Title: Coverings of $\mathbb{P}^1-{0, 1, \infty}$ with restricted "vertical" ramifications. (Moduli spaces, Galois representations and L-functions)

Author(s): Ihara, Yasutaka

Citation: 数理解析研究所講究録 (1994), 884: 105-111

Issue Date: 1994-09

URL: http://hdl.handle.net/2433/84272

Type: Departmental Bulletin Paper

Textversion: publisher
Coverings of \( \mathbb{P}^{1} \setminus \{0,1,\infty\} \) with restricted "vertical" ramifications.

Yasutaka Ihara  
RIMS, Kyoto University

Let \( S \) be any set of prime numbers, and put 
\[
\mathbb{Z}_S = \mathbb{Z}[\frac{1}{p}; p \in S],
\]
\( Q_S : \) the maximal Galois extension over \( \mathbb{Q} \) unramified outside \( S \)'s
\( S_0, \pi_1(S\mathbb{Z}_S) = \text{Gal}(Q_S/\mathbb{Q}) \). We propose to study the action of this group on
\[
\pi_1^{(S)} := \text{Ker}(\pi_1(\mathbb{P}^{1} \setminus \{0,1,\infty\}/\mathbb{Z}_S) \to \pi_1(S\mathbb{Z}_S)),
\]
where
\[
\frac{\mathbb{P}^{1} \setminus \{0,1,\infty\}}{\mathbb{Z}_S} = \text{Spec} \mathbb{Z}_S[t, \frac{1}{t}, \frac{1}{t-1}], \quad (t: \text{a variable}).
\]
In terms of Galois theory of function fields, \( \pi_1^{(S)} = \text{Gal}(M_S/Q(t)) \),
where:
\[
M \rightarrow \max \text{Gal}(\mathbb{Q}(t)/\mathbb{Q}(t)) \text{ unram. outside } t=0,1,\infty
\]
\[
\begin{array}{c}
\pi_1^{(S)} \rightarrow \text{max Gal}(\mathbb{Q}(t)/\mathbb{Q}(t)) \text{ in } M \text{ in which } v_2(l \notin S)
\end{array}
\]

Here, \( v_2 \) is the unique extension of the \( l \)-adic valuation of \( \mathbb{Q} \) to \( \mathbb{Q}(t) \) such that \( l \) is a prime element and the residue class of \( t \) is transcendental over \( \mathbb{F}_2 \).

1) Although the titles are not the same, this is a resume of my talk at the conference on March 28, 94.
We have the following two short exact sequences

\[(*) \quad 1 \rightarrow \text{Gal}(M/\mathcal{O} M) \rightarrow \text{Gal}(M/\mathcal{O}(t)) \rightarrow \text{Gal} \left( M_{S}/\mathcal{O}(t) \right) \rightarrow 1 \]

\[\quad \text{F}_2 \text{ (free profinite)} \quad \text{rank} \quad 2 \quad \text{S} \]

\[(**) \quad 1 \rightarrow \text{Gal}(M_{S}/\mathcal{O}(t)) \rightarrow \text{Gal}(\mathcal{O}S/\mathcal{O}) \rightarrow 1 \]

The most basic question is, perhaps, whether (*) is useful in the (future) study of \( \text{Gal}(\mathcal{O}S/\mathcal{O}) \). I cannot say anything about this now. Here, I state some results of my "first thought" related to (**).

**Terminology:** 
- "S-number": integers whose prime factors all belong to \( S \);
- "S-group": finite group whose order is an S-number;
- "pro-S group": proj. limit of S-groups (1|S| = 1 \Rightarrow \text{pro-nilpotent},
1|S| = 2 \Rightarrow \text{pro-solvable}).
- \( F_2^{pro-S} \): the pro-S completion of the free group of rank 2, i.e., the projective limit of all finite S-groups appearing as quotients of \( F_2 \).
Statement of results:

(i) Ramification indices of $t=0,1,\infty$ in any finite subextensions of $M_s/Q_s(t)$ are $S$-numbers. 3)

(ii) For any open subgroup $H \subset \pi_1^{<S>}$, its abelianization $H^{ab}$ is a direct product of a pro-$S$ group and a finite group.

These two are saying that $\pi_1^{<S>}$ as a quotient of $\widehat{F_2}$ is not so big. The next (iii) says something to the opposite direction.

(iii) $\text{Gal}(M_s^*/M_s)$, the kernel in (x), contains no non-trivial $S$-group as its quotient. In particular, $\widehat{\pi_2} \to \pi_2$ factors through $\pi_1^{<S>}$ as $\widehat{F_2} \to \pi_2 \to F_2$ pro-$S$ (both $\to$ are surjective).

About the exact sequence (x): (iv) The standard Puiseux embedding $\mathcal{M} \to \mathcal{O}[t] = \bigcup_{i \geq 0} \mathcal{O}[t^{1/2^i}]$ and $M_s \to Q_s[t]$, and $M_s$ is stable under the coefficientwise $\text{Gal}(Q_s(t))$-action on $Q_s[t]$. This $\text{Gal}(Q_s(t))$-action on $M_s$ gives a nice splitting of (x).

Remarks: If $S = \emptyset$, then $\Omega_s = \Omega$, $M_s = \Omega\mathbb{T}$ and $\pi_2^{<S>} = \{1\}$.

When $S = \{p\}$, I do not know whether $\pi_1^{<p>} \to F_2$ pro-$p$ or not.

When $S = \{2, p\}$ or $\{p, \infty\}$, $\pi_1^{<S>}$ is not a pro-$S$ group.

When $S = \{\text{all primes } l\}$, then $Q_s = \overline{Q}$, $M_s = M$, and $\pi_1^{<S>} = \widehat{F_2}$.

---

3) This property depends on the choice of three points on $P^1_t$, $t=0,1,\infty$.

If they were, e.g., $t=0,1,2,\infty$, then this property would not hold (unless $S \geq 2,3$).
Main ingredients for proofs.

(i) As T. Saito noted, (i) is obtained directly from the generalized Abhyankar lemma ([SGA 1] Exp XIII).

(ii) This proof relies on a result of Coleman [Co]. More precisely, it is reduced to the following statement which is (essentially) in [Co]:

Let $A$ be an abelian variety over a number field $k$, and $S$ be any set of primes of $k$. Assume $A$ has good reduction outside $S$. For each positive integer $n$ with $(n, S) = 1$, let $A[n]$ denote the group of all $n$-torsion points of $A(k)$, and $K[n]$ be its subgroup generated by the kernel of reduction mod $v$ in $A[n]$, where $v$ runs over all prime divisors of the field $k(A[n])$ dividing $n$. Then the order of $A[n]/K[n]$ is bounded by a positive number which depends only on $A$ and $k$ (in fact, only on $A \otimes \overline{k}$).

(iii) The proof relies on standard arguments of Grothendieck's ([SGA 4]) on descent of étale coverings; the only additional points to be checked are:

(a) For any finite subextension $L/Q_5(t)$ in $M_5$, the integral closure of $\overline{L}/\mathbb{Z}_5$ in $L$ is regular outside $S$ (including points above $t=0,1,\infty$ as long as they are not above $S'$).
(b) The pro-$S$ completion of the fundamental group of a compact Riemann surface of genus $> 1$ has trivial compact.

The assertion (a) can be checked easily, while (b) is proved in [Na].

(iv) The point is to prove the $\mathbb{Q}_S$-rationality of places of $M_S$ above $t \to 0, 1, \infty$. This follows by using the purity of branch loci on suitable Fermat curves whose exponents are $S$-numbers.

Some open problems:

(Problem I) Characterize $\pi_1^{\text{ess}}$ as quotient of $\hat{F}_2$.

Related questions:

(Q1) Is $\pi_1^{\text{ess}}$ the biggest quotient of $\hat{F}_2$ on which $\text{Gal}(\bar{\mathbb{Q}}_S)$ acts trivially?

(Q2) Is the center of $\pi_1^{\text{ess}}$ trivial?

(Q3) In connection with the result (ii), let $H^a_0$ denote the coprime-to-$S$ part of the torsion subgroup of $H^a$. Then what can one say about the group $\frac{H^a_0}{H^a}$?
(Problem II) Is the homomorphism
\[ \Phi_S : \text{Gal}(\bar{\mathbb{Q}} / \mathbb{Q}) \rightarrow \text{Aut} \pi_1^{(S)} \]
defined by the splitting (iv) of the exact sequence (xx) injective?

When \( S = \{ \text{all primes} \} \), \( \Phi_S \) is injective by the well-known injectivity of Belyi for the Galois representation \( \text{Gal}(\bar{\mathbb{Q}} / \mathbb{Q}) \rightarrow \text{Out} \pi_1(\mathbb{P}^1 - \{0,1,\infty\} / \mathbb{Q}) \).

I do not know at present whether \( \Phi_S \) is injective in any other cases, e.g., even when \( S = \{ \text{all primes} \} - \{p\} \).

In general, let \( \mathbb{Q}^*_S \) \( (\mathbb{Q} \subset \mathbb{Q}^*_S \subset \mathbb{Q}_S) \) denote the field corresponding to the kernel of \( \Phi_S \). What we know about \( \mathbb{Q}^*_S \):

(\#) \( \mathbb{Q}^*_S \) contains all higher circular \( S \)-units (the obvious generalization of higher circular \( L \)-units in [A-I]).

(\###) Let \( n \geq 1 \), and \( S = S_n = \{ p ; p \text{ divides } n(n-1) \} \). Assume \( \pi_1^{(S)} \) is center-free. Then \( \mathbb{Q}^*_S \) contains the splitting field of the equation \[ x^{n-2} + 2x^{n-3} + 3x^{n-4} + \ldots + (n-1)x + n = 0 \].
References:

[AJ] G. Anderson-Y. Ihara; Pro-l branched coverings of $\mathbb{P}^1$
and higher circular l-units; Ann of Math 128 (1988), 271-293.


[Na] H. Nakamura; Galois rigidity of pure sphere braid groups

[SGA 1] A. Grothendieck; Revêtements étals et groupe fondamental,
SLN 224.