数理解析研究所講究録
第 886 巻 1994 年 10-28

ON THE FUNCTION $E_\sigma(T)$

Kohji MATSUMOTO (松本耕二)

Department of Mathematics, Faculty of Education,
Iwate University, Ueda, Morioka 020, Japan

The error term function for the mean square of the Riemann zeta-function $\zeta(s)$ in the strip $\frac{1}{2} < \sigma (= \mathrm{Re}\, s) < 1$, defined by

$$E_\sigma(T) = \int_0^T |\zeta(\sigma+it)|^2 dt - \zeta(2\sigma)T - (2\pi)^{2\sigma-1}\frac{\zeta(2-2\sigma)}{2-2\sigma}T^{2-2\sigma},$$

was first introduced by the author[28] in 1989, so it has relatively short history. However, much subsequent researches have followed after [28], and now, we can draw the basic picture of the behaviour of this function. Originally, the function $E_\sigma(T)$ was introduced as the analogue of the error term function $E(T)$ on the line $\sigma = \frac{1}{2}$, which is defined by

$$E(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^2\, dt - T(\log\frac{T}{2\pi} + 2\gamma - 1)$$

(where γ denotes Euler's constant). Now we have almost all results on $E_\sigma(T)$, which are naturally expected to be obtained analogously to the case of $E(T)$; and the remaining problems seem to be rather difficult. In other words, the first step of research of $E_\sigma(T)$ is now going to be completed. Therefore, it seems that this volume is a place appropriate to summarize the results which have been obtained, and discuss the problems which should be challenged.

1

First we list up the results proved by the author[28].

(A) The explicit formula of Atkinson-type for $E_\sigma(T)$ $(\frac{1}{2} < \sigma < \frac{3}{4})$.

(B) $E_\sigma(T) = O(T^{1/(1+4\sigma)} \log^2 T)$ $(\frac{1}{2} < \sigma < \frac{3}{4})$.

(C) $\int_2^T E_\sigma(t)^2\,dt = c_1(\sigma)T^{5/2-2\sigma} + O(T^{7/4-\sigma}\log T)$ $(\frac{1}{2} < \sigma < \frac{3}{4})$.

(D) $E_\sigma(T) = \Omega(T^{3/4-\sigma})$ $(\frac{1}{2} < \sigma < \frac{3}{4})$.

(E) The observation of the singular behaviour of $E_\sigma(T)$ on the line $\sigma = \frac{3}{4}$.

The formula (A) can be proved analogously to the original argument of Atkinson[1]. The results (B) and (C) can be deduced from (A), by applying the methods of Jutila[14] and Heath-Brown [5], respectively. The result (D) is a direct corollary of (C). The restriction $\frac{1}{2} < \sigma < \frac{3}{4}$ comes from the criterion of the convergence of Oppenheim's Voronoi-type formula, and the new phenomenon (E) was discovered in connection with this restriction.

All of these results (A)-(E) have been improved in subsequent researches. First of all, it is obviously unsatisfactory that there is the restriction $\frac{1}{2} < \sigma < \frac{3}{4}$. The region $\frac{3}{4} \le \sigma < 1$ was first cultivated by Motohashi[37](1990), who proved that the estimate (B) holds for any σ satisfying $\frac{1}{2} < \sigma < 1$. (The article [37] is unpublished, but the contents of [37] are included in Ivić's lecture note[8].)

Next, in Chapters 2 and 3 of the lecture note mentioned above, Ivić carried out a detailed study of $E_\sigma(T)$. In Chapter 2, Ivić first presented the detailed proofs of the above (A)-(D) and the result of Motohashi[37], and then, he tried to give further improvements on upper-bounds of $E_\sigma(T)$ $(\frac{1}{2} < \sigma < 1)$, by combining the idea of Motohashi [37] with the theory of exponent pairs. The main theorem is Theorem 2.11 of [8], and, as corollaries, the following estimates are deduced:

$$(1.1)\quad E_\sigma(T) \ll T^{1-\sigma} \qquad (\tfrac{1}{2} < \sigma < 1),$$

$$(1.2)\quad E_\sigma(T) \ll T^{(51-56\sigma)/65+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \le \tfrac{3}{4}),$$

$$(1.3)\quad E_\sigma(T) \ll T^{(57-60\sigma)/62+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \le \tfrac{11}{12}).$$

However, the author pointed out that there is an error in the proof of Theorem 2.11 in [8]. This gap has essentially been recovered quite recently by Ivić-Matsumoto [13], in which the correct proofs of the above (1.1)-(1.3) are given. We will discuss the details later.

In Chapter 3 of [8], Ivić introduced the function

$$G_\sigma(T) = \int_2^T (E_\sigma(t) - B(\sigma))dt \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4}).$$

Here, $B(\sigma)$ is the quantity which appeared in Ivić's this research, and independently, in the joint research[31,II] of Meurman and the author. At first this quantity was introduced as the following complicated expression:

$$B(\sigma) = \zeta(2\sigma-1)\Gamma(2\sigma-1)\int_0^\infty \left\{\frac{\Gamma(1-\sigma-iu)}{\Gamma(\sigma-iu)} + \frac{\Gamma(1-\sigma+iu)}{\Gamma(\sigma+iu)} - 2u^{1-2\sigma}\sin(\pi\sigma)\right\}du$$
$$+\frac{\pi(1-2\sigma)\zeta(2-2\sigma)(2\pi)^{2\sigma-1}}{\Gamma(2\sigma)\sin(\pi\sigma)}$$

((3.3) of Ivić[8]), but now it is known that

$$B(\sigma) = -2\pi\zeta(2\sigma-1)$$

(see Appendix of Matsumoto-Meurman[31,II]). In Chapter 3 of [8], Ivić developed a detailed study of $G_\sigma(T)$, and, as a consequence, he proved

$$(1.4)\quad E_\sigma(T) = \Omega_\pm(T^{3/4-\sigma}) \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4}).$$

Here we recall the meaning of notations. The notation $f(x) = \Omega_+(g(x))$ (resp. $f(x) = \Omega_-(g(x))$) means that there exist a constant $c > 0$ and a sequence $\{x_n\}$ with $x_n \to \infty$, such that

$f(x_n) > cg(x_n)$ (resp. $f(x_n) < -cg(x_n)$) holds for any n. The notation $f(x) = \Omega_{\pm}(g(x))$ means that both $f(x) = \Omega_{+}(g(x))$ and $f(x) = \Omega_{-}(g(x))$ are valid, and $f(x) = \Omega(g(x))$ means $|f(x)| = \Omega_{+}(g(x))$.

Obviously Ivić's (1.4) gives an improvement on (D).

In the same chapter of Ivić's lecture note, the estimate

$$(1.5) \quad G_{\sigma}(T) = O(T^{5/4-\sigma}) \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4})$$

is proved((3.39) of [8]), which clarifies the meaning of the quantity $B(\sigma)$. In fact, from this estimate and the definition of $G_{\sigma}(T)$, it immediately follows that

$$(1.6) \quad \int_2^T E_{\sigma}(t)dt = B(\sigma)T + O(T^{5/4-\sigma}) \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4}).$$

This formula implies that in a sense, $B(\sigma)$ is a "mean value" of $E_{\sigma}(T)$ (as Matsumoto-Meurman[31,II] pointed out independently). Incidentally, Ivić also proved $G_{\sigma}(T) = \Omega_{\pm}(T^{5/4-\sigma})$ in [8], hence with (1.5), he completely determined the order of $G_{\sigma}(T)$.

The aim of Matsumoto-Meurman's paper[31,II], which we mentioned above several times, is to improve the error estimate in (C). Put

$$F_{\sigma}(T) = \int_2^T E_{\sigma}(t)^2 dt - c_1(\sigma)T^{5/2-2\sigma} \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4}).$$

Then, the main result of [31,II] is

$$(1.7) \quad F_{\sigma}(T) = O(T) \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4}),$$

which obviously improves (C). In Sept. 1989, a symposium on analytic number theory was held at Amalfi, Italy, and both Meurman and the author attended there. In a private conversation at Amalfi, Meurman showed an interest in the author's work[28]. Therefore, after returning to Japan, the author sent him a reprint of [28]. In his response Meurman suggested the possibility of improving (C) by using the method of his paper[33]. This was the starting point of the joint research of Meurman and the author, and when Ivić visited Japan at the end of 1990, the estimate $F_{\sigma}(T) = O(T\log^4 T)$, slightly weaker than (1.7), had been obtained. The author gave a talk on this result at Nihon University, in front of Ivić. This is the result mentioned in the Notes of Chapter 2 of Ivić[8].

The improved form (1.7) is proved in Matsumoto-Meurman [31,II]. In the same paper, the conjecture

$$(1.8) \qquad F_\sigma(T) \sim B(\sigma)^2 T \qquad (\tfrac{1}{2} < \sigma < \tfrac{3}{4})$$

is proposed, and if this conjecture would be true, then (1.7) would be best-possible. See also [29][30]. The basis which supports the conjecture (1.8) is not so firm, but for example, the following heuristic argument is possible. From (C) we have

$$\int_2^T (E_\sigma(t) - \alpha)^2 dt \sim c_1(\sigma) T^{5/2-2\sigma}$$

for any real α. Let

$$A_\sigma(T) = \int_2^T (E_\sigma(t) - \alpha)^2 dt - c_1(\sigma) T^{5/2-2\sigma}.$$

One natural candidate for α which minimizes $A_\sigma(T)$ is $B(\sigma)$, the "mean value" of $E_\sigma(T)$. Putting $\alpha = B(\sigma)$, and noting (1.6), it follows that

$$\begin{aligned} A_\sigma(T) &= \int_2^T E_\sigma(t)^2 dt - 2B(\sigma)\int_2^T E_\sigma(t)dt + B(\sigma)^2(T-2) - c_1(\sigma)T^{5/2-2\sigma} \\ &= F_\sigma(T) - 2B(\sigma)\{B(\sigma)T + O(T^{5/4-\sigma})\} + B(\sigma)^2 T \\ &= F_\sigma(T) - B(\sigma)^2 T + O(T^{5/4-\sigma}), \end{aligned}$$

hence (1.8) is required to minimize the order of T. Ivić has the opinion that the conjecture (1.8) is plausible, but the truth is still in mist.

2

When Ivić was staying at Japan in 1990, he stressed that a "unified approach" to mean value theory in the strip $\frac{1}{2} \le \sigma \le 1$ is desirable. His talk at Paris[9] is also based on the same principle. A "unified approach" should include the cases of $\sigma = \frac{1}{2}$ and $\sigma = 1$.

The case $\sigma = \frac{1}{2}$ is classical, and has been studied extensively from 1920s. It is easy to show that

$$\lim_{\sigma \to 1/2} E_\sigma(T) = E(T)$$

((2.3) of [8]). On the other hand, a deep study of the case $\sigma = 1$ was first carried out by a joint work of Balasubramanian, Ivić and Ramachandra[2](1992); they proved the asymptotic formula

$$\int_1^T |\zeta(1+it)|^2 dt = \zeta(2)T - \pi \log T + R(T)$$

with $R(T) = O((\log T)^{2/3}(\log\log T)^{1/3})$, and also obtained mean value results on $R(T)$. The limit of $E_\sigma(T)$ (as $\sigma \to 1$) is connected with $R(T)$ by the formula

$$\lim_{\sigma \to 1-0} \left\{ \zeta(2\sigma)T + (2\pi)^{2\sigma-1} \frac{\zeta(2-2\sigma)}{2-2\sigma}(T^{2-2\sigma} - 1) \right\} = \zeta(2)T - \pi \log T \ ,$$

as is shown in Ivić[9]. On various related mean values on the line $\sigma = 1$, see Ivić[11], Nakaya[40][41], and Balasubramanian-Ivić-Ramachandra[3].

The remaining strip $\frac{3}{4} \le \sigma < 1$ is most difficult to study. In the previous section we already mentioned that Motohashi's method[37] gives a tool of obtaining upper-bounds of $E_\sigma(T)$ for $\frac{3}{4} \le \sigma < 1$. However, in order to develop further studies , it is strongly desirable to prove Atkinson-type explicit formula in this strip. This was done by Matsumoto-Meurman[31,III](1993). The basic idea of the proof in [31,III] is also explained in [30].

Moreover, in Matsumoto-Meurman[31,III] the mean square of $E_\sigma(T)$ for $\frac{3}{4} \le \sigma < 1$ is studied, and

(2.1) $$\int_2^T E_{3/4}(t)^2 dt = c_2 T \log T + O(T(\log T)^{1/2}),$$

(2.2) $$\int_2^T E_\sigma(t)^2 dt = O(T) \qquad (\tfrac{3}{4} < \sigma < 1)$$

are proved. It is to be noted that, to prove such sharp results as (1.7), (2.1) and (2.2), we need the following three tools. First, the "averaging" idea of Meurman [33]; the same idea was also invented, independently, by Motohashi[34,IV][36]. Second, Preissmann's technique[43] of using Montgomery-Vaughan's inequality; this idea was originally introduced by Preissmann[42]. Ivić also gave the same result as in [43] (independently, but inspired by [42]) in his talk at Vancouver symposium in 1989, and in his lecture note ((2.100) of [8]). The third tool is the mean value theorem of Dirichlet polynomials.

A digressive talk. Preissmann[43] was published in 1993, but the preprint had already been completed around 1988. This delay is because [43] was submitted to J. Number Theory, and was left there (on editor's desk?) three years long. Finally Preissmann found another place to publish. It is well-known that J. Number Theory causes many similar troubles. For example, Matsumoto-Meurman[31,II] was submitted to J. Number Theory in March 1992, but there was no correspondence from the editors. Meurman wrote a letter of inquiry in March 1993, but no answer again. And finally, as the response to the author's recent inquiry(November 1993), they replied "We have no record of your paper".

We can observe that (2.1) and (2.2) establish clearly the singular property of $E_\sigma(T)$ on the line $\sigma = \frac{3}{4}$, which was first pointed out by the author[28]. In fact, the coefficient

$$c_1(\sigma) = \frac{2(2\pi)^{2\sigma-3/2}}{5-4\sigma}\frac{\zeta^2(3/2)}{\zeta(3)}\zeta(\tfrac{5}{2}-2\sigma)\zeta(\tfrac{1}{2}+2\sigma)$$

of the main term in (C) is divergent when $\sigma \to \frac{3}{4}-0$, and on $\sigma = \frac{3}{4}$ the figure of the main term is changed as in (2.1).

We do not know how to extend the conjecture (1.8) to the region $\frac{3}{4} \le \sigma < 1$. In [30] we mentioned timidly the possibility that the asymptotic relation

$$\int_2^T E_\sigma(t)^2\,dt \sim cT$$

may hold for $\frac{3}{4} < \sigma < 1$, but at present nothing is known in this region except (2.2).

Motohashi gave a different proof of Preissmann's result[43], from the viewpoint of additive divisor problem. This proof is mentioned in the Notes of Chapter 2 of Ivić[8]. In a private letter to the author, Motohashi presented the opinion that Montgomery-Vaughan's inequality gives upper-bounds only, while the standpoint of additive divisor problem can give the argument which may clarify the inner structure of $F_\sigma(T)$. In fact, the latter standpoint is naturally connecting with spectral analysis (see Jutila's article in the present Proceedings). Therefore, Motohashi has suggested that spectral analysis will be useful in the study of $F_\sigma(T)$ (and the corresponding object in the strip $\frac{3}{4} \le \sigma < 1$). But in any case, it seems that there remains a long way to the conjecture (1.8) and the real figure of $F_\sigma(T)$ hidden behind it.

3

In the above mentioned works[34,IV][36], Motohashi established the connection between the Riemann-Siegel-type formula of $\zeta^2(s)$ (due to Motohashi himself) and Atkinson's formula. And consequently, he proved the "smoothed" version of Atkinson's formula. His argument includes an alternative proof of (a slight improvement of) the original formula of Atkinson. He suggested one more different proof of Atkinson's formula in [39].

On this occasion we mention some other various versions and generalizations of Atkinson's method. An analogy of Atkinson's formula near $\sigma = \frac{1}{2}$ was considered by Laurinčikas[26]. Let ℓ_T be real , tends to infinity monotonically when T tends to infinity. In [26], Laurinčikas proved the Atkinson-type formula for the integral

$$\int_0^T |\zeta(\sigma_T + it)|^2\, dt \qquad (\sigma_T = \tfrac{1}{2} + \ell_T^{-1}).$$

If we fix a T, his result is nothing but the formula proved by the author[28], and actually his error estimate $O(\log^2 T)$ is weaker than the author's $O(\log T)$. However, in a private communication to the author, Laurinčikas wrote that his error term can be made as $O(\min(\ell_T/2, \log T)\log T)$, hence in case $\ell_T \equiv$ const., it can be reduced to $O(\log T)$. (Note that his result is proved under the additional condition $\ell_T \le c\log T$.) His main concern is the case $\ell_T \to \infty$ as $T \to \infty$, because the motivation of his work lies in his researches on the value-distribution of $\zeta(s)$.

Generalizations of Atkinson's method to Dirichlet L-functions were studied by Meurman and Motohashi in the middle of 1980s. Meurman[32] proved the Atkinson-type formula for

$$\sum_{\chi \bmod q} \int_0^T | L(\tfrac{1}{2} + it, \chi) |^2 \, dt,$$

where $L(s,\chi)$ is the Dirichlet L-function associated with the Dirichlet character χ, and the summation runs over all characters χ of modulus q. Motohashi[34,II][34,III] treated the mean square of individual L-functions (The details are given in [35]). Recently, Laurinčikas[27] obtained an analogue of Meurman's result[32] near the critical line.

Motohashi[34,I] discovered that Atkinson's method can be modified so as to be useful for the study of the sum $\sum_{\chi \bmod q} | L(s,\chi) |^2$, and this idea has been developed and deepened by Katsurada-Matsumoto[17][18] and Katsurada[16,II][16,III] (The results proved in [16,III] are announced in Katsurada[15]; see also his summarizing article[16,IV]). In this case Atkinson's method is effective not only in the critical strip, but also on the whole plane. For instance, one of the results in [18] is the asymptotic expansion of $\sum_{\chi \bmod q} | L(1,\chi) |^2$ with respect to q, which is far better than the former results (going back to Paley and Selberg; the hitherto best result was due to Zhang). Katsurada-Matsumoto[19][20][21] found that the same method can be applied to the (discrete and also continuous) mean squares of Hurwitz zeta-functions $\zeta(s,\alpha)$ with respect to the parameter α. For the details, see Katsurada's article in the present Proceedings.

Motohashi's very important works (partly with Ivić) on the fourth power mean of $\zeta(s)$ and additive divisor problem can also be regarded as a variant of Atkinson's method (see his expository paper[39]). In 1980s, there were two main streams in mean value theory of $\zeta(s)$; Atkinson's method, and applications of Kuznetsov's trace formula. Motohashi's theory is a splendid combination of these two principles. Recently, in the frame of his theory, Motohashi considered mean squares of other types of Dirichlet series. See [38] and his article in the present Próceedings.

Motohashi noticed that the critical property of the line $\sigma = \frac{3}{4}$ also appears in the fourth power moment situation. See also Ivić[12].

Lastly in this section, we mention Kiuchi's recent results, in which some singular situation again appears on the line $\sigma = \frac{3}{4}$. The mean square of the error term in the approximate functional equation of $\zeta^2(s)$ was first considered by Kiuchi-Matsumoto[25] (1992), and then studied further by Kiuchi[22], Ivić[10] and so on. (Note that these works are based on Motohashi's aforementioned work on the Riemann-Siegel-type formula for $\zeta^2(s)$.) Recently, Kiuchi[23] discovered that the main term of this mean square changes its figure at $\sigma = \frac{3}{4}$, in the same manner as in the case of $E_\sigma(T)$. (Quite recently, the author refined the result of Kiuchi[23] in case $\frac{1}{2} < \sigma \le 1$, which in particular includes the proof of the fact analogous to the conjecture (1.8).)

Kiuchi's another paper [24] considered the analogue of Ivić's result[7] on the integral

$$\int_0^T E(t)^2 \,|\, \zeta(\tfrac{1}{2} + it) \,|^2 \, dt$$

in case $\frac{1}{2} < \sigma < 1$, and observed the possibility that the shape of the asymptotic formula may change on the line $\sigma = \frac{5}{8}$. Is it true that such singular properties of $E_\sigma(T)$ appear at $\sigma = \frac{5}{8}$, $\sigma = \frac{7}{8}$, and at any rational points whose denominators are powers of 2? And does it imply the chaotic property of the behaviour of $E_\sigma(T)$, and therefore, the behaviour of $\zeta(s)$? Obviously it is too early to discuss such questions seriously. But these observations might tell us that, from the viewpoint of mean value theorems, we now catch the first sign of the abyss of zeta-function theory, which may be infinitely deep.

4

In the classical case of $\sigma=\frac{1}{2}$, the best known Ω-results for $E(T)$ are due to Hafner-Ivić[4], which assert

$$(4.1)\qquad E(T)=\Omega_+(T^{1/4}(\log T)^{1/4}(\log\log T)^{(3+\log 4)/4}\exp(-c_3\sqrt{\log\log\log T}))$$

$$(4.2)\qquad E(T)=\Omega_-(T^{1/4}\exp(c_4(\log\log T)^{1/4}(\log\log\log T)^{-3/4})).$$

How about $E_\sigma(T)$? We already mentioned that in the strip $\frac{1}{2}<\sigma<\frac{3}{4}$, Ivić improved the author's result (D) to obtain (1.4). On the line $\sigma=\frac{3}{4}$, it immediately follows from (2.1) that

$$(4.3)\qquad E_{3/4}(T)=\Omega(\sqrt{\log T}).$$

To obtain stronger Ω-results, a natural way is trying to develop the argument analogous to that of Hafner-Ivić in the strip $\frac{1}{2}<\sigma<\frac{3}{4}$. As for Ω_--case this method indeed works well, and we can prove

$$(4.4)\qquad E_\sigma(T)=\Omega_-(T^{3/4-\sigma}\exp(c_5(\sigma)(\log\log T)^{\sigma-1/4}(\log\log\log T)^{\sigma-5/4})) \qquad (\tfrac{1}{2}<\sigma<\tfrac{3}{4})$$

(Ivić-Matsumoto[13]). If we formally substitute $\sigma=\frac{1}{2}$ into this result, then it coincides with (4.2), so we can say that (4.4) completely corresponds to Hafner-Ivić's result.

The simple analogue of Hafner-Ivić's argument is not successful for Ω_+-case. Nevertheless, Matsumoto-Meurman [31,III] succeeded to prove

$$(4.5)\qquad E_\sigma(T)=\Omega_+(T^{3/4-\sigma}(\log T)^{\sigma-1/4}) \qquad (\tfrac{1}{2}<\sigma<\tfrac{3}{4}).$$

Putting $\sigma=\frac{1}{2}$ formally in this result, we obtain slightly weaker result than (4.1), but the difference is just a power of $\log\log T$; hence we may say that (4.5) is almost equivalent to the analogue of (4.1). Thus we have both Ω_+ and Ω_--results which supersede Ivić's (1.4).

Since Ivić knew the result (4.5) in the preprint of [31,III], he claimed repeatedly in his letters that an improvement of Ω_--result is surely possible too, and it should be done by Meurman or the author. This pressure of Ivić was the initial driving force of the Ω_--part of the joint research[13].

In the region $\frac{3}{4} < \sigma < 1$, we have no Ω-result. Meurman has the opinion that it is quite difficult to obtain any Ω-result in this region. On the line $\sigma = \frac{3}{4}$, the only Ω-result we have known is (4.3). Is it possible to improve this to obtain, for example,

$$(4.6) \qquad E_{3/4}(T) = \Omega_{\pm}(\sqrt{\log T})$$

or such? When a symposium on number theory was held at Lillafüred, Hungary, in June 1993, in a private discussion with Ivić, the author mentioned the problem of proving (4.6), as an example of remaining problems which may be accessible. However, frankly speaking, the author has no new idea of attacking (4.6). The only thing the author can say now is that (4.6) is probably rather easier than any Ω-result in $\frac{3}{4} < \sigma < 1$, which seems to be extremely difficult.

5

Besides the proof of Ω_--result (4.4), Ivić-Matsumoto[13] carries out a study on upper-bounds of $E_\sigma(T)$. As we mentioned earlier, this corrects the argument in Ivić's lecture note[8], and gives correct proofs of (1.1)-(1.3) (and indeed better estimates). The basic principle is the same as in Ivić[8]; combining the idea of Motohashi[37] with the theory of exponent pairs. Let (κ,λ) be an arbitrary exponent pair. The following two general estimates are proved in the first version of Ivić-Matsumoto[13].

THEOREM A. *Let* $\frac{1}{2} < \sigma < 1$, *and assume*

$$(5.1) \qquad \sigma \le \min\left\{1 + \frac{\kappa - \lambda}{2}, \frac{1+\lambda}{2} - \frac{\kappa}{4}\right\}.$$

Then the estimate

$$(5.2) \qquad E_\sigma(T) \ll T^{(1-2\sigma+\kappa+\lambda)/(2\lambda+1)+\varepsilon}$$

holds.

THEOREM B. *Let* $\frac{3}{4} \le \sigma < 1$, *and assume*

(5.3) $\quad 1 + (\kappa - \lambda)/2 = \sigma$.

Then the estimate

(5.4) $\quad E_\sigma(T) \ll T^{(2\lambda-1)/(4\sigma+4\lambda-2\kappa-3)+\varepsilon}$

holds.

Remark 1. Theorem A is a "corrected" version of Theorem 2.11 of Ivić[8]. However, a referee of Ivić-Matsumoto[13] suggested a way how to recover the original statement of Theorem 2.11 of [8]. A revised version of [13] is now in preparation.

Remark 2. Under the condition (5.3), we have

$$2\lambda - 1 = \lambda + (\lambda - 1) = (-2\sigma + 2 + \kappa) + (\lambda - 1) = 1 - 2\sigma + \kappa + \lambda$$

and

$$4\sigma + 4\lambda - 2\kappa - 3 = 2(2\sigma - 2 - \kappa + \lambda) + 2\lambda + 1 = 2\lambda + 1,$$

therefore the exponent of T in (5.4) is equal to the exponent in (5.2).

If (κ, λ) satisfies $\lambda = \kappa + \frac{1}{2}$, then

$$(\kappa_0, \lambda_0) = (\kappa + (\tfrac{1}{2} - \kappa)(4\sigma - 3),\ \tfrac{1}{2} + \kappa - \kappa(4\sigma - 3)) \qquad (\tfrac{3}{4} \le \sigma < 1)$$

is also an exponent pair, and Theorem B can be applied because $1 + (\kappa_0 - \lambda_0)/2 = \sigma$. The consequence is

THEOREM C. *If* $\frac{3}{4} \le \sigma < 1$ *and* (κ, λ) *satisfies* $\lambda = \kappa + \frac{1}{2}$, *then*

(5.5) $\quad E_\sigma(T) \ll T^{4\kappa(1-\sigma)/(1+4\kappa-4\kappa\sigma)+\varepsilon}$.

Remark 3. The T^ε-factors in the above theorems are all replaced by certain powers of $\log T$ in [13], but here we omit this point for simplicity.

Applying Theorem A to the famous exponent pair $(\frac{9}{56} + \varepsilon, \frac{37}{56} + \varepsilon)$ of Bombieri-Iwaniec-Huxley-Watt, we obtain (1.2). Applying Theorem C to the same exponent pair, we obtain

(5.6) $\quad E_\sigma(T) \ll T^{9(1-\sigma)/(23-9\sigma)+\varepsilon} \qquad (\frac{3}{4} \le \sigma < 1).$

The estimates (1.2) and (5.6) combined clearly improve (1.1). The estimate (1.2) can be improved if we use the exponent pair $(\frac{89}{570}+\varepsilon, \frac{89}{570}+\frac{1}{2}+\varepsilon)$, obtained recently by Huxley[6]. In [13], an estimate better than (1.3) is also proved.

The estimate

(5.7) $\quad E_\sigma(T) \ll T^{2(1-\sigma)/3+\varepsilon} \qquad (\frac{1}{2} < \sigma < 1),$

far stronger than (1.1), is also included in Theorems A-C. This corresponds to the classical estimate $E(T) \ll T^{1/3+\varepsilon}$ for $\sigma = \frac{1}{2}$. The consequence (5.7) is not included in the first version of [13], but it was stated and proved in the author's talk at Kyoto Symposium, Oct. 1993. To prove (5.7) we merely note that applying Theorem A to the classical pair $(\frac{1}{14},\frac{11}{14})$ (the usefulness of this pair was first suggested by Meurman), we have

$$E_\sigma(T) \ll T^{(13-14\sigma)/18+\varepsilon} \qquad (\tfrac{1}{2} < \sigma \le \tfrac{9}{14}),$$

which is stronger than (5.7) for $\frac{1}{2} < \sigma \le \frac{9}{14}$. The remaining region is covered by (1.2) and (5.6).

Some other choices of pairs, such as $(\frac{1}{30},\frac{26}{30})$, give better estimates for σ near $\frac{1}{2}$. In fact, various choices of exponent pairs would give various estimates of $E_\sigma(T)$, and some of which would improve the above estimates in some range of σ. The referee of [13] suggested a way of choosing a series of pairs, which gives good estimates when σ is near 1. It is also possible to give slight improvements, by using the theory of two-dimensional exponent pairs. However, as usual, obtainable results are far from the estimate which is expected to be true. We can conjecture

(5.8) $\quad E_\sigma(T) \ll \begin{cases} T^{3/4-\sigma+\varepsilon} & (\frac{1}{2} < \sigma < \frac{3}{4}) \\ T^{\varepsilon} & (\frac{3}{4} \le \sigma < 1), \end{cases}$

supported by Ω-results ((1.4),etc.). If we assume the very strong conjecture that $(\varepsilon, \frac{1}{2}+\varepsilon)$ would be an exponent pair for any $\varepsilon > 0$, then (5.8) would follow from Theorems A and C. But the latter conjecture would even lead the Lindelöf hypothesis $\zeta(\frac{1}{2}+it) \ll (1+|t|)^{\varepsilon}$, and so it is almost hopeless to obtain a proof of it in the near future. The conjecture (5.8) corresponds to the classical (and is believed to be quite difficult) conjecture

$$E(T) \ll T^{1/4+\varepsilon}$$

on the line $\sigma = \frac{1}{2}$. It is to be noted that (5.8) again indicates the critical property of the line $\sigma = \frac{3}{4}$, on which the behaviour of $E_{\sigma}(T)$ changes.

Sometimes it is observed that the two estimates $E(T) \ll T^{\vartheta+\varepsilon}$ and $\zeta(\frac{1}{2}+it) \ll (1+|t|)^{\vartheta/2+\varepsilon}$ $(\vartheta > 0)$ can be obtained in similar manners. It is natural to expect that the same connection may exist between $E_{\sigma}(T)$ and $\zeta(\sigma+it)$ $(\frac{1}{2} < \sigma < 1)$. This is just a phenomenon, and not an established principle; but if we trust this observation, we can formulate the conjecture, corresponding to (5.8), that

$$(5.9) \qquad \zeta(\sigma+it) \ll \begin{cases} (1+|t|)^{3/8-\sigma/2+\varepsilon} & (\frac{1}{2} < \sigma < \frac{3}{4}) \\ (1+|t|)^{\varepsilon} & (\frac{3}{4} \le \sigma < 1), \end{cases}$$

and these estimates would give the real order of the magnitude of $\zeta(\sigma+it)$. The latter half of the conjecture means that

$$(5.10) \qquad \mu(\sigma) = \begin{cases} \frac{3}{8} - \frac{\sigma}{2} & (\frac{1}{2} < \sigma < \frac{3}{4}) \\ 0 & (\frac{3}{4} \le \sigma < 1), \end{cases}$$

where

$$\mu(\sigma) = \limsup_{t \to \infty} \frac{\log|\zeta(\sigma+it)|}{\log t}.$$

It is probably Ivić[12] who first stated this conjecture explicitly.

The conjecture (5.8) on $E_{\sigma}(T)$ is supported by Ω-results, while (5.10) has no such reinforcing fact. Moreover, the Lindelöf hypothesis implies

$$(5.11) \qquad \zeta(\sigma+it) \ll (1+|t|)^{\varepsilon} \qquad (\tfrac{1}{2} \le \sigma < 1),$$

hence (5.10) contradicts with the Lindelöf hypothesis, therefore with the Riemann hypothesis. Nevertheless, it is perhaps not a wise way to throw over (5.10) immediately. It may be permitted to say that the certainty of the Lindelöf hypothesis (5.11) in case $\frac{1}{2} \leq \sigma < \frac{3}{4}$ is not so complete as in case $\frac{3}{4} \leq \sigma < 1$. When the author gave a talk on the contents of Matsumoto-Meurman[31,III] at Göttingen, Germany, in Sept. 1992, Jutila raised a question, in which he mentioned the possibility that $\zeta(s)$ may actually have zeros on the line $\sigma = \frac{3}{4}$, and consequently it may follow that $\mu(\frac{1}{2}) \geq \frac{1}{8}$. In a different context, Motohashi[39] also presents a doubt about the Riemann hypothesis, from the viewpoint of mean value theory.

After the author's talk at Kyoto Symposium, Elliott said (probably as a joke) "Now the Riemann hypothesis is more famous than the conjecture (5.10). But 2000 years later, the Riemann hypothesis will be a conjecture of 2100 years ago, and (5.10) of 2000 years ago, so there will be no big difference!" We may interpret that this opinion of Elliott includes the conjecture that the Riemann hypothesis will not be settled in the coming 2000 years. If this conjecture would be true, it would also be a long time later when one knows whether (5.10) is true or not. Since the author is not so bold as to discuss mathematics of the 40th century, it is better to stop here.

REFERENCES

[1] F.V.Atkinson, The mean-value of the Riemann zeta function, Acta Math. 81 (1949) 353-376.

[2] R.Balasubramanian, A. Ivić and K.Ramachandra, The mean square of the Riemann zeta-function on the line $\sigma = 1$, L'Enseignement Math. 38 (1992) 13-25.

[3] R.Balasubramanian, A. Ivić and K.Ramachandra, An application of the Hooley-Huxley contour, Acta Arith. 65 (1993) 45-51.

[4] J.L.Hafner and A. Ivić, On the mean-square of the Riemann zeta-function on the critical line, J. Number Theory 32 (1989)

151-191.

[5] D.R.Heath-Brown, The mean value theorem for the Riemann zeta-function, Mathematika 25 (1978) 177-184.

[6] M.N.Huxley, Exponential sums and the Riemann zeta-function IV, Proc. London Math. Soc. (3)66 (1993) 1-40.

[7] A. Ivić, On some integrals involving the mean square formula for the Riemann zeta-function, Publ. Inst. Math. (Belgrade) 46 (60) (1989) 33-42.

[8] A. Ivić, Mean values of the Riemann zeta function, Lecture Note Ser. 82, Tata Inst. Fund. Res., Bombay, distributed by Springer, 1991.

[9] A. Ivić, La valuer moyenne de la fonction zeta de Riemann, in "Séminaire de Théorie des Nombres, Paris, 1990-1991", S.David (ed.), Birkhäuser, 1993, pp.115-125.

[10] A. Ivić, Power moments of the error term in the approximate functional equation for $\zeta^2(s)$, Acta Arith. 65 (1993) 137-145.

[11] A. Ivić, The moments of the zeta-function on the line $\sigma = 1$, Nagoya Math. J., to appear.

[12] A. Ivić, Some problems on mean values of the Riemann zeta-function, preprint.

[13] A. Ivić and K.Matsumoto, On the error term in the mean square formula for the Riemann zeta-function in the critical strip, preprint (first version, 1993).

[14] M.Jutila, Riemann's zeta-function and the divisor problem, Ark. Mat. 21 (1983) 75-96.

[15] M.Katsurada, Asymptotic expansions of the mean square of Dirichlet L-functions, Proc. Japan Acad. 68A (1992) 219-222.

[16] M.Katsurada, Asymptotic expansions of the mean values of Dirichlet L-functions II, in "Analytic Number Theory and Related Topics", K.Nagasaka (ed.), World Scientific, 1993, pp. 61-71; III, Manuscripta Math., to appear; IV, Sūrikaiseki Kenkyūsho Kōkyūroku 837, Kyoto Univ. (1993) 72-83.

[17] M.Katsurada and K.Matsumoto, Asymptotic expansions of the mean values of Dirichlet L-functions, Math. Z. 208 (1991) 23-39.

[18] M.Katsurada and K.Matsumoto, The mean values of Dirichlet L-functions at integer points and class numbers of cyclotomic fields, Nagoya Math. J., to appear.

[19] M.Katsurada and K.Matsumoto, Discrete mean values of Hurwitz zeta-functions, Proc. Japan Acad. 69A (1993) 164-169.

[20] M.Katsurada and K.Matsumoto, Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions, Proc. Japan Acad. 69A (1993) 303-307.

[21] M.Katsurada and K.Matsumoto, Explicit formulas and asymptotic expansions for certain mean square of Hurwitz zeta-functions I, preprint.

[22] I.Kiuchi, An improvement on the mean value formula for the approximate functional equation of the square of the Riemann zeta-function, J. Number Theory 45 (1993) 312-319.

[23] I.Kiuchi, The mean value formula for the approximate functional equation of $\zeta^2(s)$ in the critical strip, Arch. Math., to appear.

[24] I.Kiuchi, An integral involving the error term of the mean square for the Riemann zeta-function in the critical strip, Math. J. Okayama Univ., to appear.

[25] I.Kiuchi and K.Matsumoto, Mean value results for the approximate functional equation of the square of the Riemann zeta-function, Acta Arith. 61 (1992) 337-345.

[26] A.Laurinčikas, The Atkinson formula near the critical line, in "New Trends in Probab. and Statist.", F.Schweiger and E. Manstavičius (eds.), VSP/TEV, 1992, pp.335-354.

[27] A.Laurinčikas, The Atkinson formula for L-functions near the critical line, Liet. Mat. Rinkinys, to appear.

[28] K.Matsumoto, The mean square of the Riemann zeta-function in the critical strip, Japanese J. Math. 15 (1989) 1-13.

[29] K.Matsumoto, Mean values of error terms in the theory of the Riemann zeta-function (in Japanese), Proc. Sympos. Analytic Number Theory and Related Topics, Gakushuin Univ., 1991, pp. 101-110.

[30] K.Matsumoto, The mean square of the Riemann zeta-function in the strip $\frac{1}{2} < \sigma < 1$ (in Japanese), Sūrikaiseki Kenkyūsho Kōkyūroku 837, Kyoto Univ. (1993) 150-163.

[31] K.Matsumoto and T.Meurman, The mean square of the Riemann zeta-function in the critical strip II, preprint 1991, now submitted to Acta Arith. (see "A digressive talk" in Section 2); III, Acta Arith. 64 (1993) 357-382.

[32] T.Meurman, A generalization of Atkinson's formula to L-functions, Acta Arith. 47 (1986) 351-370.

[33] T.Meurman, On the mean square of the Riemann zeta-function, Quart. J. Math. Oxford (2)38 (1987) 337-343.

[34] Y.Motohashi, A note on the mean value of the zeta and L-functions I, Proc. Japan Acad. 61A (1985) 222-224; II, ibid. 61A (1985) 313-316; III, ibid. 62A (1986) 152-154; IV, ibid. 62A (1986), 311-313.

[35] Y.Motohashi, On the mean square of L-functions, unpublished manuscript, 1986 (originally intended to include in [36]; now we can call it as "a missing chapter").

[36] Y.Motohashi, Lectures on the Riemann-Siegel formula, Ulam Seminar, Dept. of Math., Colorado Univ., Boulder, 1987.

[37] Y.Motohashi, The mean square of $\zeta(s)$ off the critical line, unpublished manuscript, 1990.

[38] Y.Motohashi, On the mean square of the product of the zeta and L-functions, Sūrikaiseki Kenkyūsho Kōkyūroku 837, Kyoto Univ. (1993) 57-62.

[39] Y.Motohashi, The Riemann zeta-function and the non-Euclidean Laplacian (in Japanese), Sūgaku 45 (1993) 221-243; English transl.: Sūgaku Expositions, to appear.

[40] H.Nakaya, The negative power moment of the Riemann zeta-function on the line $\sigma = 1$, preprint.

[41] H.Nakaya, Some mean squares in connection with $\zeta(1+it)$, Proc. Japan Acad. 69A (1993) 275-277.

[42] E.Preissmann, Sur la moyenne quadratique du terme de reste du problème du cercle, C. R. Acad. Sci. Paris 306 (1988) 151-154.

[43] E.Preissmann, Sur la moyenne de la fonction zêta, in "Analytic Number Theory and Related Topics", K.Nagasaka (ed.), World Scientific, 1993, pp.119-125.