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AN APPLICATION OF QUANTITATIVE SUBSPACE THEOREM

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1. Introduction

We give here one kind of generalization of the Schmidt-Schlickewei quantitative subspace theorem, using the argument of M. Ru and P. -M. Wong. We consider linear forms not only of the same number as variables but also of a larger number than the number of variables.

Let K be a normal extention of $\mathbf{Q}$ of degree d over $\mathbf{Q}$. Let $M(K)$ be the set of non-equivalent places of K : for $v \in M(K)$, denote by $|\cdot|_v$ the corresponding absolute value normalized such that for $a \in \mathbf{Q}$, $|a|_v = |a|$ if v lies above the archimedean prime of $\mathbf{Q}$, and $|p|_v = 1/p$ if v lies above the rational prime p. Let $M_\infty(K)$ be the set of archimedean places of K.

We consider $S \subset M(K)$ (not necessarily containing $M_\infty(K)$), of cardinality $s < \infty$. Let K_v be the completion of K with repect to $|\cdot|_v$. Put $d_v = [K_v : \mathbf{Q}_v]$ for the local degree. For $a \in K$, we put $\| a \| = | a |^{\frac{d_v}{d}}$.

For $\mathbf{a} = (a_0, \cdots, a_n) \in K^{n+1}, v \in M(K)$, write

$$| \mathbf{a} |_v = (| a_0 |_v^2 + \cdots + | a_n |_v^2)^{\frac{1}{2}}$$

if v is archmedean, and

$$| \mathbf{a} |_v = \max_{0 \le i \le n} | a_i |_v$$

if v is nonarchmedean. Let us denote $\| \mathbf{a} \|_v = | \mathbf{a} |_v^{\frac{d_v}{d}}$. We define the height of $\mathbf{a}$ by

$$H(\mathbf{a}) = \prod_{v \in M(K)} \| \mathbf{a} \|_v$$

and write $h(\mathbf{a}) = \log H(\mathbf{a})$.

It is well-known that the definition of $H(\mathbf{a})$ is independent of a choice of field where $\mathbf{a}$ lies, and also that $H(c\mathbf{a}) = H(\mathbf{a})$ for $c \in K - \{0\}$. Given a linear form $L(\mathbf{x}) = a_0 x_0 + \cdots + a_n x_n$

with coefficients $a_0, \cdots a_n, \in K$ not all zero, we put $H(L) = H(\mathbf{a})$ for $\mathbf{a} = (a_0 \cdots a_n)$ as the height of L. For $v \in M(K)$ we denote $\| L \|_v = \| \mathbf{a} \|_v$.

2. Quantitative subspace theorem

A higher dimensional case of Roth's theorem, that we call the subspace theorem, is established by W. M. Schmidt for archimedean places. A quantitative version is also derived by himself, and extended by H. P. Schlickewei to nonarchimedean places. See for the historical survey in [Schl] [Schm 1] [Schm 2]. We apply here the theorem of Schlickewei in [Schl] which is stated as follows.

Theorem 2.1 (Schlickewei).
Let K, S as above. Suppose that for each $v \in S$ we are given $n+1$ linearly independent linear forms $L_1^{(v)}, \cdots, L_{n+1}^{(v)}$ in $n+1$ variables with coefficients in K. Let $0 < \delta < 1$. Consider the inequality

$$\prod_{v \in S} \prod_{i=1}^{n+1} \frac{\| L_i^{(v)}(\mathbf{x}) \|_v}{\| L_i^{(v)} \|_v \| \mathbf{x} \|_v} < H(\mathbf{x})^{-n-1-\delta}.$$

Then there exists proper subspaces $S_1, \cdots, S_{t_1}$ of K^{n+1} with

$$t_1 = [(8sd)^{2^{34(n+1)d} s^6 \delta^{-2}}]$$

such that every solution $\mathbf{x} \in K^{n+1}$ lies in

$$\bigcup_{i=1}^{t_1} S_i \bigcup D$$

where

$$D = \{\mathbf{x} \in K^{n+1} \quad ; \quad H(\mathbf{x}) < \max((n+1)!^{\frac{9}{\delta}}, \quad H(L_i^{(v)})^{\frac{9d(n+1)s}{\delta}} (v \in S, i = 1, \cdots, n+1))\}.$$

3. Preliminaries

We recall here the definition of subgeneral position and Nochka weight following [R-W]. Let $1 \le k \le n < q$ be rational integers. Consider nonzero distinct q linear forms in $k+1$ variables with coefficients in K. For each linear form $L_i(\mathbf{x}) = a_{i0}x_0 + \cdots + a_{ik}x_k$, put $\mathbf{a}_i = \mathbf{a}(L_i) = (a_{i0}, \cdots, a_{ik}) \in K^{k+1}$ $(1 \le i \le q)$. The linear forms $L_1, \cdots, L_q$ are called in n-subgeneral position if any distinct $n+1$ elements of the set $\{\mathbf{a}_1, \cdots, \mathbf{a}_q\}$ span K^{k+1}. We see that n-subgeneral position is equivalent to general position when $n = k$.

Now we define Nochka weight (cf [R-W]).

Let $1 \le k \le n < q$ be rational integers and $L_1, \cdots, L_q$ be linear forms in $k+1$ variables with coefficients in K, supposed to be in n-subgeneral position. We denote the dimension of the linear span over K of a subset $B \subset A := \{\mathbf{a}_1, \cdots, \mathbf{a}_q\}$ by $d(B)$. Put $P(B) = (\sharp B,\, d(B))$ which is regarded as a point in $\mathbf{R}^2$. For two points $P_1 = (x_1,\, y_1)$, $P_2 = (x_2,\, y_2)$ in $\mathbf{R}^2$ with $x_1 \neq x_2$., we write $\sigma(P_1,\, P_2) = \frac{y_1 - y_2}{x_1 - x_2}$. Proposition 2.1 in [R-W] (under some corrections) allows us to show that there exists a sequence of subsets

$$A = B_{s+1} \supset B_s \supset B_{s-1} \cdots \supset B_1 \supset B_0 = \emptyset$$

where the sequence of numbers $\sigma(P(B_{i+1}),\, P(B_i))$ $(0 \leq i \leq s)$ is uniquely determined.

If an element $\mathbf{a} \in A$ lies in $B_{i+1} - B_i\, (0 \leq i \leq s)$, we put $\omega(\mathbf{a}) = \sigma(P(B_{i+1}),\, P(B_i))$, which is called Nochka weight. For simplicity, we write $\sigma(P(B_{i+1}),\, P(B_i)) = \sigma_i$. Several properties of Nochka weight are presented in [R-W].

4. Results

For simplicity, we restrict here $K \subset \mathbf{R}$ and consider $S = \{\infty\}$; one archmedean place of K defined by $\mid x \mid_\infty = \max(x, -x)$. Put $\mid x \mid_\infty = \mid x \mid$.

For $1 \leq k \leq n < q$, consider linear forms $L_1, \cdots, L_q$ in $k+1$ variables with coefficients in K, supposed to be in n-subgeneral position. Write $\mathbf{a}_i = (a_{i0}, \cdots, a_{ik})$ a coefficient vector of L_i respectively $(1 \leq i \leq q)$. Then for all $\mathbf{x} \in \mathbf{R}^{k+1}$ we claim

$$\sharp \left\{ i\, ;\; \frac{\parallel L_i(\mathbf{x}) \parallel}{\parallel L_i \parallel \parallel \mathbf{x} \parallel} < c_0 \right\} \leq n$$

with

$$c_0 = \frac{1}{2} \min_{\mathbf{a}_i \nparallel \mathbf{a}_j} \left(1 - \frac{\mid (\mathbf{a}_i, \mathbf{a}_j) \mid}{\mid \mathbf{a}_i \mid \mid \mathbf{a}_j \mid} \right)^{\frac{d}{2}}$$

where $(\mathbf{a}_i, \mathbf{a}_j) = a_{0i}a_{0j} + \cdots + a_{ki}a_{kj}$.

Using Theorem 2.1, we get the following quantitative statement of Theorem 3.3 of [R-W].

Theorem 4.1.

Let K, S as above.

Let $1 \leq k \leq n < q$ be rational integers and $L_1, \cdots, L_q$ be linear forms in $k+1$ variables with coefficients in K, supposed to be in n-subgeneral position. Let $\omega_i = \omega(\mathbf{a}_i)$ be the associated Nochka weight with L_i $(1 \leq i \leq q)$. Let $0 < \delta < 1$. Consider the inequality

$$\sum_{i=1}^{q} \omega_i \log \left(\frac{\parallel L_i \parallel \parallel \mathbf{x} \parallel}{\parallel L_i(\mathbf{x}) \parallel} \right) > (k + 1 + \delta) \log \mid \mathbf{x} \mid .$$

Then there exists proper subspaces $S_1, \cdots, S_{t_2}$ of K^{k+1} with

$$t_2 = [32 d^{2^{34(k+1)d}\delta^{-2}}]$$

such that every solution $\mathbf{x} \in \mathbf{Z}^{k+1}$ *with* $L_i(\mathbf{x}) \neq 0$ *for all* $1 \leq i \leq q$ *lies in*

$$\bigcup_{i=1}^{t_2} S_i \bigcup D_1 \bigcup D_2$$

where

$$D_1 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \quad ; \quad |\, \mathbf{x} \,| < \exp\left(\frac{2c_1}{\delta}\right) \right\},$$

$$D_2 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \quad ; \quad H(x) < \max(\,(k+1)!^{\frac{18}{\delta}}, \quad H(L_i)^{\frac{18d(k+1)}{\delta}}\,) \right\}$$

and

$$c_1 = \frac{(q-n)(k+1)}{n+1} \log \frac{1}{c_0}.$$

Outline of the proof of Theorem 4.1

We follow the argument of Ru-Wong. Take c_0 as above. For $1 \leq i \leq q$, put $E_i(\mathbf{x}) = \frac{\|L_i\| \, \|\mathbf{x}\|}{\|L_i(\mathbf{x})\|}$. Then we have $\sharp I(\mathbf{x}) \leq n$ where $I(\mathbf{x}) = \left\{ i; \log E_i(\mathbf{x}) \geq \log \frac{1}{c_0} \right\}$. Lemma 3.1 of [R-W] implies that there exists a set $J(\mathbf{x})$ of cardinality $k+1$ such that $\{\mathbf{a}_i; i \in J(\mathbf{x})\}$ are linearly independent and

$$\prod_{i \in I(\mathbf{x})} E_i(\mathbf{x})^{\omega_i} \leq \prod_{i \in J(\mathbf{x})} E_i(\mathbf{x})$$

with $\omega_i = \omega(\mathbf{a}_i)$ for $L_i(\mathbf{x}) = (\mathbf{a}_i, \mathbf{x})$. Therefore

$$\prod_{i \in J(\mathbf{x})} E_i(\mathbf{x}) \leq \max_I \prod_{i \in I} E_i(\mathbf{x})$$

where I runs over the family of all subsets of $\{1, \cdots, q\}$ with $\sharp I = k+1$ and $\{\mathbf{a}_i; i \in I\}$ linearly independent. Using the property $\omega_i \leq \frac{k+1}{n+1}$ of Nochka weight, we obtain

$$\sum_{i=1}^{q} \omega_i \log E_i(\mathbf{x}) = \sum_{i \in I(\mathbf{x})} \omega_i \log E_i(\mathbf{x}) + c_1 \leq \max_I \sum_{i \in I(\mathbf{x})} \log E_i(\mathbf{x}) + c_1$$

with $c_1 = \frac{(q-n)(k+1)}{n+1} \log \frac{1}{c_0}$. Thus the solutions $\mathbf{x} \in \mathbf{Z}^{k+1}$ outside of $L_1, \cdots, L_q$ of the inequality

$$\sum_{i=1}^{q} \omega_i \log E_i(\mathbf{x}) \ > \ (k+1+\delta) \log |\, \mathbf{x} \,|$$

are contained in the solutions $\mathbf{x} \in \mathbf{Z}^{k+1}$ outside of $L_1, \cdots, L_q$ of the inequality

(4.2) $$\max \sum_{i \in I}^{q} \log E_i(\mathbf{x}) \ > \ (k+1+\delta) \log |\, \mathbf{x} \,| - c_1.$$

Then the solutions of (4.2) are contained in the union of the set D_1 and the set of the solutions of

$$\max \sum_{i \in I}^{q} \log E_i(\mathbf{x}) > (k + 1 + \frac{\delta}{2}) \log | \mathbf{x} |, \tag{4.3}$$

because the solutions of (4.2) with $\log | \mathbf{x} | \geq \frac{2c_1}{\delta}$ satisfies

$$(k + 1 + \delta) \log | \mathbf{x} | - c_1 \geq (k + 1 + \frac{\delta}{2}) \log | \mathbf{x} | .$$

Now we apply Theorem 2.1 for $k+1$ variables to solve (4.3) which establishes our statement.

Applying this theorem, we obtain a quantitative statement of Theorem 3.5 of [R-W] as follows.

Theorem 4.4.
Let $1 \leq k \leq n < q$ with $q > 2n - k + 1$ be rational integers and $L_1, \cdots, L_q$ be linear forms in $k + 1$ variables with coefficients in K, in n-subgeneral position.
Let $0 < \delta < 1$. Consider the inequality

$$\sum_{i=1}^{q} \log \frac{\| L_i \| \| \mathbf{x} \|}{\| L_i(\mathbf{x}) \|} > (2n - k + 1 + \delta) \log | \mathbf{x} | .$$

Then there exists proper subspaces $S_1, \cdots, S_{t_3}$ of K^{k+1} with

$$t_3 = [128 d^{2^{34(k+1)d}(2n+1)^2 \delta^{-2}}]$$

such that every solution $\mathbf{x} \in \mathbf{Z}^{k+1}$ with $L_i(\mathbf{x}) \neq 0$ for all $1 \leq i \leq q$ lies in

$$\bigcup_{i=1}^{t_3} S_i \bigcup D_3 \bigcup D_4$$

where

$$D_3 = \left\{ \mathbf{x} \in \mathbf{Z}^{k+1} \quad ; \quad | \mathbf{x} | < \exp\left(\frac{4(2n+1)c_1}{\delta}\right) \right\},$$

$$D_4 = \{ \mathbf{x} \in \mathbf{Z}^{k+1} \quad ; H(x) < \max((k+1)!^{\frac{36(2n+1)}{\delta}}, \quad H(L_i)^{\frac{36d(2n+1)(k+1)}{\delta}} \; (i = 1, \cdots, q)) \}$$

with c_1 in Theorem 4.1.

Outline of the proof of Theorem 4.4

Put $\theta = \frac{1}{\sigma_s}$. Then $\theta = \frac{q-2n+k-1}{\omega_1+\cdots+\omega_q-(k+1)}$ and $\frac{n+1}{k+1} \leq \theta \leq \frac{2n-k+1}{k+1} \leq 2n+1$ by Theorem 4.3 (1) (3) of [R-W]. For all $v \in M(K)$ and for $\mathbf{x}$ outside of zeroes of L_i, we claim that

$$\log \frac{\| L_i \|_v \| \mathbf{x} \|_v}{\| L_i(\mathbf{x}) \|_v} > 0$$

which derives

$$\log E_i(\mathbf{x}) \leq \sum_{v \in M(K)} \log \frac{\| L_i \|_v \| \mathbf{x} \|_v}{\| L_i(\mathbf{x}) \|_v} = h(\mathbf{a}_i) + h(\mathbf{x}),$$

because we see $_{v\in M(K)} \log \| L_i(\mathbf{x}) \|_v = 0$ by the product formula. For $\varepsilon_1 > 0$, we get that a point $\mathbf{x}$ either is contained in D_5 or satisfies $\log E_i(\mathbf{x}) \leq (1+\varepsilon_1) h(\mathbf{x})$ where

$$D_5 = \left\{ \mathbf{x} \quad ; \quad h(\mathbf{x}) < \frac{\max_{1\leq i \leq q} h(\mathbf{a}_i)}{\varepsilon_1} \right\}.$$

Therefore for $\mathbf{x} \notin D_5$, we have

$$\begin{aligned} &\sum_{1\leq i\leq q} \log E_i(\mathbf{x}) \\ &= \sum_{1\leq i\leq q} (1-\theta\omega_i) \log E_i(\mathbf{x}) + \theta \sum_{1\leq i\leq q} \omega_i \log E_i(\mathbf{x}) \\ &\leq (1+\varepsilon_1)\, h(\mathbf{x}) \sum_{1\leq i\leq q} (1-\theta\omega_i) + \theta \sum_{1\leq i\leq q} \omega_i \log E_i(\mathbf{x}). \end{aligned}$$

Consider $\mathbf{x} \in \mathbf{Z}^{k+1}$ with $L_i(\mathbf{x}) \neq 0$. By theorem 4.1, the inequality

$$\begin{aligned} \sum_{1\leq i\leq q} \log E_i(\mathbf{x}) \leq (1+\varepsilon_1)\, h(\mathbf{x}) &\left(q - \theta \sum_{1\leq i\leq q} \omega_i \right) \\ &+ \theta (k+1+\delta_1) \log |\, \mathbf{x} \,| \end{aligned}$$

holds for all

$$\mathbf{x} \notin \bigcup_{i=1}^{t_2} S_i \bigcup D_1 \bigcup D_2 \bigcup D_5,$$

with $\delta = \delta_1$ in theorem 4.1. Since we have $h(\mathbf{x}) \leq \log |\, \mathbf{x} \,|$ for $\mathbf{x} \in \mathbf{Z}^{k+1}$, using the property of θ mentioned above, $\mathbf{x}$ satisfies

$$\sum_{1\leq i\leq q} \log E_i(\mathbf{x}) \leq (1+\varepsilon_1)(2n-k+1-\theta(k+1))\log|\,\mathbf{x}\,|$$
$$+\,\theta(k+1+\delta_1)\log|\,\mathbf{x}\,|$$
$$=(2n-k+1+\delta_1\theta+\varepsilon_1(2n-k+1-\theta(k+1)))\log|\,\mathbf{x}\,|\,.$$

For any $0<\delta<1$, take $\delta_1=\varepsilon_1=\frac{\delta}{2(2n+1)}$. Then $\varepsilon_1\leq\frac{\delta}{2(2n-k+1-\theta(k+1))}$ if $2n-k+1-\theta(k+1)\neq 0$, and otherwise we have $\varepsilon_1(2n-k+1-\theta(k+1))=0$. This implies $\delta\geq\delta_1\theta+\varepsilon_1(2n-k+1-\theta(k+1))$ which shows that the solutions $\mathbf{x}\in\mathbf{Z}^{k+1}$ with $L_i(\mathbf{x})\neq 0$ for all $1\leq i\leq q$ of the inequality in the statement of the Theorem lie in the desired region.

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