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Kyoto University
AN APPLICATION OF QUANTITATIVE SUBSPACE THEOREM

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1. Introduction

We give here one kind of generalization of the Schmidt-Schlickewei quantitative subspace theorem, using the argument of M. Ru and P. -M. Wong. We consider linear forms not only of the same number as variables but also of a larger number than the number of variables.

Let $K$ be a normal extention of $\mathbb{Q}$ of degree $d$ over $\mathbb{Q}$. Let $M(K)$ be the set of non-equivalent places of $K$: for $v \in M(K)$, denote by $| \cdot |_v$ the corresponding absolute value normalized such that for $a \in \mathbb{Q}$, $|a|_v = |a|$ if $v$ lies above the archimedean prime of $\mathbb{Q}$, and $|p|_v = 1/p$ if $v$ lies above the rational prime $p$. Let $M_\infty(K)$ be the set of archimedean places of $K$.

We consider $S \subset M(K)$ (not necessarily containing $M_\infty(K)$), of cardinality $s < \infty$. Let $K_v$ be the completion of $K$ with respect to $| \cdot |_v$. Put $d_v = [K_v : \mathbb{Q}_v]$ for the local degree. For $a \in K$, we put $\|a\| = |a|^{d_v/d}$.

For $a = (a_0, \cdots, a_n) \in K^{n+1}$, $v \in M(K)$, write

$$|a|_v = (|a_0|^2_v + \cdots + |a_n|^2_v)^{1/2}$$

if $v$ is archimedean, and

$$|a|_v = \max_{0 \leq i \leq n} |a_i|_v$$

if $v$ is nonarchimedean. Let us denote $\|a\|_v = |a|^{d_v/d}_v$. We define the height of $a$ by

$$H(a) = \prod_{v \in M(K)} \|a\|_v$$

and write $h(a) = \log H(a)$.

It is well-known that the definition of $H(a)$ is independent of a choice of field where $a$ lies, and also that $H(ca) = H(a)$ for $c \in K - \{0\}$. Given a linear form $L(x) = a_0 x_0 + \cdots + a_n x_n$
with coefficients $a_0, \ldots, a_n \in K$ not all zero, we put $H(L) = H(a)$ for $a = (a_0 \cdots a_n)$ as the height of $L$. For $v \in M(K)$ we denote $\| L \|_v = \| a \|_v$.

2. Quantitative subspace theorem

A higher dimensional case of Roth’s theorem, that we call the subspace theorem, is established by W. M. Schmidt for archimedean places. A quantitative version is also derived by himself, and extended by H. P. Schlickewei to nonarchimedean places. See for the historical survey in [Schl] [Schm 1] [Schm 2]. We apply here the theorem of Schlickewei in [Schl] which is stated as follows.

**Theorem 2.1 (Schlickewei).**

Let $K$, $S$ as above. Suppose that for each $v \in S$ we are given $n + 1$ linearly independent linear forms $L_1^{(v)}, \ldots, L_{n+1}^{(v)}$ in $n + 1$ variables with coefficients in $K$. Let $0 < \delta < 1$. Consider the inequality

$$\prod_{v \in S} \prod_{i=1}^{n+1} \frac{\| L_i^{(v)}(x) \|_v}{\| L_i^{(v)} \|_v \| x \|_v} < H(x)^{-n-1-\delta}.$$ 

Then there exists proper subspaces $S_1, \ldots, S_{t_1}$ of $K^{n+1}$ with

$$t_1 = [(8sd)^{2s(n+1)d}e^{s\delta^{-2}}]$$

such that every solution $x \in K^{n+1}$ lies in

$$\bigcup_{i=1}^{t_1} S_i \cup D$$

where

$$D = \{ x \in K^{n+1} ; \quad H(x) < \max((n+1)!^{\frac{\delta}{2}} \cdot H(L_i^{(v)})^{\frac{9d(n+1)x}{\delta}} (v \in S, i = 1, \ldots, n+1)) \}.$$ 

3. Preliminaries

We recall here the definition of subgeneral position and Nochka weight following [R-W]. Let $1 \leq k \leq n < q$ be rational integers. Consider nonzero distinct $q$ linear forms in $k + 1$ variables with coefficients in $K$. For each linear form $L_i(x) = a_{i0}x_0 + \cdots + a_{ik}x_k$, put $a_i = a(L_i) = (a_{i0}, \ldots, a_{ik}) \in K^{k+1}$ $(1 \leq i \leq q)$. The linear forms $L_1, \ldots, L_q$ are called in $n$-subgeneral position if any distinct $n + 1$ elements of the set $\{a_1, \ldots, a_q\}$ span $K^{k+1}$. We see that $n$-subgeneral position is equivalent to general position when $n = k$.

Now we define Nochka weight (cf [R-W]).

Let $1 \leq k \leq n < q$ be rational integers and $L_1, \ldots, L_q$ be linear forms in $k + 1$ variables with coefficients in $K$, supposed to be in $n$-subgeneral position. We denote the dimension of
the linear span over $K$ of a subset $B \subset A := \{a_1, \ldots, a_q\}$ by $d(B)$. Put $P(B) = (B, d(B))$.

For two points $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$ in $\mathbb{R}^2$ with $x_1 \neq x_2$, we write $\sigma(P_1, P_2) = \frac{y_1 - y_2}{x_1 - x_2}$. Proposition 2.1 in [R-W] (under some corrections) allows us to show that there exists a sequence of subsets

$$A = B_{s+1} \supset B_s \supset B_{s-1} \cdots \supset B_1 \supset B_0 = \emptyset$$

where the sequence of numbers $\sigma(P(B_{i+1}), P(B_i))$ $(0 \leq i \leq s)$ is uniquely determined.

If an element $a \in A$ lies in $B_{i+1} - B_i (0 \leq i \leq s)$, we put $\omega(a) = \sigma(P(B_{i+1}), P(B_i))$, which is called Nochka weight. For simplicity, we write $\sigma(P(B_{i+1}), P(B_i)) = \sigma_i$. Several properties of Nochka weight are presented in [R-W].

4. Results

For simplicity, we restrict here $K \subset \mathbb{R}$ and consider $S = \{\infty\}$; one archimedean place of $K$ defined by $|x|_\infty = \max(x, -x)$. Put $|x|_\infty = |x|$. For $1 \leq k \leq n < q$, consider linear forms $L_1, \ldots, L_q$ in $k+1$ variables with coefficients in $K$, supposed to be in $n$-subgeneral position. Write $a_i = (a_{i0}, \ldots, a_{ik})$ a coefficient vector of $L_i$ respectively $(1 \leq i \leq q)$. Then for all $x \in \mathbb{R}^{k+1}$ we claim

$$\# \left\{ i; \frac{\| L_i(x) \|}{\| L_i \| \| x \|} < c_0 \right\} \leq n$$

with

$$c_0 = \frac{1}{2} \min_{a_i, a_j} \left( 1 - \frac{|(a_i, a_j)|}{|a_i| |a_j|} \right)^{\frac{1}{2}}$$

where $(a_i, a_j) = a_{i0}a_{0j} + \cdots + a_{ik}a_{kj}$.

Using Theorem 2.1, we get the following quantitative statement of Theorem 3.3 of [R-W].

**Theorem 4.1.**

Let $K, S$ as above.

Let $1 \leq k \leq n < q$ be rational integers and $L_1, \ldots, L_q$ be linear forms in $k+1$ variables with coefficients in $K$, supposed to be in $n$-subgeneral position. Let $\omega_i = \omega(a_i)$ be the associated Nochka weight with $L_i$ $(1 \leq i \leq q)$. Let $0 < \delta < 1$. Consider the inequality

$$\sum_{i=1}^{q} \omega_i \log \left( \frac{\| L_i \| \| x \|}{\| L_i(x) \|} \right) > (k + 1 + \delta) \log |x|.$$

Then there exists proper subspaces $S_1, \ldots, S_{t_2}$ of $K^{k+1}$ with

$$t_2 = \lfloor 32d^{24(k+1)d^2-2} \rfloor$$
such that every solution $x \in \mathbb{Z}^{k+1}$ with $L_i(x) \neq 0$ for all $1 \leq i \leq q$ lies in

$$
\bigcup_{i=1}^{t_2} S_i \cup D_1 \cup D_2
$$

where

$$
D_1 = \left\{ x \in \mathbb{Z}^{k+1} ; \quad |x| < \exp \left( \frac{2c_1}{\delta} \right) \right\},
$$

$$
D_2 = \left\{ x \in \mathbb{Z}^{k+1} ; \quad H(x) < \max((k+1)!^{\frac{18}{\delta}}, H(L_i)^{\frac{18d(k+1)}{\delta}}) \right\}
$$

and

$$
c_1 = \frac{(q-n)(k+1)}{n+1} \log \frac{1}{c_0}.
$$

**Outline of the proof of Theorem 4.1**

We follow the argument of Ru-Wong. Take $c_0$ as above. For $1 \leq i \leq q$, put $E_i(x) = \frac{||L_i||||x||}{||L_i(x)||}$. Then we have $\|I(x)\| \leq n$ where $I(x) = \left\{ i ; \log E_i(x) \geq \log \frac{1}{c_0} \right\}$. Lemma 3.1 of [R-W] implies that there exists a set $J(x)$ of cardinality $k+1$ such that $\{a_i ; i \in J(x)\}$ are linearly independent and

$$
\prod_{i \in I(x)} E_i(x)^{\omega_i} \leq \prod_{i \in J(x)} E_i(x)
$$

with $\omega_i = \omega(a_i)$ for $L_i(x) = (a_i, x)$. Therefore

$$
\prod_{i \in J(x)} E_i(x) \leq \max_I \prod_{i \in I} E_i(x)
$$

where $I$ runs over the family of all subsets of $\{1, \cdots, q\}$ with $\|I\| = k+1$ and $\{a_i ; i \in I\}$ linearly independent. Using the property $\omega_i \leq \frac{k+1}{n+1}$ of Nochka weight, we obtain

$$
\sum_{i=1}^{q} \omega_i \log E_i(x) = \sum_{i \in I(x)} \omega_i \log E_i(x) + c_1 \leq \max_I \sum_{i \in I(x)} \log E_i(x) + c_1
$$

with $c_1 = \frac{(q-n)(k+1)}{n+1} \log \frac{1}{c_0}$. Thus the solutions $x \in \mathbb{Z}^{k+1}$ outside of $L_1, \cdots, L_q$ of the inequality

$$
\sum_{i=1}^{q} \omega_i \log E_i(x) > (k+1+\delta) \log |x|
$$

are contained in the solutions $x \in \mathbb{Z}^{k+1}$ outside of $L_1, \cdots, L_q$ of the inequality

$$
(4.2) \quad \max_{i \in I} \sum_{i \in I} \log E_i(x) > (k+1+\delta) \log |x| - c_1.
$$
Then the solutions of (4.2) are contained in the union of the set $D_1$ and the set of the solutions of

$$(4.3) \quad \max \sum_{i \in \ell} \log E_i(x) > (k + 1 + \frac{\delta}{2}) \log |x|,$$

because the solutions of (4.2) with $\log |x| \geq \frac{2c_1}{\delta}$ satisfies

$$ (k + 1 + \delta) \log |x| - c_1 \geq (k + 1 + \frac{\delta}{2}) \log |x|. $$

Now we apply Theorem 2.1 for $k+1$ variables to solve (4.3) which establishes our statement.

Applying this theorem, we obtain a quantitative statement of Theorem 3.5 of [R-W] as follows.

**Theorem 4.4.**

Let $1 \leq k \leq n < q$ with $q > 2n - k + 1$ be rational integers and $L_1, \ldots, L_q$ be linear forms in $k+1$ variables with coefficients in $K$, in $n$-subgeneral position.

Let $0 < \delta < 1$. Consider the inequality

$$ \sum_{i=1}^{q} \log \frac{\| L_i \| \| x \|}{\| L_i(x) \|} > (2n - k + 1 + \delta) \log |x|. $$

Then there exists proper subspaces $S_1, \ldots, S_{t_3}$ of $K^{k+1}$ with

$$ t_3 = [128d^{23d(k+1)d(2n+1)^2\delta^{-2}}] $$

such that every solution $x \in Z^{k+1}$ with $L_i(x) \neq 0$ for all $1 \leq i \leq q$ lies in

$$ \bigcup_{i=1}^{t_3} S_i \cup D_3 \cup D_4 $$

where

$$ D_3 = \left\{ x \in Z^{k+1} : |x| < \exp \left( \frac{4(2n+1)c_1}{\delta} \right) \right\}, $$

$$ D_4 = \{ x \in Z^{k+1} : H(x) < \max((k + 1)! \frac{36d(2n+1)}{\delta}, H(L_i) \frac{36d(2n+1)(k+1)}{\delta}(i = 1, \ldots, q)) \} $$

with $c_1$ in Theorem 4.1.
Outline of the proof of Theorem 4.4

Put $\theta = \frac{1}{\sigma_{\sim}Q}$. Then $\theta = \frac{q-2n+k-1}{\omega_{1}+\cdots+\omega_{q}-(k+1)}$ and $\frac{n+1}{k+1} \leq \theta \leq \frac{2n-k+1}{k+1} \leq 2n+1$ by Theorem 4.3 (1) (3) of [R-W]. For all $v \in M(K)$ and for $x$ outside of zeroes of $L_i$, we claim that

$$\log \frac{\|L_i\|_v \|x\|_v}{\|L_i(x)\|_v} > 0$$

which derives

$$\log E_i(x) \leq \sum_{v \in M(K)} \log \frac{\|L_i\|_v \|x\|_v}{\|L_i(x)\|_v} = h(a_i) + h(x),$$

because we see $v \in M(K) \log \|L_i(x)\|_v = 0$ by the product formula. For $\epsilon_1 > 0$, we get that a point $x$ either is contained in $D_5$ or satisfies $\log E_i(x) \leq (1 + \epsilon_1)h(x)$ where

$$D_5 = \left\{ x \mid h(x) < \frac{\max_{1 \leq i \leq q} h(a_i)}{\epsilon_1} \right\}.$$ 

Therefore for $x \not\in D_5$, we have

$$\sum_{1 \leq i \leq q} \log E_i(x)$$

$$= \sum_{1 \leq i \leq q} (1 - \theta \omega_i) \log E_i(x) + \theta \sum_{1 \leq i \leq q} \omega_i \log E_i(x)$$

$$\leq (1 + \epsilon_1)h(x) \sum_{1 \leq i \leq q} (1 - \theta \omega_i) + \theta \sum_{1 \leq i \leq q} \omega_i \log E_i(x).$$

Consider $x \in \mathbb{Z}^{k+1}$ with $L_i(x) \neq 0$. By theorem 4.1, the inequality

$$\sum_{1 \leq i \leq q} \log E_i(x) \leq (1 + \epsilon_1)h(x) \left( q - \theta \sum_{1 \leq i \leq q} \omega_i \right)$$

$$+ \theta (k + 1 + \delta_1) \log |x|$$

holds for all

$$x \not\in \bigcup_{i=1}^{t_2} S_i \cup D_1 \cup D_2 \cup D_5,$$

with $\delta = \delta_1$ in theorem 4.1. Since we have $h(x) \leq \log |x|$ for $x \in \mathbb{Z}^{k+1}$, using the property of $\theta$ mentioned above, $x$ satisfies
\[
\sum_{1 \leq i \leq q} \log E_i(x) \leq (1 + \varepsilon_1)(2n - k + 1 - \theta(k + 1))\log |x| \\
+ \theta(k + 1 + \delta_1)\log |x| \\
= (2n - k + 1 + \delta_1\theta + \varepsilon_1(2n - k + 1 - \theta(k + 1)))\log |x|.
\]

For any \(0 < \delta < 1\), take \(\delta_1 = \varepsilon_1 = \frac{\delta}{2(2n+1)}\). Then \(\varepsilon_1 \leq \frac{\delta}{2(2n-k+1-\theta(k+1))}\) if \(2n - k + 1 - \theta(k + 1) \neq 0\), and otherwise we have \(\varepsilon_1(2n - k + 1 - \theta(k + 1)) = 0\). This implies \(\delta \geq \delta_1\theta + \varepsilon_1(2n - k + 1 - \theta(k + 1))\) which shows that the solutions \(x \in \mathbb{Z}^{k+1}\) with \(L_i(x) \neq 0\) for all \(1 \leq i \leq q\) of the inequality in the statement of the Theorem lie in the desired region.

**References**


