Title
Representations and Pseudo-representations (Moduli spaces, Galois representations and L-functions)

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Citation
数理解析研究所講究録 (1994), 884: 24-26

Issue Date
1994-09

URL
http://hdl.handle.net/2433/84281

Type
Departmental Bulletin Paper

Textversion
displayed
Representations and Pseudo-representations

(Abstract)

by Henri Carayol

(I) Representations over local rings ([C])

Let $G$ be an abstract group and $R$ a local ring with maximal ideal $m$ and residue field $F$. We define a $d$-dimensional representation of $G$ over $R$ as usual, i.e. as an homomorphism:

$$\rho : G \rightarrow GL_d(R);$$

two such representations are called equivalent if one is conjugate of the other by some $M \in GL_d(R)$. The residual representation $\bar{\rho} : G \rightarrow GL_d(F)$ is obtained by reducing modulo $m$.

Our first result is the following:

**Theorem 1.** — Suppose $\rho$ and $\rho'$ are two $d$-dimensional representations of $G$ over $R$. Assume:

(a) $\forall g \in G$, trace $\rho(g) = \text{trace} \, \rho'(g)$, 
(b) $\bar{\rho}$ is absolutely irreducible;

then $\rho$ and $\rho'$ are equivalent.

My paper [C] also contains some "Schur-type" result, which allows, under suitable hypothesis, to realize a representation over a subring where the trace takes its values. As a consequence, we give a construction of Galois representations associated to some modular forms defined over rings. This kind of results can now be viewed as corollaries of a theorem of Louise Nyssen on pseudo-representations, which I will explain in the next paragraph.

(II) Pseudo-representations

Pseudo-representations were first introduced in dimension 2 by Andrew Wiles, as a sort of substitute for representations; they played a crucial role in the construction, using congruences between
modular forms, of some \( \ell \)-adic Galois representations ([W]). Taylor ([T]) generalized them to any dimension.

A pseudo-representation of dimension \( d \) of a group is a function on this group which satisfies the formal properties of the trace of a representation: two of those properties are obvious, and the third one reflects a certain polynomial identity on matrix rings ([P]). More precisely:

**Definition.** — Let \( G \) be a group and \( R \) a (commutative) ring. A \( d \)-dimensional pseudo-representation of \( G \) over \( R \) is a map \( T : G \to R \) which satisfies:

(a) \( T(1) = d \),

(b) \( \forall x, y \in G, \ T(xy) = T(yx) \),

(c) \( \forall x_1, \ldots, x_{d+1} \in G, \ \sum_{\sigma \in S_{d+1}} \epsilon(\sigma) T_{\sigma}(x_1, \ldots, x_{d+1}) = 0 \),

where \( \epsilon(\sigma) \) denotes the signature of \( \sigma \), and where \( T_{\sigma} \) is defined as follows: if \( \sigma \) is decomposed into a product of disjoint cycles (including fixed points viewed as 1-cycles):

\[
\sigma = (i_1^{i_2} \cdots i_1^{k_1}) \cdots (i_m^{i_1} \cdots i_m^{k_m})
\]

\[
T_{\sigma}(x_1, \ldots, x_{d+1}) = T(x_{i_1^{i_2} \cdots i_1^{k_1}}) \cdots T(x_{i_m^{i_1} \cdots i_m^{k_m}})
\]

(this makes unambiguous sense thanks to (b)).

The trace of any representation is a pseudo-representation, and according to [T] the converse is also true over an algebraically closed field of characteristic 0. Because theorem 1 asserts that we have a good theory for those representations over local rings which reduce to absolutely irreducible representations, it seems reasonable to compare both notions in this context:

**Theorem 2 [N].** — Let \( T \) be a \( d \)-dimensional pseudo-representation of a group \( G \) over an henselian separated local ring \( R \). We assume that its reduction \( \overline{T} \) modulo the maximal ideal is the trace of some absolutely irreducible \( d \)-dimensional representation over the residue field. Then \( T \) itself is the trace of a \( d \)-dimensional representation of \( G \) over \( R \) (well-defined up to equivalence according to theorem 1).
Note: A recent preprint of K. Saito ([S]) contains related results in the case of 2-dimensional representations.

(III) References


[S] K. Saito. — *Representation varieties of a finitely generated group into SL_2 or GL_2*, preprint RIMS Kyoto University.


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