$

Then $s(x)$ is nonnegative, nondecreasing and continuous in $[0,\infty)$ with $s(0)=0$.

Moreover from a lemma due to Zagier[Me2], it follows that

(27) $$s(t)\leqq\sinh^2\sqrt{at}\ ,\ a=2\pi/\sqrt{3}\ ,$$

for $0\leqq t<\sqrt{3}\pi/2$. Then we have the following lemma.

Lemma 7([G&M2]). Suppose that a and c are positive constants, and f is in $M(\hat{C})\backslash\{id.\}$ with two distinct fixed points and multiplier $\exp(2\alpha(f)+2i\theta)$. If $m(f^k)^2\geqq c$ for any k in N and

(28) $$s(a)\leqq c/8+\sinh^2\alpha(f)-\sinh^2 a,$$

then $\rho(f)\geqq 2a$.

The following is an immediate result from Jϕrgensen's inequality.

Lemma 8. Let <f,g> be a non-elementary discrete group, then

(29) $$m(f)m(g)\geqq 4(\sqrt{2}-1).$$

If $m(f)m(g)<4(\sqrt{2}-1)$ for a discrete group <f,g> ,then <f,g> is a elementary group. Next lemma gives more simple result for generators if each generative elements have the following restrictive conditions.

Lemma 9. Let <f,g> be a discrete group with $m(f)^2 < 4(\sqrt{2}-1)$ and $m(g)^2 \leqq 4(\sqrt{2}-1)$, then $fix(f) = fix(g)$.

The following proposition is an important role in this paper which is due to P.Waterman[W].

Proposition 4. Let G be a discrete group. Then there is a mobius transformation h for G satisfying

$$(30) \qquad m(f) \geqq 4(\sqrt{2}-1)$$

for any element f in $hGh^{-1}\setminus\{id.\}$.

Theorem 8. Let M be a complete hyperbolic 3-manifold, then

$$V(M) > 0.001.$$

Proof. It is known that M is represented by H^3/Γ for a torsion free Kleinian group Γ. If Γ contains parabolic transformations, then $V(M) \geqq \sqrt{3}/4$([Mel]). From now on, we consider that Γ is a purely loxodromic transformations group. we set $2\alpha' = \inf[2\alpha(f); f \in \Gamma\setminus\{id.\}]$ where $\exp(2\alpha(f)+2i\theta)$ is a multiplier of f. If $2\alpha' \leqq 0.10857$, then Theorem 7 and Remark shows that M has a collar with the volume not less than 0.001. Otherwise, we consider a conjugate group for Γ stated in Proposition 4 and reset this conjugate group Γ, then we have $m(f)^2 \geqq 4(\sqrt{2}-1)$ for f in $\Gamma\setminus\{id.\}$. Now we seek η satisfying the equation $s(\eta)=(\sqrt{2}-1)/2 + \sinh^2\alpha' - \sinh^2\eta$. we have $2\eta \geqq 0.11576$. Thus $\rho(f) \geqq 0.11576$, thus the Dirichlet fundamental polyhedron centered at j for Γ contains a ball B with hyperbolic radius 0.11576. Then we have the result.

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