数理解析研究所講究録
第 888 巻 1994 年 89-95



# Recent topics on the compressible Euler equation

Tetu Makino*, Kiyoshi Mizohata** and Seiji Ukai**

(July, 1994)

Department of Liberal Arts
Osaka Sangyo University*
and
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology**

## 1. Introduction

In this paper we shall describe recent topics on the compressible Euler equation and the relativistic Euler equation for the multidimensional case. The compressible Euler equation for an isentropic gas in $\mathbf{R}^n$ is given by

$$(1.1) \qquad \begin{cases} \rho_t + \sum_{j=1}^{n} \dfrac{\partial}{\partial x_j}(\rho u_j) = 0 , \\ (\rho u_i)_t + \sum_{j=1}^{n} \dfrac{\partial}{\partial x_j}(\rho u_i u_j + \delta_{ij} p) = \rho f_i , \quad (i = 1, 2, \cdots, n) \end{cases}$$

with the equation of state

$$(1.2) \qquad p = a^2 \rho^\gamma ,$$

where the density $\rho$, velocity $\mathbf{u} =^t (u_1, u_2, \cdots, u_n)$ and pressure $p$ are functions of $x \in \mathbf{R}^n$ and $t \geq 0$, while $f$ is a given external force and $a > 0$ and $\gamma \geq 1$ are given constants.

This is a typical example of conservation laws and has attracted many mathematicians. In particular, for the one dimensional case ( n=1 ), the Cauchy problem for (1.1) with (1.2) has been studied extensively.

On the other hand, little is known for the case $n \geq 2$. No global solutions are known to exist, but only local classical solutions, in the full generality.

In 1992, we have presented global weak solutions first for the case $n \geq 2$. We have done this, however, only for the case of spherically symmetry with $\gamma = 1$ in the domain outside a unit ball.

However, we could not content ourselves since our class of initial data does not contain stationary solutions. In 1993 we have shown that if we use a non-uniform mesh chosen carefully, Glimm's method still works and gives global weak solutions for the initial data in the class containing the stationary solutions. We shall explain these results and related topics in section 2. It was observed in [13] that these stationary solutions are stable if initial data are sufficiently close to these solutions.

For the case $\gamma > 1$, our method can't be applied. T. Makino and S. Takeno have proved, by using compensated compactness method, the existence of temporally local weak solution for (1.1) with spherical symmetry in [2]. Recently we have known that Glimm and Chen have succeeded to prove the existence of global weak solution for (1.1) with spherical symmetry. For viscous barotropic gas M. Okada and T. Makino [10] have proved the existence of global solutions.

Next, let us consider the relativistic Euler equation describing a motion of perfect fluid in special relativity. The relativistic Euler equation for an isentropic gas in the Minkowski space-time is given by

$$
(1.3)\quad
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho c^2+p}{c^2}\frac{1}{1-\frac{u^2}{c^2}}-\frac{p}{c^2}\right)+\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\frac{\rho c^2+p}{c^2}\frac{u_i}{1-\frac{u^2}{c^2}}\right)=0\,,\\
&\frac{\partial}{\partial t}\left(\frac{\rho c^2+p}{c^2}\frac{v_i}{1-\frac{u^2}{c^2}}\right)+\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\rho c^2+p}{c^2}\frac{u_iu_j}{1-\frac{u^2}{c^2}}+\delta_{ij}p\right)=0,\ (i=1,2,3),
\end{aligned}
$$

with the equation of state

$$(1.4)\qquad p=\sigma^2\rho\,,$$

where the density $\rho$, velocity $\mathbf{u}={}^t(u_1,u_2,u_3)$ and pressure $p$ are functions of $x\in\mathbf{R}^3$ and $t\geq 0$, while $c$ is the speed of the light and $\sigma$ is the speed of the sound which are constants. According to the relativistic theory, $\sigma$ never exceeds $c$. Especially, the case $\sigma^2=\frac{c^2}{3}$ is important in the context of the physics. If $c\to\infty$, (1.3) reduces to the classical compressible Euler equation (1.1).

In 1993, Smoller and Temple [11] have constructed uniformly bounded weak solutions for 1 dimensional case by using Glimm's method. After constructing approximate solutions, they have showed that the variation of $\log\rho$ is monotone decreasing.

In 1994, T.Makino and S.Ukai [3] have proved, by using Lax's theorem, the existence of the local classical solutions for (1.3). Recently we have constructed global weak solutions for (1.3) with spherical symmetry. we shall explain this result in section 3.

## 2. Global weak solution for the compressible Euler equation

In this section we shall present our recent results on the compressible Euler equation with spherical symmetry. We look for solutions of the form

$$(2.1)\qquad \rho=\rho(t,|x|),\ \vec{u}=\frac{x}{|x|}\cdot u(t,|x|).$$

Then, denoting $r=|x|$, (1.1) with $f=0$ becomes

$$
(2.2)\quad
\begin{aligned}
&\rho_t+\frac{1}{r^{n-1}}(r^{n-1}\rho u)_r=0\,,\\
&\rho(u_t+uu_r)+p_r=0\,.
\end{aligned}
$$

This equation has a singularity at r=0. To avoid the difficulty caused by this singularity, we simply deal with the boundary value problem for (2.2) in the domain $1\leq r<\infty$ ( the exterior of a sphere) with the boundary condition $u(t,1)=0$, which is identical, under the assumption (2.1), to the usual boundary condition $\vec{n}\cdot\vec{u}=0$ for (1.1) where $\vec{n}$ is the unit normal to the boundary.

Put $\tilde{\rho}=r^{n-1}\rho$. Then we get from (2.2)

$$(2.3)\qquad \begin{aligned} &\tilde{\rho}_t + (\tilde{\rho}\, u)_r = 0\,,\\ &u_t + u\, u_r + \frac{a^2\gamma\tilde{\rho}_r}{\tilde{\rho}^{2-\gamma}\, r^{(n-1)(\gamma-1)}} = \frac{a^2\gamma(n-1)\tilde{\rho}^{\gamma-1}}{r^n\,\cdot\, r^{(n-1)(\gamma-2)}}. \end{aligned}$$

Introduce the Lagrangian mass coordinates

$$(2.4)\qquad \tau = t\,,\quad \xi = \int_1^r \tilde{\rho}(t,r)\,dr\,.$$

Then $\xi > 0$ as long as $\tilde{\rho} > 0$ for $r > 1$, and (2.3) is reformulated as

$$(2.5)\qquad \begin{aligned} &\tilde{\rho}_\tau + \tilde{\rho}^2\, u_\xi = 0\,,\\ &u_\tau + \frac{a^2\gamma\tilde{\rho}_\xi}{\tilde{\rho}^{1-\gamma}\, r^{2\gamma-2}} = \frac{a^2\gamma(n-1)\tilde{\rho}^{\gamma-1}}{r^n\,\cdot\, r^{(n-1)(\gamma-2)}}. \end{aligned}$$

Put $v = 1/\tilde{\rho}$ and note that the inverse transformation to (2.4) is given by

$$(2.6)\qquad t = \tau,\ r = 1 + \int_0^\xi v(t,\zeta)d\zeta.$$

Then after changing $\tau$ to t and $\xi$ to x, (2.5) is written as

$$(2.7)\qquad \begin{aligned} &v_t - u_x = 0\,,\\ &u_t + \left(\frac{a^2}{v^\gamma}\right)_x \cdot \frac{1}{r^{(n-1)(\gamma-1)}} = \frac{a^2\gamma(n-1)v^{1-\gamma}}{r^n\,\cdot\, r^{(n-1)(\gamma-2)}}, \end{aligned}$$

where $r$ is now defined by $r = 1 + \int_0^x v(t,\zeta)d\zeta$. Now we restrict ourselves to the case $\gamma = 1$. Then (2.7) becomes

$$(2.8)\qquad \begin{aligned} &v_t - u_x = 0\,,\\ &u_t + \left(\frac{a^2}{v}\right)_x = \frac{K}{1 + \int_0^x v(t,\zeta)d\zeta}\,. \end{aligned}$$

where $K = a^2(n-1)$. Let us consider the initial boundary value problem for (2.8) in $t \geq 0,\ x \geq 0$ with the following boundary and initial conditions.

$$(2.9)\qquad u(0,x) = u_0(x)\,,\quad v(0,x) = v_0(x)\,,\quad for\ x > 0,$$

$$(2.10)\qquad u(t,0) = 0\,,\quad for\ t > 0.$$

**Theorem 2.1.** *Suppose that $u_0(x)$ and $v_0(x)$ are of bounded variation, and that $v_0(x) \geq \delta_0 > 0$ for all $x > 0$ with some positive constant $\delta_0$. Then (2.8), (2.9) and (2.10) have a global weak solution.*

This theorem can be proved by following Nishida's argument [9] based on Glimm's method [1]. For the detail, see [4]. Note that the principal part of (2.8) coincides exactly with the one-dimensional compressible Euler equation, but this coincidence does not occur for $\gamma > 1$.

**Remark 1.** Note that it is not obvious that this result also implies the existence of global weak solutions of (1.1). If solutions in Lagrangian coordinate are smooth functions, we can show that $\mathbf{u}$ and $\rho$ deduced from these solutions satisfy (1.1) by using the chain rule. But if solutions are weak solutions, we must be more careful. In [8] K. Mizohata has proved that weak solutions in Lagrangian coordinate are weak solutions in Eulerian coordinate at least they are spherically symmetric and that vice versa. Instead of using the chain rule, we use the fact that the Lagrangian transformation is a bi-Lipschitz homeomorphism to prove that $(\mathbf{u}, \rho)$ is also weak solution of (1.1). This is the main idea of Wagner [4]. He has showed the equivalence for the Cauchy problem in one space dimension. In [8], K. Mizohata has given the detailed proof of this equivalence for the more general case.

**Remark 2.** K. Mizohata has extended this result to the case in which the gravitational force $f = -\rho M/r^2$ appears in the right-hand side of the second equation of (2.2). For the detail, see [7].

It is clear that the equation (2.2) admits the stationary solutions $\rho = \bar{\rho} = constant > 0$ , $u = 0$. The corresponding stationary solutions for (2.8) are

$$v = \frac{1}{\bar{\rho}\left(1 + \frac{n}{\bar{\rho}}x\right)^{1-\frac{1}{n}}} , \quad u = 0 . \tag{2.11}$$

For these stationary solutions, we have $\inf v = 0$, i.e., $v$ is not bounded away from zero. Now, if we attempt to enlarge the class of the initial data of Theorem 2.1 so that it includes these stationary solutions, we readily encounter a difficulty in construction of the approximate solutions used in Glimm's method, [1], that is, the mesh lengths $\Delta x$ and $\Delta t$ should be chosen so that $\Delta x/\Delta t > a/\inf v$ in view of the Courant-Friedrichs-Lewy condition, whereas this could not be possible as long as the mesh is supposed to be uniform. In [5] we have showed that if we use a non-uniform mesh chosen carefully, Glimm's method still works and gives global weak solutions for the initial data in the class containing the stationary solutions (2.11).

More precisely, we deal with the initial data $u^0$ and $v^0$ satisfying the following conditions:

$$\begin{aligned} &v_0 \, , \ |u_0| \, , \ T.V.u_0 \, , \ T.V.v_0 \ \le \ C_0 \, , \\ &\frac{\delta_0}{(1+x)^{1-\epsilon}} \ \le \ v_0(x) \, , \end{aligned} \tag{2.12}$$

where $C_0$, $\delta_0$ and $\epsilon$ are positive constants independent of $x$ and $0 < \epsilon \le 1$. By using a non-uniform mesh method, we have succeeded to enlarge the class of the initial data so that it includes these stationary solutions. For the detail, see [5].

**Theorem 2.2.** *If $u_0$ and $v_0$ satisfy (2.12), then the initial-boundary value problem (2.8), (2.9) and (2.10) admit a global weak solution $(u, v)$.*

## 3. Global weak solution for the relativistic Euler equation

In this section we shall present our result on the relativistic Euler equation with spherical symmetry. Similarly we look for solutions of the form

$$\rho = \rho(t, |x|), \ \vec{u} = \frac{x}{|x|} \cdot u(t, |x|). \tag{3.1}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then (1.3) becomes

$$
(3.2)\quad
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho c^2 + p}{c^2}\frac{1}{1-\frac{u^2}{c^2}} - \frac{p}{c^2}\right) + \frac{\partial}{\partial r}\left(\frac{\rho c^2 + p}{c^2}\frac{u}{1-\frac{u^2}{c^2}}\right) + \frac{\rho c^2 + p}{c^2}\frac{1}{1-\frac{u^2}{c^2}}\frac{2u}{r} = 0\,,\\
&\frac{\partial}{\partial t}\left(\frac{\rho c^2 + p}{c^2}\frac{u}{1-\frac{u^2}{c^2}}\right) + \frac{\partial}{\partial r}\left(\frac{\rho c^2 + p}{c^2}\frac{u^2}{1-\frac{u^2}{c^2}} + p\right) + \frac{\rho c^2 + p}{c^2}\frac{u}{1-\frac{u^2}{c^2}}\frac{2u^2}{r} = 0\,.
\end{aligned}
$$

We again simply deal with the boundary value problem for (3.2) in the domain $1 \le r < \infty$ (outside a unit ball) with the boundary value condition $u(t,1) = 0$ to avoid the singularity caused at the origin. Introduce Lagrangian coordinate

$$
(3.3)\qquad x = \int_1^r \rho r^2 J dr\,,
$$

where

$$
(3.4)\qquad J = 1 + \frac{(c^2+\sigma^2)u^2}{c^2(c^2-u^2)}\,.
$$

Putting

$$
(3.5)\qquad W = \frac{1}{Jq},\ V = \frac{\frac{c^2+\sigma^2}{c^2-\sigma^2}}{J}u = \frac{(c^2+\sigma^2)c^2}{c^4+\sigma^2u^2}u\,,\ q = r^2\rho\,.
$$

Then (3.2) becomes,

$$
(3.6)\qquad
\begin{aligned}
W_t \;-\; V_x \;&=\; 0\,,\\
V_t \;+\; \left(\frac{\sigma^2}{J^2W}\right)_x \;&=\; \frac{2\sigma^2}{Jr}\,,
\end{aligned}
$$

where r is now defined by $r = 1 + \int_0^x W dx$. Let us consider the initial boundary value problem for (3.6) in $t \ge 0$, $x \ge 0$ with the following initial and boundary conditions.

$$
(3.7)\qquad q(0,x) \;=\; q_0(x),\quad u(0,x) \;=\; u_0(x),\quad x > 0\,,
$$

$$
(3.8)\qquad v(t,0) = 0\,.
$$

Our main result is as follows.

**Theorem 3.1.** *Suppose that* $\log q_0(x)$ *and* $\log\dfrac{c+u_0(x)}{c-u_0(x)}$ *are of bounded variation. Then there exists a global weak solution for (3.6), (3.7) and (3.8) satisfying*

$$
(3.9)\qquad |u| < c\,,\quad q > 0\,.
$$

We shall briefly explain the outline of proof. First, consider Riemann problem for the homogeneous equation corresponding to (3.6) which is given by

$$
(3.10)\qquad
\begin{aligned}
W_t \;-\; V_x \;&=\; 0\,,\\
V_t \;+\; \left(\frac{\sigma^2}{J^2W}\right)_x \;&=\; 0\,,
\end{aligned}
$$

Then we can show that all shock curves for (3.10) have the same figure in the plane of Riemann invariants. Fortunately, their figure is similar to the figure of shock curves of the classical case. This fact was also discovered by Smoller and Temple [11] for the one dimensional case of (1.3). We construct approximate solutions via the Glimm's difference scheme. More precisely, we construct our approximate solutions of the form

$$\{\textit{solution of Riemann problem for (3.10)}\} + \{\textit{inhomogeneous term}\} \times t\ .$$

Next we estimate the total variation of the approximate solutions. We do it by analyzing the waves in the plane of Riemann invariants $z_1$ and $z_2$ where they are given by

$$\begin{aligned} z_1 &= \log q + \frac{c^2+\sigma^2}{2\sigma c} \log \frac{c+u}{c-u}\ , \\ z_2 &= -\log q + \frac{c^2+\sigma^2}{2\sigma c} \log \frac{c+u}{c-u}\ . \end{aligned} \tag{3.11}$$

Using the geometry of shock waves, we estimate the variation of $\log q$. This is the main idea of Smoller and Temple [9]. But in our case, we must be more careful since there is an inhomogeneous term in (3.6). To obtain our desired uniform estimates, we use the transformation

$$u \;=\; c \tanh w \;=\; c\,\frac{e^w - e^{-w}}{e^w + e^{-w}}, \tag{3.12}$$

Instead of estimating $u$ itself, we estimate $w$. Fortunately, it follows that

$$\log \frac{c+u}{c-u} \;=\; 2w\ . \tag{3.13}$$

The transformation (3.12) plays a crucial role in our paper. Using (3.12) and (3.13), we can obtain uniform estimates of the approximate solutions and thus we can construct global weak solution. For the detail, see [6].

## 4. Concluding remarks

Conservation laws are important nonlinear PDEs since many problems in science have conserved quantities. Especially, the Euler equation of gas dynamics is a typical example of it and many mathematician have studied it. Consequently, much of interesting theory of conservation laws have been discovered by studying the Euler equation.

But although there are many results for the one dimensional case, there are few results for the multi-dimensional case. In this case it is indeed difficult to obtain desired uniform estimates for a system of conservation laws. If we look for the solutions with spherical symmetry, the problems become a one-dimensional problem and thus we can use one-dimensional methods and theory. But in this case we encounter a singularity at the origin. Unfortunately we do not yet succeed to deal with this singularity. To avoid this singularity, we considered the problems outside a unit ball. We thus obtained, by using one-dimensional methods and our new methods, several results for the multi-dimensional case.

We hope that by considering the problems with spherical symmetry, we can find solution to the problem for the general multi-dimensional case.

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