Elliptic \( r \)-matrix systems, affine Lie algebras, and projective representations of the braid group of the torus

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1. Classical \( r \)-matrices

**Def.** A classical \( r \)-matrix is a meromorphic function \( r(z) \) in a neighborhood of the origin in \( \mathbb{C} \) with values in \( g \otimes g \) (where \( g \) is a simple Lie algebra) which satisfies the classical Yang-Baxter equation:

\[
[r_{12}(z_1-z_2), r_{13}(z_1-z_3)] + [r_{12}(z_2-z_3), r_{23}(z_2-z_3)] + [r_{13}(z_1-z_3), r_{23}(z_2-z_3)] = 0
\]

in \( U(g)^{\otimes 3} \). Here \( r_{12} = r \otimes 1 \), \( r_{23} = 1 \otimes r \), \( r_{13} = S_{12}(1 \otimes r)S_{12} \), \( S_{12} \) - the permutation of 1-st and 2-nd factor.
2. Why look at classical $r$-matrices?

(A) $R_h(z) -$ quantum $R$-matrix (solution to the quantum Yang-Baxter equation) $\Rightarrow$
if $R_h(z) = 1 + h r(z) + O(h^2)$, then $r(z)$ is a classical $r$-matrix (Sklyanin).

(B) Let $V_1, \ldots, V_n$ be $g$-modules. $\pi_i : g \to \text{End}(V_i)$ - the corresponding Lie algebra homomorphisms, $r_{ij}(z) = \pi_i \otimes \pi_j (r(z))$ $r_{ij} : \text{End}(V_i \otimes \cdots \otimes V_n).$
Consider the 1-form
$$\omega = \frac{1}{\kappa} \sum_{i < j} r_{ij}(z_i - z_j) \text{d}(z_i - z_j), \quad \kappa \in \mathbb{C}$$
Observe $\text{d} \omega = 0$.

Claim $r(z)$ satisfies the classical YB equation $\iff [\omega, \omega] = 0$ (Cherednik).
This condition is equivalent to the zero curvature condition $\text{d} \omega + [\omega, \omega] = 0$ for the connection $\nabla = \text{d} + \omega$.

3. Classical $r$-matrix systems

We see that $r(z)$ satisfies the classical YB equation iff the system of differential
equations \( df + \omega f = 0 \) with respect to a \( V_1 \otimes \cdots \otimes V_m \)-valued function \( f(z_1, \ldots, z_n) \) has \( \text{dim} (V_1 \otimes \cdots \otimes V_m) \) linearly independent solutions. This system is called the \( r \)-matrix system corresponding to \( r \).

4. Classification of classical \( r \)-matrices

In 1982 A. Belavin and V. Drinfeld showed that if \( r(z) \) is a classical \( r \)-matrix which is invertible as a map \( g^* \to g \) for a generic \( z \in \mathbb{C} \) then \( r(z) \), up to an equivalence relation is of one of the three types: rational, trigonometric, and elliptic. Elliptic non-degenerate \( r \)-matrices exist only for \( g = sl_N \), and they were found by Belavin and Drinfeld. Trigonometric \( r \)-matrices exist for any \( g \) and were classified by Belavin and Drinfeld in 1982. There is no classification of rational \( r \)-matrices.
5. Examples of $r$-matrices

A. An example of a rational $r$-matrix is the Yang's $r$-matrix

$$ r_{\text{rat}}(z) = \frac{\Omega}{z} , $$

$$ \Omega = \sum_i x_i \otimes x_i , \{x_i\} \text{ is an orthonormal basis of } g \text{ with respect to the Killing form } \langle , \rangle . \text{ The corresponding } r \text{-matrix system is the system of Knizhnik-Zamolodchikov equation,} $$

$$ x \frac{\partial f}{\partial x_i} = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} f , \quad \Omega_{ij} = \pi_i \otimes \pi_j(\Omega) $$

B. Let $h \subset g$ be a Cartan subalgebra, $g = n^- \oplus h \oplus n^+$ be the Cartan decomposition. Let $e_\alpha$ be a $h$-eigenbasis of $n^+$, $f_\alpha$ be the dual basis of $n^-$ with respect to $\langle , \rangle , \alpha \in \Delta^+$. Let $\{H_i\}$ be an orthonormal basis of $h$ with respect to $\langle , \rangle$.

Set

$$ \Omega^+ = \sum_\alpha e_\alpha \otimes f_\alpha + \frac{1}{2} \sum_i H_i \otimes H_i $$

$$ \Omega^- = \sum_\alpha f_\alpha \otimes e_\alpha + \frac{1}{2} \sum_i H_i \otimes H_i $$
Then the simplest trigonometric \( r \)-matrix is
\[
R_{\text{trig}}(z) = \frac{\Omega^+ e^z + \Omega^-}{e^z - 1}
\]
The corresponding \( r \)-matrix system is the system of trigonometric KZ equations.

**Fact:** Trigonometric KZ can be reduced to rational (usual) KZ by a simple transformation.

Let \( g = sl_N(\mathbb{C}) \), and let \( \beta, \gamma \in \text{Aut}(g) \) be two commuting elements with no common invariant nonzero elements in \( g \). Any such pair can be reduced by a conjugation to
\[
\beta = \text{Ad}
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad \gamma = \text{Ad}
\begin{pmatrix}
\varepsilon^1 & 0 & 0 \\
0 & \varepsilon & 0 \\
0 & 0 & \varepsilon^{-N}
\end{pmatrix}
\]
where \( \varepsilon \) is a primitive \( N \)-th root of unity. Let \( \zeta(2|\tau) \) be the Weierstrass elliptic \( \zeta \)-function:
\[
\zeta(z|\tau) = \frac{i}{2} z + \lim_{M \to \infty} \sum_{-M < m, \rho < M} \left[ \frac{1}{z - m - \rho \tau + \frac{\tau}{(m + \rho \tau)^2}} \right]
\]
\((\tau \in \mathbb{C}^*)\)
Then
\[ R_{\text{ell}}(z) = \Omega \xi(z|\tau) + \sum_{0 \leq m, p \leq N-1 \atop m^2 + p^2 > 0} (1 \otimes \beta^p y^{-m})(\Omega) \times \left[ \xi \left( z - \frac{m+p\tau}{N} \right| \tau \right) - \xi \left( -\frac{m+p\tau}{N} \right| \tau \right) \]

is an elliptic \( R \)-matrix. Belavin and Drinfeld showed that any elliptic \( R \)-matrix is (up to a constant factor) equivalent to this one. It is the quasiclassical limit of the Baxter-Belavin quantum \( R \)-matrix of the (generalized) 8-vertex model. \( R_{\text{ell}}(z) \) is uniquely defined by the following conditions:

1) \( R_{\text{ell}}(z) \) is meromorphic in \( \mathbb{C} \) and has no other singularities than simple poles at \((m+p\tau)/N\), \( m, p \in \mathbb{Z} \) with residues \((1 \otimes \beta^p y^{-m})(\Omega)\);

2) \( R_{\text{ell}}(z)_{12} = -R_{\text{ell}}(-z)_{21} \) (unitarity condition)
The lattice of periods of $\mathbb{R}ee (\mathbb{Z})$ is spanned by $1$ and $\Gamma$, and its lattice of poles is spanned by $1/N$ and $\Gamma/N$.

6. Statement of the problem

Rational and trigonometric KZ equations come from representation theory. They are satisfied by the correlation functions of the Wess-Zumino-Witten conformal field theory, which are matrix elements of intertwining operators between modules over affine Lie algebras. A natural question is:

Does there exist such an interpretation for the elliptic $r$-matrix system?

7. Twisted realization of the affine Lie algebra $\mathfrak{sl}_N(\mathbb{C})$

$\mathfrak{g} = \mathfrak{sl}_N(\mathbb{C}), \quad \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c$

$[a(t) + \lambda c, \ b(t) + \mu c] = [a(t), \ b(t)]$

$+ \frac{1}{2\pi i} \oint_{|t|=1} \langle a'(t), \ b(t) \rangle \ t^{-1} dt \cdot c$
(normalization of $< , >$: $< x , x > = 2$, where $x$ is the highest root).

Let $\varepsilon \in \mathbb{C}^{\frac{2\pi i k}{N}}$ be a primitive $N$-th root of unity. Set

$$C = \begin{pmatrix} 1 & \varepsilon^{-1} & 0 \\ 0 & \ldots & \varepsilon^{1-N} \end{pmatrix}, \quad Y = \text{Ad} \ C, \quad Y(a) = C a C^{-1}$$

$$\hat{g}_y \subset \hat{g}: \quad \hat{g}_y = \{ a(t) + \lambda C \in \hat{g} \mid a(\varepsilon t) = Y(a(t)) \}$$

**Basis of $\hat{g}$**: $E_{i, i}$, $i \neq j$, $1 \leq i, j \leq N$, $H_i = E_{i, i} - E_{i+1, i+1}$, $1 \leq i \leq N-1$.

**Basis of $\hat{g}$**: $E_{i, j} \otimes t^m$, $H_i \otimes t^m$, $C$

**Basis of $\hat{g}_y$**: $E_{i, j} \otimes t^{i-i+mN}$, $H_i \otimes t^{mN}$, $C$

**Lemma**: $\hat{g}_y \cong \hat{g}$

**Proof**. \begin{align*}
\phi (E_{i, j} \otimes t^{i-i+mN}) &= E_{i, j} \otimes t^m \\
\phi (H_i \otimes t^{mN}) &= H_i \otimes t^m, \quad m \neq 0 \\
\phi (H_i) &= H_i - \frac{C}{N}, \quad \phi (C) = \frac{C}{N}
\end{align*}

gives an isomorphism.
8. Modules over $\hat{\mathfrak{g}}_\gamma$

Polarization: $\hat{\mathfrak{g}}_\gamma = \hat{\mathfrak{g}}^+ \oplus (\mathfrak{h} \oplus \mathfrak{c} \cdot \mathfrak{c}) \oplus \hat{\mathfrak{g}}^-$

$\hat{\mathfrak{g}}^+$ is spanned by $a(t) \in \hat{\mathfrak{g}}_\gamma$ with $a(0) = 0$.

$\hat{\mathfrak{g}}^-$ is spanned by $a(t) \in \hat{\mathfrak{g}}_\gamma$ with $a(\infty) = 0$.

Verma modules over $\hat{\mathfrak{g}}_\gamma$:

$$M_{\lambda, k} = \text{Ind}_{\hat{\mathfrak{g}}^+ \oplus \mathfrak{h} \oplus \mathfrak{c} \cdot \mathfrak{c}}^{\hat{\mathfrak{g}}_\gamma} X_{\lambda, k}$$

$X_{\lambda, k}$ - 1-dim $\hat{\mathfrak{g}}^+ \oplus \mathfrak{h} \oplus \mathfrak{c} \cdot \mathfrak{c}$ module generated by $v_{\lambda, k} \in X_{\lambda, k}$

$\hat{\mathfrak{g}}_\gamma v_{\lambda, k} = 0$, $\mathfrak{h} v_{\lambda, k} = \langle \lambda, h \rangle v_{\lambda, k}$

$C v_{\lambda, k} = k v_{\lambda, k}$.

(under the isomorphism $\phi$, $M_{\lambda, k}$ goes to $M_{\lambda, k}$, standard Verma module over $\mathfrak{g}_\gamma$ with $\Lambda = \lambda + k \rho$, $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$, $k = N k$.

Evaluation representations of $\hat{\mathfrak{g}}_\gamma$:

$V$ - finite dimensional $\mathfrak{g}_N$ module

$V(z) = V \otimes \mathbb{C}[z, z^{-1}]$
\( \hat{\gamma} \) acts in \( V(z) \):
\[
[(a(t) + \lambda c) \cdot \nu](z) = a(z) \nu(z), \quad z \in \mathbb{C}^*. 
\]

\( C : V \to V \) is defined since
\( C \in \text{GL}_N \quad (C = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon^{1-N} \end{pmatrix}) \).

\( V_C(z) \) def \{ \nu(z) \in V(z) \mid \nu(z\varepsilon) = C \nu(z) \} \)

Observe: \( V_C(z) \) is stable under the action of \( \hat{\gamma} \).

Fact: Under the isomorphism \( \phi \), \( V_C(z) \) over \( \hat{\gamma} \) goes to \( V(z) \) over \( \mathbb{C} \).

Grading: Let \( \mathcal{I} \) be the operator of principal gradation:
\[
[\mathcal{I}, a(t) + \lambda c] = t a'(t), \quad a(t) + \lambda c \in \hat{\gamma}.
\]

Set \( \hat{\gamma}_N = \hat{\gamma} \oplus CC \). Operator \( \mathcal{I} \) naturally acts on \( \hat{\gamma}_N \)-modules:

On \( V_C(z) \) : \( \mathcal{I} \nu(z) = \frac{d}{dz} \nu(z) \)

On \( M_{\lambda, k} \) : \( \mathcal{I} X = \frac{\langle \lambda, \lambda \rangle}{2(k+1)} + \deg(X) \)
where \( X \) is homogeneous.

\( z^\Delta V_c(z) \) - the same module as \( V_c(z) \), but it acts as

\[
\Delta v = z \frac{dv}{dz} + \Delta v \quad (\Delta \in \mathbb{C})
\]

9. Intertwiners

We want to study \( \hat{\Phi} \)-intertwining operators

\[
\hat{\Phi}(z) : M_{\lambda, k} \rightarrow M_{\nu, k} \hat{\otimes} z^\Delta V_c(z)
\]

where

\[
\Delta = \frac{\langle \nu, \nu \rangle - \langle \lambda, \lambda \rangle}{2(k + 1)}
\]

and \( M \hat{\otimes} V \) denotes the completed tensor product.

**Theorem** \( \hat{\Phi}(z) \) (for generic parameters) are in 1-1 correspondence with vectors in \( V \) of weight \( \lambda - \nu \), and \( \Phi(z) \) is uniquely determined by the highest matrix coefficient \( \langle \nu^*, \Phi(z) V_{\lambda, k} \rangle \in V^{\lambda - \nu} \)


10. Products of intertwiners

Suppose we have

\[ \Phi_1(z_1) : M_{\lambda_1, k} \rightarrow M_{\lambda_0, k} \otimes z_1^{\Delta_1} V_1(z_1) \]
\[ \Phi_2(z_2) : M_{\lambda_2, k} \rightarrow M_{\lambda_1, k} \otimes z_2^{\Delta_2} V_2(z_2) \]
\[ \vdots \]
\[ \Phi_n(z_n) : M_{\lambda_n, k} \rightarrow M_{\lambda_{n-1}, k} \otimes z_n^{\Delta_n} V_n(z_n) \]

**Question**: Can we make sense of the product \( \Phi_1(z_1) \cdots \Phi_n(z_n) \)?

(assuming now \( z_1, \ldots, z_n \in C^* \))

**Answer**: Yes, we can if

\[ |z_1| > |z_2| > \cdots > |z_n| \]

Then every matrix element of this product is a convergent power series.

Thus, we have an operator

\[ \Phi = \Phi_1 \Phi_2 \cdots \Phi_n : M_{\lambda_n, k} \rightarrow \hat{M}_{\lambda_0, k} \otimes V_1 \otimes \cdots \otimes V_n \]

(\( \hat{M} \) is the completion of \( M \) with respect to the principal gradation) which depends on \( z_1, \ldots, z_n \) and is defined.
in the region $|z_1| > |z_2| > \cdots > |z_n|$. Note that it will, in general, be a multivalued function since it will have a factor $\Delta_i = \frac{\langle \lambda_i, \lambda_{i-1} \rangle - \langle \lambda_i, \lambda_i \rangle}{2(k+1)}$.

In KZ theory we study the highest matrix element $\langle V_{\lambda_0, k}, \phi V_{\lambda_n, k} \rangle$ of $\phi$ and prove that it satisfies the trigonometric KZ equation.

11. The B-operator

In the case $\mathfrak{g} = \mathfrak{sl}_N$, the Dynkin diagram of $\widehat{\mathfrak{g}}$ admits an outer automorphism of order $N$. It corresponds to an outer automorphism $\beta : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$. It restricts to the previously introduced $\beta$ on $\mathfrak{g} \subset \widehat{\mathfrak{g}}$. Define the map

$$B : M_{\lambda, k} \to M_{\beta(\lambda), k}$$

$$(\beta(\lambda)(h) = \lambda(\beta^{-1}(h)))$$

by $B V_{\lambda, k} = V_{\beta(\lambda), k}$

$$\beta \alpha v = \beta(\alpha) B v, \quad \alpha \in \widehat{\mathfrak{g}},$$

then $B^n = 1$. 

12. Trace

Assume that \( \lambda_n = \beta(\lambda_0) \). Fix \( q, |z| < 1 \)

Consider the trace

\[
F(z_1, \ldots, z_n|q) = \operatorname{Tr}_{M_{\lambda_0, k}} (\phi_1(z_1) \cdots \phi_n(z_n) B q^{-2})
\]

Claim: This is convergent when

\[
|z_1| > |z_2| > \cdots > |z_n| > |qz_1|
\]

So, it defines a holomorphic function in this region. (Notation: \( q = e^{2\pi i \tau} \)).

13. Main theorem

\[
(k+1) z_j \frac{\partial F}{\partial z_j} = \sum_{j \neq i} \frac{1}{2\pi i N} \mathcal{R}_{\lambda}(\frac{\log z_j - \log z_i}{2\pi i} | N \tau) F
\]

This is equivalent to the elliptic \( \mathcal{R} \)-matrix system – we just have to make a change of variable \( y_j = C \log z_j \).

Claim: For generic parameters, if we take all possible \( \lambda_0, \lambda_1, \ldots, \lambda_n \) and all possible intertwiners with these \( \lambda_i \), we get very nice basis of the space of solutions of the elliptic \( \mathcal{R} \)-matrix system.
14. The moduli equation.

The defined traces satisfy one more differential equation - of the form
\[ g \frac{\partial F}{\partial g} = \text{(something known)} \cdot F \]

Before we write it down, introduce some notations.

New variables: \[ y_j = \frac{N \log z_j}{2\pi i} \]
\[ \tau = \frac{\log z}{2\pi i}, \quad x = k + 1, \quad \rho(y|\tau) = \frac{1}{N^2} \text{Re} \left( \frac{y}{N} \right) \]

Then, we have the elliptic $n$-matrix system
\[ x \frac{\partial F}{\partial y_e} = \sum_{j \neq e} \rho_{ej} (y_e - y_j | \tau) \cdot F \]

Let \[ s(y|\tau) = \int_0^y \frac{\partial \rho(x|\tau)}{\partial \tau} \, dx \]

(the integral is well-defined since all singularities of $\frac{\partial F}{\partial \tau}$ are second order poles with residue 0). Also, $\Lambda(\tau)$ is a $g \otimes g$ valued "modular" function which for $g = \frac{N \tau}{2\pi}$ equals
\[2\pi i \angle (\tau) = -\sum_{m \geq 0} \frac{q^{2m+1} (1+q^{4m+2})}{(1-q^{4m+2})^2} (e \otimes e + f \otimes f) - \sum_{m \geq 0} \frac{2q^{4m+2}}{(1-q^{4m+2})^2} (e \otimes f + f \otimes e) + \frac{1}{2} \left( \frac{1}{8} + \sum_{m \geq 0} \frac{q^{2m}}{(1+q^{2m})^2} \right) h \otimes h\]

Then,

**Theorem.** The function \( F \) satisfies the differential equation

\[ \lambda \frac{\partial F}{\partial \tau} = \sum_{i,j=1}^{n} L_{ij}(\tau) F + \sum_{j<i} S_{ij}(y_i-y_j | \tau) F \]

(Here \( L_{ij}(\tau) \) is defined according to the rule \((a \otimes b)_{ii} = \pi_i(a) \pi_i(b), \) \( a, b \in \mathbb{G}, \) \( \pi_i : \mathbb{G} \to \text{End}(V_i) \))

So, we get an extended system of \( n+1 \) consistent differential equation. Let us call it the elliptic KZ equations.

**Corollary.** \( \text{Tr} |_{M_{0,k}} (Bq^{-2}) = 1 \)
15. Examples

$n = 1$ — then we only have the module equation, but it is already interesting.

$V = V_2 = 2$-d representation of $sl_2$.

Then we have two possible operators:

$\phi^+(z) : M_{-\frac{1}{2}}, k \rightarrow M_{\frac{1}{2}}, k \otimes V$

$\phi^-(z) : M_{-\frac{1}{2}}, k \rightarrow M_{-\frac{1}{2}}, k \otimes V$

We can consider two solutions.

$T_{\pm}(z) = \text{Tr} \left\{ M_{\pm\frac{1}{2}}, k \left( \phi^\pm(z) B q^{-z} \right) \right\}$

Solving the elliptic KZ, we get

$T_{\pm}(q) = \zeta(q^2)^{3/4k} \nu_{\pm}$

$\zeta(q) = q^{1/4} \prod_{m \geq 1} (1 - q^m) \quad \text{(Dedekind } \eta \text{-function)}$

16. Fundamental solution

Let $V$ be a finite dimensional $g$-module, $v \in V$ a vector of weight $\mu$. Let

$\phi^\mu(z) : M_{\lambda}, k \rightarrow M_{\lambda + \mu}, k \otimes V(z)$

be the intertwiner for which

$\langle u^\mu_{\lambda + \mu}, k, \phi^\mu(z) u^\lambda_{\lambda}, k \rangle = \varepsilon^\Delta u (1 + O(\varepsilon))$

$\varepsilon \rightarrow 0$
Def 1. The fundamental solution of the elliptic $KZ$ equations is the $End(V_1 \otimes \cdots \otimes V_n)$-valued function $F$ of $z_1, \ldots, z_n, \varrho$ such that

$F(z_1, \ldots, z_n; \varrho) u = \left. T_{z_n} \right|_{M^{\lambda_0}_{a, k}} \left( \phi^u_{\varrho}(z_1) \cdots \phi^u_{\varrho}(z_n) B_{\varrho}^{-\delta} \right)$

where $\lambda_0 = (\beta - 1)^{-1}(\mu)$.

Def 2. Fundamental solution is the solution $F$ with values in $End(V_1 \otimes \cdots \otimes V_n)$ which has the property $F \sim z_1^{D_1} \cdots z_n^{D_n}$, $\varrho \to 0$, $z_i/z_{i+1} \to \infty$, where $D_j \in End(V_1 \otimes \cdots \otimes V_n)$ are defined by

$D_j(u_1 \otimes \cdots \otimes u_n) = \frac{\langle \lambda_{j-1}, \lambda_{j-1} \rangle - \langle \lambda_j, \lambda_j \rangle}{2(\delta + 1)} u_1 \otimes \cdots \otimes u_n$

$\lambda_j = (\beta - 1)^{-1}(\sum_{i=1}^n \chi_i) + \sum_{i=1}^j \chi_i$

$X_j$ are the weights of $u_j \in V_j$.

These definitions are equivalent.
17. Modular Invariance

Fact. The $y_j$-part of the elliptic KZ system is modular invariant, i.e., does not change under

$$\hat{y}_i = \frac{y_i}{c\tau + d}, \quad \hat{\tau} = \frac{a\tau + \beta}{c\tau + d}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$$

$$\Gamma(N) = \{ A \in SL_2(\mathbb{Z}) \mid A \equiv Id \text{ mod } N \}$$

(congruence subgroup)

However, the $\tau$-equation is not completely invariant. If you apply $A \in \Gamma(N)$, you have to subtract a linear combination of $y_j$-equations with suitable coefficients, and then you get almost the original equation:

$$x \frac{\partial F}{\partial \tau} = \sum_{j=1}^{n} L_{y_j}(\tau) F + \sum_{j<i} S_{ij} (y_i - y_j) F$$

$$+ \sum_{i=1}^{n} \frac{cS_i}{2N(c\tau + d)} F$$

where $S_i$ is the value of the Casimir of $G_j$ in $V_i$. Therefore, we have a corresponding invariance statement about solutions.
claim. The fundamental solution \( \mathcal{F} \) transforms under a modular transformation \( A \in (a \; b ; c \; d) \in \Gamma(N) \) as follows:

\[
\mathcal{F}(\hat{y}_1, \ldots, \hat{y}_m | \tau) = (ct+d)^{1/2N} \sum_i \delta_i X \mathcal{F}(y_1, \ldots, y_m | \tau) \chi(A)
\]

where \( X(A) \) is a projective representation of \( \Gamma(N) \) in \( V_1 \otimes \cdots \otimes V_m \). Thus, our construction assigns to every set of representations \( V_1, \ldots, V_m \) of the an action of \( \Gamma(N) \) in their tensor product. This action can be suitably extended to an action of \( \text{SL}_2(\mathbb{Z}) \) in \( \text{End}(V_1 \otimes \cdots \otimes V_m) \). It is difficult and interesting to compute it.

18. Monodromy

The monodromy of the elliptic KZ system with respect to the \( z \)-variable is easily computable since it reduces to interchanging of intertwines, the same as for the usual KZ equations.
For $\mathfrak{sl}_2$, we have the "exchange relations"
\[
\Phi^{\lambda_1 \lambda_0} (z_1) \Phi^{\lambda_2 \lambda_1} (z_2) = \mathcal{A}^* \sum_{\nu, \nu_0} R_{\lambda_1 \lambda_0} (\lambda_2, \lambda_0) \nu_0 \nu_0 \sigma \Phi^{\nu_0 \lambda_0} (z_2) \Phi^{\lambda_2 \nu_0} (z_1)
\]
where $R$ is the constant quantum $R$-matrix (without a spectral parameter). Using this relation, we can easily give an explicit formula for what happens to the fundamental solution $\mathcal{F}$ when $y_i$ interchanges with $y_i + 1$ (multiplication by an $R$-matrix) or when $y_i$ goes around a cycle on the torus $\mathbb{C}/\langle 1, \tau \rangle$ (a product of $R$-matrices and diagonal matrices). Thus, we can compute the monodromy group of the elliptic KZ equations. Note that this group will be (in general) the braid group of the torus but its central extension, since the elliptic KZ equations are not a local system on $(\mathbb{C}/\langle 1, \tau \rangle)^n$ but only a projective local system. It has to do with the fact that although the automorphisms $\beta$ and $\gamma$ of $\mathfrak{sl}_N$ defined earlier commute with each other, the corresponding
matrices $B$ and $C$ in $GL_N$ do not. They generate a Heisenberg group of order $N^3$. This group plays an important role in the theory of the elliptic KZ equations, in particular, it gives rise to a 2-cocycle on the braid group.

It is interesting that if all $V_i$ are fundamental representations, and $n$, $N$ are coprime, the monodromy representation of the extended braid group of the torus arising from the elliptic KZ equation factors through the Cherednik "Double Affine Hecke Algebra".