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Generalized Poisson integrals on unbounded domains and their applications

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PART 1. Introduction and the case of half-spaces

Let \( \mathbb{R} \) and \( \mathbb{R}_+ \) be the set of all real numbers and all positive real numbers, respectively. The boundary and the closure of a set \( S \) in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) (\( n \geq 2 \)) are denoted by \( \partial S \) and \( \overline{S} \), respectively. We also introduce the spherical coordinates \((r, \theta)\), \( \theta = (\theta_1, \theta_2, \ldots, \theta_{n-1}) \), in \( \mathbb{R}^n \) which are related to the cartesian coordinates \((X, y)\), \( X = (x_1, x_2, \ldots, x_{n-1}) \) by the formulas

\[
x_1 = r \left( \prod_{j=1}^{n-1} \sin \theta_j \right) \quad (n \geq 2), \quad y = r \cos \theta_1,
\]

and if \( n \geq 3 \),

\[
x_{n+1-k} = r \left( \prod_{j=1}^{k-1} \sin \theta_j \right) \cos \theta_k \quad (2 \leq k \leq n-1),
\]

where

\( 0 \leq r < +\infty, \ 0 \leq \theta_j \leq \pi \ (1 \leq j \leq n-2; \ n \geq 3), \ -2^{-1}\pi < \theta_{n-1} \leq 2^{-1}\pi \ (n \geq 2) \).

The unit sphere (the unit circle, if \( n = 2 \)) and the upper half unit sphere \( ((1, \theta_2, \ldots, \theta_{n-1}) \in \mathbb{R}^n; 0 \leq \theta_1 < \frac{\pi}{2}) \) (the upper half unit circle \( ((1, \theta_1) \in \mathbb{R}^2; -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}) \), if \( n = 2 \)) in \( \mathbb{R}^n \) are denoted by \( S^{n-1} \) and \( S^*_+ \), respectively. The half-space

\[ ((X, y) \in \mathbb{R}^n; X \in \mathbb{R}^{n-1}, y > 0) = ((r, \theta) \in \mathbb{R}^n; \theta \in S^*_+, 0 < r < +\infty) \]

is denoted by \( T_n \).
Given a domain $D \subset \mathbb{R}^n$ and a continuous function $g$ on a subset $S \subset \partial D$, we say that $h$ is a solution of the (classical) Dirichlet problem on $D$ with $g$, if $h$ is harmonic in $D$ and

$$\lim_{P \to Q \in D} h(P) = g(Q)$$

for every $Q \in S$. If $D$ is a bounded domain and $g$ is a bounded function on $\partial D$, then the existence of a solution of the Dirichlet problem and its uniqueness is completely known (e.g. see [10, Theorem 5.21]). Otherwise we may suppose that $D$ is always an unbounded domain by using the Kelvin transformation.

When $D$ is the typical unbounded domain $\mathbb{T}_n$, the following results are known. Let $g(X)$ be a continuous function on $\partial \mathbb{T}_n = \mathbb{R}^{n-1}$ satisfying (1) with a non-negative integer $l$:

$$\int_{\mathbb{R}^{n-1}} \frac{|g(X)|}{1+|X|^{n+l}} \, dX < +\infty.$$  

Then Armitage [1, Theorem 2] gave the explicit form of a solution of the Dirichlet problem on $\mathbb{T}_n$ with $g$ (also see Siegel [15, P.1 and p.7]). Further, for any continuous function $g(X)$ on $\partial \mathbb{T}_n$ Finkelstein and Scheinberg [7] showed the existence of a solution of the Dirichlet problem on $\mathbb{T}_n$ with $g$ and Gardiner [8] gave the solution explicitly.

About the uniqueness of solutions of the Dirichlet problem on $\mathbb{T}_n$, Helms [11, p.42 and p.158] states that even if $g(X)$ is a bounded continuous function on $\partial \mathbb{T}_n$, the solution of the Dirichlet problem on $\mathbb{T}_n$ with $g$ is not unique and to obtain the unique solution $H(P)$ ($P=(x,y)\in \mathbb{T}_n$) we must specify the behavior of $H(P)$ as $y \to +\infty$. In connection with this remark, Siegel [15, Theorems 1 and 3] gave the following result to more restricted boundary function $g$ than (1).
Let \( l \) be a non-negative integer. If \( g(x) \) (\( x \in \partial T_n = \mathbb{R}^{n-1} \)) is a continuous function on \( \partial T_n \) such that
\[
|g(x)| \leq G(x) \quad (x \in \mathbb{R}^{n-1}, |x| = x > 0)
\]
for a continuous function \( G(x) \) (\( x \in \mathbb{R} \)), \( G(x) = G(-x) \),
\[
\int_{-\infty}^{+\infty} \frac{|G(x)|}{1 + |x|^{n+l}} \, dx < +\infty,
\]
then there exists a solution \( H(T_n, l; g)(P) \) of the Dirichlet problem on \( T_n \) with \( g \) satisfying
\[
H(T_n, l; g)(P) = o(r^{l+1}/\cos \theta_1) \quad (r \to +\infty)
\]
\( (P=(r, \theta) \in T_n, \, \theta=(\theta_1, \theta_2, \ldots, \theta_{n-1})) \).

If \( h(P) \) is a solution of the Dirichlet problem on \( T_n \) with \( g \) such that
\[
h(P) = o(r^{l+1}/\cos \theta_1) \quad (P=(r, \theta) \in T_n),
\]
then
\[
h(P) = H(T_n, l; g)(P) + \Gamma(h)(P) \quad (P \in T_n),
\]
where \( \Gamma(h)(P) \) is a harmonic polynomial (of \( P=(x_1, x_2, \ldots, x_{n-1}, y) \in \mathbb{R}^n \)) of degree \( l \) vanishing on
\[
\partial T_n = \{(x_1, x_2, \ldots, x_{n-1}, 0) \in \mathbb{R}^n; \, (x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \}.
\]

To answer the question of Siegel [15, p. 8] Yoshida [19] proved

**Theorem Y1** [19, Theorems 1 and 2]. Let \( g(Q) \) be a continuous function on \( \partial T_n \) (\( n \geq 2 \)) satisfying (1) with a non-negative integer \( l \). Then there exists a solution \( H(T_n, g; l)(P) \) of the Dirichlet problem with \( g \) satisfying
\[
\lim_{r \to \infty} r^{-l-1} \int_{S_{n-1}} H(T_n, l; g)(r, \theta) \cos \theta_1 \, d\sigma_\theta = 0
\]
\( (P=(r, \theta) \in T_n, \, \theta=(\theta_1, \theta_2, \ldots, \theta_{n-1})) \),
where \( d\sigma_\theta \) is the surface element of \( S^{n-1} \). If \( h(P) \) is a solution of
the Dirichlet problem on \( T_n \) with \( g \) satisfying
\[
\lim_{r \to \infty} r^{-l-1} \int_{S^{n-1}_+} h^+(r, \theta) \cos \theta \, d\sigma_\theta = 0,
\]
then
\[
h(P) = H(T_n, l; g)(P) + \pi(h)(P), \quad \pi(h)(P) = \begin{cases} 
\chi \pi^*(h)(P) & (l \geq 1) \\
0 & (l = 0)
\end{cases}
\]
for every \( P = (x, y) \in T_n \), where \( \pi^*(h)(P) \) is a polynomial of \( P = (x_1, x_2, \ldots, x_{n-1}, y) \in \mathbb{R}^n \) of degree at most \( l-1 \) and even with respect to the variable \( y \).
PART 2. The conical case

1. Introduction

A half-space is a special one of more general unbounded domains cones. To generalize Theorem $Y_1$ to results about cones, we shall first pay attention to Yoshida's result [18, Theorem 3] concerning the Dirichlet problem on a cone. To state it, we need some preliminaries.

Let $\Delta_n (n \geq 2)$ be the Laplace operator and $\Lambda_n$ be the spherical part of the spherical coordinates of $\Delta_n$:

$$\Delta_n = \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \Lambda_n.$$ 

Given a domain $\Omega$ on $S^{n-1} (n \geq 2)$, consider the Dirichlet problem for

$$\begin{align*}
(\Lambda_n + \lambda)F &= 0 \quad \text{on } \Omega \\
F &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

we denote the least positive eigenvalue of (2) by $\lambda(\Omega, 1)$ and the normalized positive eigenfunction corresponding $\lambda(\Omega, 1)$ by $f_1^\Omega(\theta)$. We shall denote two solutions of the equation

$$t^2 + (n-2)t - \lambda(\Omega, 1) = 0$$

by $\alpha(\Omega, 1), -\beta(\Omega, 1)$ ($\alpha(\Omega, 1), \beta(\Omega, 1) > 0$). Given a domain $\Omega$ on $S^{n-1}$, the set

$$\{(r, \theta) \in \mathbb{R}^n; (1, \theta) \in \Omega, r \in \mathbb{R}_+\}$$

and

$$\{(r, \theta) \in \mathbb{R}^n; (1, \theta) \in \partial \Omega, r \in \mathbb{R}_+\}$$

in $\mathbb{R}^n$ are denoted by $C_n(\Omega)$ and $S_n(\Omega)$, respectively. If $n=2$, then $C_2(\Omega)$ is an angular domain.
In the following, we put the strong assumption relative to $\Omega$ on $S^{n-1}$: if $n \geq 3$, $\Omega$ is a $C^2,\alpha$-domain (0 $<$ $\alpha$ $<$ 1) on $S^{n-1}$ surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [10, pp.88-89] for the definition of $C^2,\alpha$-domain).

Let $G_{C_n}(\Omega)((r_1, \theta_1), (r_2, \theta_2)) ((r_1, \theta_1), (r_2, \theta_2) \in C_n(\Omega))$ be the Green function of a cone $C_n(\Omega)$ and let $s_n$ denote the surface area $2\pi^{n/2}(\Gamma(n/2))^{-1}$ of $S^{n-1}$. The function

$$c_n^{-1}\frac{\partial}{\partial \nu}G_{C_n}(\Omega)(P, Q), \quad c_n = \begin{cases} \frac{2\pi}{(n-2)s_n}, & (n=2) \\ \frac{2\pi}{(n-2)s_n}, & (n \geq 3) \end{cases}$$

of $Q \in \partial C_n(\Omega)$ for any fixed $P \in C_n(\Omega)$ is an ordinary Poisson kernel, where $\frac{\partial}{\partial \nu}$ denotes the differentiation at $Q$ along the inward normal into $C_n(\Omega)$. Let $F(r, \theta)$ be a function on $C_n(\Omega)$. We put

$$\mu_0(F) = \lim_{r \to \infty} r^{-\alpha(\Omega, 1)} \int_{\Omega} F(r, \theta) f_1^\Omega(\theta) d\sigma_\theta,$$

and

$$\eta_0(F) = \lim_{r \to \infty} r^{-\beta(\Omega, 1)} \int_{\Omega} F(r, \theta) f_1^\Omega(\theta) d\sigma_\theta,$$

if they exist, where $d\sigma_\theta$ is the surface element on $S^{n-1}$.

**Theorem Y** [18, Theorem 3 and Lemma 3]. Let $g(Q) = g(t, \Xi)$ be a continuous function on $S_n(\Omega)$ satisfying

$$\int_0^{\infty} t^{-\alpha(\Omega, 1)-1} \left( \int_{\partial \Omega} |g(t, \Xi)| d\sigma_\Xi \right) dt < +\infty$$

and

$$\int_0^{\infty} t^{-\beta(\Omega, 1)-1} \left( \int_{\partial \Omega} |g(t, \Xi)| d\sigma_\Xi \right) dt < +\infty$$

(if $n=2$ and $\Omega = (\gamma, \delta)$, then

$$\int_{\partial \Omega} |g(t, \Xi)| d\sigma_\Xi = |g(t, \gamma)| + |g(t, \delta)|).$$

Then the Poisson integral

$$H(C_n(\Omega); g)(P) = c_n^{-1} \int_{S_n(\Omega)} g(Q) \frac{\partial}{\partial \nu}G_{C_n}(\Omega)(P, Q) d\sigma_Q$$

is a solution of the classical Dirichlet problem on $C_n(\Omega)$ with $g$ such that.
\[ \mu_0(\mid H(C_n(\Omega);g)\mid) = 0 \quad \text{and} \quad \eta_0(\mid H(C_n(\Omega);g)\mid) = 0, \]

where \( d\sigma_\Theta \) is the surface element on \( S^{n-1} \). If \( h(P) \) is any solution of the classical Dirichlet problem on \( C_n(\Omega) \) with \( g \), then all of the limits \( \mu_0(h), \eta_0(h) (\rightarrow \mu_0(h), \eta_0(h) \leq +\infty) \), \( \mu_0(\mid h\mid) \) and \( \eta_0(\mid h\mid) \)

\( \mu_0(\mid h\mid) \leq +\infty \) and \( \eta_0(\mid h\mid) \leq +\infty \),

then

\[ h(P) = H(C_n(\Omega),g)(P) + (\mu_0(h)r^{-\alpha(\Omega,1)} + \eta_0(h)r^{-\beta(\Omega,1)})f^\Omega_1(\Theta) \]

for any \( P = (r,\Theta) \in C_n(\Omega) \).

In this paper we shall show the existence of solutions of the Dirichlet problem on a cone (Theorems 1 and 2) and a type of uniqueness of them (Theorems 7 and 8) by introducing the conical generalized Poisson kernels and Poisson integrals, the special cases of which are \( H(T_n,\ell;g) \) and \( H(C_n(\Omega);g)(P) \). They generalize Theorem Y_1 to the conical case and Theorem Y_2 to more unrestricted boundary function than (4). To prove the uniqueness, we shall give two results (Theorems 5 and 6) which are the conical version of Kuran's result [12, Theorem 10]. We also generalize the results of Finkelstein and Scheinberg [7] and Gardinar [8] to the conical case (Theorems 3 and 4). Finally a result of Yoshida [19, Theorem 3] will be generalized in the conical form (Theorem 9).

2. Results about the existence of solutions.

We denote the non-decreasing sequence of positive eigenvalues of (2) by \( (\lambda(\Omega,k))_{k=1}^\infty \). In this expression we write \( \lambda(\Omega,k) \) the same
number of times as the dimension of the corresponding eigenspace.

When the normalized eigenfunction corresponding \( \lambda(\Omega, k) \) is denoted by \( f_k^\Omega(\theta) \), the set of sequential eigenfunctions corresponding to the same value of \( \lambda(\Omega, k) \) in the sequence \( \left\{ f_k^\Omega(\theta) \right\}_{k=1}^\infty \) makes an orthonormal basis for the eigenspace of the eigenvalue \( \lambda(\Omega, k) \). We can also say that for each \( \Omega \subset S^{n-1} \) there is a sequence \( (k_i) \) of positive integers such that \( k_1 = 1, \lambda(\Omega, k_1) < \lambda(\Omega, k_{i+1}) \),

\[
\lambda(\Omega, k_i) = \lambda(\Omega, k_i + 1) = \lambda(\Omega, k_i + 2) = \ldots = \lambda(\Omega, k_i + 1 - 1) \quad (i = 1, 2, 3, \ldots)
\]

and \( \left\{ f_k^\Omega, f_{k+1}^\Omega, \ldots, f_{k+1-1}^\Omega \right\} \) is an orthonormal basis for the eigenspace of the eigenvalue \( \lambda(\Omega, k_i) \) \( (i = 1, 2, 3, \ldots) \). It is well known that \( k_2 = 2 \) and \( f_2^\Omega(\theta) > 0 \) for any \( \theta \in \Omega \) (see Courant and Hilbert [5, p. 451 and p. 458]). With respect to \( (k_i) \), the following Remark 1 shows that even in the case \( \Omega = S^{n-1}_+ \) \( (n = 2, 3, 4, \ldots) \), not only the simplest case \( k_1 = 1 \) \( (i = 1, 2, 3, \ldots) \) but also other complex cases can appear.

If we note that \( \Omega \) is an \( (n-1) \)-dimensional compact Riemannian manifold with its boundary to be sufficiently regular, we know that

\[
\lambda(\Omega, k) \sim A(\Omega, n)k^{2/(n-1)} \quad (k \to +\infty)
\]

(e.g. see Cheng and Li [4]) and

\[
\sum_{\lambda(\Omega, k) \leq x} (f_k^\Omega(\theta))^2 \sim B(\Omega, n)x^{(n-1)/2} \quad (x \to +\infty)
\]

uniformly with respect to \( \theta \) (e.g. Minakshisundaram and Pleijel [13], and also Essén and Lewis [6, p.120 and pp.126-128]), where \( A(\Omega, n) \) and \( B(\Omega, n) \) are both constants depending \( \Omega \) and \( n \), respectively. Hence there exist two positive constants \( M_1, M_2 \) such that

\[
M_1 k_1^{2/(n-1)} \leq \lambda(\Omega, k) \quad (k = 1, 2, \ldots)
\]

and
\[ |f_k^\Omega(\theta)| \leq M_2 k^{1/2} \quad (\theta \in \Omega, \ k=1,2,\ldots) \]

If we denote two solutions of the equation
\[ t^2 + (n-2)t - \lambda(\Omega,k) = 0 \]
by \( \alpha(\Omega,k), -\beta(\Omega,k) \) (\( \alpha(\Omega,k), \beta(\Omega,k) > 0 \)), then we also have
\[ \alpha(\Omega,k), \beta(\Omega,k) \geq M_3 k^{1/(n-1)} \quad (k=1,2,\ldots), \]
from (5), where \( M_3 \) is a positive constant independent of \( k \). We remark that both
\[ r^\alpha(\Omega,k)f_k^\Omega(\theta) \quad \text{and} \quad r^{-\beta(\Omega,k)}f_k^\Omega(\theta) \quad (k=1,2,\ldots,\ldots) \]
are harmonic on \( C_n(\Omega) \) and vanish continuously on \( S_n(\Omega) \). For a domain \( \Omega \) and the sequence \( (k_1) \) mentioned above, by \( I(\Omega,k_{\ell}) \) we denote the set of all positive integers less than \( k_{\ell} \) \( (\ell=1,2,3,\ldots) \). In spite of the fact \( I(\Omega,k_1) = \emptyset \), the summation over \( I(\Omega,k_1) \) of a function \( S(k) \) of a variable \( k \) will be used by promising
\[ \sum_{k \in I(\Omega,k_1)} S(k) = 0. \]

REMARK 1. Suppose \( \Omega=S_{n-1}^n (n \geq 2) \). Then
\[ (6) \quad c_n^{-1} \nabla_{TN} G_{TN} \left((r,\theta),(t,\Theta)\right) = 2s_n^{-1} \sum_{k=0}^\infty c_{k,n+2} r^{k+1} t^{-k-n} \cos \theta L_{k,n+2}(\cos \gamma) \]
for any \((X,Y) = (r,\theta) \in T_n \) and \((Z,0) = (t,\Theta) \in \Theta T_n \) satisfying \( r < t \), where \( c_{k,n+2} = \binom{k+n-1}{k} \), \( L_{k,n+2} \) is the \((n+2)\)-dimensional Legendre polynomial of degree \( k \) and \( \gamma \) is the angle between \( M=(X,0) \) and \( N=(Z,0) \) defined by
\[ \cos \gamma = \frac{(M,N)}{|M||N|} \]
(see Armitage [1, Theorem E]). On the other hand, Remark 4 in Section 4 applied to \( \Omega=S_{n-1}^n \) gives the Fourier series expansion of the function.
of $\theta$ with respect to the sequence of eigenfunctions of (2).

Hence, in comparison with (2-4) we obtain

$$\alpha(S_n^{-1}, k_1) = i,$$

$$\beta(S_n^{-1}, k_1) = n+i-2 \quad (i=1,2,3,\ldots; n=2,3,4,\ldots).$$

Consider the simplest case $n=2$ i.e. $\Omega=S^1$. For $(r, \theta_1) \in \mathbb{T}_2$ and $(t, s) \in \mathbb{R}$, we see $\cos \gamma = \frac{t}{|t|} \sin \theta_1$ and hence

$$k_1 = i \quad (i=1,2,3,\ldots)$$

$$f_2^0(\theta_1) = \rho_k \cos \theta_1 L_{k-1,4}(\sin \theta_1) \quad (k=1,2,\ldots),$$

where

$$\rho_k = (\int_{-1}^1 (\cos \theta_1 L_{k-1,4}(\sin \theta_1))^2 d\theta_1)^{-1/2}.$$

Next, suppose $n=3$ i.e. $\Omega=S^2$. Then for $(r, \theta) = (x,y) \in \mathbb{T}_3$, $\theta=(\theta_1, \theta_2)$ and $(t, s) \in \mathbb{R}^2$, $s=(\pi \frac{\xi_2}{2}, \xi_2)$, we see

$$\cos \gamma = \sin \theta_1 \sin \theta_2 \sin \xi_2 + \sin \theta_1 \cos \theta_2 \cos \xi_2.$$ 

If we put

$$L_{0,5} = \Phi_{0,0} = 1$$

and

$$L_{k,5}(\sin \theta_1 \sin \theta_2 \sin \xi_2 + \sin \theta_1 \cos \theta_2 \cos \xi_2)$$

$$= \Phi_{k,0}(\theta_1, \theta_2) \cos^k \xi_2 + \Phi_{k,1}(\theta_1, \theta_2) \cos^{k-2} \xi_2 + \ldots$$

$$+ \Phi_{k,2^{-1}k}(\theta_1, \theta_2) \cos^{k-2[2^{-1}k]} \xi_2$$

$$+ \Psi_{k,0}(\theta_1, \theta_2) \cos^{k-1} \xi_2 \sin \xi_2 + \Psi_{k,1}(\theta_1, \theta_2) \cos^{k-3} \xi_2 \sin \xi_2 + \ldots$$

$$+ \Psi_{k,[k-1/2]}(\theta_1, \theta_2) \cos^{k-2[(k-1)/2]} \frac{\xi_2}{2} \sin \xi_2$$

$$\quad (k=1,2,3,\ldots),$$

then

$$k_1 = 1 + \frac{(i-1)1}{2} \quad (i=1,2,3,\ldots)$$
and
\[ f_1^\Omega(\theta) = (2n_s^{-1})^{1/2} \cos \theta_1, \]
\[ f_k^\Omega_{j+1}(\theta) = \begin{cases} 
\rho_{k_1+j} \Phi_{i-1,j}(\theta_1, \theta_2) \cos \theta_1, 
& (j=0,1,\ldots,[(i-1)/2]) \\
\rho_{k_1+j} \Psi_{i-1,j-[(i-1)/2]-1}(\theta_1, \theta_2) \cos \theta_1 
& (j=([(i-1)/2]+1,\ldots,[(i-1)/2]+[(i-2)/2]+1) \\
& (i=2,3,4,\ldots),
\end{cases} \]

where
\[ \rho_{k_1+j} = \begin{cases} 
\left( \int_{S^2} \Phi_{i-1,j}(\theta_1, \theta_2) \cos^2 \theta_1 \, d\sigma_\theta \right)^{-1/2} 
& (j=0,1,\ldots,[(i-1)/2]) \\
\left( \int_{S^2} \Psi_{i-1,j-[(i-1)/2]-1}(\theta_1, \theta_2) \cos^2 \theta_1 \, d\sigma_\theta \right)^{-1/2} 
& (j=([(i-1)/2]+1,\ldots,[(i-1)/2]+[(i-2)/2]+1) \\
& (i=2,3,4,\ldots). \] 

The Fourier coefficient
\[ \int_{\Omega} F(\theta) f_k^\Omega(\theta) \, d\sigma_\theta \]
of a function \( F(\theta) \) on \( \Omega \) with respect to the orthonormal sequence
\( \{f_k^\Omega(\theta)\} \) is denoted by \( c(F,k) \), if it exists. Now we shall define
generalized Poisson kernels of the conical type. For two
non-negative integers \( l, m \) and two points \( P=(r,\theta) \in C_n(\Omega), \)
\( Q=(r,\xi) \in S_n(\Omega) \), we put
\[ (8) \quad \nabla(C_n(\Omega), l)(P,Q) = \sum_{k \in I(\Omega, k_{l+1})} \alpha(\Omega, k) c((H^l_1)_2, k) t^{\alpha(\Omega, k) - n-1} \alpha(\Omega, k) f_k^\Omega(\theta), \]
and
\[ \nabla(C_n(\Omega), m)(P,Q) = \sum_{k \in I(\Omega, k_{m+1})} \beta(\Omega, k) c((H^m_2)_3, k) t^{\beta(\Omega, k) - n-1} \beta(\Omega, k) f_k^\Omega(\theta), \] 

where
\[(H_{\Xi})_r(\theta) = c_n^{-1} \partial G_{C_n}(\Omega)((r, \theta), (2, \Xi)) \quad (r=1, 3).\]

We introduce two functions of \(P \in C_n(\Omega)\) and \(Q = (t, \Xi) \in S_n(\Omega)\)

\[
\bar{W}(C_n(\Omega), l)(P, Q) = \begin{cases} V(C_n(\Omega), l)(P, Q) & (1 \leq t < +\infty) \\ 0 & (0 < t < 1) \end{cases}
\]

and

\[
\bar{W}(C_n(\Omega), m)(P, Q) = \begin{cases} V(C_n(\Omega), m)(P, Q) & (0 < t < 1) \\ 0 & (1 \leq t < +\infty). \end{cases}
\]

The generalized Poisson kernel \(K(C_n(\Omega), l, m)(P, Q)\) with respect to \(C_n(\Omega)\) is defined by

\[
K(C_n(\Omega), l, m)(P, Q) = c_n^{-1} \partial G_{C_n}(\Omega)(P, Q) - \bar{W}(C_n(\Omega), l)(P, Q) - \bar{W}(C_n(\Omega), m)(P, Q).
\]

In fact

\[
K(C_n(\Omega), l, 0)(P, Q) = c_n^{-1} \partial G_{C_n}(\Omega)(P, Q) - \bar{W}(C_n(\Omega), l)(P, Q) \quad (l \geq 1)
\]

and

\[
K(C_n(\Omega), 0, 0)(P, Q) = c_n^{-1} \partial G_{C_n}(\Omega)(P, Q).
\]

REMARK 2. Put \(\Omega = S^{n-1}_+\) and \(r_2 = 1\) in Remark 3 of Section 4. Then from (7) we have

\[
c_n^{-1} \partial G_{T_n}((r, \theta), (t, \Xi)) = \sum_{i=0}^{\infty} 2^{n-1+i} r t^{l-1-n} \sum_{k=k_i}^{k_{i+1}-1} c(H_{\Xi}, k) f_k^\Omega(\theta)
\]

for any \((r, \theta) \in T_n\) and any \((t, \Xi) \in \partial T_n\) \((r < t)\), which is (6). Hence we obtain

\[
2^{n+i} \sum_{k=k_i}^{k_{i+1}-1} c(H_{\Xi}, k) f_k^\Omega(\theta) = 2 s^{-1} c_i, n+2 \cos \theta_{1, i, n+2}(\cos \gamma)
\]

\[(i=0, 1, 2, \ldots).\]

Since
\[
\bar{V}(T_n, l)(P, Q) = \sum_{i=0}^{l-1} 2^{n+i_r l+1} t^{-n-i} \left( \sum_{k=k_1+1}^{k+1} c((H^1_1, k)f_k(\theta)) \right)
\]

from (7), we finally have

\[
\bar{V}(T_n, l)(P, Q) = 2s^{-1} \sum_{i=0}^{l-1} c_i, n+2 r^{i+1} t^{-i-n} \cos \theta L_i, n+2 (\cos \gamma).
\]

This shows that our kernel \( K(T_n, l, 0)(P, Q) \) \((l \geq 1)\) coincides with ones in Armitage [1], Siegel [15] and Yoshida [19].

Let \( F(P) = F(r, \theta) \) be a function on \( C_n(\Omega) \) and put

\[
N(F)(r) = \int_\Omega F(r, \theta) f_1^\Omega(\theta) d\sigma_\theta.
\]

For two non-negative integers \( p \) and \( q \) we write

\[
-\alpha(\Omega, k_{p+1}) = \lim_{r \to 0} N(F)(r) \quad \text{and} \quad \eta_q(F) = \lim_{r \to 0} N(F)(r),
\]

if they exist. Since \( k_1 = 1 \), we know that these with \( p = q = 0 \) are consistant with (3).

The following theorem is a generalization of the first part of Theorem \( Y_2 \) which is the case \( l = m = 0 \) of Theorem 1.

**THEOREM 1.** Let \( l, m \) be two non-negative integers and \( g(Q) = g(t, \Sigma) \) be a continuous function on \( S_n(\Omega) \) satisfying (9) with \( l \) and (10) with \( m \):

(9) \[
\int_{t}^{\infty} x^{-\alpha(\Omega, k_{l+1})-1} \left( \int_{\partial \Omega} g(t, \Sigma) d\sigma_\Sigma \right) dt < \infty
\]

and

(10) \[
\int_{0}^{t} x^{-\beta(\Omega, k_{m+1})-1} \left( \int_{\partial \Omega} g(t, \Sigma) d\sigma_\Sigma \right) dt < \infty.
\]

Then

\[
H(C_n(\Omega), l, m; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), l, m)(P, Q) d\sigma_Q
\]
is a solution of the classical Dirichlet problem on $C_n(\Omega)$ with $g$ satisfying
\[ \mu_l(\{|H(C_n(\Omega),\ell,m;g)|\}) = \mu_m(\{|H(C_n(\Omega),\ell,m;g)|\}) = 0. \]

To emphasize that Theorem 1 is also a natural generalization of the first part of Theorem $Y_1$, the following Theorem 2 is more desirable than Theorem 1.

**Theorem 2.** Let $g(Q)=g(t,\Xi)$ be a continuous function on $\partial C_n(\Omega)$ satisfying (9) with a non-negative integer $l$. Then
\[ H(C_n(\Omega),\ell,0;g)(P) = \int_{S_n(\Omega)} g(Q)K(C_n(\Omega),\ell,0)(P,Q)d\sigma_Q \]
is a solution of the classical Dirichlet problem on $C_n(\Omega)$ with $g$ satisfying
\[ \mu_l(\{|H(C_n(\Omega),\ell,0;g)|\}) = 0. \]

By taking $\Omega=S^{n-1}_+$, we obtain from (7)

**Corollary 1 (Yoshida [19, Theorem 1]).** Let $g(X)$ be a continuous function on $\partial T_n=\mathbb{R}^{n-1}$ satisfying (1) with a non-negative integer $l$. Then $H(T_n,\ell,0;g)(P)$ is a solution of the Dirichlet problem on $T_n$ with $g$ such that
\[ \mu_l(\{|H(T_n,\ell,0;g)|\}) = 0. \]

To solve the Dirichlet problem on $C_n(\Omega)$ with any function $g(Q)$, we shall define other Poisson kernels. Let $\phi(t)$ (resp. $\psi(t)$) be a positive continuous function of $t\geq 1$ (resp. $0<t<1$) satisfying
\( \varphi(1) = 2^{-\alpha(\Omega,1)} \) (resp. \( \psi(1) = \frac{3}{2}^{-\beta(\Omega,1)} \)).

For a domain \( \Omega \subset \mathbb{S}^{n-1} \) and the sequence \( (\alpha(\Omega,k_i))_{i=1}^{\infty} \) (resp. \( (\beta(\Omega,k_i))_{i=1}^{\infty} \)), denote the set

\[
(t \geq 1; -\alpha(\Omega,k_i) = (\log 2)^{-1}(\log(t^{n-1}\varphi(t))) \\
\text{(resp. } 0 < t \leq 1; -\beta(\Omega,k_i) = (\log \frac{3}{2})^{-1}\log(t^{n-1}\psi(t)))
\]

by \( \overline{S}(\Omega,\varphi,i) \) (resp. \( \overline{S}(\Omega,\psi,i) \)). Then \( 1 \in \overline{S}(\Omega,\varphi,1) \) (resp. \( 1 \in \overline{S}(\Omega,\psi,1) \)).

When there is an integer \( N \) such that \( \overline{S}(\Omega,\varphi,N) \neq \emptyset \) and \( \overline{S}(\Omega,\varphi,N+1) = \emptyset \) (resp. \( \overline{S}(\Omega,\psi,N) \neq \emptyset \) and \( \overline{S}(\Omega,\psi,N+1) = \emptyset \)), denote the set \( i \in \{1, 2, \ldots, N\} \) of integers by \( \overline{J}(\Omega,\varphi) \) (resp. \( \overline{J}(\Omega,\psi) \)). Otherwise, denote the set of all positive integers by \( \overline{J}(\Omega,\varphi) \) (resp. \( \overline{J}(\Omega,\psi) \)). Let \( \overline{t}(i) = \max \{ t(i); t(i) \in \overline{S}(\Omega,\varphi,i) \} \) (resp. \( \overline{t}(i) = \min \{ t(i); t(i) \in \overline{S}(\Omega,\psi,i) \} \) be the minimum (resp. maximum) of elements \( t \) in \( \overline{S}(\Omega,\varphi,i) \) (resp. \( \overline{S}(\Omega,\psi,i) \)) for each \( i \in \overline{J}(\Omega,\varphi) \) (resp. \( i \in \overline{J}(\Omega,\psi) \)). In the former case, we put \( \overline{t}(N+1) = +\infty \) (resp. \( \overline{t}(N+1) = 0 \)). Then \( \overline{t}(1) = 1 \) (resp. \( \overline{t}(1) = 1 \)).

We define \( \overline{W}(C_n(\Omega),\varphi)(P,Q) \) (\( P \in C_n(\Omega) \), \( Q = (t, \xi) \in S_n(\Omega) \)) by

\[
\overline{W}(C_n(\Omega),\varphi)(P,Q) = \begin{cases} 
0 & (0 < t < 1) \\
\frac{1}{V(C_n(\Omega),i)}(P,Q) & (t(i) \leq t \leq t(i+1); i \in \overline{J}(\Omega,\varphi)).
\end{cases}
\]

We also define \( \overline{W}(C_n(\Omega),\psi)(P,Q) \) (\( P \in C_n(\Omega) \), \( Q = (t, \xi) \in S_n(\Omega) \)) by

\[
\overline{W}(C_n(\Omega),\psi)(P,Q) = \begin{cases} 
0 & (1 < t < +\infty) \\
\frac{1}{V(C_n(\Omega),i)}(P,Q) & (1 < t(i+1) \leq t \leq t(i); i \in \overline{J}(\Omega,\psi)).
\end{cases}
\]

The Poisson kernel \( K(C_n(\Omega),\varphi,\psi)(P,Q) \) and \( K(C_n(\Omega),\psi)(P,Q) \) (\( P \in C_n(\Omega) \), \( Q \in S_n(\Omega) \)) are defined by

\[
K(C_n(\Omega),\varphi,\psi)(P,Q) = c_n^{-1} \frac{\partial}{\partial v} G_n(\Omega)(P,Q) - \overline{W}(C_n(\Omega),\varphi)(P,Q) - \overline{W}(C_n(\Omega),\psi)(P,Q)
\]

and

\[
= c_n^{-1} \frac{\partial}{\partial v} G_n(\Omega)(P,Q) - \overline{W}(C_n(\Omega),\varphi)(P,Q) - \overline{W}(C_n(\Omega),\psi)(P,Q)
\]


\[ K(C_n(\Omega), \varphi)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} C_n(\Omega)(P, Q) - \bar{w}(C_n(\Omega), \varphi)(P, Q). \]

Now we have

**THEOREM 3.** Let \( g(Q) \) be a continuous function on \( S_n(\Omega) \). Then there are two positive continuous functions \( \varphi(t) \) of \( t \geq 1 \) and \( \psi(t) \) of \( 0 < t \leq 1 \) such that

\[ H(C_n(\Omega), \varphi, \psi; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), \varphi, \psi)(P, Q) d\sigma_Q \]

is a solution of the Dirichlet problem on \( C_n(\Omega) \) with \( g \).

We can also obtain

**THEOREM 4.** Let \( g(Q) \) be a continuous function on \( \partial C_n(\Omega) \). Then there is a positive continuous function \( \varphi(t) \) of \( t \geq 1 \) such that

\[ H(C_n(\Omega), \varphi; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), \varphi)(P, Q) d\sigma_Q \]

is a solution of the Dirichlet problem on \( C_n(\Omega) \) with \( g \).

If we take \( \Omega = S_n^{n-1} \) in Theorem 4, then we have

**COROLLARY 2** (Finkelstein and Scheinberg [7] and Gardinar [8]). Let \( g(Q) \) be a continuous function on \( \partial T_n \). Then there is a positive continuous function \( \varphi(t) \) of \( t \geq 1 \) such that

\[ H(T_n, \varphi; g)(P) = \int_{\partial T_n} g(Q) K(T_n, \varphi)(P, Q) d\sigma_Q \]

is a solution of the Dirichlet problem on \( T_n \) with \( g \).
3. Results about a type of uniqueness of solutions.

The following result is just a generalization of Picard's theorem stating that if \( H \) is a positive harmonic function in the Euclidean space then \( H \) is a constant.

Let \( h(r,\theta) \) be harmonic on \( \mathbb{R}^d \) \((d \geq 2)\). If, for some positive \( t \),
\[
r^{-t-1} \mathfrak{M}(h^+)(r) \to 0 \quad (r \to +\infty),
\]
\[
\mathfrak{M}(h^+)(r) = \int_{\mathbb{S}^{d-1}} h^+(r,\theta) d\sigma_\theta,
\]
then for some positive integer \( l \) less than \( t+1 \)
\[
h(r,\theta) = C + \sum_{k=1}^{l} P_k(r,\theta) \quad ((r,\theta) \in \mathbb{R}^d),
\]
where \( C \) is a constant and \( P_k(r,\theta) = r^k Y_k(\theta) \) is a homogeneous harmonic polynomial of order \( k \) \((Y_k(\theta) \) is a spherical harmonic function of Laplace) (see e.g. Brelot [2, Appendix, §26]).

It is well known that the potential theory in \( \mathbb{R}^n \) is intimately related to the potential theory in \( \mathbb{R}^{n+2} \) and many results on harmonic functions in \( \mathbb{R}^n \) can easily obtained by a passage to \( \mathbb{R}^{n+2} \). By using this fact, Kuran proved the following theorem.

**Theorem K** (Kuran [12, Theorem 10]). Let \( h(x,y) \) \((=h(r,\theta))\) be a harmonic function on \( T_n \) such that \( h \) vanishes continuously on \( \partial T_n \).

If, for some positive \( t \),
\[
\lim_{r \to \infty} r^{-t-2} \mathfrak{M}(r,\theta) = 0, \quad \mathfrak{M}(r,\theta) = (s^+_r)^{-1} \int_{S^+_r} yh^+(r,\theta) \, dS^+_r,
\]
where \( S^+_r = \{(r,\theta) \in T_n ; \theta \in S^{n-1}_+\} \), \( s^+_r \) is the surface area of the spherical part of \( S^+_r \) and \( dS^+_r \) is the surface element of \( S^+_r \), then
\[
h = \pi
\]
in \( T_n \), where \( \pi \) is a polynomial of \((x_1,x_2,\ldots,x_{n-1},y)\) in \( \mathbb{R}^n \) of degree
less than \( t \) and even with respect to the variable \( y \).

Though his method is not applicable, we shall try to extend these results for functions defined on cones, one of which is the half-space. And we can obtain

**Theorem 5.** Let \( p, q \) be two positive integers and \( h(r, \theta) \) be a harmonic function in \( C_n(\Omega) \) vanishing continuously on \( S_n(\Omega) \). If \( h \) satisfies (11) with \( p \) and (12) with \( q \):

\[(11) \quad \mu_p(h^+) = 0\]

and

\[(12) \quad \eta_q(h^+) = 0,\]

then

\[h(r, \theta) = \sum_{k \in I(\Omega, k_{p+1})} A_k(h) r^{\alpha(\Omega, k)} f_k(\theta) + \sum_{k \in I(\Omega, k_{q+1})} B_k(h) r^{-\beta(\Omega, k)} f_k(\theta),\]

for every \((r, \theta) \in C_n(\Omega)\), where \( A_k(h) (k=1,2,\ldots,k_{p+1}-1) \) and \( B_k(h) (k=1,2,\ldots,k_{q+1}-1) \) are all constants.

In comparison with Theorem K, the following type is preferred.

**Theorem 6.** Let \( h(r, \theta) \) be a harmonic function in \( C_n(\Omega) \) vanishing continuously on \( S_n(\Omega) \). If \( h \) satisfies (11) with a positive integer \( p \), then

\[h(r, \theta) = \sum_{k \in I(\Omega, k_{p+1})} A_k(h) r^{\alpha(\Omega, k)} f_k(\theta),\]

for every \((r, \theta) \in C_n(\Omega)\), where \( A_k(h) (k=1,2,\ldots,k_{p+1}-1) \) are all constants.
When we specialize $\Omega=S^{n-1}_+$ i.e. $C_n(\Omega)=\mathbb{T}_n$, from Theorem 6 and (7), we obtain the following corollary 3. The equality
$$\vartheta(yh^+,r) = 2s_n^{-1}rN(h^+)(r)$$
shows that this is equal to Theorem K

COROLLARY 3. Let $h(r,\theta)$ be a harmonic function in $\mathbb{T}_n$ vanishing continuously on $\partial\mathbb{T}_n$. If, for some positive $t$,
$$\lim_{r \to \infty} r^{-(t+1)}N(h^+)(r) = 0,$$
then
$$h(r,\theta) = y\Pi,$$
where $\Pi$ is a polynomial in $\mathbb{R}^n$ with degree less than $t$ and even respect to the variable $y$.

By using Theorem 5, we can prove the following Theorem 7.

THEOREM 7. Let $l$, $m$ be two non-negative integers and $p$, $q$ be two positive integers satisfying $p \geq l$, $q \geq m$. Let $g(t,\Omega)$ be a continuous function on $S_n(\Omega)$ satisfying (9) with $l$ and (10) with $m$. If $h(r,\theta)$ is a solution of the Dirichlet problem on $C_n(\Omega)$ with $g$ satisfying
(13)
$$\mu_p(h^+) = 0$$
and
$$\eta_q(h^+) = 0,$$
then
$$h(r,\theta) = H(C_n(\Omega),l,m;g)(P) + \sum_{k \in \mathbb{I}(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k(\theta) + \sum_{k \in \mathbb{I}(\Omega,k_{q+1})} B_k(h)r^{-\beta(\Omega,k)}f_k(\theta).$$
for every \( P=(r,\theta) \in C_n(\Omega) \), where \( A_k(h) \) (\( k=1,2,\ldots,k_{p+1}-1 \)) and \( B_k(h) \) (\( k=1,2,\ldots,k_{q+1}-1 \)) are all constants.

If we take \( l=m=0 \) and \( p=q=1 \) in Theorem 7, then we have the following result containing the second part of Theorem \( Y_2 \).

**COROLLARY 4.** Let \( g(\Omega) \) be a continuous function on \( S_n(\Omega) \) satisfying (4). If \( h(r,\theta) \) is a solution of the Dirichlet problem on \( C_n(\Omega) \) with \( g \) satisfying

\[
\mu_1(h^+) = \eta_1(h^+) = 0,
\]

then

\[
h(r,\theta) = H(C_n(\Omega);g)(P) + A_1(h)\alpha(\Omega,1) f_1(\theta) + B_1(h)\beta(\Omega,1) f_1(\theta)
\]

for every \( P=(r,\theta) \in C_n(\Omega) \), where \( A_1(h) = \mu_0(h) \) and \( B_1(h) = \eta_0(h) \).

By Theorem 6, we also have

**THEOREM 8.** Let \( l \) be a non-negative integer and \( p \) be a positive integer satisfying \( l \leq p \). Let \( g(t,\Omega) \) be a continuous function on \( \Theta C_n(\Omega) \) satisfying (9) with \( l \). If \( h(r,\theta) \) is a solution of the Dirichlet problem on \( C_n(\Omega) \) with \( g \) satisfying (13) with \( p \), then

\[
h(r,\theta) = H(C_n(\Omega),l,0;g)(P) + \sum_{k \in \mathbb{N}} A_k(h)\alpha(\Omega,k) f_k(\theta)
\]

for every \( P=(r,\theta) \in C_n(\Omega) \), where \( A_k(h) \) (\( k=1,2,\ldots,k_{p+1}-1 \)) are all constants.

If we put \( \Omega=S_n^{n-1} \), \( l=\rho \) and \( p=\rho \) (resp. \( l=\rho-1 \) and \( p=\rho \)) (\( \rho \) is a positive integer) in Theorem 8, we obtain from Corollary 3
COROLLARY 5 (Yoshida [19, Theorem 2 (resp. Corollary 2)]). Let $\rho$ be a positive integer and $g(x)$ be a continuous function on $\partial T_n = \mathbb{R}^{n-1}$ satisfying (1) with $\rho$ (resp. (1) with $\rho-1$). If $h(P)$ is a solution of the Dirichlet problem on $T_n$ with $g$ such that
\[ \lim_{r \to \infty} r^{-(\rho+1)} N(h^+)(r) = 0, \]
then
\[ h(P) = H(T_n, \rho, 0; g)(P) + yF(h)(P) \]
(resp. $h(P) = H(T_n, \rho-1, 0; g)(P) + yF(h)(P)$),
where $F(h)(P)$ is a harmonic polynomial (of $P=(x_1, x_2, \ldots, x_{n-1}, y) \in \mathbb{R}^n$) of at most degree $\rho-1$ vanishing on $\partial T_n$ and even with respect to the variable $y$.

The following Theorem 9 also generalizes a result of Yoshida [19, Theorem 3].

THEOREM 9. If $h(r, \theta)$ is a harmonic function on $C_n(\Omega)$ and is continuous on $\overline{C_n(\Omega)}$ such that the restriction $h=h|_{\partial C_n(\Omega)}$ of $h$ to $\partial C_n(\Omega)$ satisfies
\[ \int_{t}^{\infty} \alpha(\Omega, k_{l+1})^{-1} \left( \int_{\partial \Omega} |h(t, \Xi)| d\sigma_{\Xi} \right) dt < +\infty, \]
for some non-negative integer $l$ and
\[ \lim_{r \to \infty} \frac{\log N(h^+)(r)}{\log r} < +\infty, \]
then for some positive integer $p$
\[ h(r, \theta) = H(\Omega, l, 0; h)(P) + \sum_{k \in I(\Omega, \omega^{p+1})} A_k(h) r^{\alpha(\Omega, k_{l})_{l}}(\theta) \]
\[ \alpha(\Omega, k_{l})_{l} \]
at every $P=(r, \theta) \in C_n(\Omega)$, where $A_k(h) \ (k=1, 2, \ldots, k_{p+1}-1)$ are all constants.

The Proof of all results in this part will be found in Yoshida and Miyamoto [20].
PART 3. The cylindrical case

1. Introduction

There is another typical unbounded domain which is a cylinder
\[ \Gamma_n(D) = D \times \mathbb{R} \]
with a bounded domain \( D \subset \mathbb{R}^{n-1} \). The existence and the uniqueness of solutions of the Dirichlet problem on \( \Gamma_n(D) \) with a continuous function on \( \partial \Gamma_n(D) \) are worth being inquired. In this direction, Yoshida [18] proved the following Theorem A. To state it we need some preliminaries.

Consider the Dirichlet problem
\begin{equation}
(\Delta_{n-1} + \lambda)f = 0 \quad \text{in } D \\
f = 0 \quad \text{on } \partial D
\end{equation}

for a bounded domain \( D \subset \mathbb{R}^{n-1}(n \geq 2) \), where \( \Delta_1 = d^2/dx^2 \). Let \( \lambda(D, 1) \) be the least positive eigenvalue of (14) and \( f_1^P(X) \) be the normalized eigenfunction corresponding to \( \lambda(D, 1) \). In order to make the subsequent consideration simpler, we put a strong assumption on \( D \) throughout the whole this paper: If \( n \geq 3 \), then \( D \) is a \( C^{2, \alpha} \)-domain \( (0 < \alpha < 1) \) in \( \mathbb{R}^{n-1} \) surrounded by a finite number of mutually disjoint closed hypersurfaces (for example, see Gilberg and Trudinger [9, pp.88-89] for the definition of \( C^{2, \alpha} \)-domain). Let \( G_{\Gamma_n(D)}(P_1, P_2) \) be the Green function of \( \Gamma_n(D) \) \( (P_1, P_2 \in \Gamma_n(D)) \) and \( \partial G_{\Gamma_n(D)}(P, Q)/\partial \nu \) be the differentiation at \( Q \in \partial \Gamma_n(D) \) along the inward normal into \( \Gamma_n(D) \) \( (P \in \Gamma_n(D)) \).

Given a function \( F(X, y) \) on \( \Gamma_n(D) \), we denote the function of \( y \) defined by the integral
\[ \int_D F(X, y)f_1^P(X)dX \]
by \( N(F)(y) \), where \( dX \) denotes the \((n-1)\)-dimensional volume element. We write
\[ \mu_0(N(F)) = \lim_{y \to -\infty} \exp(-\sqrt{\lambda(D, 1)y})N(F)(y) \]
and
\[ \eta_0(N(F)) = \lim_{y \to -\infty} \exp(\sqrt{\lambda(D, 1)y})N(F)(y), \]
if they exist.

Theorem A (Yoshida [18, Theorem 6]). Let \( g(Q) \) be a continuous function on \( \partial \Gamma_n(D) \) satisfying
\begin{equation}
\int_{-\infty}^{\infty} \exp(-\sqrt{\lambda(D, 1)|y|})(\int_{\partial D} |g(X, y)|d\sigma_X)dy < \infty,
\end{equation}
where $d\sigma_X$ is the surface area element of $\partial \Omega$ at $X$ and if $n = 2$ and $D = (\gamma, \delta)$, then

$$
\int_{\partial \Omega} |g(X,y)|d\sigma_X = |g(\gamma,y)| + |g(\delta,y)|.
$$

Then the Poisson integral

$$
PI_g(P) = c_n^{-1} \int_{\partial \Gamma_n(D)} g(Q) \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P,Q) d\sigma_Q
$$

is a solution of the Dirichlet problem on $\Gamma_n(D)$ with $g$, where

$$
c_n = \begin{cases} 
2\pi & (n = 2) \\
(n - 2)s_n & (n \geq 3) \ (s_n \text{ is the surface area of the unit sphere } S^{n-1})
\end{cases}
$$

and $d\sigma_Q$ is the surface area element on $\partial \Gamma_n(D)$ at $Q$. Let $h(P)$ be any solution of the Dirichlet problem on $\Gamma_n(D)$ with $g$. Then all of the limits $\mu_0(N(h)), \eta_0(N(h))$ ($-\infty < \mu_0(N(h))$, $\eta_0(N(h)) \leq \infty$), $\mu_0(N(|h|))$ and $\eta_0(N(|h|))$ ($0 \leq \mu_0(N(|h|))$, $\eta_0(N(|h|)) \leq \infty$) exist, and if

(16) \quad $\mu_0(N(|h|)) < \infty$ and $\eta_0(N(|h|)) < \infty$,

then

$$
h(P) = PI_g(P) + (\mu_0(N(h)) \exp(\sqrt{\lambda(D,1)y}) + \eta_0(N(h)) \exp(-\sqrt{\lambda(D,1)y})) f^D_1(X)
$$

for any $P = (X,y) \in \Gamma_n(D)$.\\

This Theorem A shows that under the conditions (15) and (16) the existence and a type of uniqueness of solutions for the Dirichlet problem on $\Gamma_n(D)$ can be proved, respectively.

If $n = 2$, then $\Gamma_n(D)$ is a strip. The strip $\Gamma_2((0, \pi))$ with $D = (0, \pi)$ is simply denoted by $\Gamma_2$. With respect to the Dirichlet problem on $\Gamma_2$, Widder obtained

**Theorem B** (Widder [13, Theorems 1 and 3]). If $g_i(t)$ $(i = 1, 2)$ is a continuous function on $R$ satisfying

$$
\int_{-\infty}^{\infty} |g_i(t)| \exp(-|t|) dt < \infty,
$$

then

$$
H(\Gamma_2; g_1, g_2)(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(x,t-y)g_1(t) dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\pi-x, t-y)g_2(t) dt
$$

$$(P(x,y) = \frac{\sin x}{\cosh y - \cos x})$$

is a harmonic function in $\Gamma_2$ and a continuous function on $\overline{\Gamma_2}$ such that

$$
H(\Gamma_2; g_1, g_2)(0,y) = g_1(y) \text{ and } H(\Gamma_2; g_1, g_2)(\pi,y) = g_2(y) \quad (-\infty < y < \infty).
$$
If $h(x,y)$ is a harmonic function in $\Gamma_2$ and a continuous function on $\Gamma_2$ such that

$$h(0,y) = g_1(y), \quad h(\pi, y) = g_2(y) \quad (-\infty < y < \infty)$$

and

$$\int_0^\pi |h(x,y)| dx = O(e^{bl}) \quad (|y| \to \infty),$$

then

$$h(x,y) = H(\Gamma_2; g_1, g_2)(x,y)$$

on $\Gamma_2$.

Though by a conformal mapping a strip is reduced to $T_2$ which was treated in [20] as a special case, it may be interested to be indipendently treated as a special case of cylinders.

In this paper, the first parts of Theorems A and B will be extended by defining generalized Poisson integrals with continuous functions under more unrestricted conditions than (15) and (16) (Theorem 1 and Corollary 1). We shall also prove that for any continuous function $g$ on $\Gamma_n(D)$ there is a solution of the Dirichlet problem on $\Gamma_n(D)$ with $g$ (Theorem 2 and Corollary 2). The results (Theorem 3 and Corollary 3) which generalize the second parts of Theorems A and B will be connected with a type of uniqueness of solutions for the Dirichlet problem on $\Gamma_n(D)$.

2. Statements of our results

We denote the non-decreasing sequence of positive eigenvalues of (15) by $\{\lambda(D,k)\}_{k=1}^\infty$. In this expression we write $\lambda(D,k)$ the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding $\lambda(D,k)$ is denoted by $f^D_k$, the set of sequential eigenfunctions corresponding to the same value of $\lambda(D,k)$ in the sequence $\{f^D_k\}_{k=1}^\infty$ makes an orthonormal basis for the eigenspace of the eigenvalue $\lambda(D,k)$. We can also say that for each $D \subset R^{n-1}$ there is a sequence $\{k_i\}$ of positive integers such that $k_1 = 1, \lambda(D,k_i) \leq \lambda(D,k_{i+1})$

$$\lambda(D,k_i) = \lambda(D, k_i + 1) = \lambda(D, k_i + 2) = \ldots = \lambda(D, k_{i+1} - 1)$$

and $\{f^D_{k_i}, f^D_{k_{i+1}}, \ldots, f^D_{k_{i+1} - 1}\}$ is an orthonormal basis for the eigenspace of the eigenvalue $\lambda(D, k_i)$ ($i = 1, 2, 3, \ldots$). It is well known that $k_2 = 2$ and $f^D_2(X) > 0$ for any $X \in D$ (see Courant and Hilbert [5, p.451 and p.458]). With respect to $\{k_i\}$, the following Example (2) shows that even in the case where $D$ is an open disk in $R^2$, not the simplest case $k_i = i$ ($i = 1, 2, 3, \ldots$), but complex case can appear.

When $D$ has sufficiently smooth boundary, we know that

$$\lambda(D,k) \sim A(D,n)k^{2/(n-1)} \quad (k \to \infty)$$

and

$$\sum_{\lambda(D,k) \leq x} \{f^D_k(X)\}^2 \sim B(D,n)x^{(n-1)/2} \quad (x \to \infty)$$
uniformly with respect to $X \in D$, where $A(D, n)$ and $B(D, n)$ are both constants depending on $D$ and $n$ (e.g. see Weyl [7] and Carleman [3]). Hence there exist two positive constants $M_1, M_2$ such that

$$M_1 k^{2/(n-1)} \leq \lambda(D, k) \quad (k = 1, 2, \ldots)$$

and

$$|f_k^D(X)| \leq M_2 k^{1/2} \quad (X \in D, k = 1, 2, \ldots).$$

We remark that both

$$\exp(\sqrt{\lambda(D, k)} y) f_k^D(X) \quad \text{and} \quad \exp(-\sqrt{\lambda(D, k)} y) f_k^D(X) \quad (k = 1, 2, \ldots)$$

are harmonic on $\Gamma_n(D)$ and vanish continuously on $\partial \Gamma_n(D)$.

For a domain $D$ and the sequence \{$k_j$\} mentioned above, by $I(D, k_j)$ we denote the set of all positive integers less than $k_j$ ($j = 1, 2, 3, \ldots$). In spite of the fact $I(D, k_1) = \emptyset$, the summation over $I(D, k_1)$ of a function $S(j)$ of a variable $j$ will be used by promising $\Sigma_{k \in I(D, k_1)} S(k) = 0$.

**Examples.** (1) Let $D = (0, \pi)$. Then (14) is reduced to find solutions $f(x)$ ($0 \leq x \leq \pi$) such that

$$\frac{d^2 f(x)}{dx^2} + \lambda f(x) = 0 \quad (0 < x < \pi)$$

and

$$f(0) = f(\pi) = 0$$

It is easy to see that $k_i = i$, $\lambda(D, k) = k^2$ and $f_k^D(x) = \sqrt{2 \pi} \sin kx$ ($k = 1, 2, 3, \ldots$).

(2) Let $D = \{(x, y) \in R^2; x^2 + y^2 < 1\}$. Let \{$\alpha_{n,m}$\}$_{m=1}^{\infty}$ be the increasing sequence of positive real numbers $\alpha_{n,m}$ such that

$$J_n(\alpha_{n,m}) = 0 \quad (n = 0, 1, 2, \ldots),$$

where $J_n(z)$ is the Bessel function of order $n$. If the spherical coordinates $x = r \cos \theta, y = r \sin \theta$ ($0 \leq r < 1, \ 0 \leq \theta < 2\pi$) are introduced, then $J_n(\alpha_{n,m} r) \cos n\theta$ and $J_n(\alpha_{n,m} r) \sin n\theta$ ($n \neq 0, m = 1, 2, 3, \ldots$) are two eigenfunctions corresponding to the eigenvalue $\lambda = \alpha_{n,m}^2$ (see Courant and Hilbert [5]). Since we do not know how the zeros of Bessel functions distribute, we cannot explicitly determine the sequence \{k_i\} with respect this $D$.

The Fourier coefficient

$$\int_D F(X) f_k^D(X) dX$$

of a function $F(X)$ on $D$ with respect to the orthonormal sequence \{$f_k^D(X)$\} is denoted by $c(F, k)$, if it exists. Now we shall define generalized Poisson kernels. Let $l$ and $m$ be two non-negative integers. For two points $P = (X, y) \in \Gamma_n(D)$, $Q = (X^*, y^*) \in \partial \Gamma_n(D)$, we put

$$\overline{V}(\Gamma_n(D), l)(P, Q)$$
\[ \sum_{k \in I(D, k_{i+1})} \exp \left( \sqrt{\lambda(D, k)} \right) c((H_{\lambda^*})_1, k) f_k^P(X) \exp(\sqrt{\lambda(D, k)}y) \exp(-\sqrt{\lambda(D, k)}y^*), \]

and

\[ \mathcal{V}(\Gamma_n(D), m)(P, Q) = \sum_{k \in I(D, k_{m+1})} \exp \left( \sqrt{\lambda(D, k)} \right) c((H_{\lambda^*})_1, k) f_k^P(X) \exp(-\sqrt{\lambda(D, k)}y) \exp(\sqrt{\lambda(D, k)}y^*), \]

where

\[ (H_{\lambda^*})_1(X) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}((X, 1), (X^*, 0)). \]

We remark that \( \mathcal{V}(\Gamma_n(D), l)(P, Q) \) and \( \mathcal{V}(\Gamma_n(D), m)(P, Q) \) are two harmonic functions of \( P \in \Gamma_n(D) \) for any fixed \( Q \in \partial \Gamma_n(D) \). We introduce two functions of \( P \in \Gamma_n(D) \) and \( Q = (X^*, y^*) \in \partial \Gamma_n(D) \)

\[ \mathcal{W}(\Gamma_n(D), l)(P, Q) = \begin{cases} \mathcal{V}(\Gamma_n(D), l)(P, Q) & (y^* \geq 0) \\ 0 & (y^* < 0) \end{cases} \]

and

\[ \mathcal{W}(\Gamma_n(D), m)(P, Q) = \begin{cases} \mathcal{V}(\Gamma_n(D), m)(P, Q) & (y^* \leq 0) \\ 0 & (y^* > 0) \end{cases} \]

The Poisson kernel \( K(\Gamma_n(D), l, m)(P, Q) \) with respect to \( \Gamma_n(D) \) is defined by

\[ K(\Gamma_n(D), l, m)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \mathcal{W}(\Gamma_n(D), l)(P, Q) - \mathcal{W}(\Gamma_n(D), m)(P, Q). \]

We note

\[ K(\Gamma_n(D), 0, 0)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q). \]

Let \( p, q \) be two non-negative integers and \( I(y) \) be a function on \( R \). The finite or infinite limits

\[ \lim_{y \to -\infty} \exp(-\sqrt{\lambda(D, k_{p+1})y})I(y) \quad \text{and} \quad \lim_{y \to -\infty} \exp(\sqrt{\lambda(D, k_{q+1})y})I(y) \]

are denoted by \( \mu_p(I) \) and \( \eta_q(I) \), respectively, when they exist.

**Theorem 10.** Let \( l, m \) be two non-negative integers and \( g(Q) = g(X^*, y^*) \) be a continuous function on \( S_n(D) \) satisfying

\[ \int_{-\infty}^{\infty} \exp(-\sqrt{\lambda(D, k_{i+1})y^*})(\int_{\partial D} |g(X^*, y^*)|d\sigma^{X^*})dy^* < \infty \]

and

\[ \int_{-\infty}^{\infty} \exp(\sqrt{\lambda(D, k_{m+1})y^*})(\int_{\partial D} |g(X^*, y^*)|d\sigma^{X^*})dy^* < \infty. \]

Then

\[ H(\Gamma_n(D), l, m, g)(P) = \int_{S_n(D)} g(Q) K(\Gamma_n(D), l, m)(P, Q)d\sigma_Q \]
is a solution of the Dirichlet problem on $\Gamma_n(D)$ with $g$ satisfying

$$\mu_l(N(|H(\Gamma_n(D), l, m; g)|)) = \eta_m(N(|H(\Gamma_n(D), l, m; g)|)) = 0.$$ 

If $n = 2$ and $D = (0, \pi)$, then we immediately obtain the following Corollary 6 which generalizes Theorem B.

**Corollary 6.** Let $l, m$ be two non-negative integers and $g_1(y^*), g_2(y^*)$ be two continuous functions on $R$ satisfying

\begin{equation}
\int_{-\infty}^{\infty} |g_i(y^*)| \exp(-(l + 1)y^*) dy^* < \infty \quad (i = 1, 2).
\end{equation}

Then

$$H(\Gamma_2, l, m; g_1, g_2)(x, y) = \int_{-\infty}^{\infty} g_1(y^*) K(\Gamma_2, l, m)((x, y), (0, y^*)) dy^* + \int_{-\infty}^{\infty} g_2(y^*) K(\Gamma_2, l, m)((x, y), (\pi, y^*)) dy^*$$

is a harmonic function in $\Gamma_2$ and a continuous function on $\overline{\Gamma_2}$ such that

$$H(\Gamma_2, l, m; g_1, g_2)(0, y^*) = g_1(y^*)$$

and

$$H(\Gamma_2, l, m; g_1, g_2)(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty)$$

To solve the Dirichlet problem on $\Gamma_n(D)$ with any function $g(Q)$ on $\partial \Gamma_n(D)$, we shall define another Poisson kernel. Let $\varphi(t)$ be any positive continuous function of $t \geq 0$ satisfying

$$\varphi(0) = \exp(-\sqrt{\lambda(D, 1)}).$$

For a domain $D \subset R^{n-1}$ and the sequence $\{\lambda(D, k_i)\}$, denote the set

$$\{t \geq 0; \exp(-\sqrt{\lambda(D, k_i)}) = \varphi(t)\}$$

by $S(D, \varphi, i)$. Then $0 \in S(D, \varphi, 1)$. When there is an integer $N$ such that $S(D, \varphi, N) \neq \emptyset$ and $S(D, \varphi, N + 1) = \emptyset$, denote the set $\{i; 1 \leq i \leq N\}$ of integers by $J(D, \varphi)$. Otherwise, denote the set of all positive integers by $J(D, \varphi)$. Let $t(i) = t(D, \varphi, i)$ be the minimum of elements $t$ in $S(D, \varphi, i)$ for each $i \in J(D, \varphi)$. In the former case, we put $t(N + 1) = \infty$. Then $t(1) = 0$.

We define $W(\Gamma_n(D), \varphi)(P, Q)$ ($P \in \Gamma_n(D)$, $Q = (X^*, y^*) \in S_n(D)$) by

$$W(\Gamma_n(D), \varphi)(P, Q) = \begin{cases} 
0 & (y^* < 0) \\
V(\Gamma_n(D), i)(P, Q) & (t(i) \leq y^* < t(i + 1); i \in J(D, \varphi)).
\end{cases}$$
We also define \( W(\Gamma_n(D), \varphi)(P, Q) \) \((P \in \Gamma_n(D), Q = (X^*, y^*) \in S_n(D))\) by

\[
W(\Gamma_n(D), \varphi)(P, Q) = \begin{cases} 
0 & (y^* > 0) \\
V(\Gamma_n(D), i)(P, Q) & (-t(i + 1) < y^* \leq -t(i); i \in J(D, \varphi)).
\end{cases}
\]

The Poisson kernel \( K(\Gamma_n(D), \varphi)(P, Q) \) \((P \in \Gamma_n(D), Q \in S_n(D))\) is defined by

\[
K(\Gamma_n(D), \varphi)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - W(\Gamma_n(D), \varphi)(P, Q) - W(\Gamma_n(D), \varphi)(P, Q).
\]

Now we have

**Theorem 11.** Let \( g(Q) \) be a continuous function on \( \partial \Gamma_n(D) \). Then there is a positive continuous function \( \varphi(t) \) of \( t \geq 0 \) connected with \( g \) such that

\[
H(\Gamma_n(D), \varphi; g)(P) = \int_{\partial \Gamma_n(D)} g(Q) K(\Gamma_n(D), \varphi)(P, Q) d\sigma_Q
\]

is a solution of the Dirichlet problem on \( \Gamma_n(D) \) with \( g \).

If we take \( n = 2 \) and \( D = (0, \pi) \) in Theorem 11, we obtain

**Corollary 7.** Let \( g_1(y^*) \) and \( g_2(y^*) \) be two continuous functions on \( R \). Then there is a positive continuous functions \( \varphi(t) \) of \( t \geq 0 \) such that

\[
H(\Gamma_2, \varphi; g_1, g_2)(x, y) = \int_{-\infty}^{\infty} g_1(y^*) K(\Gamma_2, \varphi)((x, y), (0, y^*)) dy^* + \int_{-\infty}^{\infty} g_2(y^*) K(\Gamma_2, \varphi)((x, y), (\pi, y^*)) dy^*
\]

is a harmonic function in \( \Gamma_2 \) and a continuous function on \( \Gamma_2 \) such that

\[
H(\Gamma_2, \varphi; g_1, g_2)(0, y^*) = g_1(y^*), \quad H(\Gamma_2, \varphi; g_1, g_2)(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty).
\]

**Theorem 12.** Let \( l, m \) be two non-negative integers and \( p, q \) be two positive integers satisfying \( p \geq l, q \geq m \). Let \( g(X^*, y^*) \) be a continuous function on \( \partial \Gamma_n(D) \) satisfying (17). If \( h(X, y) \) is a solution of the Dirichlet problem on \( \Gamma_n(D) \) with \( g \) satisfying

\[
\mu_p(N(h^+)) = 0 \quad \text{and} \quad \eta_q(N(h^+)) = 0
\]

then

\[
h(X, y) = H(\Gamma_n(D), l, m; g)(P) + \sum_{k \in I(D, k_{p+1})} A_k(h) \exp(\sqrt{\lambda(D, k)} y) f_k^P(X) + \sum_{k \in I(D, k_{q+1})} B_k(h) \exp(-\sqrt{\lambda(D, k)} y) f_k^P(X)
\]

for every \( P = (X, y) \in \Gamma_n(D) \), where \( A_k(h)(k = 1, 2, \ldots, k_{p+1} - 1) \) and \( B_k(h)(k = 1, 2, \ldots, k_{q+1} - 1) \) are all constants.
If we take \( n = 2 \) and \( D = (0, \pi) \) in Theorem 12, then we have

**Corollary 8.** Let \( l, m \) be two non-negative integers and \( p, q \) be two positive integers satisfying \( p \geq l, \ q \geq m \). Let \( g_1(y^*), \ g_2(y^*) \) be two continuous functions on \( \mathbb{R} \) satisfying (18). If \( h(x, y) \) is a harmonic function in \( \Gamma_2 \) and a continuous function on \( \overline{\Gamma_2} \) such that

\[
h(0, y^*) = g_1(y^*) \quad \text{and} \quad h(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty),
\]

and

\[
\lim_{y \to -\infty} \exp(-(p + 1)y) \int_0^\pi h(x, y) \sin x \, dx = \lim_{y \to -\infty} \exp((q + 1)y) \int_0^\pi h(x, y) \sin x \, dx = 0,
\]

then

\[
h(x, y) = H(\Gamma_2, l, m; g_1, g_2)(x, y) + \sum_{k=1}^p A_k(h) \exp(ky) \sin kx + \sum_{k=1}^q B_k(h) \exp(-ky) \sin kx
\]

for every \((x, y) \in \Gamma_2\), where \( A_k(h) \ (k = 1, 2, \ldots, p) \) and \( B_k(h) \ (k = 1, 2, \ldots, q) \) are all constants.

The proof of all results in this part will be found in Miyamoto [14].
References


