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Generalized Coherent States and Holonomic Phases

Kuzuha-Asahi 1-12-32, Hirakata City and
Department of Physics, Ritsumeikan University

松本雅生 (Masao Matsumoto)

Department of Physics, Ritsumeikan University

倉辻比呂志 (Hiroshi Kuratsuji)

1. Introduction

The main feature of quantum mechanics is the existence of the probability amplitude which underlies all the atomic processes.[1] In particular, the phase of the probability amplitude (or wave function) is crucial, since it governs the "interference "phenomena, while the square of the absolute value of the amplitude gives the probability itself. The exisitence of the phase represents the fundamental departure from the usual classical probability theory.

Recently, the role of the phase in quantum mechanics has renewed interest from various motivations. Especially, if one is concerned with the cyclic change in quantum state, then one gets the so-called geometric phase or topological phases.[2] The monumental work of the geometric phase has been done by Dirac, in which the non-integrable (path dependent) phase factor has been firstly recongnized.[3] The historical development of geometric phases has been made in two distinct aspects. The first aspect has been known in early sixties in molecular physics in connection with the consideration of adiabatic processes[4] which finally leads to the so called Berry's phase.[2] The other aspect, which seems to be less noticed than the formar, has a long history. It stems from the path integral in the representation of "generalized coherent states", in which integration paths are defined on the phase space in a generalized sence.[5,6] This approach has many physical applications, for example, of collective motions of fermion systems[7]

In this note, we shall study the latter aspect mentioned above; namely, we scrutinize the feature of the geometric phase that is inspired from the generalized coherent state.

We bear the name;"canonical phase" to this specific phase. Why do we need to understand the canonical phase in the framework of path integrals? This comes from the fact that the geometric phase reflects the global structure of the quantum system. On the other hand, the global aspect of quantum mechanics is naturally described by path integral formulation.[8] Now, the coherent state path integrals invoke the generalized coherent states(CS) $|Z\rangle$, which are parametrized by a point Z on some manifolds (in many cases complex manifolds) whose coordinates are related to some external or macroscopic parameters of the systems. The manifolds just turn out to be the generalized phase spaces mentioned above, over which the integral is carried out. By considering the propagator for a roundtrip of a point on the generalized phase space, the canonical phase is obtained naturally as the contour integral of the "connection" between two coherent states which lie infinitesimally each other.[9] The procedure looks like that used in the dynamical theory of the adiabatic phase,[10] where the the adiabaticity controls the manner of change of phase factor appearing in the quantum system which suffers from the external field that is changing adiabatically. In contrast to this, in the case of canonical phase, we have no concept of adiabaticity; instead of this, we have a substitute; that is, the dynamical principle controlling the motion of Z; namely, a cyclic path on the manifold of CS is determined by quantum action principle, which yields a dymamical equation on the parameter space.

In this paper we shall give some concrete examples of canonical phases; these are constructed from several CS whose generalized phase spaces are non-compact as well as compact. We take up the examples from the spin CS, boson CS, and Lorentz or SU(1,1)CS. The physical sytems under cosideration are well-known ones in magnetic resonance and quantum optics. The contents of sections 2 to 7 are as follows. In section 2 we review the general theory of the canonical phase. The canonical phase with use of spin(SU(2))CS is dicussed in section 3, which follows the recent work by us.[11] These two sections are the preparation to the subsequent sections 4 and 5, where we investigate the geometric character of the canonical phases for non-compact CS; the case of the boson CS (section 4) and the case of Lorentz CS (section 5). We also study the relation between the canonical phase and the semiclassical quantization scheme in section 6. Section 7 gives a summary. The details of the present study will be given elsewhere.[12]

2. General theory

First we give a concise review of the general theory of the canonical phase.[7] Using CS, the propagator starting from and ending at the state $|Z_0\rangle$ (in time T) can be written as

$$\langle Z_0|\, Pe^{-\frac{i}{\hbar}\int \hat{H}(t)\,dt}\, |Z_0\rangle = \oint e^{\frac{i}{\hbar}S(C)}\; D\mu(Z) \tag{2.1}$$

where P means time ordered products and the integral are performed over all cyclic paths. Here $D\mu(Z) \equiv \prod_{t=0}^{T} d\mu(Z(t))$, where $d\mu(Z)$ denotes the invariant measure on the generalized phase space specified by the complex vector $Z = (z_1, z_2, \cdots, z_n)$ and S is the action for each path:

$$\begin{aligned} S(C) &= \int_0^T \langle Z|\, i\hbar\frac{\partial}{\partial t} - \hat{H}(t)\,|Z\rangle\, dt \\ &= \int_0^T (\langle Z|\, i\hbar\frac{\partial}{\partial t}|Z\rangle - H(Z,Z^*))dt \equiv \int_0^T Ldt, \end{aligned} \tag{2.2}$$

where $H(Z,Z^*) \equiv \langle Z|\,\hat{H}\,|Z\rangle \equiv H(t)$ is the expectation value of the Hamiltonian.

By using the overlap of the unnormalized CS $\left|\tilde{Z}\right\rangle$: $F(Z,Z^*) = \left\langle \tilde{Z}|\tilde{Z}\right\rangle$, the Lagrangian can be cast into the form

$$L = \sum_{k=1}^{n} \frac{i\hbar}{2}\left(\frac{\partial \log F}{\partial z_k}\dot{z}_k - \frac{\partial \log F}{\partial {z_k}^*}\dot{z_k}^*\right) - H(Z,Z^*). \tag{2.3}$$

To get (2.1), we used the property of "the resolution of unity ": $\int |Z\rangle\; d\mu(Z)\,\langle Z| = \mathbf{1}$. Now we select one cyclic path $C(Z(t))$ in the generalized phase space and caluclate

$$\Gamma(C) = \oint_C \langle Z|\, i\hbar\frac{\partial}{\partial t}|Z\rangle\; dt \tag{2.4}$$

which we shall call "canonical phase ". It is a topological phase obtained in the light of the CS path integral formulation. Here we choose this path C by taking the semiclassical limit of (2.1), which amounts to

$$\delta \int \langle Z| i\hbar \frac{\partial}{\partial t} - \hat{H}(t) |Z\rangle \, dt = 0$$

which yields the equations of Z

$$i\hbar \sum_{j=1}^{n} g_{i\bar{j}} \dot{z}_j = \frac{\partial H}{\partial z_i^*}, -i\hbar \sum_{j=1}^{n} g_{i\bar{j}} \dot{z}_j^* = \frac{\partial H}{\partial z_i} \tag{2.5}$$

with

$$g_{i\bar{j}} = \frac{\partial^2 \log F}{\partial z_i \partial z_j^*},$$

which is the metric of the generalized phase space.

Some explanation is in order regarding the meaning of the choice of circuit. Taking the semiclassical limit is the condition of choosing the specific path among all the possible paths. Here the parameter controlling the limit is the Planck constant. This feature corresponds to the situation of taking the adiabatic limit for the quantum transition for which the transition takes place between the states labelled by the discrete eigenstates each of which has the same quantum number.[10] Consider a special case of the vanishing Hamiltonian; for which the solution of the equation of motion (2.5)becomes trivial. In this case, the transition amplitude is simply given as

$$\prod_{k=1}^{\infty} \langle Z_{k+1} | Z_k \rangle = \exp[i\Gamma(C)]$$

and the topological feature becomes manifest yielding the topological invariants defined over the compact phase space that are represented by the first Chern class. In the case of the non-vanishing Hamiltonian, one should evaluate the phase Γ as a function of the loop (the solution of (2.5) independent of the Hamiltonian. The problem here is that: in general it is not easy to obtain a cyclic motion that makes difficult to calculate $\Gamma(C)$. In the following sections we shall take up simple models to allow simple specific solutions for the equation of motion.

Next we will give the principle of an experimental detection method of the canonical phase. After a cyclic motion the state vector totally aquires a phase $\Phi = \frac{1}{\hbar}S(C)$(eq.(2.2)) which, by eq. (2.2), consists of two parts: $\Phi = \frac{1}{\hbar}(\Gamma - \Delta)$. Here Γ is the canonical phase and

$$\Delta(C) = \oint_C H(Z, Z^*)dt \tag{2.6}$$

is the time integral of the expectation value of the Hamiltonian which may be called the "dynamical phase ". How can we extract the effect of Γ? The answer can be given as follows. Consider two particle beams, each of which has the same Δ and in one of which Γ vanishes. Interferencing these two beams reveals the canonical phase itself.

3. The topological phase associated with the SU(2) CS

In this chapter general we shall apply the general theory to the case of the SU(2)CS. Consider a particle with spin J possessing the magnetic moment μ in a magnetic field B. Here we take $\mathbf{B}(t) = (B_0 \cos\omega t, B_0 \sin\omega t, B)$ that is a static field along the z-axis plus an additive field rotating perpendicular to it with the frequency ω, which is familiar in magnetic resonance. The system may be described by the spin (angular momentum) or SU(2) CS $|z\rangle$ — a typical CS for a compact group — defined as :

$$|z\rangle = (1 + |z|^2)^{-J} e^{z\hat{J}_+} |0\rangle, \tag{3.1}$$

where $|0\rangle = |J, -J\rangle$ satisfying $\hat{J}_z |0\rangle = -J |0\rangle$ and $\hat{J}_\pm$are usual spin operators and z takes any complex values. The Hamiltonian of the system is

$$\hat{H}(t) = -\mu\mathbf{B}(\mathbf{t})\bullet\mathbf{J}, \tag{3.2}$$

where $\mathbf{J} \equiv (\hat{J}_1, \hat{J}_2, \hat{J}_3)$ is a matrix vector satisfying $\mathbf{J} \times \mathbf{J} = i\mathbf{J}$. By using the polar coordinate $z = \tan\frac{\theta}{2}e^{-i\phi}(0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi)$ (which shows that in the case of the spin CS the generalized phase space is isomorphic to S^2) ,(3.1),(3.2) and ref[5], $H(t) \equiv H(z, z^*)$

is expressed as

$$H(z,z^*) = H(\theta,\phi) = -\mu J[B_0 \sin\theta\cos(\phi-\omega t) - B\cos\theta]. \tag{3.3}$$

The variation equation is

$$\dot{\theta} = -\frac{\mu B_0}{\hbar}\sin(\phi-\omega t), \dot{\phi} = -\frac{\mu}{\hbar}[B_0\cot\theta\cos(\phi-\omega t)+B], \tag{3.4}$$

which allows a special solution

$$\phi = \omega t,\ \ \theta = \theta_0 (= \text{const}), \tag{3.5}$$

where the following relation should be hold among the parameters θ_0, B, B_0:

$$\cot\theta_0 = -\left(\frac{B}{B_0} + \frac{\hbar\omega}{\mu B_0}\right). \tag{3.6}$$

The solution of the form (3.5) shall be called the "resonance "solution. In fact, that corresponds to the resonance solution for the case of the forced oscillation. The set of parameters (B, B_0, ω) satisfying (3.6) that gives the same θ_0 yields the same "resonance solution ". The set forms a surface in the paramter space (B, B_0, ω), which we call "invariant surface "hereafter. Note that the quantities in the right hand side of (3.6)are all given in terms of natural constants that allow to be compared with the experimental situation. From (3.6) we can imagine that this solution forms the surface in the 3-dimensional parameter space (B, B_0, θ_0) which determines the resonance condition.

Next we turn to the evaluation of the canonical phase fitting to this special solution. That leads us to

$$\Gamma(C) = 2\pi J\hbar(1-\cos\theta_0) = -J\hbar\Omega(C) \tag{3.7}$$

where $\Omega(C)$ is the solid angle that the curve C subtends at the origin of the phase space. On the other hand, the dynamical phase (2.6)is given by

$$\Delta(C) = \frac{2\pi\mu J}{\omega}(B_0\sin\theta_0 - B\cos\theta_0). \tag{3.8}$$

The important point is that the canonial phase depends only on θ_0. Therefore any point on the "invariant surface "gives the same canonical phase. But the dynamical phase is not determined only by θ.

We can make use of the above charactor of the canonical phase to detect its effects in experiments. Consider a beam splitted into two parts, each of which suffers the magnetic field whose parameters (B, B_0, ω) are on the invariant surface. Recombining these two gives the effect of the dynamical phase. Or we can set two parts, each of which has the parameters giving the same $\Delta(C)$. One of the simplest case comes true when $\Delta(C)$ vanishes. For brevety, we put the frequency fixed. Then (3.5) means $\cot\theta_0 = \frac{B_0}{B}$, which together with (3.6) results in $\omega = -\frac{\mu(B_0^2+B^2)}{\hbar B}$. In this case the beam is prepared in such a way that one of which suffers the field satisfying the above condition and does another no fields. Recombinig these two reviels only the effect of the canonical phase. The interference pattern goes as $I \propto \cos^2(\frac{\Gamma(C)}{2\hbar})$.

One can easily find that the resonance solution meat linear stability conditin locally, which is necessary to experimental detections. Similarly we can find that the solutions obtained in the next chapter also meet the condition. So we will not repeat to discuss it there.

4. Non-compact Coherent State I: Boson CS

Consider a harmonic oscillator driven by an external force. The hamiltonian is :

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + {\omega_0}^2\hat{q}^2) + F(t)\hat{q}, \tag{4.1}$$

which can be written by boson creation and anhialation operators as

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + \beta(t)\hat{a}^\dagger + \beta^*(t)\hat{a}. \tag{4.2}$$

This type of Hamiltonians appears in the problems of detecting gravitational radiation[14] and of quantum optics.[15] [16] In this section we shall take up the second one for example and discuss the possibility of finding the effect of the holonomic phase.

Consider a single mode electric field inside a cavity driven externally by a coherent driving field. If we neglect the cavity damping, we have the Hamiltonian:

$$\hat{H} = \hbar\omega_0\hat{a}^\dagger\hat{a} + i\hbar(\hat{a}^\dagger E(t)e^{-i\omega t} - \hat{a}E^*(t)e^{i\omega t}), \tag{4.3}$$

which belongs to the type of (4.2). The first term denotes the cavity mode Hamiltonian where ω_0 means the fundamental cavity resonance and does the second one Hamiltonian for the coherent driving field respectivly. Here $E(t)$ is the driving field amplitude, while ω the driving frequency. If the cavity contains the medium with unharmonicity Hamiltonian contains the term having $\hat{a}^{\dagger 2}\hat{a}^{2}$.[16] Here we will consider the case when the unharmonicity of the medium can be neglected. We also assume, for brevety, $E(t) = -\tilde{E}i$ (pure imaginary const.).

The description of the system may fall in the field ofthe boson CS: $|z\rangle = e^{-\frac{1}{2}|z|^2} e^{z\hat{a}^{\dagger}} |0\rangle$ Here z satisfies $\hat{a}\,|z\rangle = z\,|z\rangle$ and propotional to complex amplitude of the classical electromagnetic field obtained as the solution of Maxewell eqation.[15] In terms of the polar coodinate, $z = re^{i\theta}$, a series of evaluation of the quantities goes as follows.

The variation equation :
$$\dot{r} = -\tilde{E}\sin(\theta + \omega t),$$
$$r(\dot{\theta} - \omega_0) = -\tilde{E}\cos(\theta + \omega t).$$

The resonance solution: $r = r_0,\ \theta = -\omega t.$

For the solution, the relation that specifies

the "invariant surface ": $r_0 = \left|\frac{E}{\omega+\omega_0}\right|$

The canonical phase : $\Gamma(C) = 2\pi\hbar r_0^2$

The dynamical phase: $\Delta(C) = \frac{2\pi}{\omega}[\hbar(\omega_0^2 r_0^2 - 2\tilde{E}r_0)]$.

Now one of the sufficient conditions for $\Delta(C)$ to vanish is that the integrand its-self vanishes, which results in $\omega = -\frac{1}{2}\omega_0$.

Then the canonical phase is $\Gamma(C) = \frac{8\pi\hbar}{\omega_0^2}|E|^2$. The experimental detection method is as the same as that in §3, due to one mentioned at the end of §2.: The physical quantity that we observe in the experiment is the phase of the electromagnic or photon field.

5. The non-compact CS II: The Lorentz coherent state

In this section we shall deal with the holonomic phase that is connected with another class of non-compact coherent state; the SU(1,1) coherent state (See Appendix A.); alternatively we call the Lorentz coherent state, since SU(1,1) is locally isomorphic to Lorentz group of 2+1 dimension.

Now for our purpose it needs to take account of the linear operator that is just the generators of the Lorentz CS. One of the realization of this algebra is given by a set of bilnear forms of boson creation and anihilation operators for a single mode electromagnetic field :

$$\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2,\ \hat{K}_- = \frac{1}{2}\hat{a}^2,\ \hat{K}_0 = \frac{1}{4}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}). \tag{5.1}$$

In this case $k = \frac{1}{4}$ or $k = \frac{3}{4}$. Appendix A and (5.1)show that, for $k = \frac{1}{4}$, $|0\rangle$ coincides the photon vacuum and does the state of photon number being one for $k = \frac{3}{4}$. In this realization, the operator $S(\zeta) = e^{\zeta\hat{K}_+ - \zeta^*\hat{K}_-} = e^{\frac{1}{2}(\zeta(\hat{a}^\dagger)^2 - \zeta^*\hat{a}^2)}$ is nothing but a "squeeze operator "with a squeeze parameter $\tanh|\zeta|$ and a rotating angle $\phi/2$.[17]

The physical system we take up here is the composed system of cavity mode and the squeezed state generating interaction.[16] The hamiltonian for the system is

$$\hat{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar[V^*(\hat{a}^\dagger)^2 + V\hat{a}^2], \tag{5.2}$$

where V is the interaction parameter including the effect of pumping light. Here we take $V = \kappa e^{i\omega t}$: an oscillating pumping light. Using the parametrization $z = \tanh(\frac{\tau}{2})e^{-i\omega t}$, $H(t) \equiv \langle z|\,\hat{H}\,|z\rangle$ can be expressed as

$$H(t) = H(\tau,\theta) = 2\hbar k[\omega_0 \cosh\tau + 2\kappa \sinh\tau \cos(\phi - \omega t)]. \tag{5.3}$$

A series of results for this case are the following.

The variation equation : $\begin{aligned} \dot{\phi} &= 2[\omega_0 + \kappa \coth\tau \cos(\phi - \omega t)], \\ \dot{\tau} &= 4\kappa \sin(\phi - \omega t). \end{aligned}$

The "resonance solution ": $\phi = \omega t, \tau = \tau_0 (= const)$.

The invariant surface, that is the sheet in the parameter space $(\omega, \omega_0, \kappa)$ on which τ_0 is constant, is determined by : $\coth\tau_0 = \frac{\omega - 2\omega_0}{4\kappa}$.

For the path C described by this solution,

the canonical phase : $\Gamma(C) = 2\pi\hbar k(\cosh\tau_0 - 1)$.

The dynamical phase : $\Delta(C) = \frac{4\pi\hbar k}{\omega}(\omega_0 \cosh\tau_0 + \kappa \sinh\tau_0)$.

We see a close analogy to the case of spin CS and boson CS that the canonical phase depends only on the "invariant surface "and the dynamical phase does not. The situation enables us to follow the previous method for experiments. Here we state the last one in section 3. Let us set a light beam splitting into two parts, each of which has the same

$\Delta(C)$ but non-vanishig $\Gamma(C)$ only for one. That occurs when $\omega = \frac{2}{\omega_0}[(\omega_0^2 - 2\kappa^2) + \frac{2\kappa\alpha}{\sinh \tau_0}]$, where $\sinh \tau_0 = \frac{1}{\kappa^2 - \omega_0^2}[\alpha\kappa \pm \omega_0(\kappa^2 + \alpha^2 - \omega_0^2)^{1/2}]$. Here α is a positive constant. We set two parts of the beam, for one of which ω satisfies the above condition and for another one $\omega = 0$. Then recombining them leads to the interference pattern shows solely the effect of $\Gamma(C)$.

Next let us give to some comments on related points. First the meaning of the "resonance solution "can be understood in a manner analogous to the the SU(2) case by means of Bloch like vector.[12] [18] Second in order to obtain the result for the adiabatic limit, we only have to let $\omega \to 0$ in the above argument.[12]

6. The Relation to Semiclassical Quantization Scheme

The models used in the preceding sections allow us to calculate the canonical phases in a non-trivial way. The basic point is that the Hamiltonians are time- dependent through the time-dependence of the external field. The cyclic change of the external field results in the phase holonomy. This fact implies that the non-trivial canonical phase arises only for the case that the time-dependent Hamiltonian. Note that in the case of time-dependent Hamiltonian the concept of energy spectrum has no meaning. Now, what is expected for the case of time-independent Hamiltonian?

In what follows, the relation between the time independent hamiltonian and the semiclassical quantization scheme is considered. Imagine an isolated system : a system that has no interaction with external ones. Then the Hamiltonian involves no time dependence and the motion of the parameter Z that determines the canonical phase lies on the surface of the constant energy: $H(Z, Z^*) = E$. After a cyclic change the wave function acquire the phase $\Phi = \frac{1}{\hbar}(\Gamma(C) - ET)$. Note that $\Gamma(C)$ depends only on Z, whereas does ET on t. For an isolated system the wave function must be single valued with respect to Z; the phase change associated with the cyclic change is given as before and the single valuedness requires it to be unity; namely,

$$\exp[i\Gamma(C)] = 1$$

and hence we have

$$\Gamma(C) = \oint_C \langle Z|\, i\hbar \frac{\partial}{\partial t}\, |Z\rangle\, dt = 2\pi\hbar m (m : \text{integer}). \tag{6.1}$$

That means that the canonical phase factor becomes trivial for the case of an isolated system that is described by a time-independent Hamiltonian. The condition (6.1)is crucial in the sense that it can be used as the device for determining the energy spectra of the system; so-called the semiclassical quantization conditon.[19] Φ is the phase change for the classical wave function The integral contour is determined by the action principle.

In this way, we have inevitably "non-holonomy" for the case of an isolated system. The single valuedness, which manifests the non-holonomical nature, determines the energy eigenvalues, while for the system set in the time varying external field the energy eigenvalue becomes nonsense. As a result of this, the non-trivial holonomic phase appears!

7. Summary

We have shown the following results for the canonical phase, which is considered to be a kind of the holonomic phases defined in terms of the coherent state path integral:

(1) :In the case of a simple non-compact CS as well as a compact one, there exists a simple path on the generalized phase space of CS ("resonance solution ") when the external parameters satisfy a specific condition("invariant surface ").

(2) :The holonomic phase depends only on the "invariant surface ".

(3) :The charactor of (2) enables us to construct experiments in which the effect of the holonomic phase or dynamical phase are extracted from each other.

(4) : We have "non-holonomy" when we are concerned with an isolated system; non-holonomic condition is equivalent to the single valuedness which results in the semi-classical quantization rule leading to the energy spectra.

Finally, a few prospective points are in order concerning the relation between the canonical phase and quantum optics. The one is the problem of generating the light corresponding to CS, which provides wide possibilities to make the light which has no classical counterpart such as so-called squeezed states. The other is, by making use of the frequent appearence of CS in various branches of physics, we may simulate the problems of other branches (i.e. field theory and condensed matter physics) by using optical devices. Thirdly, a different kind of understanding to topological phases that is more general than given here can be found in ref.[20].[20]

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APPENDIX A

Let us review the SU(1,1)CS concisely.[6] The discrete series generators $K_i (i = 1,2,3)$ of SU(1,1) algebra satisfy the followig commutation relations :

$$[\hat{K}_0, \hat{K}_\pm] = \pm\hat{K}_\pm, \ [\hat{K}_-, \hat{K}_+] = 2\hat{K}_0, \tag{A1}$$

where $\hat{K}_\pm = \pm i(\hat{K}_1 \pm i\hat{K}_2)$ are rasing and lowering operators of a SU(1,1) state. The eigen vectors of K_0 are specified by (k,m): $\hat{K}_0 \left|k, k+m\right\rangle = (k+m)\left|k,m\right\rangle$,
where k is a real number determined by the representation of SU(1,1) algebra and m is a non-negative integer. Specifically call $|0\rangle \equiv |k, m = 0\rangle$ and we get Lorentz CS:

$$|z\rangle = e^{\zeta\hat{K}_+ - \zeta^*\hat{K}_-}|0\rangle = (1-|z|^2)^k e^{z\hat{K}_+}|0\rangle, \tag{A2}$$

where $z = \tanh|\zeta|e^{i\phi}$.

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