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KONTSEVICH'S INTEGRAL FOR THE HOMFLY POLYNOMIAL AND ITS APPLICATIONS

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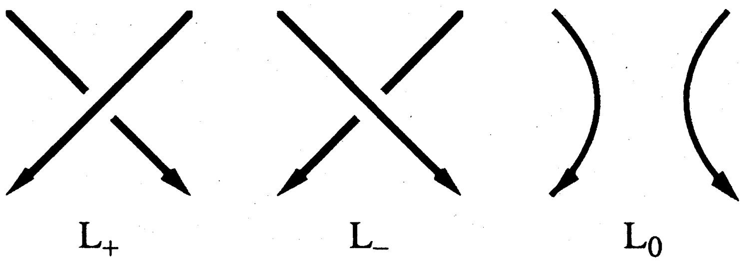
INTRODUCTION

The Homfly polynomial is a two variable extension of the Jones polynomial. It is an isotopy invariant of links and is a Laurent polynomial in two parameters ℓ and m defined by the skein relation

$$\ell^{-1} P_{L_+}(\ell,m) - \ell P_{L_-}(\ell,m) = m P_{L_0}(\ell,m),$$



where L_+, L_-, L_0 are identical except within a ball and, in this ball, they are positive crossing, negative crossing and trivial. The Homfly polynomial of the trivial knot is defined to be 1.

$$P_{\bigcirc}(\ell,m) = 1.$$

On the other hand, Zagire's multiple zeta function is a generalization of the zeta function given by

$$\zeta(s_1, s_2, \cdots, s_k) = \sum_{1 \leq m_1 < m_2 < \cdots < m_k} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}}.$$

Expressing the HOMFLY polynomial by means of Kontsevich's iterated integral, we get some relations among values of Zagier's mixed zeta functions

$$\sum_I c_J \zeta(J) = 0.$$

This is one of the applications of Kontsevich's invariant we want to show in this note. We have similar relations from the Kauffman polynomial of links. We can extend such integral for tangles and this has some application to representation theory of the Iwahori-Hecke algebras.

First of all, I want to give a little about the background of Kontsevich's integral. Kontsevich defines a knot invariant by applying an iterated integral for a knot. His theory uses the following two things. One is the theory of Vassiliev invariants and another is the theory of quasi-Hopf algebras by Drinfeld. Vassiliev's idea is very simple, but his computation is complicated and he use a spectral sequence. Then Birman and Lin gives purely combinatorial interpretation of Vassiliev construction. Kontsevich defines an invariant with values in a Hopf algebra $\mathcal{A}$, and this algebra is defined with the relation introduced in the paper of Birman-Lin, which we call the 4-term relation. On the other hand, several years ago, Kohno studied the monodromy of the Kunizinik-Zamodorochikov connection. Let (ρ, U) be the vector representation of the Lie algebra sl_m and let r be the image of the Casimir element in $\mathrm{End}(U \otimes U)$. Kohno considered the following connection.

$$\omega = \sum_{1 \leq i < j \leq n} r_{ij} \frac{dz_i - dz_j}{z_i - z_j},$$

where r_{ij} is the operator acting on $U \otimes \cdots \otimes U$ by r on the i-th and j-th component. He uses Chen's iterated integral and he found that the monodromy corresponds to the Iwahori-Hecke algebra. The monodromy is given by a braid and so he gives a representation of the braid group to the Iwahori-Hecke algebra. Inspired by this result, Drinfeld construct theory of quasi-Hopf algebras.

The amazing fact is that the 4-term relation given by Birman-Lin corresponds to the condition for the flatness of the KZ-equation. Let us replace r by an abstract operator Ω like this. Let ω denote this form then ω is flat if it satisfies

$$d\omega + \omega \wedge \omega = 0.$$

This relation is satisfied if

$$[\Omega_{ij}, \Omega_{ik} + \Omega_{jk}] = 0,$$

and this relation corresponds the relation for the Vassiliev invariant. Let us explain Ω graphically by a dashed arc like this, then this relation is illustrated like this. This correspondence may be one motivation for Kontsevich's construction.

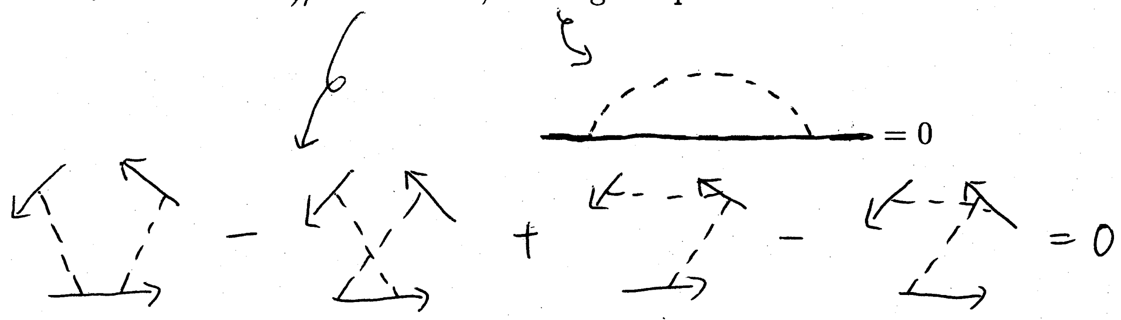
1. Kontsevich's integral

Kontsevich defines an invariant of knots by using iterated integrals, and we generalize his construction for links for later use.

1.1. Cord diagrams. Let k be a positive integer. A cord diagram on k circles is k oriented numbered circles with finitely many dashed cords marked on it. The circles are called Wilson loops. Here dashed cords just mean parings of points on the circles and nothing more. The placement of the cord has no meaning except the end points. Let $\mathcal{D}^{(k)}$ denote the collection of all cord diagrams on k circles.

Let the vector space $\mathcal{A}'^{(k)}$ be the quotient

$$\mathcal{A}'^{(k)} = \mathbf{C}\mathcal{D}^{(k)})/\text{4T–relation, framing independence relation.}$$



The module $\mathcal{A}'^{(k)}$ is graded by the number of cords. Let $A^{(k)}$ be the completion of $\mathcal{A}'^{(k)}$ by this grading.

Proposition. *$(\mathcal{A}^{(1)})^{\otimes k}$ acts on $\mathcal{A}^{(k)}$ by connected sums, where the i-th component of $(\mathcal{A}^{(1)})^{\otimes k}$ is summed to the i-th circle of a cord diagram in $\mathcal{A}^{(k)}$. Due to the 4-term relation, the above action does not depend on the place on the circle to connect diagrams. Especially, $\mathcal{A}^{(1)}$ is a commutative algebra.*

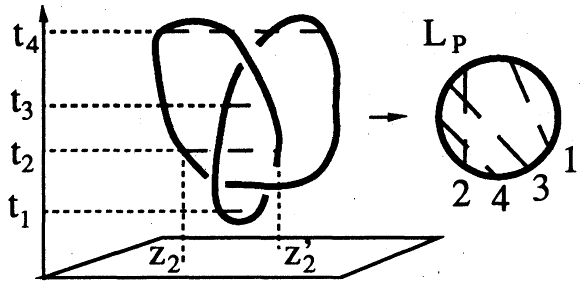
The structure of $\mathcal{A}^{(1)}$ is studied intensively by Bar-Natan and Kontsevich.

1.2. Iterated integral. Let L be a k-component link embedded in $\mathbf{R} \times \mathbf{C}$, whose components are numbered from 1 to k. We assume that L is in a general position. Let $Z(L)$ be

$$Z(L) = \sum_{n=0}^{\infty} \frac{1}{(2\pi i)^n} \sum_{\substack{P=\{(z_1,z_1'),\cdots,(z_n,z_n')\} \\ \text{horizontal configuration}}} \int_{t_1<t_2<\cdots<t_n} (-1)^{\#P_{\text{down}}} L_P \bigwedge_{i=1}^{n} \frac{dz_i(t_i) - dz_i'(t_i)}{z_i(t_i) - z_i'(t_i)} \in A^{(k)}.$$

In this equation, P is a horizontal configuration of the link L as in the figure. Every horizontal cord in P defines locally homeomorphic maps $t_i \to z_i$, $t_i \to z_i'$.

$\#P_{\text{down}}$ is the number of points z_i and z_i' at which L is oriented downwards, L_P is the image of the cord diagram in $\mathcal{A}^{(k)}$ naturally associated with L and P.



This integral has singularities at the maximal and minimal points. However, it is finite because of the framing independence relation.

Since the 4-term relation corresponds to the flatness of the KZ-equation, we have

Proposition. *If a link L' is obtained from a link L by a horizontal deformation, then $Z(L) = Z(L')$.*

Due to the factor $(-1)^{P_{\text{down}}}$, we have

Proposition. *If L' is obtained from L by a vertical move of a maximal or the minimal point, then $Z(L) = Z(L')$.*

Proposition. *For a connected sum of a knot K and a link L,*

$$Z(K\#L) = Z(K) \cdot Z(L).$$

Proof. Connect K and L vertically.

Z is invariant under horizontal deformations and vertical moves of minimal and maximal points, but not invariant by the stretching move like this. Due to the last proposition, we get

Lemma. *Let L be a link and L' be a link equal to L except this part. Then*

$$Z(L') = Z(\infty) \cdot Z(L),$$

where $Z(\infty)$ acts on this component.

Hence, if we normalize $Z(L)$ by using $Z(\infty)$, we may get an ambient isotopy invariant of L. Let

$$\kappa(L) = (Z(L)^{-s(L_1)} \otimes \cdots \otimes Z(L)^{-s(L_k)}) \cdot Z(L),$$

where L_i is the i-th component of L and $s(L_i)$ is the number of maximal points of L_i.

Theorem. *$\kappa(L)$ is an ambient isotopy invariant of links.*

We call κ Kontsevich's integral invariant.

2. Weight and Homfly polynomial

2.1. classical limit of R-matrix. Let U be the m dimensional vector space acting sl_m, and $\{e_1, e_2, \cdots, e_m\}$ be a basis of U. Let R be the R-matrix

$$R = -q \sum_i E_{i,i} \otimes E_{i,i} - \sum_{i \neq j} E_{i,j} \otimes E_{j,i} + (q^{-1} - q) \sum_{i<j} E_{i,i} \otimes E_{j,j},$$

Put $q = \exp(h)$, $R' = -q^{-m} R$ and $r = P \left. \dfrac{d(R' - R'^{-1})}{dh} \right|_{h=0}$, where P is the permutation, i.e. $P(u_1 \otimes u_2) = u_2 \otimes u_1$. r is called the classical limit of R and

$$r = 2(P - m\,\mathrm{id}).$$

Note that r is two times to the image of Casimir element in $\mathrm{End}(U \otimes U)$.

2.2. State model. We define a state model (weight system) [2] for $\mathcal{A}^{(k)}$ similar to Turaev's model in [11]. Let D be a cord diagram with k Wilson loops. A mapping $f : \{\text{arc of } D\} \to \{1, 2, \cdots, m\}$ is called a state of D, where a arc is a connected component of $\{\text{circle}\} \setminus \{\text{nord}\}$. For every state of D, we assign $r_{a_1 a_2}^{a_3 a_4}\, h$ for each cord like this. Let $W_r(D)$ ge a state sum on D defined by

$$W(D) = \sum_{\substack{f: \\ \{\text{arc}\} \to \{1,2,\cdots,m\}}} \prod_{\text{cord of } D} h\, r_{a_1 a_2}^{a_3 a_4}.$$

Proposition. *The mapping W is factored by $A^{(k)}$.*

This comes from the next two lemmas.

Lemma. *Small r satisfies the 4-term relation*

$$[r_{ij}, r_{ik} + r_{jk}] = 0 \quad (\{i, j, k\} = \{1, 2, 3\}),$$

where $r_i j \in End(U^{\otimes 3})$ acts on the i-th and j-th component of $U^{\otimes 3}$.

Lemma. $\sum_k r_{ik}^{kj} = 0.$

This comes from the normalization of R by $-q^{-m}$.

The weight W satisfies the following local relations between diagrams.

$$W(\downarrow\!\!\cdots\!\!\downarrow) = 2h(W(\times) - m\,W(\downarrow\,\downarrow)).$$

$$W(D \cup \bigcirc) = m\,W(D),$$

$$W(\bigcirc) = m.$$

For a cord diagram D with k cords, $W(D) = h^k \times$ a polynomial in m. It is easy to check that W is compatible with the 4T–relation and the framing independence relation. Hence

Proposition. *W is well-defined.*

From the last two relation, we have

Lemma.

$$W(D_1 \cup D_2) = W(D_1)\, W(D_2).$$

2.3. Invariant. Now we can construct an invariant κ_W by

$$\kappa_W(L) := \frac{W(Z(\infty))}{m}\, W(\kappa(L)).$$

We multiply the factor $\dfrac{W(Z(\infty))}{m}$ so that $\kappa_W(\bigcirc) = 1$.

We investigate property of κ_W for a disjoint union.

Proposition. *The invariant κ_W satisfies*

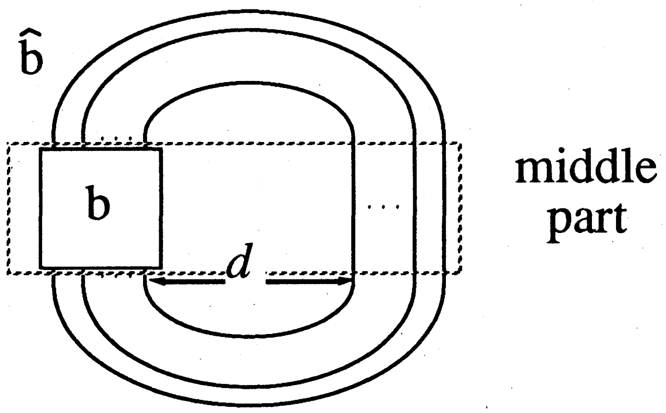
$$\kappa_W(L_1 \cup L_2) = \frac{m}{W(Z(\infty))}\, \kappa_W(L_1)\, \kappa_W(L_2).$$

Theorem. *$\kappa_W(L) = P_L(e^{mh}, e^{-h} - e^h)$ for any link L (closed braid $\hat{b}$).*

We first show the following lemma to reduce the problem to a braid.

Lemma. *Let b be a braid and $\hat{b}$ be its closure. Then $Z(\hat{b}) = Z(b) \cdot \lim_{d\to\infty} Z(\hat{b} \setminus b)$.*

Proof. $Z(b) \cdot Z(\hat{b} \setminus b)$ does not count the integral of the middle part for a configuration including a cord connecting b and the closing strings. Hence we have to show that the integral for such configuration goes to 0 if d tends to ∞. If d tends to ∞, this integral is bounded by const. $\times \log(1 + 1/d)$, while the integral for $\hat{b} \setminus b$ is bounded by const. $\times (\log d)^k$, where k is the number of cords. Hence the product of them goes to 0 if d goes to infinity. □



Proof of the theorem. We show the skein relation for $Z(b)$. Let U be the m–dimensional vector space on which the Lie algebra sl_m acts naturally. Our weight W is related to the Casimir element in $\mathrm{End}(U \otimes U)$. Kohno [6] shows the skein relation from the study of monodromies of the KZ-connection. This proves the theorem.

3. Relation between Zagier's mixed zeta values

3.1. $\kappa_W(\bigcirc \cup \bigcirc)$. The invariant κ_W is equal to the Homfly polynomial and so we have

$$\kappa_W(\bigcirc \cup \bigcirc) = \frac{e^{mh} - e^{-mh}}{e^h - e^{-h}} = \frac{\sinh m\, h}{\sinh h},$$

while κ_W satisfies

$$\kappa_W(\bigcirc \cup \bigcirc) = \frac{m}{W(Z(\infty))}\, \kappa_W(\bigcirc)\, \kappa_W(\bigcirc) = \frac{m}{W(Z(\infty))},$$

by the previous proposition. Hence

$$W(Z(\infty)) = m\, \frac{\sinh h}{\sinh m\, h}.$$

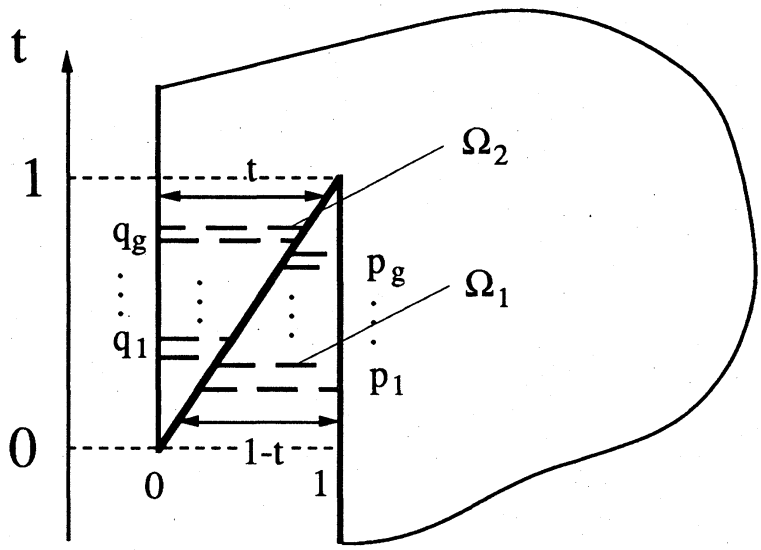
3.2. $Z(\infty)$. On the other hand, we can compute $Z(\infty)$ and $W(Z(\infty))$ by another method. Let $I = (p_1,\, q_1,\, \cdots,\, p_g,\, q_g)$, $|I| = \sum_i p_i + q_i$, and $g(I) = g$. Let $\zeta_I = \zeta(\underbrace{1, \cdots, 1}_{p_1 - 1},\, q_1 + 1,\, \cdots,\, \underbrace{1, \cdots, 1}_{p_g - 1},\, q_g + 1)$.

Theorem. $Z(\infty) = 1 + \sum_{g(I)\geq 1} \frac{(-1)^{\sum_i q_i}}{(2\pi i)^{\sum_i p_i + q_i}} \zeta_I D_I$, *where* D_I *is the configuration given in the next figure.*

Proof. $Z(\infty)$ is a regular isotopy invariant and so we compute $Z(\infty)$ for this special diagram. By an induction, we can show that the iterated integral from $t = 0$ to $t = x$ of the diagram is given by generalized dilogarithm function

$$\sum_{0<m_1<m_2<\cdots<m_k} \frac{x^{s_k}}{m_1{}^{s_1}\cdots m_k{}^{s_k}}$$

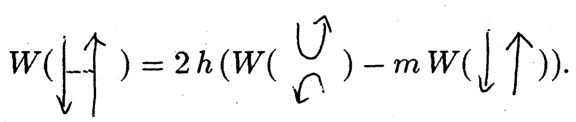
for some $s_1, \cdots, s_k$. □



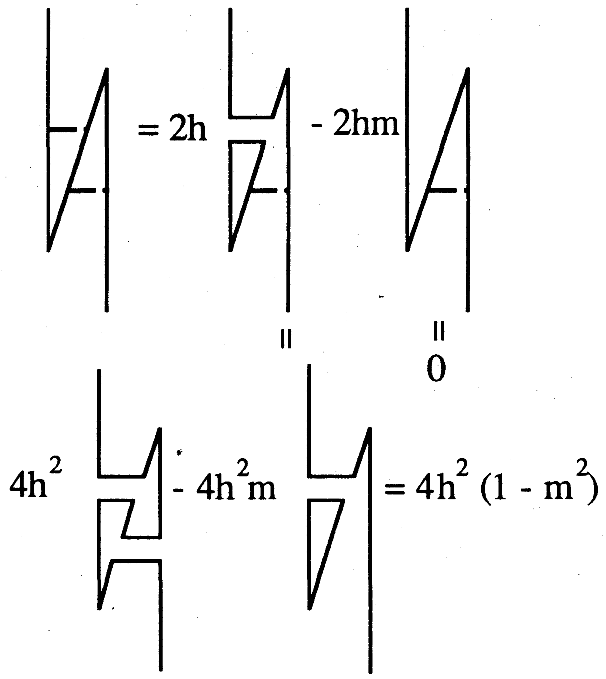
To compute $W(Z(\infty))$, we have to compute $W(D_I)$.

Proposition. $W(D_I) = \frac{(2mh)^{\sum_i p_i + q_i}}{m^{2g-1}} (1 - m^2)$.

Proof. We compute it for $D_{1,1}$. This case, the orientation of the two Wilson loops at the ends of a cord are different and so we replace a cord by this way.



First, apply this replacement to Ω_1. Then it decompose 2 diagrams. But this one is 0 because of the framing independence relation. So replace Ω_2 of the non-trivial one. Then we get $\dfrac{(2\,m\,h)^2}{m}\,(1-m^2)$. $\square$



Combining the above theorem and proposition, we get

$$W_r(Z(\infty)) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{1 \leq g(I) \leq n \\ |I|=2n}} (-1)^{-n+\sum_i p_i}\, h^{2\,n}\, m^{2\,n-2\,g(I)}\, (1-m^2)\, \frac{\zeta_I}{\pi^{2n}}.$$

3.3. Relation between values of Zagier's multiple zeta functions. Comparing the coefficient of $h^n\,m^p$ of this formula and $m\,\dfrac{\sinh h}{\sinh mh}$, we get relations between values of Zagier's multiple zeta functions. $m\,\dfrac{\sinh h}{\sinh mh}$ has the following Taylor expansion.

Since $t\,\exp(xt)/(\exp(t)-1) = \sum_{n=0}^{\infty} B_n(x)\,t^n/n!$ where $B_n(x)$ is the Bernoulli polynomial, we have

$$m\,\frac{\sinh h}{\sinh mh} = 1 + \sum_{n=1}^{\infty} B_{2n+1}\!\left(\tfrac{m+1}{2\,m}\right) \frac{(2m)^{2n+1}}{(2n+1)!}\, h^{2n}.$$

We also know that

$$B_{2n+1}(\tfrac{m+1}{2\,m}) = -\sum_{r=0}^{n} \binom{2n+1}{2r} (1-2^{1-2r})\,(2m)^{-2n-1+2r}\,B_{2r},$$

because $B_n(x+h) = \sum_{r=0}^{n} B_r(x)\,h^{n-r}$, $B_n(1/2) = -(1-2^{1-n})\,B_n$ and $B_{2\ell+1} = 0$ for any positive integer ℓ. Here B_n are the Bernoulli numbers. Hence we get

$$\frac{1}{(2n+1)!}\binom{2n+1}{2r}(2-2^{2r})\,B_{2r} =$$

$$\sum_{\substack{g(I)=n-r\\ |I|=2\,n}} (-1)^{-n+\sum_i p_i}\,\frac{\zeta_I}{\pi^{2n}} - \sum_{\substack{g(I)=n-r+1\\ |I|=2\,n}} (-1)^{-n+\sum_i p_i}\,\frac{\zeta_I}{\pi^{2n}}.$$

Examples. If $r = 0$ then

$$\zeta(2^k)/\pi^{2\,k} = 1/(2k+1)!.$$

If $r = n$ then

$$\frac{2-2^{2n}}{(2n)!}\,B_{2n} \quad = \quad (-1)^{1-n}\,\frac{1}{\pi^{2n}}\,[\zeta(2n) - \zeta(1,2n-1) + \cdots + \zeta(1^{2n-2},2)].$$

By using $B_{2n} = 2\,(2n)!\,(-1)^{n-1}\,(2\,\pi)^{-2n}\,\zeta(2n)$, we get

$$\zeta(2n) - \zeta(1,2n-1) + \cdots + \zeta(1^{2n-2},2) = 2\,(1-\frac{1}{2^{2n-1}})\,\zeta(2n).$$

Hence

$$(\frac{1}{2^{2n-2}} - 1)\,\zeta(2n) - \zeta(1,2n-1) + \cdots + \zeta(1^{2n-2},2) = 0.$$

For example, if $n = 2$, $-\frac{3}{4}\,\zeta(4) - \zeta(1,3) + \zeta(1,1,2) = \frac{1}{4}\,\zeta(4) - \zeta(1,3) = 0$ since $\zeta(1,1,2) = \zeta(4)$, and so

$$\zeta(1,3) = \frac{1}{4}\,\zeta(4) = \frac{1}{360}\,\pi^4.$$

4. Another applications

4.1. Kauffman polynomial. We also studied the Kauffman polynomial by using Kontsevich's integral. To do this, we have to generalize the integral for framed links. We have to remove the framing independence relation. Then we regularized the integral so that the integral is finite. This case, we also get relations between values of Zagier's multiple zeta functions. Using this relations and other known relation, we determined the values of $\zeta(s_1, \cdots, s_k)$ for $\sum_i s_i = 6$ case. Only $\zeta(6) = \zeta(1,1,1,1,2)$ is a rational number and others are in $\mathbf{Q} + \mathbf{Q}\zeta(3)^2$. There are many other knot invariant, so we may get much more relations between values of Zagier's multiple zeta functions.

4.2. Tangles and Quasi-Hopf algebras. As we regularized the integral for framed links, we regularized the integral for a trivial tangle with three strings of the shape N. Kontsevich's integral depend on the all strings. However, using the integral of this diagram of the shape N, we can localize the integral. Moreover, we can split the integral for a tangle into a multiple of fundamental parts of tangles. This representation resembles to theory of quasi-Hopf algebra by Drinfeld. The integral for the tangle of the shape N corresponds to the associator of a quasi-Hopf algebra. So we think our theory extracts essential part of Drinfeld's theory.

4.3. Iwahori-Hecke algebras. Combining the weight corresponding to the Casimir element of sl_m and the representation of tangles by Kontsevich's integral, we can construct a homomorphism from the Iwahori-Hecke algebra to the group ring of a symmetric group. We can give the actual image of the associator in this case.

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