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On the Singular Limits of the Boltzmann Equation

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1 Introduction

The nonlinear PDE's describing the motion of a fluid make a long list, among which are the Boltzmann equation, the Navier–Stokes and Euler equations, compressible and incompressible, to mention a few. Newton's equation of motion must be also included in this list as an equation for the microscopic description of the motion where the fluid is considered as a system of many small particles. The compressible and incompressible Navier-Stokes and Euler equations look at the fluid at the macroscopic level as a continuum while the Boltzmann equation is inbetween, at the mesoscopic level. Different nonlinear equations of different types come according to which levels are adopted for the description of the motion and to which properties of the fluid are to be investigated.

Apart from Newton's equation, however, they are derived more or less on physical intuition. Thus one of the main issues in the fluid dynamics is to reveal how these nonlinear equations are interrelated to each other and to find out the regimes of their validity which are not quite clear from their derivations. In physics, the diagram depicted in Fig.1 has been widely known, which says that, starting from Newton's equation of motion, one equation can be obtained form another at the limit value of a certain physical parameter

$$\begin{tikzcd}
 & \text{Newton} \arrow[d, "N \to \infty"] & \\
 & \text{Boltzmann} \arrow[dl, dashed, "\epsilon << 1"'] \arrow[dr, "\epsilon \to 0"] & \\
\text{C.NS} \arrow[rr, "\nu \to 0"'] \arrow[d, "M \to 0"'] & & \text{C.E} \arrow[d, "M \to 0"] \\
\text{IC.NS} \arrow[rr, "\nu \to 0"] & & \text{IC.E}
\end{tikzcd}$$

C: Compressible IC: Incompressible
NS: Navier–Stokes E: Euler

Fig. 1

contained in the latter equation. The parameters in Fig.1 are N (the number of fluid particles), ϵ (the mean free path), ν (the viscosity coefficient) and M (the Mach number).

Much has been done in the last two decades to confirm this diagram with mathematical rigor. To prove the convergence

$$A_\mu \longrightarrow B \quad as \quad \mu \to \mu^*$$

needs to prove that solutions exist to the equations A_μ uniformly for all μ near μ^*, that they converge to some limit as $\mu \to \mu^*$ and that the limit solves the equation B. Thus the diagram in Fig.1 provides numerous challenging mathematical problems. They are nice examples of problems in the theory of singular perturbation. At present, this diagram is mathematically completed, though not fully, for the Cauchy problems and the mechanism for the development of the initial layer is well revealed, whereas almost nothing is known for the initial boundary value problems where the boundary layer prevails. Some remarks and references for the case of the Cauchy problems are given in §5.

According to the above diagram, the compressible Euler equation is connected with the Boltmann equation, a fact established rather formally by

Hilbert [15], but the broken line, coming from the Chapmann–Enskog expansion (see, e.g.,[9]), does not give an asymptotic expansion in the normal sense.

The objective of this article is to show that both the incompressible Navier-Stokes and Euler equations can also be connected directly with the Boltzmann equation, not via the corresponding compressible equations, by means of suitable scalings of variables. This adds new links in the classical diagram given in Fig. 1 and implies the special role the Boltzmann equation plays in the fluid dynamics. This new observation was initiated by Sone [26] (see also Sone-Aoki [27]) for the stationary case and then by Bardos-Golse-Levermore [4] and De Masi-Esposito-Lebowitz [10] for the time dependent case. The proof of convergence was given by Bardos-UKai [5].

2 The Boltzmann Equation

The (normalized) number density $f = f(t, x, v)$ of gas particles at time $t \geq 0$ having position $x \in \mathbf{R}^3$ and velocity $v \in \mathbf{R}^3$ is governed by the Boltzmann equation,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\epsilon} Q[f, g], \tag{2.1}$$

where $\epsilon > 0$ denotes the mean free path, regarded as a parameter in the sequel, while Q, describing collisions of particles, is a bilinear symmetric integral operator in v only. The reader is referred to [8] or [9] for the explicit form of Q as well as the derivation of (2.1). If f is normalized suitably (e.g. devided by the total number N of the gas particles), then Q becomes independent of ϵ after factorized out as in (2.1).

(2.1) is an equation of motion in the mesoscopic regime and the moments of f with respect to v give the macroscopic density ρ, flow velocity u and temperature T by

$$\begin{aligned} &\rho = < 1, f >, \quad \rho u = < v, f >, \\ &\rho T = \tfrac{1}{2} < |v - u|^2, f >, \end{aligned} \tag{2.2}$$

where

$$< f, g > = \int_{\mathbf{R}^3} f(v) g(v) dv.$$

The following properties of Q are found in [8], [9], and essential in the sequel.

[Q1] *Let* $\varphi = 1, v, |v|^2$. *Then for any* $f, g > 0$,

$$< \varphi, Q[f,g] >= 0.$$

[Q2] *For any* $f > 0$,

$$< \log f, Q[f,f] >\leq 0.$$

[Q3] *The followings are equivalent.*

(a) $Q[f,f] = 0$.

(b) $< \log f, Q[f,f] >= 0$.

(c) $f = M(v)$ *where*

$$M(v) = M[\rho, u, T](v) = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-u|^2}{2T}\right), \tag{2.3}$$

with some constants $\rho > 0$, $u \in \mathbf{R}^3$, $T > 0$ *independent of* v.

The functions φ in [Q1] are called *collision invariants* while M in [Q3](c) a *Maxwellian* which represents an equilibrium state of the gas with the density ρ, the flow velocity u and the temperature T, or more precisely, it is called a *local* Maxwellian if ρ, u, T depends on t and x, and an *abolute* or *global* Maxwellian otherwise.

Much has been done on the globlal existence of solutions to the Cauchy and initial–boundary value problems for (2.1). The first global solutions are due to Ukai [29] for initials near an absolute Maxwellian and to Diperna–Lions [11] for arbitrary L^1 initials. See also [30], [11] and references therein.

3 The Compressible Limit

The gas is expected to behave like a fluid if it is dense, namely, if ϵ is sufficiently small. In fact, the compressible Euler equation is obtained from (2.1) in the limit $\epsilon \to 0$. The following theorm is adopted from [4] and goes back to Hilbert [15].

Theorem 3.1. *Write the solution of (2.1) as f^ϵ. Suppose that as $\epsilon \to 0$,*

$$
(3.1)\qquad
\begin{array}{ll}
\text{(a)} & f^\epsilon \to f^0 \quad \textit{in } \mathcal{D}_{t,x,v} \ \textit{(distribution sense)},\\
& \textit{with some limit } f^0,\\
\text{(b)} & < \psi, f^\epsilon > \to < \psi, f^0 > \quad \textit{in } \mathcal{D}_{t,x},\\
& \textit{for any test function } \psi(v) \textit{ such that } |\psi(v)| \le C(1+|v|^2),\\
\text{(c)} & < \psi \log f^\epsilon, f^\epsilon > \to < \psi \log f^0, f^0 > \quad \textit{in } \mathcal{D}_{t,x},\\
& \textit{for any test function } \psi(v) \textit{ such that } |\psi(v)| \le C(1+|v|),\\
\text{(d)} & \limsup_{\epsilon\to 0} < \log f^\epsilon, Q[f^\epsilon, f^\epsilon] > \le < \log f^0, Q[f^0, f^0] > .
\end{array}
$$

Then, the limit f^0 must be a Maxwellian M given by (2.3) and ρ, $u = (u_1, u_2, u_3)$, T involved in this M, being functions of t and x, must solve the compressible Euler equation,

$$
(3.2)\qquad
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0,\\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p = 0,\\
(\rho E)_t + \nabla \cdot (\rho E u + p u) = 0,
\end{cases}
$$

where $u \otimes u = (u_i u_j)$, and

$$p = \rho T, \quad E = \frac{1}{2}|u|^2 + \frac{3}{2}T,$$

are the pressure and energy per unit mass respectively.

It should be noted that (3.1), combined with (2.2), implies

$$\rho^\epsilon = < 1, f^\epsilon > \to \rho = < 1, f^0 >,$$

and so on.

Proof of Theorem 3.1. Take the limits of the inner products $< \phi, (2.1) >$ to deduce

$$(3.3)\qquad < \phi, f^0 >_t + \nabla < v\phi, f^0 > = 0,$$

by the aid of [Q1] and (3.1)(b), and of $\epsilon < \log f^\epsilon, (2.1) >$ to deduce

$$< \log f^0, Q[f^0, f^0] > \ge 0,$$

by (3.1)(c)(d). The latter then holds with equality due to [Q2], and so f^0 must be a Maxwellian due to [Q3](b). Now (3.3), together with (2.2), reduces to (3.2).

The convergence hypothesis (3.1) was substantiated first by Nishida [25] for the Cauchy problem, using the abstract Cauchy Kowalevskaya theorem developed in [24]. Roughly speaking, he showed that if the initial data is analytic and near an absolute Maxwellian, then (3.1)(a) takes place in a norm strong enough to assure the rest of (3.1), locally in time. In general the convergence is not uniform near $t = 0$ due to the development of the initial layer. A necessary and sufficient condition for the uniform convergence up to $t = 0$ was found later by Ukai-Asano [32] to be that the initial data is itself to be a local Maxwellian. Caflisch [7] solved a reversed problem, proving that if (3.2) has a sufficiently smooth (but not necessarily analytic) solution on some time interval and if M^E is the Maxellian corresponding to this solution, then solutions to (2.1) with the initial data $M^E|_{t=0}$ exist for all small $\epsilon > 0$ and converge to M^E as $\epsilon \to 0$, both uniformly on the same time interval.

4 The Incompressible Limits

The incompressible Navier-Stokes and Euler equation can be also obtained as the limit of the Boltzmann equation. Transform (2.1) with the scalings

$$t = \frac{t'}{\epsilon^\alpha}, \qquad f = M_0 + \epsilon^\beta M_0^{1/2} g, \tag{4.1}$$

where $\alpha, \beta > 0$ and M_0 is any absolute Maxwellian. It turns out that we are looking at how a nearly equilibrium fluid behaves after transient effects diminish. It was shown in [4], [10] that differnt choices of the scaling powers α and β result in different incompressible limits.

After (4.1), (2.1) reduces, dropping ' for t, to

$$\frac{1}{\epsilon^\alpha}\frac{\partial g}{\partial t} + v \cdot \nabla_x g = \frac{1}{\epsilon} L g + \frac{1}{\epsilon^{1-\beta}} \Gamma[f, f], \tag{4.2}$$

where L is a linear operator and Γ a symmetric bilinear operator, given by

$$Lg = 2M_0^{-1/2} Q[M_0, M_0^{1/2} g], \quad \Gamma[f, g] = M_0^{-1/2} Q[M_0^{1/2} f, M_0^{1/2} g], \tag{4.3}$$

respectively. In the below we choose $M_0 = M[1,0,1](v)$, without loss of generality, which is possible by a suitable scaling and translation of v. Moreover, we assume Grad's cutoff hard potential [14] for the operator Q.

Theorem 4.1. ([4], [10]). *Let α, $\beta > 0$ and write the solution of (4.2) as g^ϵ. Suppose that as $\epsilon \to 0$,*

$$\begin{array}{lll} & \text{(a)} & g^\epsilon \to g^0 \ \ \textit{in } \mathcal{D}_{t,x,v} \ \ \textit{(distribution sense)}, \\ & & \textit{with some limit } g^0, \\ (4.4) & \text{(b)} & < \psi, g^\epsilon > \to < \psi, g^0 > \ \ \textit{in } \mathcal{D}_{t,x}, \\ & \text{(c)} & < \psi, \Gamma[g^\epsilon, g^\epsilon] > \to < \psi, \Gamma[g^0, g^0] > \ \ \textit{in } \mathcal{D}_{t,x}, \\ & & \textit{both for any test function } \psi(v) \textit{ such that} \\ & & |\psi(v)| \le C(1+|v|^3), \end{array}$$

Then, the limit g^0 must be of the form

$$g^0 = \{\eta + u \cdot v + \frac{1}{2}\theta(|v|^2 - 3)\}M_0(v)^{1/2}. \tag{4.5}$$

Here the coefficients $\eta \in \mathbf{R}$, $u \in \mathbf{R}^3$, $\theta \in \mathbf{R}$ are functions of t and x and satisfy

$$\nabla(\eta + \theta) = 0, \quad \nabla \cdot u = 0. \tag{4.6}$$

They satisfy furhter equations which differ according to the choice of α and β.

(1) $\alpha = \beta = 1$.

$$\begin{aligned} &u_t - \nu\Delta u + u \cdot \nabla u + \nabla p = 0, \\ &\theta_t - \kappa\Delta\theta + u \cdot \nabla\theta = 0. \end{aligned} \tag{4.7}$$

(2) $\alpha = 1$ *and* $\beta > 1$.

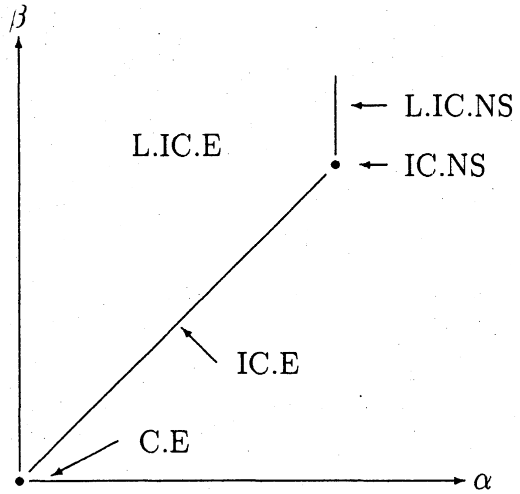
$$u_t - \nu\Delta u + \nabla p = 0, \quad \theta_t - \kappa\Delta\theta = 0. \tag{4.8}$$

(3) $0 < \alpha = \beta < 1$.

$$u_t + u \cdot \nabla u + \nabla p = 0, \quad \theta_t + u \cdot \nabla\theta = 0. \tag{4.9}$$

(4) $0 < \alpha < 1$ *and* $\alpha < \beta$,

$$u_t + \nabla p = 0, \quad \theta_t = 0. \tag{4.10}$$



L: Linear
C: Compressible IC: Incompressible
NS: Navier–Stokes E: Euler

Fig. 2

(5) *No more equations for other choices of α, β.*
In the above, p is a suitable function while ν, κ are positive constants given by

(4.11) $$\nu = -\frac{1}{3} < \Psi, L^{-1}\Psi >, \quad \kappa = -\frac{1}{10} < \Phi, L^{-1}\Phi >,$$

with

(4.12) $$\Psi = v \otimes v - \frac{1}{3}|v|^2 I, \quad \Phi = (\frac{1}{2}|v|^2 - \frac{5}{2})v.$$

Notice that the case $\alpha = \beta = 0$ reduces to Theorem 3.1. The first equation in (4.6) is the *Bousinessq* equation. The first equation of (4.7) with the second of (4.6) is the incompressible Navier-Stokes equation and the second equation of (4.7) is the heat convection equation. The constants ν and κ are the viscosity coefficient and heat diffusitivity respectively, and the functions Φ, Ψ are *Barnett* functions. Also, the first equation of (4.9) with the second of (4.6) is the incompressible Euler equation, and (4.8) and (4.10) are the linearized versions of (4.7) and (4.9) respectively. Fig.2 summerizes the conclusions of Theorem 4.1.

Since $\rho = 1$ for M_0 of our choice, we have, $\rho^\epsilon =< 1, f^\epsilon >= 1 + \epsilon^\beta \eta^\epsilon$ with

$$\eta^\epsilon =< 1, M_0^{1/2} g^\epsilon > \to \eta,$$

and similarly, $T^\epsilon = 1 + \epsilon^\beta \theta^\epsilon$ and $\theta^\epsilon \to \theta$, both by (4.4)(b).

The convergence hypothesis (4.4) is to be verified. We state the result for the case (1) but similar results can be obtained for other cases. We shall consider the Cauchy problem to (4.2) with the initial condition

$$g^\epsilon|_{t=0} = g_0, \tag{4.13}$$

in which g_0 does not depend on ϵ. Roughly speaking, g^ϵ converges globally in time and strongly if g_0 is small. Also, the initial layer is found to exist. Define the space

$$X = \{g(x,v) | \sup_{v\in\mathbf{R}^3}(1+|v|^3)||g(\cdot,v)||_{H^3(\mathbf{R}^3_x)} < \infty\}, \tag{4.14}$$

and denote its norm by $||\cdot||$. The following three theorems are found in Bardos–Ukai [5].

Theorem 4.2. *Let $\alpha = \beta = 1$. There exists a positive number c_0 and the following holds for all $g_0 \in X$ with $||g_0|| \le c_0$.*
(1) *For each $\epsilon \in (0,1]$, there exists a unique global solution $g^\epsilon \in C([0,\infty);X)$ satisfying*

$$||g^\epsilon(t)|| \le C, \tag{4.15}$$

with a constant $C > 0$ independent of both ϵ and t.
(2) *As $\epsilon \to 0$,*

$$g^\epsilon \to g^0 \quad \begin{array}{l} \textit{weakly* in } L^\infty(0,\infty;X), \textit{ and,} \\ \textit{uniformly for } (t,x,v) \in [\delta_0, T_0]\times K \times \mathbf{R}^3 \\ \textit{for any } T_0 > \delta_0 > 0 \textit{ and for any compact } K \subset \mathbf{R}^3. \end{array} \tag{4.16}$$

(3) $g_0 \in C([0,\infty);X)$.

The convergence (2) is strong enough to assure all of (4.4), and (3) means, in particular, the continuity of g^0 up to $t=0$, which does not come from (2) since $\delta_0 > 0$, and entrains that for the coefficients in (4.5),

$$(\eta, u, \theta) \in C([0,\infty); H^3(\mathbf{R}^3_x)). \tag{4.17}$$

Put

$$(\eta_0, u_0, \theta_0) = < (1, v, \frac{1}{2}(|v|^2 - 3)), M_0^{1/2} g_0 > .$$

and define the projection P_0 by

$$P_0 g_0 = \{a + b \cdot v - \frac{a}{2}(|v|^2 - 3)\} M_0^{1/2}, \tag{4.18}$$

with

$$a = \frac{1}{2}(\eta_0 - \theta_0), \quad b = P u_0, \tag{4.19}$$

where P is the projection to the divergence-free subspace.

Theorem 4.3. (1) $g^0|_{t=0} = P_0 g_0$.
(2) (u, θ) *is a unique strong global solution to the Cauchy problem for (4.7) coupled with the second equation of (4.6) and with the initial condition,*

$$(u, \theta)|_{t=0} = (b, -a). \tag{4.20}$$

In (2) of Theorem 4.2, $\delta_0 > 0$ for general initials, that is, the uniform convergence breaks down near $t = 0$ and the initial layer develops. However,

Theorem 4.4. $\delta_0 = 0$ *if and only if* $g_0 = P_0 g_0$.

5 Remarks concerning the diagram

1. Newton to Boltzmann.
   The idea goes back to Grad [13], which is now called the Boltzmann-Grad limit. The first convergence proof was given by Lanford III, [22], on a short time interval of several mean free times. The global in time convergence was discussed by Illner–Pluvireti [16].

2. Boltzmann to Compressible Euler. See §3 for the references.

3. Boltzmann to Compressible Navier–Stokes.
   This follows formally by the so-called Chapmann-Enskog expansion (see [9]), which, thought, is not the asymptotic expansion in the normal sense. Kawashima–Matsumura–Nishida [19] proved that for initials near an absolute Maxwellian, $f^\epsilon \to M[\rho, u, T]$ as $t \to \infty$, (ρ, u, T) solving the compressible Navier-Stokes equation with the viscosity coefficient and heat diffusivity propotional to ϵ.

4. Compressible Navier–Stokes to Incompressible Navier–Stokes.
   The time local convergence is discussed for divergence free initials in Klainermann–Majda [20].

5. Compressible Navier–Stokes to Compressible Euler.
   For the time local convergence, see Kawashima [18]. No initial layer develops.

6. Compressible Euler to Incompressible Euler.
   For the divergence free initials, the time local convergence is discussed on the Cauchy problem by Klainerman–Majda [21] and on the initial boundary value problem by Agemi [1], Ebin [12], see also da Veiga [6]. Since the boundary conditions are the same for both cases, no boundary layer appears. The initial layer appears, on the other hand, for non-divergence initials, see Ukai [31], Asano [2].

7. Incompressible Navier–Stokes to Incompressible Euler.
   For the Cauchy problem, see Kato [17]. The boundary layer problem for the incompressible Navier–Stokes equation is one of the most important issues in the fluid dynamics, in connection to the nature of the turbulance, but almost nothing is known about this. See Asano [3] for the treatment in the space of analytic functions, and Matsui [23] for an example of the boundary layer. See Tani [28] for the slip boundary condition for which the boundary layer does not develop.

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