数理解析研究所講究録
第 886 巻 1994 年 128-148

Hecke-eigenfunctions on the space of rational binary quadratic forms and periods of Maass wave forms

Fumihiro Sato (佐藤文広)

Department of Mathematics, Rikkyo University
Nishi-Ikebukuro, Toshimaku, Tokyo 171, Japan

§0 Introduction

Since the present paper is a continuation of the joint work [6] with Y.Hironaka, we begin by summarizing what we did in [6] .

Let $X = Sym(2,\mathbb{Q})^{nd} = \{x \in M(2,\mathbb{Q}) \mid {}^t x = x, \det x \neq 0\}$, $G = GL_2^+(\mathbb{Q})$, and $\Gamma = SL_2(\mathbb{Z})$. In [6] , we considered the function spaces

$$\begin{aligned} \mathcal{C}^\infty(\Gamma\backslash X) &= \{\phi : X \to \mathbb{C} \mid \phi(\gamma\cdot x) = \phi(x)\ (\gamma \in \Gamma)\}, \\ \mathcal{S}(\Gamma\backslash X) &= \{\phi \in \mathcal{C}^\infty(\Gamma\backslash X) \mid \mathrm{Supp}(\phi) \text{ consists of a finite number of } \Gamma\text{-orbits}\} \end{aligned}$$

and studied the action of the Hecke algebra $\mathcal{H} = \mathcal{H}(G,\Gamma)$ on $\mathcal{S}(\Gamma\backslash X)$ and $\mathcal{C}^\infty(\Gamma\backslash X)$. In particular, we determined the $\mathcal{H}$-module structure of $\mathcal{S}(\Gamma\backslash X)$ and all $\mathcal{H}$-eigen functions in $\mathcal{C}^\infty(\Gamma\backslash X)$.

Since $\mathcal{C}^\infty(\Gamma\backslash X)$ can be regarded as the set of all invariants of proper equivalence classes of rational binary quadratic forms, we call an element in $\mathcal{C}^\infty(\Gamma\backslash X)$ an *(abstract) class invariant.* One of our results in [6] is the eigenfunction expansion of abstract class invariants. Therefore $\mathcal{H}$-eigen class invariants are quite interesting and should be important in the arithmetic of binary quadratic forms (or quadratic number fields). The results in [6] showed that the zeta functions of binary quadratic forms are the most fundamental class invariants in the sense that the zeta functions contain all necessary information to determine the $\mathcal{H}$-module structures of $\mathcal{C}^\infty(\Gamma\backslash X)$ and $\mathcal{S}(\Gamma\backslash X)$. In particular, we can construct a standard basis of each $\mathcal{H}$-eigen space starting from the zeta functions. However it is still interesting to find an arithmetic method of constructing $\mathcal{H}$-eigen class invariants. In [6] , we presented two examples of arithmetically defined $\mathcal{H}$-eigen class invariants:

1. the residue of the Dirichlet series
$$\sum_{n=1}^{\infty} \frac{\cot \pi n \alpha}{n^s} \qquad \alpha = \text{a real quadratic number}$$
at $s = 1$ viewed as a function of α (due to Arakawa [1]);
2. the Hirzebruch sum, which is defined with the continued fraction expansion of a real quadratic number (due to Lu [8]).

These two examples of eigen class invariants essentially coincide with each other and reduced to a certain special value of the zeta functions of binary quadratic forms (Arakawa [1], [2]).

In the present paper, we give another construction of Hecke-eigen class invariants starting from Hecke-eigen Maass forms. Namely, using the period integral of Maass forms, we define an $\mathcal{H}$-homomorphism of the space of even Maass forms into $\mathcal{C}^\infty(\Gamma\backslash X)$. Hence the periods of Hecke-eigen forms provide Hecke-eigen class invariants. Applying the results in [6] to the periods of Maass forms, we can see that properties of Hecke-eigen abstract class invariants are closely related to several important facts in the theory of the theta correspondence (Maass correspondence) of Maass wave forms (cf. [7]).

The present paper is organized as follows. In §1, we recall the result in [6] on the determination of $\mathcal{H}$-eigen class invariants (Theorem 1.1). We also calculate the action of the Hecke operators on functions on $\mathbb{Z} - \{0\}$ obtained by taking an average of values of class invariants over the set of Γ-equivalence classes with fixed discriminant (Theorem 1.2). In §2.1, the action of Hecke operators on periods of Maass forms is examined. In §2.2, we discuss the relation of Theorem 1.2 and the theta correspondence between Maass forms of weight 0 and Maass forms of weight 1/2. In §2.3, we prove an expression of zeta functions attached to $\mathcal{H}$-eigen class invariants as a linear combination of Euler products related to quadratic number fields (and $\mathbb{Q} \oplus \mathbb{Q}$). For the periods of Maass forms, the expression is essentially equivalent to the definition of the Shimura correspondence (for Maass forms) based on Fourier coefficients.

§1 Hecke-eigen class invariants

1.1 Let

$$\begin{aligned} X &= \left\{x \in M(2,\mathbb{Q}) \mid {}^t x = x,\ \det x \neq 0\right\}, \\ G &= GL_2^+(\mathbb{Q}) = \{g \in GL_2(\mathbb{Q}) \mid \det g > 0\}, \\ \Gamma &= SL_2(\mathbb{Z}). \end{aligned}$$

Then G acts on X by

$$g \cdot x = (\det g)^{-1} \cdot g x\, {}^t g \quad (g \in G, x \in X).$$

Put

$$\mathcal{C}^\infty(\Gamma\backslash X) = \{\Phi : X \to \mathbf{C} \mid \Phi(\gamma \cdot x) = \Phi(x)\ (\gamma \in \Gamma)\}.$$

We call a function in $\mathcal{C}^\infty(\Gamma\backslash X)$ an *(abstract) class invariant.* We denote by $\mathcal{H} = \mathcal{H}(G, \Gamma)$ the Hecke algebra of G with respect to Γ, which acts on $\mathcal{C}^\infty(\Gamma\backslash X)$ as follows:

$$[\Gamma g \Gamma] * \Phi(x) = \sum_i \Phi(g_i \cdot x), \quad \Gamma g \Gamma = \bigcup_i \Gamma g_i \text{ (disjoint union)}.$$

Note that the action of any double coset containing a scalar matrix is trivial.

1.2 In [6] we have determined all Hecke-eigen abstract class invariants. Let us recall briefly the result in [6] .

Denote by K a quadratic number field or $\mathbf{Q} \oplus \mathbf{Q}$ and let $D = D_K$ be its discriminant. We understand that $D_{\mathbf{Q}\oplus\mathbf{Q}} = 1$. Let $\mathcal{O}_{f,K}$ be the order of K of conductor f and $Cl_{f,K}$ the narrow ideal class group of $\mathcal{O}_{f,K}$. Let $\mathfrak{X}_K(f)$ be the character group of $Cl_{f,K}$. If f_1 divides f_2, then, using the canonical mapping $Cl_{f_2,K} \to Cl_{f_1,K}$, we consider $\mathfrak{X}_K(f_1)$ as a subgroup of $\mathfrak{X}_K(f_2)$. Let $\mathfrak{X}_K(f)^{pr}$ be the subset of primitive characters in $\mathfrak{X}_K(f)$ and put

$$\mathfrak{X}_K = \bigcup_{f \in \mathbf{N}} \mathfrak{X}_K(f)^{pr}.$$

For a $\chi \in \mathfrak{X}_K$, we denote by f_χ the conductor of χ.

Denote by $\mathrm{disc}(x)$ the discriminant of $x \in X$:

$$\mathrm{disc}(x) = b^2 - 4ac, \quad x = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

For a non-zero rational number d, we put

$$X_d = \begin{cases} \{x \in X \mid \mathrm{disc}(x) = d\}, & \text{if } d > 0, \\ \{x \in X \mid \mathrm{disc}(x) = d,\ x = \text{positive definite}\}, & \text{if } d < 0. \end{cases}$$

For a Γ-stable subset Y of X, set

$$\mathcal{C}^\infty(\Gamma\backslash Y) = \{\Phi \in \mathcal{C}^\infty(\Gamma\backslash X) \mid \mathrm{Supp}(\Phi) \subset Y\}.$$

Then the decomposition

$$(1.1)\quad \mathcal{C}^\infty(\Gamma\backslash X) = \prod_{D<0} \prod_{t \in \mathbf{Q}_+^\times} \{\mathcal{C}^\infty(\Gamma\backslash X_{t^2D}) \cup \mathcal{C}^\infty(\Gamma\backslash(-X_{t^2D}))\} \times \prod_{D>0} \prod_{t \in \mathbf{Q}_+^\times} \mathcal{C}^\infty(\Gamma\backslash X_{t^2D})$$

is a direct product decomposition as $\mathcal{H}$-module. Note that $\mathcal{C}^\infty(\Gamma\backslash X_{t^2D})$ is isomorphic to $\mathcal{C}^\infty(\Gamma\backslash X_D)$ by the mapping $\Phi \mapsto \Phi'(x) = \Phi(tx)$ $(t > 0)$ and $\mathcal{C}^\infty(\Gamma\backslash(-X_{t^2D}))$ is isomorphic to $\mathcal{C}^\infty(\Gamma\backslash X_{t^2D})$ by the mapping $\Phi \mapsto \Phi'(x) = \Phi(-x)$. Hence it is sufficient to study the $\mathcal{H}$-module structure only for $\mathcal{C}^\infty(\Gamma\backslash X_D)$.

Let

$$X^{pr}_{f,K} = \left\{ x = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in X \;\middle|\; \begin{array}{l} a,b,c \in \mathbb{Z}, (a,b,c)=1 \\ \mathrm{disc}(x) = f^2 D \end{array} \right\}.$$

Namely $X^{pr}_{f,K}$ is the set of half-integral primitive binary quadratic forms of conductor f. We say that the conductor of $x \in X$ is equal to f if $tx \in X^{pr}_{f,K}$ for some $t \in \mathbb{Q}^\times$. Denote by f_x the conductor of x. It is well-known that $\Gamma\backslash X^{pr}_{f,K}$ can be canonically identified with $Cl_{f,K}$ and has a group structure. In the following, we do not distinguish these two groups and consider a character in $\mathfrak{X}_K(f)$ as a character of $\Gamma\backslash X^{pr}_{f,K}$. We denote by $h_{f,K}$ the class number $|Cl_{f,K}|$.

Let ch_x be the characteristic function of $[x] := \Gamma\cdot x$ for $x \in X$. For $\chi \in \mathfrak{X}_K$ and $T \in X^{pr}_{f,K}$, take a common multiple f_1 of f_χ and f, and put

$$p_\chi(\mathrm{ch}_{\frac{1}{f}T}) = \frac{1}{h_{f_1,K}} \sum_{[S]\in Cl_{f_1,K}} \chi([S])\mathrm{ch}_{\frac{1}{f}(T\cdot S)},$$

where $T\cdot S$ stands for a representative of the product in $Cl_{f,K}$ of $[T]$ and the image of $[S]$ under the canonical map $Cl_{f_1,K} \to Cl_{f,K}$. Then the right hand side is independent of the choice of such an f_1; hence we get a linear operator p_χ on $\mathcal{C}^\infty(\Gamma\backslash X_D)$. Since p_χ $(\chi \in \mathfrak{X}_K)$ are $\mathcal{H}$-endomorphisms and satisfy

$$p_\chi \circ p_\psi = \begin{cases} p_\chi & \text{if } \chi = \psi, \\ 0 & \text{if } \chi \neq \psi \end{cases}$$

([6, Lemma 2.3 (i)]), we obtain the following direct product decomposition

$$\mathcal{C}^\infty(\Gamma\backslash X_D) = \prod_{\chi\in\mathfrak{X}_K} \mathcal{C}^\infty(\Gamma\backslash X_D)_\chi$$

as $\mathcal{H}$-module, where $\mathcal{C}^\infty(\Gamma\backslash X_D)_\chi = p_\chi(\mathcal{C}^\infty(\Gamma\backslash X_D))$.

For a $\chi \in \mathfrak{X}_K$ and a multiple f of f_χ, set

$$c_{\chi,f} = \frac{1}{h_{f,K}} \sum_{[S]\in Cl_{f,K}} \chi([S])\mathrm{ch}_{\frac{1}{f}S}. \tag{1.2}$$

Then $c_{\chi,f}$ $(f \in \mathbb{N})$ span the space $\mathcal{C}^\infty(\Gamma\backslash X_D)_\chi$.

For any $\Lambda = (\lambda_p)_{p:\mathrm{prime}}$, $(\lambda_p \in \mathbb{C}/\frac{2\pi i}{\log p}\mathbb{Z})$, we define an algebra homomorphism $\xi_\Lambda : \mathcal{H} \to \mathbb{C}$ by

$$\begin{aligned} \xi_\Lambda(T_p) &= p^{1/2}(p^{\lambda_p} + p^{-\lambda_p}), & T_p &= \left[\Gamma\begin{pmatrix} p & \\ & 1 \end{pmatrix}\Gamma\right] \\ \xi_\Lambda(T_{p,p}) &= 1, & T_{p,p} &= \left[\Gamma\begin{pmatrix} p & \\ & p \end{pmatrix}\Gamma\right] \end{aligned}$$

for each rational prime p.

Theorem 1.1 ([6, Theorem 6]) (i) *If $\xi : \mathcal{H} \to \mathbf{C}$ is an algebra homomorphism obtained as a system of eigenvalues of some Hecke-eigen class invariant, then $\xi = \xi_\Lambda$ for some Λ.*

(ii) *Put*

$$\mathcal{C}^\infty(\Gamma\backslash X_D)_{\chi,\Lambda} = \left\{\Phi \in \mathcal{C}^\infty(\Gamma\backslash X_D)_\chi \mid f * \Phi = \xi_\Lambda(f)\Phi, \ ({}^\forall f \in \mathcal{H})\right\}.$$

Then we have

$$\dim \mathcal{C}^\infty(\Gamma\backslash X_D)_{\chi,\Lambda} = 1$$

and the space $\mathcal{C}^\infty(\Gamma\backslash X_D)_{\chi,\Lambda}$ is spanned by the function

$$\omega_{\chi,\Lambda} = \frac{1}{[\mathcal{O}_K^1 : \mathcal{O}_{f_\chi,K}^1]} \sum_{\substack{f \\ f_\chi | f}} h_{f,K}\, \psi_{\chi,f/f_\chi}(\Lambda)\, c_{\chi,f},$$

$$\psi_{\chi,f/f_\chi}(\Lambda) = \prod_{p|\frac{f}{f_\chi}} \psi_{\chi,p^{e_p}}(\lambda_p), \quad e_p = \mathrm{ord}_p(f/f_\chi),$$

$$\psi_{\chi,p^e}(\lambda_p)$$
$$= \begin{cases} p^{-\frac{e}{2}} \dfrac{p^{(e+1)\lambda_p} - p^{-(e+1)\lambda_p}}{p^{\lambda_p} - p^{-\lambda_p}} - \chi(\mathfrak{p}) p^{-(e+1)/2} \dfrac{p^{e\lambda_p} - p^{-e\lambda_p}}{p^{\lambda_p} - p^{-\lambda_p}} & \text{if } \chi_{K,f_\chi}(p) = 0 \\ \dfrac{p^{-\frac{e}{2}}}{(1+p^{-1})(p^{\lambda_p} - p^{-\lambda_p})} \left\{p^{(e-1)\lambda_p}(p^{2\lambda_p} - p^{-1}) - p^{-(e-1)\lambda_p}(p^{-2\lambda_p} - p^{-1})\right\} & \text{if } \chi_{K,f_\chi}(p) = -1 \\ \dfrac{p^{-\frac{e}{2}}}{(1-p^{-1})(p^{\lambda_p} - p^{-\lambda_p})} \left\{p^{e\lambda_p}(p^{\lambda_p} + p^{-1-\lambda_p} - (\chi(\mathfrak{p}) + \overline{\chi}(\mathfrak{p}))p^{-\frac{1}{2}}) \right. & \\ \qquad \left. - p^{-e\lambda_p}(p^{-\lambda_p} + p^{-1+\lambda_p} - (\chi(\mathfrak{p}) + \overline{\chi}(\mathfrak{p}))p^{-\frac{1}{2}})\right\} & \text{if } \chi_{K,f_\chi}(p) = 1, \end{cases}$$

where $\chi_{K,f_\chi}(p) = \left(\frac{f_\chi^2 D}{p}\right)$ and

$$\chi(\mathfrak{p}) = \begin{cases} \chi([\mathbf{Z}(p,p) + \mathbf{Z}(1, f_\chi)]) & \text{if } D = 1 \\ \chi([\mathfrak{p} \cap \mathcal{O}_{f_\chi,K}]) & \text{if } D \neq 1 \text{ and } (p) = \mathfrak{p}\overline{\mathfrak{p}} \text{ in } K. \end{cases}$$

The functions ψ_{χ,p^e} satisfy the following recursion formula:

$$(1.3) \qquad (p^{\lambda_p} + p^{-\lambda_p})\psi_{\chi,p^e}(\lambda_p) = p^{1/2}\psi_{\chi,p^{e+1}}(\lambda_p) + p^{-1/2}\psi_{\chi,p^{e-1}}(\lambda_p).$$

Note that the recursion formula is of the form precisely the same as that of the recursion formula satisfied by the Fourier coefficients $a(p^e)$ of a Hecke-eigen Maass wave form with the eigenvalue $p^{\lambda_p}+p^{-\lambda_p}$ (cf. §2, (2.1)).

1.3 Hecke algebra action on functions of discriminant

We say that a function $\Phi\in\mathcal{C}^\infty(\Gamma\backslash X)$ is *homogeneous of degree* 0, if $\Phi(tx)=\Phi(x)$ for any $t\in\mathbb{Q}^\times$. Let $\mathcal{C}^\infty(\Gamma\backslash X)^0$ be the space of functions in $\mathcal{C}^\infty(\Gamma\backslash X)$ homogeneous of degree 0. Let $\mathbb{Z}^*=\mathbb{Z}-\{0\}$ and denote by $C(\mathbb{Z}^*)$ the space of $\mathbb{C}$-valued functions on $\mathbb{Z}^*$. We define a linear mapping $\rho:\mathcal{C}^\infty(\Gamma\backslash X)^0\to C(\mathbb{Z}^*)$ by setting

$$\rho(\Phi)(n)=|n|^{-3/4}\sum_{\substack{x\in\Gamma\backslash X_{\mathbb{Z}}\\ \mathrm{disc}(x)=n}}\Phi(x)\quad(n\in\mathbb{Z}^*),$$

where $X_{\mathbb{Z}}$ is the set of half-integral 2 by 2 symmetric matrices.

Let μ be the class invariant defined by

$$\mu(x)=\begin{cases}[\mathcal{O}_K^1:\mathcal{O}_{f_x,K}^1] & \text{if } \mathrm{disc}(x) \text{ is not a square,}\\ 1 & \text{if } \mathrm{disc}(x) \text{ is a square,}\end{cases}$$

where f_x is the conductor of x. We introduce a new action $\star$ of $\mathcal{H}$ on $\mathcal{C}^\infty(\Gamma\backslash X)$ by setting

$$f\star\Phi(x)=\mu(x)(f*(\mu^{-1}\Phi))(x)\quad(f\in\mathcal{H},\Phi\in\mathcal{C}^\infty(\Gamma\backslash X)). \tag{1.4}$$

The definition of the $\star$-action may look quite technical; however the action on the characteristic function ch_x of $\Gamma\cdot x$ is quite simple. In fact we have the following ([6, Lemma 2.4]):

$$[\Gamma g\Gamma]\star\mathrm{ch}_x=\sum_i\mathrm{ch}_{g_i\cdot x},\quad\Gamma g\Gamma=\bigcup_i\Gamma g_i\quad(\text{disjoint union}).$$

We define an action of $\mathcal{H}$ also on $C(\mathbb{Z}^*)$. For a rational prime p and a $b\in C(\mathbb{Z}^*)$, put

$$\begin{aligned}T_p*b(n)&=p^{3/2}b(np^2)+\left(\frac{n}{p}\right)b(n)+p^{-1/2}b(\frac{n}{p^2}),\\ T_{p,p}*b(n)&=b(n),\end{aligned} \tag{1.5}$$

where $\left(\frac{n}{p}\right)$ is the Legendre symbol. We understand that $\left(\frac{n}{p}\right)=0$ if p divides n. Since T_p and $T_{p,p}^{\pm1}$ generate the Hecke algebra $\mathcal{H}$, the identity (1.5) defines an action of $\mathcal{H}$ on $C(\mathbb{Z}^*)$. The mapping $\rho:\mathcal{C}^\infty(\Gamma\backslash X)^0\to C(\mathbb{Z}^*)$ has the following compatibility with the $\mathcal{H}$-action.

Theorem 1.2 *For any odd prime p and any $\Phi\in\mathcal{C}^\infty(\Gamma\backslash X)^0$, we have*

$$\rho(T_p\star\Phi)=T_p*\rho(\Phi).$$

Remarks. (1) The action of $p^{-1/2}T_p$ on $C(\mathbb{Z}^*)$ is of the same form as that of the action of the Hecke operator T_{p^2} on the Fourier coefficients of Maass wave forms of $\frac{1}{2}$-weight (cf. (2.3), [7]). Therefore we write

$$T_{p^2}b(n) = p\,b(np^2) + p^{-1/2}\left(\frac{n}{p}\right)b(n) + p^{-1}b(\frac{n}{p^2}). \tag{1.6}$$

We explain some implication of the theorem above in the theory of automorphic forms in §2.

(2) For $p = 2$, as we can see from the proof below, we have the following:

$$\rho(T_2 \star \Phi)(f^2D) = T_2 * \rho(\Phi)(f^2D) \quad \text{unless } 2\nmid f \text{ and } 4|D.$$

The proof of the theorem above is based on the following proposition, which describes the $\star$-action of $\mathcal{H}$ on $\mathcal{C}^\infty(\Gamma\backslash X_D)$ completely.

Proposition 1.3 *For $\chi \in \mathfrak{X}_K$, let $c_{\chi,f}$ be the function in $\mathcal{C}^\infty(\Gamma\backslash X_D)_\chi$ defined by (1.2). We understand $c_{\chi,f} \equiv 0$ unless $f_\chi|f$. Then*

$$T_p \star c_{\chi,f} = (p - \chi_{K,f}(p))c_{\chi,fp} + (1 - \delta\left(\frac{f}{p}\right))\left(\sum_{N(\mathfrak{p})=p} \bar{\chi}([\mathfrak{p} \cap \mathcal{O}_{f_\chi}])\right)c_{\chi,f} + \delta\left(\frac{f}{pf_\chi}\right)c_{\chi,f/p},$$

where $\chi_{K,f}(p) = \left(\frac{Df^2}{p}\right)$ and $\delta(a) = 1$ or 0 according as $a \in \mathbb{Z}$ or $\notin \mathbb{Z}$.

Proposition 1.3 is proved essentially in [6, pp.134–135]. In the special case where χ is the trivial character, we have the following:

Corollary 1.4 *For a positive integer f, put*

$$c_f(x) = \frac{1}{h_{K,f}} \sum_{[S]\in Cl_f} ch_{\frac{1}{f}S}(x).$$

Then

$$T_p \star c_f = \begin{cases} p\,c_{fp} + c_{\chi,f/p} & \text{if } p|f, \\ (p - \chi_K(p))c_{fp} + (1 + \chi_K(p))c_f & \text{if } p\nmid f, \end{cases}$$

where $\chi_K(p) = \left(\frac{D}{p}\right)$.

Proof of Theorem 1.2. For an $x \in X$, put

$$K_x = \begin{cases} \mathbb{Q}(\sqrt{\mathrm{disc}(x)}) & \text{if } \mathrm{disc}(x) \text{ is not a square,} \\ \mathbb{Q} \oplus \mathbb{Q} & \text{if } \mathrm{disc}(x) \text{ is a square.} \end{cases}$$

Let f_x be the conductor of x. For a $\Phi \in \mathcal{C}^\infty(\Gamma \backslash X)^0$, we put

$$pr_1\Phi(x) = \frac{1}{h_{f_x,K_x}} \sum_{[S] \in Cl_{f_x,K_x}} \Phi([S]).$$

Then, by [6, pp. 133–135], we have

$$pr_1(f \star \Phi) = f \star (pr_1(\Phi)) \quad (f \in \mathcal{H}).$$

We also have $\rho(\Phi) = \rho(pr_1(\Phi))$. Hence it is enough to to prove Theorem 1.2 for functions satisfying $\Phi = pr_1(\Phi)$. Define a function $c_{f,K}$ by

$$c_{f,K}(x) = \begin{cases} \dfrac{1}{h_{f,K}} & \text{if } x \in \mathbb{Q}^\times X_{f,K}^{pr} \\ 0 & \text{otherwise.} \end{cases}$$

Then, taking a representative $x_{f,K}$ of $X_{f,K}^{pr}$ for each f and K, we have

$$\Phi = \sum_K \sum_{f=1}^{\infty} h_{f,K} \Phi(x_{f,K}) c_{f,K}.$$

By Corollary to Proposition 1.3,

$$\begin{aligned} T_p \star \Phi &= \sum_K \sum_{f=1}^{\infty} h_{f,K} \Phi(x_{f,K}) T_p \star c_{f,K} \\ &= \sum_K \sum_{f=1}^{\infty} h_{f,K} \Phi(x_{f,K}) \\ &\quad \times \left\{ (p - \chi_{K,f}(p)) c_{fp,K} + \left(1 - \delta\left(\frac{f}{p}\right)\right)(1 + \chi_K(p)) c_{f,K} + \delta\left(\frac{f}{p}\right) c_{f/p,K} \right\}. \end{aligned}$$

Put $\tau_D = 2$ or 1 according as $D < 0$ or $D > 0$. We write $n = f^2 D$, where D is a fundamental discriminant, and let $K = \mathbb{Q}(\sqrt{D})$ $(D \neq 1)$ or $\mathbb{Q} \oplus \mathbb{Q}$ $(D = 1)$. Then we have

$$\begin{aligned} \rho(T_p \star \Phi)(n) &= \tau_D \, |n|^{-3/4} \sum_{d|f} \sum_{x \in Cl_{d,K}} T_p \star \Phi(x) \\ &= \tau_D \, |n|^{-3/4} \sum_{d|f} h_{d,K} \, T_p \star \Phi(x_{d,K}) \end{aligned}$$

$$= \tau_D |n|^{-3/4} \sum_{d|f} h_{d,K} \left\{ \left(1 - \delta\left(\frac{d}{p}\right)\right)(1+\chi_K(p))\Phi(x_{d,K}) \right.$$
$$\left. + \frac{h_{pd,K}}{h_{d,K}}\Phi(x_{pd,K}) + \delta\left(\frac{d}{p}\right)(p - \chi_{K,\frac{d}{p}}(p))\frac{h_{\frac{d}{p},K}}{h_{d,K}}\Phi(x_{\frac{d}{p},K}) \right\}$$
$$= \tau_D |n|^{-3/4} \sum_{d|f} \left\{ \left(1 - \delta\left(\frac{d}{p}\right)\right)(1+\chi_K(p))h_{d,K}\Phi(x_{d,K}) \right.$$
$$\left. + h_{pd,K}\Phi(x_{pd,K}) + \delta\left(\frac{d}{p}\right)(p - \chi_{K,\frac{d}{p}}(p))h_{\frac{d}{p},K}\Phi(x_{\frac{d}{p},K}) \right\}.$$

Suppose that $p\not| f$. Then $p \not| d$ and $\delta\left(\frac{d}{p}\right) = 0$. Hence

$$\rho(T_p \star \Phi)(n) = \tau_D |n|^{-3/4} \chi_K(p) \sum_{d|f} h_{d,K}\Phi(x_{d,K}) + \tau_D |n|^{-3/4} \sum_{d|fp} h_{d,K}\Phi(x_{d,K})$$
$$= p^{3/2}\rho(\Phi)(np^2) + \chi_K(p)\rho(\Phi)(n). \tag{1.7}$$

Since $p \not| f$, we have $\chi_K(p) = \left(\frac{D}{p}\right) = \left(\frac{Df^2}{p}\right) = \left(\frac{n}{p}\right)$, By assumption, p is odd, hence $p^2 \not| D$. This implies that $p^2 \not| n$ and $\rho(\Phi)(n/p^2) = 0$. Thus we obtain Theorem 2 in the case $p \not| f$. Next we consider the case $p|f$. Then we have

$$\rho(T_p \star \Phi)(n) = \tau_D |n|^{-3/4} \sum_{\substack{d|f \\ p\not| d}} (1+\chi_K(p))h_{d,K}\Phi(x_{d,K}) + \tau_D |n|^{-3/4} \sum_{\substack{d|fp \\ p|d}} h_{d,K}\Phi(x_{d,K})$$
$$+ \tau_D |n|^{-3/4} \sum_{d|\frac{f}{p}} (p - \chi_{K,d}(p))h_{d,K}\Phi(x_{d,K})$$
$$= p^{3/2}\rho(\Phi)(np^2) + p^{-1/2}\rho(\Phi)(n/p^2).$$

This proves Theorem 1.2 completely. ∎

§2 Periods of Maass forms

2.1 Let $\mathfrak{H} = \{z \in \mathbf{C} \mid \Im z > 0\}$. Then the group $GL_2^+(\mathbf{R})$ acts on $\mathfrak{H}$ by linear fractional transformation. We put $\Gamma = SL_2(\mathbf{Z})$ as in §1. For $k = 0$ or $1/2$, put

$$\Delta_k = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - kiy\frac{\partial}{\partial x}.$$

Let $L^2(\Gamma\backslash\mathfrak{H})$ be the space of measurable functions on $\Gamma\backslash\mathfrak{H}$ square integrable with respect to the invariant measure $\frac{dx\,dy}{y^2}$. Put

$$\mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) = \left\{ \phi \in L^2(\Gamma\backslash\mathfrak{H}) \;\middle|\; \begin{array}{l} \Delta_0\phi + \lambda(1-\lambda)\phi = 0, \phi(-\bar{z}) = \phi(z) \\ \int_0^1 \phi(x+iy)\,dx = 0 \end{array} \right\}.$$

A function in $\mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H},\lambda)$ is called an *even Maass wave form* (of weight 0). A function ϕ in $\mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H},\lambda)$ has an Fourier expansion of the form

$$\phi(z) = \sum_{n\neq 0} a(n) W_{0,\lambda-\frac{1}{2}}(4\pi|n|\,y)e(nx),$$

where $e(x) = \exp(2\pi i x)$ and $W_{\kappa,\mu}(z)$ is the Whittaker function, which is given by

$$W_{\kappa,\mu}(z) = \frac{z^{\kappa}e^{-z/2}}{\Gamma(\mu+\frac{1}{2}-\kappa)}\int_0^{\infty} e^{-t}t^{\mu-\kappa-\frac{1}{2}}\left(1+\frac{t}{z}\right)^{\mu+\kappa-\frac{1}{2}} dt$$
$$(\mathrm{Re}(\mu+\tfrac{1}{2}-\kappa) > 0, |\arg z| < \pi).$$

Since ϕ is assumed to be even, we have $a(n) = a(-n)$. The Hecke algebra $\mathcal{H}$ acts on the space $\mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H},\lambda)$ by

$$[\Gamma g\Gamma] * \phi(z) = \sum_i \phi(g_i\cdot z), \quad \Gamma g\Gamma = \bigcup_i \Gamma g_i \quad \text{(disjoint union)}.$$

The mapping $\phi \mapsto p^{-1/2}T_p * \phi$ coincides with the Hecke operator introduced by Maass [9]. Let

$$p^{-1/2}T_p * \phi(z) = \sum_{n\neq 0} b(n) W_{0,\lambda-\frac{1}{2}}(4\pi|n|\,y)e(nx),$$

be the Fourier expansion. Then the action of T_p is expressed in terms of Fourier coefficients as follows:

(2.1) $$b(n) = p^{1/2}a(np) + p^{-1/2}a\left(\frac{n}{p}\right).$$

For $\epsilon = \pm$, we put

$$I_\epsilon = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \epsilon = +, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \epsilon = -, \end{cases}$$

and

$$H_\epsilon = SO(I_\epsilon) = \begin{cases} SO(2) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| \theta\in\mathbf{R}\right\} & \epsilon = +, \\ SO(1,1) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| a\in\mathbf{R}\right\} & \epsilon = -. \end{cases}$$

We normalize the Haar measure $d\mu_\epsilon$ on H_ϵ by

$$d\mu_\epsilon = \begin{cases} \dfrac{1}{4\pi^{1/2}}d\theta & \epsilon = +, \\ \dfrac{da}{a} & \epsilon = -. \end{cases}$$

For an $x \in X$, we write

$$x = \begin{cases} t_x g_x \cdot I_+, & t_x \in \mathbf{R}^\times, g_x \in SL_2(\mathbf{R}) \quad \text{if } \operatorname{disc}(x) < 0, \\ t_x g_x \cdot I_-, & t_x \in \mathbf{R}_+^\times, g_x \in SL_2(\mathbf{R}) \quad \text{if } \operatorname{disc}(x) > 0. \end{cases}$$

We define the period mapping $\mathcal{M} : \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) \to \mathcal{C}^\infty(\Gamma\backslash X)^0$ by

$$\mathcal{M}(\phi)(x) = \int_{g_x^{-1}\Gamma_x g_x \backslash H_\epsilon} \phi(g_x h \cdot i)\, d\mu_\epsilon(h), \quad (x \in X, \phi \in \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda)),$$

where $\epsilon = \operatorname{sgn}(-\operatorname{disc}(x))$ and $\Gamma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}$. Since ϕ is cuspidal, the integral $\mathcal{M}(\phi)(x)$ is absolutely convergent and defines a function in $\mathcal{C}^\infty(\Gamma\backslash X)^0$. We also consider the following slight modification $\mathcal{P}$ of $\mathcal{M}$:

$$\begin{array}{rcl} \mathcal{P} : \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) & \longrightarrow & \mathcal{C}^\infty(\Gamma\backslash X)^0. \\ \phi & \longmapsto & \frac{1}{\mu(x)}\mathcal{M}(\phi)(x) \end{array}$$

Theorem 2.1 (i) *We consider $\mathcal{C}^\infty(\Gamma\backslash X)^0$ as an $\mathcal{H}$-module under the $\star$-action. Then the mapping*

$$\mathcal{M} : \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) \longrightarrow \mathcal{C}^\infty(\Gamma\backslash X)^0$$

is an $\mathcal{H}$-homomorphism:

$$\mathcal{M}(f * \phi) = f \star \mathcal{M}(\phi) \quad (f \in \mathcal{H}).$$

(ii) *We consider $\mathcal{C}^\infty(\Gamma\backslash X)^0$ as an $\mathcal{H}$-module under the $*$-action. Then the mapping*

$$\mathcal{P} : \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) \longrightarrow \mathcal{C}^\infty(\Gamma\backslash X)^0$$

is an $\mathcal{H}$-homomorphism:

$$\mathcal{P}(f * \phi) = f * \mathcal{P}(\phi) \quad (f \in \mathcal{H}).$$

Proof. By (1.4), the first assertion is equivalent to the second. Let us prove the second assertion. It is sufficient to prove it for $f = [\Gamma g \Gamma]$. Let

$$\Gamma g \Gamma = \bigcup_i \Gamma g_i$$

be the right coset decomposition. Put

$$\Gamma_x' = \Gamma_x \cap \left(\bigcap_i g_i^{-1} \Gamma g_i \right).$$

Then, Γ'_x is a subgroup of Γ_x of finite index. Since we can take $g_{g_i \cdot x} = p^{-1/2} g_i g_x$, by the definition of the period and the action of the Hecke-algebra, we have

$$\begin{aligned} \mathcal{M}([\Gamma g \Gamma] * \phi)(x) &= \int_{g_x^{-1}\Gamma_x g_x \backslash H_\pm} \sum_i \phi(g_i g_x h \cdot i)\, d\mu_\pm(h) \\ &= \frac{1}{[\Gamma_x : \Gamma'_x]} \sum_i \int_{g_x^{-1}\Gamma'_x g_x \backslash H_\pm} \phi(g_i g_x h \cdot i)\, d\mu_\pm(h) \\ &= \sum_i [g_i^{-1}\Gamma_{g_i \cdot x} g_i : \Gamma_x] \int_{(g_i g_x)^{-1}\Gamma_{g_i \cdot x}(g_i g_x)\backslash H_\pm} \phi(g_i g_x h \cdot i)\, d\mu_\pm(h) \\ &= \sum_i [g_i^{-1}\Gamma_{g_i \cdot x} g_i : \Gamma_x] \mathcal{M}(\phi)(g_i \cdot x). \end{aligned}$$

By [6, (1.2) and Lemma 1.1], the right hand side is equal to

$$\sum_i \frac{\mu(x)}{\mu(g_i \cdot x)} \mathcal{M}(\phi)(g_i \cdot x).$$

Hence we obtain

$$\mathcal{P}([\Gamma g \Gamma] * \phi)(x) = \sum_i \mathcal{P}(\phi)(g_i \cdot x) = [\Gamma g \Gamma] * \mathcal{P}(\phi)(x).$$

This proves the theorem. ∎

By Theorems 1.2 and 2.1, we have the following

Corollary 2.2 *We have*

$$\rho(\mathcal{M}(T_p * \phi)) = T_p * \rho(\mathcal{M}(\phi)) \quad (\phi \in \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda))$$

for any odd prime p.

Theorem 2.3 *Suppose that* $\phi \in \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda)$ *is an even Hecke-eigen Maass form and satisfies*

$$T_p * \phi = \beta_p \phi$$

for any rational prime p. Then $\mathcal{M}(\phi)$ *(resp.* $\mathcal{P}(\phi)$*) is a Hecke-eigen class invariant under the* $\star$*- (resp.* $*$*-) action:*

$$T_p \star \mathcal{M}(\phi) = \beta_p \mathcal{M}(\phi), \quad T_p * \mathcal{P}(\phi) = \beta_p \mathcal{P}(\phi) \quad \textit{for any prime } p.$$

Moreover, if we define $\Lambda = (\lambda_p)$ *by* $\beta_p = p^{1/2}\left(p^{\lambda_p} + p^{-\lambda_p}\right)$*, then*

$$\mathcal{P}(\phi) = \sum_K \frac{1}{h_K} \sum_{\chi \in \mathfrak{X}_K} \frac{1}{f_\chi} \prod_{p | f_\chi} L_p\left(1, \left(\frac{D_K}{\cdot}\right)\right) \left\{ \sum_{[S] \in Cl_{f_\chi, K}} \overline{\chi([S])} \mathcal{M}(\phi)(S) \right\} \tilde{\omega}_{\chi, \Lambda},$$

where $\tilde{\omega}_{\chi,\Lambda}$ is a function obtained from the function $\omega_{\chi,\Lambda}$ given in Theorem 1.1 by extending it to a function of homogeneous of degree 0 supported on $\mathbb{Q}^\times X_{D_K}$.

Proof. By the previous theorem, it is obvious that $\mathcal{M}(\phi)$ (resp. $\mathcal{P}(\phi)$) is a Hecke-eigen class invariant under $\star$- (resp. $*$-) action. Since $\mathcal{P}(\phi)$ is homogeneous of degree 0, by (1.1) and Theorem 1.1, we have

$$\mathcal{P}(\phi) = \sum_K \sum_{\chi \in \mathfrak{X}_K} a_{\chi,K}\, \tilde{\omega}_{\chi,\Lambda}$$

for some constants $a_{\chi,K}$. Let S_χ be the element in $X^{pr}_{f_\chi,K}$ that represents the unit element of $Cl_{f_\chi,K}$. Then

$$\begin{aligned} a_{\chi,K} &= p_\chi(\mathcal{P}(\phi))(S_\chi) \\ &= \frac{1}{h_{f_\chi,K}} \sum_{[S]\in Cl_{f_\chi,K}} \overline{\chi([S])}\mathcal{P}(\phi)(S) \\ &= \frac{1}{h_{f_\chi,K}[\mathcal{O}_K^1 : \mathcal{O}_{f_\chi,K}^1]} \sum_{[S]\in Cl_{f_\chi,K}} \overline{\chi([S])}\mathcal{M}(\phi)(S). \end{aligned}$$

Since

(2.2)
$$h_{f,K} = \frac{f\, h_K}{[\mathcal{O}_K^1 : \mathcal{O}_{f,K}^1]} \prod_{p|f} \left(1 - \left(\frac{D_K}{p}\right) p^{-1}\right),$$

we obtain

$$a_{\chi,K} = \frac{1}{h_K f_\chi} \prod_{p|f_\chi} L_p\left(1, \left(\frac{D_K}{\cdot}\right)\right) \sum_{[S]\in Cl_{f_\chi,K}} \overline{\chi([S])}\mathcal{M}(\phi)(S).$$

∎

Corollary 2.4 *Under the same assumption as in the theorem above, we have*

$$T_{p^2}\rho(\mathcal{M}(\phi)) = p^{-1/2}\beta_p\, \rho(\mathcal{M}(\phi)) \quad \textit{for any odd prime } p$$

(for the definition of T_{p^2}, see (1.6)).

2.2 For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, we put

$$J(\gamma, z) = \epsilon_d^{-1} \left(\frac{c}{d}\right) \left(\frac{cz+d}{|cz+d|}\right)^{1/2},$$

where

$$\epsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4 \\ \sqrt{-1} & d \equiv 3 \pmod 4 \end{cases}$$

and $\left(\frac{c}{d}\right)$ has the same meaning as in [13]. Let

$$\mathfrak{S}^+_{1/2}(\Gamma_0(4)\backslash\mathfrak{H},\mu)=\left\{F:\mathfrak{H}\to\mathbf{C}\ \middle|\ \begin{array}{l} F(\gamma\cdot z)=J(\gamma,z)F(z)\quad(\forall\gamma\in\Gamma_0(4))\\ \Delta_{1/2}F+\mu(1-\mu)F=0,\ LF=F\\ \int_0^1 F(x+iy)\,dx=0,\ \int_{\Gamma_0(4)\backslash\mathfrak{H}}|F(z)|^2\frac{dx\,dy}{y^2}<\infty\end{array}\right\},$$

where

$$LF(z)=\frac{1}{4}e^{i\pi/4}\left(\frac{z}{|z|}\right)^{-1/2}\sum_{\nu \bmod 4}F\left(\frac{-1+4\nu z}{16z}\right).$$

We call an $F\in\mathfrak{S}^+_{1/2}(\Gamma_0(4)\backslash\mathfrak{H},\mu)$ a Maass cusp form of weight $\frac{1}{2}$. A Maass cusp form F in $\mathfrak{S}^+_{1/2}(\Gamma_0(4)\backslash\mathfrak{H},\mu)$ has a Fourier expansion of the form

$$F(z)=\sum_{n\neq0}\rho(n)W_{\frac{1}{4}\mathrm{sgn}(n),\mu-\frac{1}{2}}(4\pi\,|n|\,y)e(nx).$$

For each odd prime p, the action of the Hecke operator T_{p^2} is defined by

(2.3) $T_{p^2}F(z)$

$$=\sum_{n\neq0}\left\{p\,\rho(np^2)+p^{-1/2}\left(\frac{n}{p}\right)\rho(n)+p^{-1}\rho\left(\frac{n}{p^2}\right)\right\}W_{\frac{1}{4}\mathrm{sgn}(n),\mu-\frac{1}{2}}(4\pi\,|n|\,y)e(nx).$$

Let us recall the Maass correspondence between $\mathfrak{S}^+_0(\Gamma\backslash\mathfrak{H},\lambda)$ and $\mathfrak{S}^+_{1/2}(\Gamma_0(4)\backslash\mathfrak{H},\mu)$ (cf. [7]). Put

$$Q=\begin{pmatrix}0&0&-2\\0&1&0\\-2&0&0\end{pmatrix},\quad R=\begin{pmatrix}2&0&0\\0&1&0\\0&0&2\end{pmatrix}.$$

Let $r:SL_2(\mathbf{R})\to GL_3(\mathbf{R})$ be the second symmetric tensor representation:

$$r\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\right)=\begin{pmatrix}a^2&ab&b^2\\2ac&ad+bc&2bd\\c^2&cd&d^2\end{pmatrix}.$$

The image of $SL_2(\mathbf{R})$ coincides with the identity component of $SO(Q)_{\mathbf{R}}$.

Let

$$\Theta(z,g)=y^{3/4}\sum_{x\in\mathbf{Z}^3}e((xQ+iyR)[r(g)^{-1}x])\quad(z=x+iy\in\mathfrak{H},g\in SL_2(\mathbf{R}))$$

be the Siegel Theta series. Then $\Theta(z,g)$ has the following properties:

(i) $\Theta(\gamma\cdot z, g) = J(\gamma, z)\Theta(z, g)$, $\gamma \in \Gamma_0(4)$;
(ii) $\Theta(z, \gamma g k) = \Theta(z, g)$, $(\gamma \in \Gamma, k \in SO(2))$;
(iii) $\Theta\left(z, \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \eta^{1/2} & 0 \\ 0 & \eta^{-1/2} \end{pmatrix}\right)$ is an even function of ξ.

Theorem 2.5 *For a $\phi \in \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda)$, put*

$$\Theta(\phi)(z) = \int_{\Gamma\backslash SL_2(\mathbb{R})} \phi(g)\Theta(z, g)\, dg.$$

Then,
(i) $\Theta(\phi)$ *is in* $\mathfrak{S}_{1/2}^+(\Gamma_0(4)\backslash\mathfrak{H}, \mu)$ *for* $\mu = \frac{2\lambda+1}{4}$ *and the mapping*

$$\Theta : \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) \longrightarrow \mathfrak{S}_{1/2}^+(\Gamma_0(4)\backslash\mathfrak{H}, \mu)$$

is compatible with the action of $\mathcal{H}$. *Namely we have*

$$\Theta(p^{-1/2}T_p\phi) = T_{p^2}\Theta(\phi) \quad \textit{for any odd prime } p.$$

(ii) *Let*

$$\Theta(\phi)(z) = \sum_{n\neq 0} \rho(n) W_{\frac{1}{4}\mathrm{sgn}(n),\mu-\frac{1}{2}}(4\pi\,|n|\,y)e(nx).$$

be the Fourier expansion. Then, under a suitable normalization of the Haar measure dg on $SL_2(\mathbb{R})$, *we have*

$$\rho(n) = |n|^{-3/4} \sum_{\substack{x\in X_{\mathbb{Z}} \\ \mathrm{disc}(x)=n}} \mathcal{M}(\phi)(x) = \rho(\mathcal{M}(\phi))(n).$$

(for the definition of $\rho(\mathcal{M}(\phi))$, *see* §1.3.)

A proof of the theorem above can be found in, e.g., [7] except the compatibility of Θ with the $\mathcal{H}$-action (see also [4], [10], and [14] in the holomorphic case). The compatibility with the $\mathcal{H}$-action is an immediate consequence of Corollary 2.2. The following commutative diagram summarizes the argument leading to the compatibility with the $\mathcal{H}$-action:

$$\begin{array}{ccc} \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda) & \xrightarrow{\ \Theta\ } & \mathfrak{S}_{1/2}^+(\Gamma_0(4)\backslash\mathfrak{H}, \mu) \\ \mathcal{M}\ \big\downarrow\ \text{period} & & \big\downarrow\ \text{Fourier coefficients} \\ \mathcal{C}^\infty(\Gamma\backslash X)^0 & \xrightarrow{\ \rho\ } & C(\mathbb{Z}^*). \end{array}$$

The compatibility of the mapping $\mathcal{M}$ (resp. ρ) with the $\mathcal{H}$-action is given by Theorem 2.1 (1) (resp. Theorem 1.2).

Recall that the proof of Theorem 1.2 is based on Proposition 1.3, and the proof of Proposition 1.3 in [6] is based on two lemmas of Shintani ([14, Lemmas 2.3, 2.4]), which are key lemmas of his proof of the compatibility of the theta correspondence with the Hecke operators in the case of holomorphic modular forms. Thus the diagram above reveals the properties of the $\mathcal{H}$-action on $C^\infty(\Gamma\backslash X)$ lying behind Shintani's proof.

2.3 Zeta functions with coefficients $\mathcal{M}(\phi)$

Let $\mathcal{S}(Sym(2,\mathbb{Q}))$ be the space of Schwartz-Bruhat functions on $Sym(2,\mathbb{Q})$, namely, functions f satisfying the conditions

(2.4) there exist lattices L_1 and L_2 such that $\mathrm{Supp}(f) \subset L_1$ and $f(x)$ is constant on each coset modulo L_2.

We identify $Sym(2,\mathbb{Q})$ with its dual vector space via the symmetric bilinear form $\langle x, x^*\rangle = \mathrm{tr}(xwx^*w^{-1})$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

For $f_0 \in \mathcal{S}(Sym(2,\mathbb{Q}))$, we define its Fourier transform $\widehat{f_0}$ as follows. For $x^* \in Sym(2,\mathbb{Q})$, take a lattice L in $Sym(2,\mathbb{Q})$ such that the value of $f_0(x)$ is determined by the coset of x modulo L and x^* is contained in the dual lattice

$$L^* = \{x^* \in Sym(2,\mathbb{Q}) \mid \langle x^*, L\rangle \subset \mathbb{Z}\}.$$

Put

$$\widehat{f_0}(x^*) = v(L)^{-1} \sum_{x \in Sym(2,\mathbb{Q})/L} f_0(x)\, e^{2\pi i <x,x^*>},$$

where $v(L) = \int_{Sym(2,\mathbb{R})/L} dx$. Then $\widehat{f_0}(x^*)$ is independent of the choice of L and defines a function in $\mathcal{S}(Sym(2,\mathbb{Q}))$, which is the Fourier transform of f_0.

For an $f_0 \in \mathcal{S}(Sym(2,\mathbb{Q}))$, take a congruence subgroup $\Gamma_0 \subset SL_2(\mathbb{Z})$ satisfying

$$f_0(\gamma x\, {}^t\gamma) = f_0(x) \quad (\gamma \in \Gamma_0).$$

Put

$$v(\Gamma_0) = \int_{\Gamma_0\backslash\mathfrak{H}} \frac{dx\,dy}{y^2}.$$

For $\phi \in \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H},\lambda)$ and $f_0 \in \mathcal{S}(Sym(2,\mathbb{Q}))$, we define the zeta functions by setting

$$\xi_\epsilon(\phi, f_0; s) = \frac{1}{v(\Gamma_0)} \sum_{\substack{x\in\Gamma_0\backslash X \\ \mathrm{sgn}\,\mathrm{disc}(x)=\epsilon}} \frac{f_0(x)\eta(x)\mathcal{M}(\phi)(x)}{|\mathrm{disc}(x)|^s}, \quad \epsilon = \pm, \tag{2.5}$$

where $\eta(x) = [\Gamma_x : \Gamma_{0,x}]$. The zeta functions ξ_ϵ are absolutely convergent for $\mathrm{Re}(s) > \frac{3}{2}$ and do not depend on the choice of Γ_0. In [12, §6.2], we have studied analytic properties of ξ_ϵ in the case where f_0 is the characteristic function of a lattice in $Sym(2, \mathbf{Q})$. The general theory of zeta functions with automorphic forms developed in [12] can be applied to ξ_ϵ for arbitrary f_0 and we can obtain the following theorem:

Theorem 2.6 *The zeta functions $\xi_\pm(\phi, f_0; s)$ have analytic continuations to entire functions of s of finite order and satisfy the functional equation.*

$$\begin{pmatrix} \xi_+(\phi, f_0; \frac{3}{2} - s) \\ \xi_-(\phi, f_0; \frac{3}{2} - s) \end{pmatrix} = 2^{2(s-1)}\pi^{\frac{1}{2}-2s}\,\Gamma\left(s + \frac{\lambda - 1}{2}\right)\Gamma\left(s - \frac{\lambda}{2}\right) \times \begin{pmatrix} \cos(\pi s) & \frac{\pi^{3/2}\Gamma(1-\frac{\lambda}{2})^2}{2^{\lambda+3}\Gamma(1-\lambda)}\sin\left(\frac{\pi\lambda}{2}\right) \\ \frac{2^{\lambda+3}\Gamma(1-\lambda)}{\pi^{3/2}\Gamma(1-\frac{\lambda}{2})^2}\cos\left(\frac{\pi\lambda}{2}\right) & \sin(\pi s) \end{pmatrix} \begin{pmatrix} \xi_+(\phi, \hat{f}_0; s) \\ \xi_-(\phi, \hat{f}_0; s) \end{pmatrix}.$$

Problem (Converse theorem?). It is quite natural to ask whether the functional equations in Theorem 2.6 characterize the image of the period mapping $\mathcal{M}$ in $\mathcal{C}^\infty(\Gamma\backslash X)^0$. Let $\Phi \in \mathcal{C}^\infty(\Gamma\backslash X)^0$ and consider the Dirichlet series

$$\xi_\pm(\Phi, f_0; s) = \frac{1}{v(\Gamma_0)} \sum_{\substack{x\in\Gamma_0\backslash X \\ \mathrm{sgn}\,\mathrm{disc}(x)=\pm}} \frac{f_0(x)\eta(x)\Phi(x)}{|\mathrm{disc}(x)|^s}, \quad (f_0 \in \mathcal{S}(Sym(2, \mathbf{Q}))). \tag{2.6}$$

Suppose that $\xi_\pm(\Phi, f_0; s)$ converge absolutely for sufficiently large $\mathrm{Re}(s)$ and the conclusion of Theorem 2.6 holds for all $f_0 \in \mathcal{S}(Sym(2, \mathbf{Q}))$. Then one can ask:

Is there any $\phi \in \mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H}, \lambda)$ such that $\Phi = \mathcal{M}(\phi)$?

Now we consider the following special case of the zeta functions (2.6):

$$\xi_\epsilon(\Phi, X_{\mathbf{Z}}; s) = \xi_\epsilon(\Phi, f_{X_{\mathbf{Z}}}; s),$$

where $f_{X_{\mathbf{Z}}}$ is the characteristic function of the lattice $X_{\mathbf{Z}}$ of half-integral 2 by 2 symmetric matrices.

Theorem 2.7 *Let $\Phi \in \mathcal{C}^\infty(\Gamma\backslash X)^0$ be a Hecke-eigen class invariant under the $\star$-action and $p^{1/2}(p^{\lambda_p} + p^{-\lambda_p})$ the eigenvalue of T_p. Put*

$$L(\Phi; s) = \prod_p \frac{1}{1 - (p^{\lambda_p} + p^{-\lambda_p})p^{-s} + p^{-2s}}.$$

Suppose that $\xi_\epsilon(\Phi, f_0; s)$ *converge absolutely when* $\mathrm{Re}(s)$ *is sufficiently large. Then we have*

$$v(\Gamma)\xi_\epsilon(\Phi, X_{\mathbf{Z}}; s) = \zeta(2s)L\left(\Phi; 2s - \frac{1}{2}\right) \sum_{\substack{K \\ \operatorname{sgn} D_K = \epsilon}} \frac{\rho(\Phi)(D_K)}{D_K^{s-\frac{3}{4}}} \cdot \zeta_K(2s)^{-1}, \tag{2.7}$$

where $\zeta(s)$ *is the Riemann zeta function and*

$$\zeta_K(s) = \begin{cases} \text{the Dedekind zeta function of } K \text{ if } K \text{ is a quadratic number field,} \\ \zeta(s)^2 \text{ if } K = \mathbf{Q} \oplus \mathbf{Q}. \end{cases}$$

Proof. By the same argument as in the proof of Theorem 2.3, any $\star$-eigen class invariant Φ in $\mathcal{C}^\infty(\Gamma\backslash X)^0$ is of the form

$$\frac{\Phi(x)}{\mu(x)} = \sum_K \frac{1}{h_k} \sum_{\chi \in \mathfrak{X}_K} \frac{1}{f_\chi} \prod_{p | f_\chi} L_p\left(1, \left(\frac{D_K}{\cdot}\right)\right) \left\{ \sum_{[S] \in Cl_{f_\chi, K}} \overline{\chi([S])} \Phi(S) \right\} \tilde{\omega}_{\chi, \Lambda}(x).$$

Put

$$\Phi_0(x) = \mu(x) \sum_K \frac{1}{h_K} \left(\sum_{[S] \in Cl_{1,K}} \Phi(S) \right) \tilde{\omega}_{\chi_{0,K}, \Lambda}(x),$$

where $\chi_{0,K}$ is the trivial character of $Cl_{1,K}$. Then we have

$$\xi_\epsilon(\Phi, X_{\mathbf{Z}}; s) = \xi_\epsilon(\Phi_0, X_{\mathbf{Z}}; s).$$

Hence, we obtain

$$\begin{aligned} v(\Gamma)\xi_\epsilon(\Phi, X_{\mathbf{Z}}; s) &= \tau_\epsilon \sum_{d=1}^\infty \sum_{\substack{D \\ \operatorname{sgn} D = \epsilon}} \sum_{f=1}^\infty \sum_{[S] \in Cl_{f,K}} \frac{\Phi_0(dS)}{|\mathrm{disc}(dS)|^s} \\ &= \sum_{d=1}^\infty \frac{1}{d^{2s}} \sum_{\substack{D \\ \operatorname{sgn} D = \epsilon}} \frac{\rho(\Phi)(D)}{D^{s-3/4}} \sum_{f=1}^\infty \frac{1}{f^{2s}} \cdot \frac{h_{f,K}[\mathcal{O}_K^1 : \mathcal{O}_{f,K}^1]}{h_K} \tilde{\omega}_{\chi_{0,K}, \Lambda}(S_{f,K}), \end{aligned}$$

where $\tau_\epsilon = 1$ of 2 according as $\epsilon = +$ or $-$, and $S_{f,K}$ is a representative of $X_{f,K}^{pr}$. By the definition of $\tilde{\omega}_{\chi,\Lambda}$ (Theorem 1.1),

$$\tilde{\omega}_{\chi_{0,K}, \Lambda}(S_{f,K}) = \psi_{\chi_{0,K}, f}(\Lambda).$$

Therefore, the class number formula (2.2) yields the identity

$$\begin{aligned} v(\Gamma)\xi_\epsilon(\Phi, X_{\mathbf{Z}}; s) &= \zeta(2s) \sum_{\substack{D \\ \operatorname{sgn} D = \epsilon}} \frac{\rho(\Phi)(D)}{D^{s-3/4}} \sum_{f=1}^\infty \frac{\psi_{\chi_{0,K}, f}(\Lambda)}{f^{2s-1}} \cdot \prod_{p | f} \left(1 - \left(\frac{D}{p}\right) p^{-1}\right) \\ &= \zeta(2s) \sum_{\substack{D \\ \operatorname{sgn} D = \epsilon}} \frac{\rho(\Phi)(D)}{D^{s-3/4}} \prod_p \left\{ 1 + \left(1 - \left(\frac{D}{p}\right) p^{-1}\right) \sum_{e=1}^\infty \psi_{\chi_{0,K}, p^e}(\lambda_p) p^{-e(2s-1)} \right\}. \end{aligned}$$

The recursion formula (1.3) implies the relation

$$1+\left(1-\left(\frac{D}{p}\right)p^{-1}\right)\sum_{e=1}^{\infty}\psi_{\chi_{0,K},p^e}(\lambda_p)\,T^e=\frac{(1-p^{-1}T)(1-\left(\frac{D}{p}\right)p^{-1}T)}{1+(p^{\lambda_p}+p^{-\lambda_p})p^{-1/2}T+p^{-1}T^2}.$$

This proves the theorem. ∎

In the theorem above, let us assume that $\Phi=\mathcal{M}(\phi)$ for some Hecke-eigen Maass form $\phi\in\mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H},\lambda)$ satisfying

(2.8) $$p^{-1/2}T_p*\phi=\alpha_p\phi$$

for any rational prime p. Then $L(\mathcal{M}(\phi);s)$ coincides with the L-function

$$L(\phi,s)=\prod_p\frac{1}{1-\alpha_pp^{-s}+p^{-2s}}$$

of ϕ introduced by Maass [9]. Hence we have the following.

Corollary 2.8 *Let $\phi\in\mathfrak{S}_0^+(\Gamma\backslash\mathfrak{H},\lambda)$ be a Hecke-eigen Maass form satisfying* (2.8). *Then we have*

(2.9) $$v(\Gamma)\xi_\epsilon(\phi,X_{\mathbb{Z}};s)=\zeta(2s)L\left(\phi,2s-\frac{1}{2}\right)\sum_{\substack{K\\ \operatorname{sgn}D_K=\epsilon}}\frac{\rho(\mathcal{M}(\phi))(D_K)}{D_K^{s-\frac{3}{4}}}\cdot\zeta_K(2s)^{-1}.$$

Remarks. (1) Let us consider the subseries

$$v(\Gamma)\xi_K(\phi,X_{\mathbb{Z}};s)=\sum_{f=1}^{\infty}\sum_{\substack{x\in\Gamma\backslash X_{\mathbb{Z}}\\ \operatorname{disc}(x)=f^2D_K}}\frac{\mathcal{M}(\phi)(x)}{|\operatorname{disc}(x)|^s}$$

of $\xi_\epsilon(\phi,f_{X_{\mathbb{Z}}};s)$ corresponding to K. For simplicity, we put $\rho(n)=\rho(\mathcal{M}(\phi))(n)$. Then we have

$$v(\Gamma)\xi_K(\phi,f_{X_{\mathbb{Z}}};s)=\frac{1}{D_K^{s-3/4}}\sum_{f=1}^{\infty}\frac{\rho(f^2D_K)}{f^{2s-3/2}}.$$

Moreover the term in the right hand side of (2.9) corresponding to K is

$$\frac{\rho(D_K)}{D_K^{s-\frac{3}{4}}}\cdot\frac{\zeta(2s)L\left(\phi,2s-\frac{1}{2}\right)}{\zeta_K(2s)}.$$

Hence we have

$$\rho(D_K)L(\phi,s)=L\left(s+\frac{1}{2},\left(\frac{D_K}{\cdot}\right)\right)\sum_{f=1}^{\infty}\frac{\rho(f^2D_K)}{f^{s-1}}.$$

This is the Maass form version of the formula relating the Fourier coefficients of forms of half-integral weight and the Fourier coefficients of forms of integral weight (cf. [13] for the holomorphic case, and [7, Proposition 4.1] for the Maass form case). Thus the structure of the $\mathcal{H}$-module $\mathcal{C}^\infty(\Gamma\backslash X)$ is closely related to the fact that the Dirichlet series $\sum_{n=1}^{\infty}\frac{\rho(n)}{n^s}$ given by the Mellin transform of a Hecke-eigen form of half-integral weight does not have Euler product, but the subseries $\sum_{f=1}^{\infty}\frac{\rho(f^2D_K)}{f^s}$ does have.

(2) In [3], Datskovski obtained a formula similar to (2.9) in the case where ϕ is a constant function on $SL_2(\mathbb{R})$ ([3, Theorem 7.2]). In this case we must remove the subseries $\xi_{\mathbb{Q}\oplus\mathbb{Q}}(\phi, f_{X_{\mathbb{Z}}}; s)$ from $\xi_\epsilon(\phi, f_{X_{\mathbb{Z}}}; s)$ to obtain converging Dirichlet series. The proof of the theorem above applies also to this non-cuspidal case and the theorem remains to hold if we remove the terms corresponding to $K = \mathbb{Q}\oplus\mathbb{Q}$. The Hecke-eigenvalue α_p of a constant function is equal to $p^{1/2}+p^{-1/2}$ and $L(\phi, s) = \zeta(2s)\zeta(2s-1)$. Hence our result is consistent with Datskovski's. Datskovski proved a similar result also in the case where the base field is an algebraic number field with class number 1 ([3, Theorem 7.1]), or more generally with odd class number ([3, Theorem 7.3]). Hironaka [5] extended the results in [6] to the case where the class number of the base field is equal to 1. Using her results, we can obtain a generalization of the theorem above to Hilbert modular case under the same assumption on the class number of the base field, which covers also [3, Theorem 7.1].

References

[1] T.Arakawa, Dirichlet series $\sum_{n=1}^{\infty}\frac{\cot \pi n\alpha}{n^s}$, Dedekind sums and Hecke L-functions for real quadratic fields, *Comment. Math. Univ. St. Pauli* **37**(1988), 209–235.

[2] T.Arakawa, A note on Hirzebruch sum, *Comment. Math. Univ. St. Pauli* **42**(1993), 81–92.

[3] B.Datskovski, A mean value theorem for class numbers of quadratic extensions, *Contemporary Math.* **143**(1993), 179–242.

[4] W.Duke, Hyperbolic distribution problems and half-integral weight Maass forms, *Invent. Math.* **92**(1988), 73–90.

[5] Y.Hironaka, Fourier-Eisenstein transform on the space of binary quadratic forms over an algebraic number field with class number 1, unpublished (1992).

[6] Y.Hironaka and F.Sato, Fourier-Eisenstein transform and Plancherel formula for rational binary quadratic forms, *Nagoya Math. J.* **128**(1992), 121–151.

[7] S.Katok and P.Sarnak, Heegner points, cycles and Maass forms, Preprint, 1992(?).

[8] H.Lu, Hirzebruch sums and Hecke operators, *J. Number Theory* **38**(1991), 185–195.

[9] H.Maaß, Über die neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen, *Math. Ann.* **121**(1949), 141–183.

[10] H.Maaß, Über die räumliche Verteilung der Punkte in Gittern mit indefinter Metrik, *Math. Ann.* **138**(1959), 287–315.

[11] F.Sato, On functional equations of zeta distributions, *Adv. Studies in pure Math.* **15**(1989), 465–508.

[12] F.Sato, Zeta functions of prehomogeneous vector spaces with coefficients related to periods of automorphic forms, to appear in *K.G.Ramanathan memorial issue of Proc. Ind. Acad.*

[13] G.Shimura, On modular forms of half-integral weight, *Ann. Math.* **97**(1973), 440–481.

[14] T.Shintani, On construction of holomorphic cuspforms of half-integral weight, *Nagoya Math. J.* **58**(1975), 83–126.

[15] T.Shintani, On zeta functions associated with the vector space of quadratic forms, *J. Fac. Sci. Univ. Tokyo* **22**(1975), 25–65.