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Kyoto University
Inversion Formulas arising in Inverse Boundary Value Problems

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1. Results. We formulate two inverse problems, which are analogous to the inverse conductivity problem [10].

Notation. $\Omega$ is a bounded domain of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$; $ds$ is the standard measure on $\partial\Omega$; $\nu$ is the unite outer normal vector field on $\partial\Omega$; $X = \{H^{1/2}(\partial\Omega)\}^2$ and $Y = H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$; $B(X,X^*)$ is the Banach space of all bounded linear maps from $X$ to its dual $X^*$ and $B(Y,Y^*)$ that of all bounded linear maps from $Y$ to its dual $Y^*$; $\nabla u$ is the Jacobian matrix of a vector valued function $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$ on $\Omega$ and $\text{Sym} \nabla u$ its symmetric part; $\nabla^2 w$ is the Hessian matrix of a scalar function $w$ on $\Omega$; $a \otimes b = (a_i b_j)$ for two vectors $a = (a_i)$, $b = (b_j)$; $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; $A_{11} = e_1 \otimes e_1$; $A_{12} = e_1 \otimes e_2 + e_2 \otimes e_1$; $A_{22} = e_2 \otimes e_2$; $\partial_z = \frac{\partial}{\partial x_2} - z \frac{\partial}{\partial x_1}$ for each $\approx \in C$.

Let $C = (C_{ijkl}(x))_{i,j,k,l=1^2}$ be a fourth-order tensor field over $\Omega$ with components $C_{ijkl} \in L^\infty(\Omega)$. We denote by $C(x)A$ the 2 $\times$ 2 matrix $(\sum_{k,l} C_{ijkl}(x) a_{kl})$ for each $x \in \Omega$ and 2 $\times$ 2 matrix $A = (a_{k,l})$. We call $C$ an elasticity tensor field if

$$ C_{ijkl} = C_{klij} = C_{lkij} $$

hold for each $i,j,k,l = 1,2$ and there exists a positive number $\delta$ such that

$$ C(x)A \cdot A \equiv \sum C_{ijkl}(x) a_{k,l} a_{i,j} \geq \delta |A|^2 $$

holds for almost all $x \in \Omega$ and all real symmetric 2 $\times$ 2 matrix $A = (a_{ij})$.

For each elasticity tensor field $C$ we define $L_C$, which is a second order system of partial differential operators acting $\{H^1(\Omega)\}^2$, via

$$ L_C u = \left( \sum \frac{\partial}{\partial x_j} (C_{11kl}(x) \frac{\partial u^k}{\partial x_l}) \right) \left( \sum \frac{\partial}{\partial x_l} (C_{12kl}(x) \frac{\partial u^k}{\partial x_l}) \right), u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in \{H^1(\Omega)\}^2. $$

The associated Dirichlet-to-Neumann map $\Pi_C \in B(X, X^*)$ is defined by

$$ \Pi_C(\varphi) = \{C(x)\text{Sym} \nabla u|_{\partial\Omega}, \varphi \in X, $$

where $u \in \{H^1(\Omega)\}^2$ is the unique solution to

$$ L_C u = 0 \quad \text{in} \quad \Omega $$

Typeset by $\LaTeX$. 
\[ u|_{\partial\Omega} = \varphi. \]

\( \Pi_{C}(\varphi)ds \) is the force exerted across \( ds \) which deforms \( \Omega \) into \( \Omega + u \).

On the other hand, for each elasticity tensor field \( M \) we define \( L_{M} \), which is a fourth-order partial differential operator acting on \( H^{2}(\Omega) \), via

\[ L_{M}w = \sum \frac{\partial^{2}}{\partial x_{i}\partial x_{j}}\{M_{ijkl}(x)\frac{\partial^{2}w}{\partial x_{k}\partial x_{l}}\}, w \in H^{2}(\Omega). \]

The associated Dirichlet-to-Neumann map \( \Pi_{M}^{*} \in B(Y, Y^{*}) \) is defined by

\[ \Pi_{M}^{*}(\varphi) = \left( -\left\{ \frac{\partial}{\partial\tau}M_{\tau}(w) + Q(w) \right\} M_{\iota}'(w) \right)|_{\partial\Omega}, \phi \in Y, \]

where \( w \in H^{2}(\Omega) \) is the unique solution to

\[ L_{M}w = 0 \ \text{in} \ \Omega \]

\[ \left( \begin{array}{c} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial \nu} \end{array} \right)|_{\partial\Omega} = \phi, \]

\[ M_{\nu}(w) = M(x)\nabla^{2}w \cdot \nu \otimes \nu, M_{\tau}(w) = M(x)\nabla^{2}w \cdot \nu \otimes \tau, \]

\[ Q(w) = \sum \frac{\partial}{\partial x_{\beta}}\{(M(x)\nabla^{2}w)_{\alpha\beta}\}\nu_{\alpha}, \tau = \begin{pmatrix} -\nu_{2} \\ \nu_{1} \end{pmatrix}. \]

\( \Pi_{M}^{*}(\emptyset) \) is the external force applied to \( \partial\Omega \) which deforms \( \Omega \) into the graph of \( w \); \( M_{\nu}(w) \) is the bending moment; the first component of \( \Pi_{M}^{*}(\phi) \) is the vertical reaction at \( \partial\Omega \).

This talk is concerned with the following:

**Inverse Problems.**

I. Determine \( C \) from \( \Pi_{C} \);
II. Determine \( M \) from \( \Pi_{M}^{*} \).

The elasticity tensor field is said to be isotropic if there exist \( \lambda, \mu \in L^\infty(\Omega) \), which are called the Lamé parameters, such that

\[ C(x)A = \lambda(x)Trace(A)I_{2} + 2\mu(x)A \]

holds for almost all \( x \in \Omega \) and all real symmetric \( 2 \times 2 \)-matrix \( A \). Since isotropic \( C \) uniquely determines its Lamé parameters we write \( C_{(\lambda, \mu)} \) and \( \Pi(\lambda, \mu) \) instead of \( C \) and \( \Pi_{C} \), respectively.

The first problem for isotropic \( C \) was taken up by the author [2], Akamatsu-Nakamura-Steinberg[1], Nakamura-Uhlmann [8]. In particular, Nakamura-Uhlmann [8] proved that if \( \lambda \) and \( \mu \) are smooth on \( \overline{\Omega} \) and sufficiently close to constants, then \( \Pi(\lambda, \mu) \) uniquely determines \( (\lambda, \mu) \). In [9] they treated the problem of determining \( D^{\alpha}C|_{\partial\Omega}, |\alpha| = 0, 1, \cdots \) from \( \Pi_{C} \) modulo smoothing operators on \( \partial\Omega \), where \( C \) is not necessary isotropic and restricted to being in a class of anisotropic elasticity tensor fields, respectively.
The second problem for isotropic $M$ was taken up by the author [3]. In [3] it is proved that if the Lamé parameters $\lambda, \mu$ of $M$ are smooth and sufficiently close to constants on $\Omega$, then $\Pi^{*}_{(\lambda, \mu)}$ together with $D^{\alpha} \lambda |_{\partial \Omega}$, $|\alpha| = 0, 1$ and $D^{\beta} \mu |_{\partial \Omega}$, $|\beta| = 0, 1, 2, 3$ uniquely determine $(\lambda, \mu)$.

In this talk first we shall point out that I and II are equivalent to each other on the simply connected $\Omega$; second we consider the Fréchet derivative $d\Pi_{C}$ and $d\Pi_{M}^{*}$ at anisotropic $C$ and $M$, respectively; we shall study a relationship between them and give a characterization of the injectivity of $d\Pi_{C}$ by the Stroh eigenvalues of $C$.

For each elasticity tensor field $C$ denote by $[C]$ the symmetric $3 \times 3$-matrix

$$[C] = \begin{pmatrix} C_{1111} & C_{1112} & C_{1122} \\ C_{1211} & C_{1212} & C_{1222} \\ C_{2211} & C_{2212} & C_{2222} \end{pmatrix}. $$

We can define the transform $C^{*}$ of $C$ characterized by

$$[C]^{-1} = PJ[C^{*}]JP$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

For the detail of the properties of this transform we refer the readers to [5] and [6]. It follows from the definition that $(C^{*})^{*} = C$ and $(C_{(\lambda, \mu)})^{*} = C_{(\lambda^{*}, \mu^{*})}$ with $\lambda^{*} = -\frac{\lambda}{4\mu(\lambda + \mu)}$, $\mu^{*} = \frac{1}{4\mu}$. We prove in §2.

Theorem 1[6, Theorem A]. Let $\Omega$ be simply connected. Then

$$\Pi_{C_{1}} = \Pi_{C_{2}} \iff \Pi_{C_{1}^{*}}^{*} = \Pi_{C_{2}^{*}}^{*}. $$

As a corollary we have immediately

Corollary 2. Let $\Omega$ be simply connected. Then

$$\Pi_{(\lambda_{1}, \mu_{1})} = \Pi_{(\lambda_{2}, \mu_{2})} \iff \Pi_{(\lambda_{1}^{*}, \mu_{1}^{*})}^{*} = \Pi_{(\lambda_{2}^{*}, \mu_{2}^{*})}^{*}. $$

This connects the work done by Nakamura-Uhlmann [8] to that done by the author [3]. Theorem 1 shows the equivalence of I and II on any simply connected $\Omega$.

The following is a linearized version of Theorem 1.

Theorem 3[6, Theorem C]. Let $\Omega$ be simply connected and $M = C^{*}$. Then $\ker d\Pi_{C}$ is topologically linear isomorphic to $\ker d\Pi_{M}^{*}$ under the relative topology from $L^{\infty}(\Omega)$.

In the theorem stated below it is not assumed that $\Omega$ is simply connected.

Theorem 4[6, Theorem D]. Let $C$ be homogeneous and $M = C^{*}$. Then,

$$\ker d\Pi_{C} = 0 \iff \ker d\Pi_{M}^{*} = 0 \iff D(P_{M}) \neq 0 $$

where $D(P_{M})$ is the discriminant of the polynomial

$$P_{M}(\tau) = M \left( \begin{pmatrix} 1 \\ \tau \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tau \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \tau \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \tau \end{pmatrix} \right);$$
there is an explicit formula of the left inverse of $d\Pi_{M}^{*}$.

This is proved under $C_{1112} = C_{1222} = 0$ in [5] and therein we wrote down explicitly the left inverse of $d\Pi_{C}$ for such $C$ with $D(P_{C^{-1}}) \neq 0$; the author does not have the explicit formula of the left inverse $d\Pi_{C}$ for general $C$ with $D(P_{C^{-1}}) \neq 0$; the roots of the algebraic equation $P_{C^{-1}}(\tau) = 0$ are called the Stroh eigenvalues of $C$(see [5]).

In the next section we will give the proofs of Theorems 1 ~ 4.

2 Proofs. Throughout this section $(\cdot)_{j}$ stands for partial differentiation with respect to $x_{j}$ for each $j = 1, 2$.

**Proof of Theorem 1.** We study the relationship between three function spaces

\[ \mathcal{P}_{C} \equiv \{ u \in H^{1}(\Omega, C^{2}) | \mathcal{L}_{C}u = 0 \text{ in } \Omega \}, \]

\[ \mathcal{S}_{C^{-1}} \equiv \{ s \in L^{2}(\Omega, Sym(C^{2})) | \sum_{M} s_{\alpha\beta,M} = 0, 2\mathbf{C}^{-1}s_{12,12} = (\mathbf{C}^{-1}s)_{11,22} + (\mathbf{C}^{-1}s)_{22,11} \text{ in } \Omega \}, \]

\[ \mathcal{A}_{C^{-1}} \equiv \{ w \in H^{2}(\Omega, C) | L_{C^{-1}}w = 0 \text{ in } \Omega \}. \]

We can easily check that the map

\[ f : \mathcal{P}_{C} \ni u \mapsto s = CSym\nabla u \in \mathcal{S}_{C^{-1}} \]

is well defined. On the other hand, for the check of the well definedness of the map

\[ g : \mathcal{A}_{C^{-1}} \ni w \mapsto s = -J'\nabla^{2}wJ' \in \mathcal{S}_{C^{-1}}, \]

we needs the following

**Lemma 1**[6, Lemma A]. For any function $w$, put

\[ s = -J'\nabla^{2}wJ'. \]

Then

\[ \sum_{M} s_{\alpha\beta,M} = 0 \]

and

\[ (\mathbf{C}^{-1}s)_{11,22} + (\mathbf{C}^{-1}s)_{22,11} - 2(\mathbf{C}^{-1}s)_{12,12} = L_{C^{-1}}w. \]

We claim

**Lemma 2**[6, Lemma B]. Let $\Omega$ be simply connected. Then both $f$ and $g$ are surjective.

The proof of this lemma is based on two facts stated below.

Let $E = (E_{ij}(x))_{i,j=1,2}$ be a second-order symmetric tensor field on $\Omega$. Then if

\[ 2E_{12,12} = E_{11,22} + E_{22,11} \]
holds, there exists a vector valued function $u$ such that
\[ E = Sym \nabla u, \]
and vice versa; the equation
\[ \sum_{\beta} s_{\alpha\beta,\beta} = 0 \]
is equivalent to
\[ d(s_{11} dx_2 - s_{12} dx_1) = 0, d(s_{21} dx_2 - s_{22} dx_1) = 0. \]

Now we can give the proof of Theorem 1. Applying Green's theorem to $\Pi_{C_1} = \Pi_{C_2}$ and using Lemma 2, we obtain that
\[ \Pi_{C_1} = \Pi_{C_2} \]
\[ \Leftrightarrow \]
\[ \forall u_j \in P_{C_j}, \int_{\Omega} (C_1 - C_2) Sym \nabla u_1 \cdot Sym \nabla u_2 dx = 0 \]
\[ \Leftrightarrow \]
\[ \forall s_j \in S_{C_j^{-1}}, \int_{\Omega} (C_2^{-1} - C_1^{-1}) s_1 \cdot s_2 dx = 0. \]

Here we note that for any $H = (H_{ijkl}(x))$ satisfying $H_{ijkl} = H_{klij} = H_{klji}$ there exists a unique $H^\dagger$ such that
\[ [H^\dagger] = J[H]J. \]

Then we have
\[ Hs_1 \cdot s_2 = H^\dagger \nabla^2 w_1 \cdot \nabla^2 w_2 \]
for $s = -J' \nabla^2 w_j J'$. Furthermore, we see that
\[ (C^{-1})^\dagger = C^*. \]

Therefore $\Pi_{C_1} = \Pi_{C_2}$ is equivalent to
\[ \forall w_j \in A_{C_j^*}, \int_{\Omega} \{(C_2^{-1})^\dagger - (C_1^{-1})^\dagger\} \nabla^2 w_1 \cdot \nabla^2 w_2 dx = 0 \]
\[ \Leftrightarrow \]
\[ \forall w_j \in A_{C_j^*}, \int_{\Omega} (C_2^* - C_1^*) \nabla^2 w_1 \cdot \nabla^2 w_2 dx = 0 \]
\[ \Leftrightarrow \]
\[ \Pi_{C_1^*} = \Pi_{C_2^*}. \]

Q.E.D.

Since the proof of Theorem 3 can be done in the same way we omit the proof.

**Proof of Theorem 4.** At first we prove
Proposition 1 [4, Theorem A.] Let $M$ be homogeneous. Then

$$\ker d\Pi_M = 0 \iff D(P_M) \neq 0;$$

there is an explicit formula of the left inverse of $d\Pi_M$ for such $M$.

By this proposition we see that the set of all homogeneous elasticity tensor fields is divided into two groups. This classification just coincides with that done by Lekhnitskii[7].

Proof of Proposition 1. We can write $P_M(\tau)$ in the form

$$M_{2222}(\tau - \alpha)(\tau - \overline{\alpha})(\tau - \beta)(\tau - \overline{\beta})$$

with some $\alpha, \beta$ satisfying $\text{Im } \alpha \cdot \text{Im } \beta > 0$. Hence

$$D(P_M) \neq 0 \iff \alpha \neq \beta$$

and $L_M$ can be factorized as follows:

$$M_{2222} \partial_{\alpha} \partial_{\overline{\alpha}} \partial_{\beta} \partial_{\overline{\beta}}.$$

Hence if $P_M(z) = 0$, the function

$$\exp\{-ic(x_1 + zx_2)\} \quad (c \in \mathbb{C})$$

is a solution of $L_M w = 0$.

(i) $\iff$

Assume $\alpha \neq \beta$. Let $\xi \in \mathbb{R}^2 \setminus \{0\}$ and

$$\{z_1, z_2\} = \{\alpha, \overline{\alpha}\}, \{\alpha, \beta\}, \{\alpha, \overline{\beta}\}, \{\beta, \overline{\alpha}\}, \{\beta, \overline{\beta}\}, \{\overline{\alpha}, \beta\}, \{\overline{\alpha}, \overline{\beta}\}.$$  

Then

$$E_\xi(x; z_1, z_2) := \exp\{-i\frac{\xi_2 - z_1 \xi_1}{z_2 - z_1}(x_1 + z_2 x_2)\}$$

is a solution of $L_M w = 0$ and

$$E_\xi(x; z_1, z_2)E_\xi(x; z_2, z_1) = e^{-ix \cdot \xi}$$

holds. Let $d\Pi_M^*(H) = 0$. This is equivalent to

$$\int_\Omega H(x) \nabla^2 u \cdot \nabla^2 v dx = 0$$

for any $u, v$; the solutions of $L_M w = 0$ in $\Omega$. Substitute $E_\xi(x; z_1, z_2)$ and $E_\xi(x; z_2, z_1)$ for $u, v$. Then we obtain

$$\text{SAH}(\xi) = 0$$
where

\[
S = \begin{pmatrix}
1 & \alpha + \overline{\alpha} & \alpha^2 + \overline{\alpha^2} & \alpha \overline{\alpha} & \alpha^2 \overline{\alpha} & \alpha^2 \overline{\alpha^2} \\
1 & \alpha + \overline{\beta} & \alpha^2 + \overline{\beta^2} & \alpha \overline{\beta} & \alpha^2 \overline{\beta} & \alpha^2 \overline{\beta^2} \\
1 & \alpha + \beta & \alpha^2 + \beta^2 & \alpha \beta & \alpha^2 \beta & \alpha^2 \beta^2 \\
1 & \beta + \overline{\beta} & \beta^2 + \overline{\beta^2} & \beta \overline{\beta} & \beta^2 \overline{\beta} & \beta^2 \overline{\beta^2} \\
1 & \alpha + \overline{\beta} & \alpha^2 + \beta^2 & \alpha \overline{\beta} & \alpha^2 \overline{\beta} & \alpha^2 \overline{\beta^2} \\
1 & \alpha + \overline{\beta} & \alpha^2 + \beta^2 & \alpha \overline{\beta} & \alpha^2 \overline{\beta} & \alpha^2 \overline{\beta^2}
\end{pmatrix},
\]

\[
A \tilde{H}(\xi) = \begin{pmatrix}
\tilde{H}A_{11} \cdot A_{11} \\
\tilde{H}A_{11} \cdot A_{12} \\
\tilde{H}A_{11} \cdot A_{22} \\
\tilde{H}A_{12} \cdot A_{12} \\
\tilde{H}A_{12} \cdot A_{22} \\
\tilde{H}A_{22} \cdot A_{22}
\end{pmatrix},
\]

and

\[
\tilde{H}(\xi) = \int_{\Omega} e^{-ix \cdot \xi} H(x) dx \quad (\xi \in \mathbb{R}^2).
\]

Since

\[
det \ S = -((\alpha - \overline{\alpha})(\alpha - \overline{\beta})(\alpha - \beta)(\overline{\alpha} - \overline{\beta})(\overline{\alpha} - \beta)(\overline{\beta} - \beta))^2 \quad (\text{(4, Lemma A)}),
\]

we can conclude \( A \tilde{H}(\xi) = 0 \) and it is possible to write down the left inverse of \( d \Pi_M^*(H) \) explicitly. The result is as follows. Put

\[
C(\xi; \{z_1, z_2\}) := \left\{ \frac{(z_2 - z_1)^2}{(\xi_2 - z_1 \xi_1)(\xi_2 - z_2 \xi_1)} \right\}^2,
\]

\[
D(\xi; \{z_1, z_2\}) := C(\xi; \{z_1, z_2\}) \int_{\Omega} d\Pi_M(H) \left( \frac{u|_{\partial\Omega}}{\partial u|_{\partial\Omega}} \right) \cdot \left( \frac{v|_{\partial\Omega}}{\partial v|_{\partial\Omega}} \right) ds
\]

for \( u = E_\xi(x; z_1, z_2) \) and \( v = E_\xi(x; z_2, z_1) \). Then

\[
D(0; \{z_1, z_2\}) := \lim_{\xi \to 0} D(\xi; \{z_1, z_2\})
\]

exists and the left inverse of \( d \Pi_M^* \) is given by the following formula:

\[
A \tilde{H}(\xi) = S^{-1}D(\xi)
\]

where

\[
D(\xi) := \begin{pmatrix}
D(\xi; \{\alpha, \overline{\alpha}\}) \\
D(\xi; \{\alpha, \overline{\beta}\}) \\
D(\xi; \{\alpha, \beta\}) \\
D(\xi; \{\beta, \overline{\beta}\}) \\
D(\xi; \{\beta, \beta\}) \\
D(\xi; \{\overline{\alpha}, \overline{\beta}\}) \\
D(\xi; \{\overline{\alpha}, \beta\}) \\
D(\xi; \{\overline{\beta}, \beta\}) \\
D(\xi; \{\overline{\alpha}, \overline{\beta}\})
\end{pmatrix}.
\]

(iii) \( \Rightarrow \)
Let $\text{Im} \ z \neq 0$. We note that the identity
\[
\nabla^2 w = \frac{1}{(\overline{z} - z)^2} \left\{ \partial^2_{\overline{z}} w A'_{11} + ( -\partial_z \partial_{\overline{z}} w A'_{12} + \partial^2_z w A'_{22} ) \right\} \quad ([4, 1.13])
\]
holds for any scalar function $w$ where
\[
\begin{align*}
A'_{11} &= \left( \begin{array}{c} 1 \\ z \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ z \end{array} \right), \\
A'_{12} &= \left( \begin{array}{c} 1 \\ z \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ \overline{z} \end{array} \right) + \left( \begin{array}{c} 1 \\ \overline{z} \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ z \end{array} \right), \\
A'_{22} &= \left( \begin{array}{c} 1 \\ \overline{z} \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ \overline{z} \end{array} \right).
\end{align*}
\]
This yields
\[
H(x) \nabla^2 u \cdot \nabla^2 v = \frac{1}{(\overline{z} - z)^4} \sigma^t (HA) \sigma \left( \begin{array}{c} \partial^2_z u \\ \partial^2_{\overline{z}} u \end{array} \right) \cdot \left( \begin{array}{c} \partial^2_{\overline{z}} v \\ \partial^2_z v \end{array} \right)
\]
where
\[
\sigma = \left( \begin{array}{ccc} 1 & 2 & 1 \\ z & z + \overline{z} & \overline{z} \\ z^2 & 2z\overline{z} & \overline{z}^2 \end{array} \right),
\]
\[
HA = \begin{pmatrix} HA_{11} \cdot A_{11} & HA_{11} \cdot A_{12} & HA_{11} \cdot A_{22} \\ HA_{12} \cdot A_{12} & HA_{12} \cdot A_{22} \\ HA_{22} \cdot A_{22} & HA_{22} \cdot A_{22} \end{pmatrix} \in \text{Sym}(\mathbb{R}^3).
\]
Now assume $z = \alpha = \beta$. If we take $H$ such that
\[
\sigma^t (HA) \sigma = \left( \begin{array}{ccc} \partial^3 \varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial^3_{\overline{z}} \varphi \end{array} \right) \quad (\Longleftrightarrow HA = 2 \text{Re} \left\{ \partial^3_{\overline{z}} \varphi \left( \begin{array}{c} \overline{z}^2 \\ -2z \end{array} \right) \otimes \left( \begin{array}{c} \overline{z}^2 \\ -2z \end{array} \right) \right\})
\]
with some $\varphi$ satisfying $D^\alpha \varphi = 0$ for $|\alpha| = 0, 1, 2$ on $\partial \Omega$, integration by parts tells us that
\[
\int_\Omega H(x) \nabla^2 u \cdot \nabla^2 v dx = \frac{1}{(\overline{z} - z)^4} \int_\Omega (\partial^3_{z} \varphi \partial^2_z u \partial^2_{\overline{z}} v + \partial^3_{\overline{z}} \varphi \partial^2_{\overline{z}} u \partial^2_{z} v) dx
\]
\[
= \frac{2}{(\overline{z} - z)^4} \int_\Omega (\partial_z \varphi \partial_z \partial^2_{\overline{z}} u \partial^2_{\overline{z}} v + \partial_{\overline{z}} \varphi \partial_{\overline{z}} \partial^2_{z} u \partial^2_{z} v) dx = 0,
\]
where $\partial^2_{z} \partial^2_{\overline{z}} u = \partial^2_{\overline{z}} \partial^2_{z} v = 0$ in $\Omega$. Since $L_M = M_{2222} \partial^2_{z} \partial^2_{\overline{z}}$, we have $H \in \ker d\Pi_M^*$. Q.E.D.

In [4, Proposition A], we gave how to find such $\varphi$ appeared above for each $H \in \ker d\Pi_M^*$ when $\Omega$ is simply connected.
Proposition 2 [5, Theorem B.] Let $C$ be homogeneous. Then

$$
D(P_{C\cdot}) = 0 \implies \ker d\Pi_{C} \neq 0.
$$

Proof. By assumption, we can write $P_{C\cdot}(\tau)$ in the form

$$
C_{2222}^{*}(\tau - z)^{2}(\tau - \overline{z})^{2}
$$

with $\text{Im} \ z \neq 0$. Then we can show that

$$
K_{z} := \{ H | \sigma^{t}(HA)\sigma = \begin{pmatrix} \partial_{z}^{2}\varphi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial_{\overline{z}}^{2}\varphi \end{pmatrix}, D^{\alpha}\varphi|_{\partial\Omega} = 0 \text{ for } |\alpha| = 0, 1, 2 \} \subset \ker d\Pi_{C}
$$

and

$$
\ker d\Pi_{C} \subset \ker d\Pi_{C}^{*}.
$$

These are proved as follows.

First we show

$$
H \in \ker d\Pi_{C}
$$

if and only if

$$
\int_{\Omega} H(x) \text{Sym}\nabla_{C}\mathcal{F}_{C}w_{1} \cdot \text{Sym}\nabla_{C}\mathcal{F}_{C}w_{2} \, dz = 0
$$

for any $w_{j} = (w_{j}^{1}, w_{j}^{2})^{t}$ satisfying $L_{C}\cdot w_{j} = 0$ in $\Omega(j = 1, 2)$, where

$$
L_{C}\mathcal{F}_{C} = \mathcal{F}_{C}L_{C} \sim L_{C}\cdot I_{2}.
$$

Second we write $C$ in terms of $z$ in the form

$$
C \sim C_{z}(\theta) \quad ([5, \text{Proposition 3}]),
$$

where

$$
AC_{z}(\theta) \sim \begin{pmatrix} t^{2} \\ -st \\ 2t(\theta - \frac{1}{2}) \\ 4t(1 - \theta + \frac{s^{2}}{4t}) \\ -s \\ 1 \end{pmatrix},
$$

$$
t = z \cdot \overline{z}, s = z + \overline{z}, \frac{s^{2}}{4t} < \theta < 1.
$$

Note that we ignored the nonzero constant multiplication factor.

Third we show that for any $w = (w^{1}, w^{2})$, the factorization

$$
\text{Sym}\nabla_{C}w \sim \partial_{z}^{2}uA_{11}' - \frac{1}{2} \partial_{z}\partial_{\overline{z}}(u + v)A_{12}' + \partial_{\overline{z}}^{2}vA_{22}' \quad ([5, \text{Proposition 5}])
$$
holds where
\[
\begin{pmatrix} u \\ v \end{pmatrix} \sim \begin{pmatrix} \theta - \frac{s^2}{4t} & 2 - \theta - \frac{s^2}{4t} \\ -2 - \theta - \frac{s^2}{4t} & -(\theta - \frac{s^2}{4t}) \end{pmatrix} \begin{pmatrix} \partial_{\overline{z}}w_z \\ \partial_{z}w_{\overline{z}} \end{pmatrix},
\]
\[
\begin{pmatrix} w_z \\ w_{\overline{z}} \end{pmatrix} = \begin{pmatrix} 1 \\ z \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.
\]

Fourth we show that \( H \in \ker d\Pi_C \) is equivalent to
\[
\int_{\Omega} \sigma^t(\mathbf{HA})\sigma \left( -\frac{1}{2} \partial_z \partial_{\overline{z}}(u \partial_z \frac{2}{z}u \partial_{z}v + v) \right) dx = 0 \quad ([5, \text{Proposition 6}])
\]
for any \( u, u', v, v' \) the solutions of \( L_C \cdot w = 0 \) in \( \Omega \). From this we obtain immediately \( \text{Ker}_z \subset \ker d\Pi_C \). Finally put \( u = v \) and \( u' = v' \). Then we obtain \( \ker d\Pi_C \subset \ker d\Pi_C^* \).

**Q.E.D.**

**Proposition 3** [6] Let \( C \) be homogeneous. Then
\[
D(P_C^*) \neq 0 \implies \ker d\Pi_C = O.
\]

**Proof.** Take a open ball \( B \) such that \( \Omega \subset B \). Then
\[
\ker d\Pi_C(\text{on } \Omega) \subset \ker d\Pi_C(\text{on } B)
\]
by zero extension of \( H \in \ker d\Pi_C(\text{on } \Omega) \) outside \( \Omega \). Since \( B \) is simply connected, Theorem 2 and Proposition 1 imply
\[
\ker d\Pi_C(\text{on } B) \simeq \ker d\Pi_{C^*}(\text{on } B) = O
\]
and hence \( \ker d\Pi_C = O \) on \( \Omega \). Q.E.D.

This completes the proof of Theorem 4.

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**REFERENCES**


