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The mean square of Hecke L-series attached to holomorphic cusp-forms

by

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The aim of the present paper is to indicate a new approach to the mean square of Hecke L-functions attached to holomorphic cusp-forms, which is simpler than the one developed by A. Good in his important works [1][2] and is in fact able to yield a deeper result. We restrict ourselves to the case of $SL(2,\mathbb{Z})$ the full modular group, mainly for the sake of simplicity. The extension to an arbitrary congruence subgroup should not cause any essential difficulty, save for the technical complexity induced by the possible presence of many inequivalent cusps.

Thus, let A be a holomorphic cusp-form of an even integral weight k (≥ 12). We denote its Fourier expansion by

$$A(z)=\sum_{n=1}^{\infty}a(n)e(nz),$$

where $e(\cdot)=\exp(2\pi i\,\cdot)$ and z is on the upper half-plane as usual. We assume that A is a simultaneous eigen-function of all Hecke operators acting on the space spanned by all holomorphic cusp-forms of the weight k. Namely, we have, for any positive integer n, the relation

$$\frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b=1}^{d}\left(\frac{a}{d}\right)^{k/2}A\Big(\frac{az+b}{d}\Big)=t(n)A(z)$$

with a certain real number $t(n)$. This implies that

$$a(n)=a(1)t(n)n^{\frac{1}{2}(k-1)}.$$

The L-function and the Hecke series attached to A are defined by

$$L_A(s) = \sum_{n=1}^{\infty} a(n) n^{-s}, \qquad \mathrm{Re}(s) > \tfrac{1}{2}(k+1),$$

$$H_A(s) = \sum_{n=1}^{\infty} t(n) n^{-s}, \qquad \mathrm{Re}(s) > 1,$$

respectively. Both sums converge absolutely in the indicated ranges of the variable s, and continuable to entire functions.

Then our problem is to consider the mean value

$$I_A(T) = \int_0^T |H_A(\tfrac{1}{2} + it)|^2 dt.$$

Good [2] has already established the asymptotic formula

$$I_A(T) = TP(\log T,\, A) + E_A(T),$$

where $P(\cdot, A)$ is a linear polynomial and

$$E_A(T) \ll_A T^{2/3} (\log T)^c.$$

We shall show briefly an alternative proof of this estimate. An advantage of our argument over Good's is in that ours can yield an estimate of the mean square of the error term $E_A(T)$ that seems essentially the best possible.

Our basic idea is an adaptation, to the present situation, of the argument that we have developed in [5] for the analysis of the fourth power moment of the Riemann zeta-function:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt.$$

Thus we start from the expression

$$I_A(u, v; G) = (G\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} H_A(u + it) H_A(v - it) e^{-(t/G)^2} dt,$$

where $G > 0$ is arbitrary. This is an entire function of the complex variables u, v. In particular we have

$$I_A(P_T; G) = (G\sqrt{\pi})^{-1} \int_{-\infty}^{\infty} |H_A(\tfrac{1}{2} + i(T + t))|^2 e^{-(t/G)^2} dt,$$

where P_T is the point $(\frac{1}{2}+iT,\ \frac{1}{2}-iT)$ with an arbitrary $T>0$. We are going to establish an explicit formula for $I_A(P_T;G)$.

We have, on the other hand,

$$I_A(u,v;G)=|a(1)|^{-2}J_A(u+\tfrac{1}{2}(k-1),v+\tfrac{1}{2}(k-1);G)$$

with

$$J_A(u,v;G)=(G\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}L_A(u+it)L_{\bar{A}}(v-it)e^{-(t/G)^2}dt,$$

where $\bar{A}(z)$ denotes the complex conjugate of $A(-\bar{z})$. In the region of absolute convergence we have

$$\begin{aligned}J_A(u,v;G)&=\sum_{n,m\geq 1}\frac{a(n)\overline{a(m)}}{n^u m^v}\exp\big(-(\frac{G}{2}\log\frac{m}{n})^2\big)\\&=R_A(u+v)+J_A^{(1)}(u,v;G)+J_{\bar{A}}^{(1)}(v,u;G).\end{aligned}$$

Here

$$R_A(s)=\sum_{n=1}^{\infty}|a(n)|^2 n^{-s}$$

is the Rankin zeta-function for the form A, and

$$J_A^{(1)}(u,v;G)=\sum_{n,m\geq 1}\frac{a(n)\overline{a(n+m)}}{n^u(n+m)^v}\exp\big(-(\frac{G}{2}\log\frac{n+m}{n})^2\big).$$

To transform this double sum we introduce the function

$$w(x;u,G)=(1-x)^{-u}\exp\big(-(\frac{G}{2}\log(1-x))^2\big),\qquad 0\leq x\leq 1,$$

as well as its Mellin transform

$$\tilde{w}(s;u,G)=\int_0^1 x^{s-1}w(x;u,G)dx.$$

Since we have

$$\tilde{w}(s;u,G)=\frac{(-1)^p}{s(s+1)\cdots(s+p-1)}\int_0^1 x^{s+p-1}(\frac{\partial}{\partial x})^p w(x;u,G)dx,$$

the function $\tilde{w}(s;u,G)$ is regular except for $s=0,-1,-2,\ldots$, and of rapid decay in the half plane $\mathrm{Re}(s)>-B$ with an arbitrary fixed positive B; moreover it is entire with respect to u.

Then we have, by Mellin's inversion formula,

$$\begin{aligned} J_A^{(1)}(u,v;G) &= \sum_{n,m\geq 1} \frac{a(n)\overline{a(n+m)}}{(n+m)^{u+v}} w(\frac{m}{n+m};u,G) \\ &= \frac{1}{2\pi i} \sum_{n,m\geq 1} \frac{a(n)\overline{a(n+m)}}{(n+m)^{u+v}} \int_{(\eta)} (\frac{m}{n+m})^{-s} \tilde{w}(s;u,G)ds \\ &= \frac{1}{2\pi i} \int_{(\eta)} \{\sum_{m\geq 1} m^{-s} D_A(u+v-s,m)\} \tilde{w}(s;u,G)ds. \end{aligned}$$

Here the function D_A is defined by

$$D_A(s,m) = \sum_{n=1}^{\infty} \frac{a(n)\overline{a(n+m)}}{(n+m)^s}.$$

Checking the absolute convergence, we see that the last expression for $J_A^{(1)}(u,v;G)$ holds on the condition, e.g.,

$$\eta > 1, \quad \mathrm{Re}(u+v) > k+\eta.$$

Next, we need a spectral decomposition of $D_A(s,m)$. For this sake we introduce the Poincaré series

$$P_m(z,s) = \sum_{g\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} (\mathrm{Im} g(z))^s e(mg(z)),$$

where Γ_∞ is the stabilizer of the point at infinity. Then we have

$$D_A(s+k-1,m) = \frac{(4\pi)^{s+k-1}}{\Gamma(s+k-1)} \langle P_m(\cdot,s), |A|^2\rangle_k, \qquad \mathrm{Re}(s) > 1.$$

The inner product $\langle a\,, b\rangle_k$ is defined to be

$$\int_{\mathcal{F}} a(z)\overline{b(z)} y^k d\mu(z),$$

where $\mathcal{F}$ and $d\mu(z)$ are the usual fundamental region of $SL(2,\mathbb{Z})$ and the Poincaré metric on the upper half plane, respectively. We shall also use the convention

$$\langle a\,, b\rangle = \langle a\,, b\rangle_0.$$

At this stage we have to introduce some rudimental notion from the theory of non-holomorphic cusp-forms. Thus, let $\{\lambda_j = \varkappa_j^2 + \frac{1}{4};\ \varkappa_j > 0,\ j \geq 1\} \cup \{0\}$ be the discrete spectrum of the hyperbolic Laplacian acting on the Hilbert space of all real-analytic functions that are $SL(2,\mathbb{Z})$-automorphic and square integrable with respect to $d\mu$ over the region $\mathcal{F}$. Let ψ_j be the eigen-wave corresponding to λ_j so that it has the Fourier expansion

$$\psi_j(z) = \sqrt{y}\sum_{n\neq 0} \rho_j(n) K_{i\varkappa_j}(2\pi|n|y)e(nx),$$

where K_ν is the K-Bessel function of order ν. We may assume the set $\{\psi_j\}$ forms an orthonormal system, and moreover each ψ_j is a Maass wave. The latter means that ψ_j is a simultaneous eigen-function of all Hecke operators. Namely, we have, for any positive integer n,

$$\frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b=1}^{d}\psi_j\left(\frac{az+b}{d}\right) = t_j(n)\psi_j(z)$$

as well as

$$\psi_j(-\bar{z}) = \varepsilon_j\psi_j(z)$$

with a certain real number $t_j(n)$ and $\varepsilon_j = \pm 1$. In particular we have

$$\rho_j(n) = \rho_j(1)t_j(n), \qquad t_j(-n) = \varepsilon_j t_j(n).$$

We need also to introduce the Hecke series $H_j(s)$ attached to ψ_j. This is defined initially to be the sum

$$\sum_{n=1}^{\infty} t_j(n)n^{-s}, \qquad \mathrm{Re}(s) > 1$$

and continued to an entire function.

Now we may return to the issue of the spectral decomposition of the function $D_A(s,m)$. We have, for $\mathrm{Re}(s) > 1$,

$$\langle P_m(\cdot,s), |A|^2\rangle_k = \sum_{j=1}^{\infty}\langle P_m(\cdot,s),\psi_j\rangle\langle |A|^2, \overline{\psi_j}\rangle_k$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle P_m(\cdot,s), E(\cdot,\tfrac{1}{2}+ir)\rangle\langle |A|^2, E(\cdot,\tfrac{1}{2}-ir)\rangle_k\, dr,$$

where $E(z,s)$ is the Eisenstein series for $SL(2,\mathbb{Z})$. We have

$$\langle P_m(\cdot,s),\psi_j\rangle = \sqrt{\pi}(4\pi m)^{\frac{1}{2}-s}\overline{\rho_j(m)}\Gamma(s-\tfrac{1}{2}+i\varkappa_j)\Gamma(s-\tfrac{1}{2}-i\varkappa_j)/\Gamma(s),$$

$$\langle P_m(\cdot,s),E(\cdot,\tfrac12+ir)\rangle = 2^{2(1-s)}\pi(m\pi)^{\frac12-s-ir}\sigma_{2ir}(m)\frac{\Gamma(s-\frac12+ir)\Gamma(s-\frac12-ir)}{\Gamma(s)\Gamma(\frac12-ir)\zeta(1-2ir)},$$

and

$$\langle |A|^2,E(\cdot,\tfrac12-ir)\rangle_k = \pi^{\frac12+ir}\frac{R_A^*(\frac12+ir)}{\Gamma(\frac12+ir)\zeta(1+2ir)},$$

where

$$R_A^*(s) = 2^{2(1-k-s)}\pi^{1-k-2s}\Gamma(s)\Gamma(s+k-1)\zeta(2s)R_A(s+k-1).$$

As is well-known, the modified Rankin zeta-function $R_A^*(s)$ has simple poles at $s=0$, 1, and is regular elsewhere satisfying the functional equation $R_A^*(s)=R_A^*(1-s)$.

Collecting these, we find that for $\mathrm{Re}(s)>1$

$$D_A(s+k-1,m) =$$

$$\frac{(4\pi)^k m^{\frac12-s}}{2\Gamma(s)\Gamma(s+k-1)}\Big\{\sum_{j=1}^{\infty}\overline{\rho_j(1)}t_j(m)\langle\psi_j,|A|^2\rangle_k\Gamma(s-\tfrac12+i\varkappa_j)\Gamma(s-\tfrac12-i\varkappa_j)$$

$$+\frac12\int_{-\infty}^{\infty}\frac{m^{-ir}\sigma_{2ir}(m)\Gamma(s-\frac12+ir)\Gamma(s-\frac12-ir)R_A^*(\frac12+ir)}{|\Gamma(\frac12+ir)\zeta(1+2ir)|^2}dr\Big\}.$$

On the right side the convergence is very rapid. To see it we invoke

Lemma.
For any positive X we have

$$\sum_{\varkappa_j\le X}|\rho_j(1)t_j(m)|^2e^{-\pi\varkappa_j}\ll X^2+m^{\frac12+\varepsilon}$$

and

$$\sum_{\varkappa_j\le X}|\langle\psi_j,|A|^2\rangle_k|^2e^{\pi\varkappa_j}+\int_{-X}^{X}\frac{|R_A^*(\frac12+ir)|^2}{|\zeta(1+2ir)|^2}e^{2\pi|r|}dr\ll X^{2k},$$

where ε is an arbitrary fixed positive constant.

The first statement is due to Kuznetsov [4] and the second to Good [2].

We now insert the above spectral decomposition of $D_A(\cdot,m)$ into the relation

$$J_A^{(1)}(u+\tfrac12(k-1),v+\tfrac12(k-1);G) =$$

$$\frac{1}{2\pi i}\int_{(\eta)}\{\sum_{m\ge1}m^{-s}D_A(u+v-s+k-1,m)\}\tilde w(s;u+\tfrac12(k-1),G)ds,$$

where $\mathrm{Re}(u+v) > 1+\eta > 2$. The resulting sums and integrals are all absolutely convergent. Then, after some housekeeping, we obtain

$$J_A^{(1)}(u+\tfrac{1}{2}(k-1),\, v+\tfrac{1}{2}(k-1); G) =$$
$$-\, i(4\pi)^{k-1}\sum_{j=1}^{\infty}\overline{\rho_j(1)}H_j(u+v-\tfrac{1}{2})\langle\psi_j, |A|^2\rangle_k \Psi_k(\varkappa_j; u, v; G)$$
$$-\frac{1}{2}i(4\pi)^{k-1}\int_{-\infty}^{\infty}\frac{R_A^*(\frac{1}{2}+ir)\zeta(u+v-\frac{1}{2}+ir)\zeta(u+v-\frac{1}{2}-ir)}{|\Gamma(\frac{1}{2}+ir)\zeta(1+2ir)|^2}\Psi_k(r; u, v; G)dr,$$

where

$$\Psi_k(r; u, v; G) = \int_{(\eta)}\frac{\Gamma(u+v-s-\frac{1}{2}+ir)\Gamma(u+v-s-\frac{1}{2}-ir)}{\Gamma(u+v-s)\Gamma(u+v-s+k-1)}\tilde{w}(s; u+\tfrac{1}{2}(k-1), G)ds.$$

We have to continue analytically this spectral decomposition to a neighborhood of the point P_T.

For this sake we transform $\Psi_k(r; u, v; G)$. We replace $\tilde{w}(s; u+\frac{1}{2}(k-1), G)$ by its defining integral over the interval $0 \le x \le 1$. The resulting double integral is absolutely convergent. Then, after exchanging the order of integration, we get a Barnes integral inside. It is equal to

$$\frac{i}{\tan(\pi ir)}x^{u+v-\frac{1}{2}+ir}\frac{\Gamma(\frac{1}{2}+ir)\Gamma(\frac{3}{2}-k+ir)}{\Gamma(1+2ir)}F(\tfrac{1}{2}+ir, \tfrac{3}{2}-k+ir; 1+2ir; x)$$
$$+ \text{(the same expression but with } -r \text{ instead of } r),$$

where F is the hypergeometric function. To this we apply the functional equation

$$F(a, b; c; z) = (1-z)^{c-a-b}F(c-a, c-b; c; z),$$

getting

$$\frac{i}{\tan(\pi ir)}x^{u+v-\frac{1}{2}+ir}(1-x)^{k-1}\frac{\Gamma(\frac{1}{2}+ir)\Gamma(\frac{3}{2}-k+ir)}{\Gamma(1+2ir)}F(\tfrac{1}{2}+ir, k-\tfrac{1}{2}+ir; 1+2ir; x)$$
$$+ \text{(the same expression but with } -r \text{ instead of } r).$$

We then replace this F by its Euler integral representation. It gives the formula

$$\Psi_k(r; u, v; G) = i\frac{\Gamma(\frac{3}{2}-k+ir)}{\tan(\pi ir)\Gamma(\frac{1}{2}+ir)}\int_0^1 x^{u+v-\frac{3}{2}+ir}(1-x)^{\frac{1}{2}(k-1)-u}\exp\big(-(\tfrac{G}{2}\log(1-x))^2\big)$$
$$\times\int_0^1 (y(1-y))^{-\frac{1}{2}+ir}(1-xy)^{\frac{1}{2}-ir-k}dydx$$
$$+ \text{(the same expression but with } -r \text{ instead of } r).$$

In this we change the variables by replacing x by $x/(1+x)$ and y by $1-y$ so that we have

$$\Psi_k(r;u,v;G) = i\frac{\Gamma(\frac{3}{2}-k+ir)}{\tan(\pi ir)\Gamma(\frac{1}{2}+ir)}\int_0^\infty x^{u+v-\frac{3}{2}+ir}(1+x)^{\frac{1}{2}(k-1)-v}\exp\big(-(\frac{G}{2}\log(1+x))^2\big)$$
$$\times\int_0^1 (y(1-y))^{-\frac{1}{2}+ir}(1+xy)^{\frac{1}{2}-ir-k}dydx$$
$$+ \text{(the same expression but with } -r \text{ instead of } r).$$

But, the last integral over y is equal to

$$\frac{\Gamma(\frac{1}{2}+ir)}{2\pi i\,\Gamma(k-\frac{1}{2}+ir)}\int_{-\infty i}^{\infty i}\frac{\Gamma(\frac{1}{2}+ir+s)\Gamma(k-\frac{1}{2}+ir+s)}{\Gamma(1+2ir+s)}\Gamma(-s)x^s ds,$$

where the path separates the poles of $\Gamma(-s)$ and those of the other factors in the integrand to the right and the left, respectively. Inserting this into the last expression for $\Psi_k(r;u,v;G)$, we get a double integral that converges absolutely. Then, exchanging the order of integration again, we obtain

$$\Psi_k(r;u,v;G) = \frac{\Xi_k(-ir;u,v;G)-\Xi_k(ir;u,v;G)}{2\sin(\pi ir)\Gamma(k-\frac{1}{2}+ir)\Gamma(k-\frac{1}{2}-ir)},$$

where

$$\Xi_k(\xi;u,v;G) = \int_{-\infty i}^{\infty i}\frac{\Gamma(u+v-\frac{1}{2}-s+\xi)}{\Gamma(\frac{3}{2}-u-v+s+\xi)}$$
$$\times\Gamma(s+1-u-v)\Gamma(k+s-u-v)\tilde{w}_*(s;v-\tfrac{1}{2}(k-1),G)ds$$

with $\tilde{w}_*(s;\omega,G)$ being the Mellin transform of

$$w_*(x;\omega,G) = (1+x)^{-\omega}\exp\big(-(\frac{G}{2}\log(1+x))^2\big).$$

In the last integral the path separates the poles of $\Gamma(u+v-\frac{1}{2}-s+\xi)$ and those of $\Gamma(s+1-u-v)\Gamma(k+s-u-v)\tilde{w}_*(s;v-\frac{1}{2}(k-1),G)$ to the right and the left, respectively. We note that $\tilde{w}_*(s;\omega,G)$, as a function of s, has poles at non-positive integers, and is regular elsewhere; moreover it is of rapid decay in any fixed vertical strip.

Now, the last formula for $\Psi_k(r;u,v;G)$ gives, for those u, v such that $\mathrm{Re}(u+v)$ is sufficiently large,

$$J_A^{(1)}(u+\tfrac{1}{2}(k-1),v+\tfrac{1}{2}(k-1);G)$$
$$=\frac{1}{2}(4\pi)^{k-1}\sum_{j=1}^\infty\frac{\overline{\rho_j(1)}H_j(u+v-\frac{1}{2})\langle\psi_j,|A|^2\rangle_k}{\sinh(\pi\varkappa_j)|\Gamma(k-\frac{1}{2}+i\varkappa_j)|^2}\{\Xi_k(i\varkappa_j;u,v;G)-\Xi_k(-i\varkappa_j;u,v;G)\}$$
$$+2(4\pi)^{k-2}\int_{(0)}\frac{R_A^*(\frac{1}{2}+\xi)\zeta(u+v-\frac{1}{2}+\xi)\zeta(u+v-\frac{1}{2}-\xi)\Xi_k(\xi;u,v;G)}{\tan(\pi\xi)\Gamma(k-\frac{1}{2}+\xi)\Gamma(k-\frac{1}{2}-\xi)\zeta(1+2\xi)\zeta(1-2\xi)}d\xi$$
$$=J_A^{(1,d)}(u,v;G)+J_A^{(1,c)}(u,v;G),$$

say. This reduces the problem of the required analytic continuation to the investigation of the analytical properties of the function $\Xi_k(\xi; u, v; G)$. Such a study has already been developed in our former paper [5]; in fact we have treated there a situation closely related to the case $k = 1$, though we have $k \geq 12$ here. Thus by an easy modification of the argument in the fourth section of [5] we find that $\Xi_k(\xi; u, v; G)$ is (i) a meromorphic function of three complex variables ξ, u, v whose polar set is

$$\left\{(\xi, u, v);\ \text{either } \xi = -j - \tfrac{1}{2} \text{ or } \xi + u + v = -j' + \tfrac{1}{2} \text{ with integers } j, j' \geq 0\right\}$$

and (ii) of rapid decay with respect to ξ when it remains in any fixed vertical strip and (u, v) in any fixed compacta of $\mathbb{C}^2$. This observation and the above lemma give immediately that $J_A^{(1,d)}(u, v; G)$ is meromorphic over the entire $\mathbb{C}^2$, and regular in a neighborhood of the point P_T. On the other hand the analytic continuation of $J_A^{(1,c)}(u, v; G)$ can be achieved in much the same way as in the corresponding part of the fourth section of [5], though in the present situation we have to invoke some analytical properties of the Rankin zeta-function $R_A(s)$. Thus, in the above expression for $J_A^{(1,c)}(u, v; G)$ we shift the contour to the far right; one should note that $\Xi_k(\xi; u, v; G)$ is regular for $\mathrm{Re}(\xi) \geq 0$. Among the poles we encounter in this process the point $\xi = u+v-\frac{3}{2}$, which comes from the factor $\zeta(u+v-\frac{1}{2}-\xi)$, is the most significant. In fact, because of the analytic continuation we may confine (u, v) in the vicinity of P_T in the resulting expression and shift back the contour to the original; those residues coming from the poles except for $\xi = -u-v+\frac{3}{2}$, which belongs to the factor $\zeta(u+v-\frac{1}{2}+\xi)$, cancel out. Hence in the vicinity of P_T the function $J_A^{(1,c)}(u, v; G)$ has the expression as the sum of the residues at the points $\xi = \pm(u + v - \frac{3}{2})$ and the same integral as in the original expression of $J_A^{(1,c)}(u, v; G)$ but with a different (u, v).

Collecting the above discussion we obtain the following explicit formula for $I_A(P_T; G)$:

Theorem.
For any positive G and T we have

$$\begin{aligned}(G\sqrt{\pi})^{-1} &\int_{-\infty}^{\infty} |H_A(\tfrac{1}{2} + i(T + t))|^2 e^{-(t/G)^2} dt \\ &= M_A(T, G) + \sum_{j=1}^{\infty} \frac{\langle \overline{\rho_j(1)\psi_j}, |A|^2\rangle_k H_j(\frac{1}{2})}{\sinh(\pi\varkappa_j)|\Gamma(k - \frac{1}{2} + i\varkappa_j)|^2}\Theta_k(\varkappa_j; T, G) \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{R_A^*(\frac{1}{2} + ir)|\zeta(\frac{1}{2} + ir)|^2}{\tanh(\pi r)|\Gamma(k - \frac{1}{2} + ir)\zeta(1 + 2ir)|^2}\Theta_k(r; T, G) dr.\end{aligned}$$

Here $M_A(T,G)$ is essentially a linear polynomial of $\log T$, and

$$\Theta_k(r;T,G) = -(4\pi)^k \int_0^\infty (x(x+1))^{\frac{1}{2}k-1} \cos(T\log(1+\frac{1}{x})) \exp(-(\frac{G}{2}\log(1+\frac{1}{x}))^2)\Lambda_k(x,r)dx;$$

$$\Lambda_k(x,r) = \mathrm{Im}\left[x^{\frac{1}{2}-k-ir}\frac{\Gamma(k-\frac{1}{2}+ir)\Gamma(\frac{1}{2}+ir)}{\Gamma(1+2ir)}F(k-\tfrac{1}{2}+ir,\tfrac{1}{2}+ir;1+2ir;-\frac{1}{x})\right].$$

This should be compared with [5, Theorem]. The similarity between them is rather remarkable.

Naturally we have to make the statement on $M_A(T,G)$ more precise. As can be seen from the above explanation of the analytic continuation of $J_A^{(1,c)}(u,v;G)$, it is actually the value at $(u,v) = P_T$ of the expression

$$|a(1)|^{-2}\left\{R_A(\tau+k) - \frac{i(4\pi)^{k-1}R_A^*(\tau)\tan(\pi\tau)}{\Gamma(\tau+k-1)\Gamma(k-\tau)\zeta(2-2\tau)}\Phi_k(u,v;G)\right\},$$

where $\tau = u+v-1$ and

$$\Phi_k(u,v;G) = \Xi_k(\tau-\tfrac{1}{2};u,v;G) + \Xi_k(\tau-\tfrac{1}{2};v,u;G) - \Xi_k(\tfrac{1}{2}-\tau;u,v;G) - \Xi_k(\tfrac{1}{2}-\tau;v,u;G).$$

To compute this we write the principal part of $R_A^*(s)$ at the origin as

$$R_A^*(s) = C_A^{(-1)}s^{-1} + O(1).$$

We observe first that the last two summands in $\Phi_k(u,v;G)$ are regular at P_T; thus their contribution is

$$|a(1)|^{-2}\frac{i(4\pi)^{k-1}\pi C_A^{(-1)}}{\Gamma(k-1)\Gamma(k)\zeta(2)}(\Xi_k(\tfrac{1}{2};P_T;G) + \Xi_k(\tfrac{1}{2};P_{-T};G)).$$

By the definition of Ξ_k we have

$$\begin{aligned}\Xi_k(\tfrac{1}{2};P_{\pm T};G) &= -\int_{(\frac{1}{2})}\Gamma(-s)\Gamma(s+k-1)\tilde{w}_*(s;1-\tfrac{1}{2}k\mp iT,G)ds\\ &= 2\pi i\Gamma(k-1)\int_0^\infty(1+x)^{\pm iT}f_k(x,G)dx,\end{aligned}$$

where

$$f_k(x,G) = \frac{(1+x)^{\frac{1}{2}(k-1)} - (1+x)^{\frac{1}{2}(1-k)}}{x(1+x)^{\frac{1}{2}}}\exp(-(\frac{G}{2}\log(1+x))^2).$$

Hence we have

$$\Xi_k(\tfrac{1}{2};P_T;G)+\Xi_k(\tfrac{1}{2};P_{-T};G)=4\pi i\Gamma(k-1)\int_0^\infty \cos(T\log(1+x))f_k(x,G)dx$$
$$\ll \frac{G}{T(G+T)},$$

which is negligible. On the other hand, we have

$$\Xi_k(\tau-\tfrac{1}{2};u,v;G)=\int_{-i\infty}^{i\infty}\frac{\Gamma(2\tau-s)}{\Gamma(s)}\Gamma(s-\tau)\Gamma(s+k-1-\tau)\tilde{w}_*(s;v-\tfrac{1}{2}(k-1),G)ds$$
$$=2\pi i\frac{\Gamma(\tau)\Gamma(\tau+k-1)}{\Gamma(2\tau)}\tilde{w}_*(2\tau;v-\tfrac{1}{2}(k-1),G)+\Xi_k^*(\tau-\tfrac{1}{2};u,v;G).$$

The last term has the same integral representation as $\Xi_k(\tau-\frac{1}{2};u,v;G)$ but with the contour $\mathrm{Re}(s)=\frac{1}{2}$; note that we may now assume that τ is close to 0. It is obviously regular at P_T, and we have

$$\Xi_k^*(-\tfrac{1}{2};P_T;G)=-\Xi_k(\tfrac{1}{2};P_T;G).$$

Hence its contribution is negligible.

Thus we have

$$M_A(T,G)=|a(1)|^{-2}\lim_{(u,v)\to P_T}\{R_A(\tau+k)+S_A(u,v;G)\}+O\Big(\frac{G}{T(T+G)}\Big),$$

where

$$S_A(u,v;G)=\frac{(4\pi)^k\Gamma(\tau)R_A^*(\tau)\tan(\pi\tau)}{2\Gamma(2\tau)\Gamma(k-\tau)\zeta(2-2\tau)}\Delta_k(u,v;G);$$
$$\Delta_k(u,v;G)=\tilde{w}_*(2\tau;u-\tfrac{1}{2}(k-1),G)+\tilde{w}_*(2\tau;v-\tfrac{1}{2}(k-1),G).$$

By the definition of $\tilde{w}_*$ we have, for (u,v) close to P_T,

$$\Delta_k(u,v;G)=\frac{1}{2\tau}\int_0^\infty x^{2\tau}\delta_k(x;u,v;G)dx,$$

where

$$\delta_k(x;u,v;G)=-\frac{\partial}{\partial x}\Big[((1+x)^{\frac{1}{2}(k-1)-u}+(1+x)^{\frac{1}{2}(k-1)-v})\exp\big(-(\frac{G}{2}\log(1+x))^2\big)\Big].$$

Hence in the vicinity of $\tau=0$ we have

$$S_A(u,v;G)=\frac{(4\pi)^{k+1}C_A^{(-1)}}{4\Gamma(k)\zeta(2)}\cdot\frac{1}{\tau}+O(1),$$

The main term has to cancel the principal part of $R_A(\tau+k)$ at $\tau=0$, as indeed it does. This fact implies that we may take the above limit on the condition $\tau=0$. Then we find immediately that there are two constants $p_A^{(0)}$, $p_A^{(1)}$ such that

$$M_A(T,G)=p_A^{(0)}+p_A^{(1)}m_k(T,G),$$

where

$$m_k(T,G)=\int_0^\infty \delta_k(x;P_T;G)\log x\,dx.$$

We have

$$\begin{aligned}m_k(T,G)&=-2\mathrm{Re}\int_0^\infty \frac{\partial}{\partial x}\Big[((1+x)^{\frac12 k-1+iT}\exp\big(-(\frac{G}{2}\log(1+x))^2\big)\Big]\log x\,dx\\ &=-\frac{2}{G\sqrt{\pi}}\mathrm{Re}\Big[\Big(\frac{\partial}{\partial\alpha}\Big)_{\alpha=0}\int_0^\infty x^\alpha\int_{-\infty}^\infty(\tfrac12 k-1+i(T+t))(1+x)^{\frac12 k-2+i(T+t)}e^{-(t/G)^2}dtdx\Big].\end{aligned}$$

In the inner integral we move the contour to $\mathrm{Im}(t)=L$ with a large $L>0$; then we may exchange the order of integration. Thus we have

$$\begin{aligned}m_k(T,G)&=-\frac{2}{G\sqrt{\pi}}\mathrm{Re}\int_{\mathrm{Im}(t)=L}\Big\{\frac{\Gamma'}{\Gamma}(\tfrac12 k+i(T+t))+\cot(i(T+t))+\gamma\Big\}e^{-(t/G)^2}dt\\ &=-\frac{2}{G\sqrt{\pi}}\mathrm{Re}\int_{-\infty}^\infty\frac{\Gamma'}{\Gamma}(\tfrac12 k+i(T+t))e^{-(t/G)^2}dt-2\gamma+O(e^{-(T/G)^2}),\end{aligned}$$

where γ is the Euler constant.

Collecting the above discussion on $M_A(T,G)$ we find that there exist two constants $d_0(A)$, $d_1(A)$ such that

$$M_A(T,G)=d_1(A)\log T+d_0(A)+O(T^{-1})$$

uniformly for $0<G<\frac12 T(\log T)^{-\frac12}$, provided T is sufficiently large.

We may now turn to some immediate consequences of our theorem. The analysis developed in the sixth section of [5] can easily be modified to yield an asymptotic evaluation of $\Theta_k(r;T,G)$. Actually we have

$$\Theta_k(r;T,G)\sim-\frac{(4\pi)^{k+1}}{4\sqrt{2T}}r^{k-\frac32}\cos\Big(r\log\frac{r}{4eT}\Big)e^{-(Gr/2T)^2}$$

providing $T^{\varepsilon} \leq G \leq \frac{1}{2}T(\log T)^{-\frac{1}{2}}$ with a fixed $\varepsilon > 0$. Then, after an integration over T of the formula in the theorem, we recover Good's estimate of $E_A(T)$. Also, the argument of Ivić and Motohashi [3] can be extended to the present situation, and we obtain without extra difficulty

$$\int_0^V E_A(T)^2 dT \ll V^2(\log V)^c.$$

Further, the argument of the present paper can be modified so as to extend to Hecke series the result of our recent paper [6] on the meromorphy of the function

$$\int_1^\infty |\zeta(\tfrac{1}{2}+it)|^4 t^{-\xi} dt.$$

In fact we can show that

$$\int_1^\infty |H_A(\tfrac{1}{2}+it)|^2 t^{-\xi} dt$$

can be continued to a function meromorphic over the entire complex plane. As in the case of the zeta-function the critical line is $\mathrm{Re}(\xi) = \frac{1}{2}$. For, if we have

$$\sum_{\varkappa_j=\mu} \langle \overline{\rho_j(1)}\psi_j\,, |A|^2\rangle_k H_j(\tfrac{1}{2}) \neq 0$$

then $\xi = \frac{1}{2} + \mu i$ is a simple pole. Moreover this implies that

$$E_A(T) = \Omega_{\pm}(\sqrt{T}).$$

However, we have not yet established such a non-vanishing theorem.

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