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Sharp characters and their generalizations

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1. Blichfeldt's Theorem

Let $G$ be a finite group and $\chi$ a virtual character of $G$. Let $L$ be the set of values of $\chi$. For $l \in L$, we define the number $B(l)$ as follows:

$$B(l) = \frac{a(l)}{|G|} \prod_{l' \in L-\{l\}} (l-l'),$$

where $a(l)$ denotes the number of elements $x$ in $G$ with $\chi(x) = l$.

Ninety years ago, Blichfeldt [B] proved that $B(l)$ is an algebraic integer for any $l \in L$. Our first aim is to extend this result. We will show that the numbers $B(l)$ ($l \in L$) are in fact the values of a virtual character $\tilde{\chi}$ of $G$, constructed from $\chi$ in a definite manner.

More precisely, we have the following

**Theorem 1.** Let $\tilde{\chi}$ be a class function on $G$ defined by $\tilde{\chi}(x) = B(\chi(x))$ for $x \in G$.

Then $\tilde{\chi}$ is a virtual character of $G$.

Since the value of a group character is a sum of roots of unity, it is clear that Theorem 1 implies Blichfeldt's Theorem mentioned above.

**Proof of Theorem 1.** (Outline) For $x \in G$, we let $f_x$ denote the monic polynomial of least degree whose set of roots is $L-\{\chi(x)\}$. Let $f$ be the average of $f_x$ over $G$:

$$f = \frac{1}{|G|} \sum_{x \in G} f_x.$$

Then we have the following
Claim. $f$ is a monic polynomial with integral coefficients of degree $|L| - 1$.

In fact, the coefficients of $f$ are expressed by integral linear combinations of $(\chi^i, 1_G)$ $i = 0, 1, \ldots$ and symmetric functions of the elements in $L$. For example, if $L = \{n, l, k\}$ then we have $f(X) = X^2 - ((n+l+k) - (\chi, 1_G))X + ((nl+lk+kn) - (n+l+k)(\chi, 1_G) + (\chi^2, 1_G))$.

Now Theorem 1 follows easily from Claim since $\tilde{\chi} = f(\chi)$.

Remark. The above $f$ is the polynomial of least degree with $f(l) = B(l)$ for every $l \in L$, that is, the Lagrange interpolation polynomial through the points $((l, B(l)) | l \in L)$.

One of the typical properties of $\tilde{\chi}$ is that it does not take the value 0. So we can define the class function $1/\tilde{\chi}$. By direct calculation, we obtain

**Proposition 2.** $(\chi^{-i}, 1/\tilde{\chi}) = 0$ for $i = 0, 1, \ldots, |L| - 2$.

Using Proposition 2 ($i=0$), we have the following divisibility conditions.

**Proposition 3.** For any $l \in L$, $B(l)$ divides $a(l) \prod_{l' \in L - \{l\}} B(l')$ in the ring of algebraic integers. In particular, if $\chi$ is a character of degree $n$, then $B(n)$ divides $\prod_{l \in L - \{n\}} B(l)$.

2. Sharp characters of finite groups

Under the same notation as in Section 1, we will define sharp triples for group characters.

**Definitions.** The triple $(G, \chi, l)$ is called a sharp triple if $B(l)$ is a unit in the ring of algebraic integers. The pair $(G, \chi)$ is called a sharp pair if $(G, \chi, \chi(1))$ is a sharp triple.
The concept of sharp pairs was first introduced by Cameron and Kiyota [CK], and their definition of sharp pairs is slightly different from ours. But at least in case \( \chi \) is a faithful character of \( G \), these two definitions are the same. So the concept of sharp triples is a natural generalization of that of sharp pairs.

We will give some examples of sharp triples.

**Example 1.** Let \( G \) be cyclic and \( \chi \) be a faithful linear character of \( G \). Then \((G, \chi, l)\) is sharp for every \( l \in \text{Im} \chi \).

**Example 2.** Let \( G \) be a sharply \( t \)-transitive permutation group and \( \pi \) be the associated permutation character. Then \((G, \pi, l-2)\) is a sharp triple, and \((G, \pi)\) is a sharp pair.

The following Lemmas are easy to prove. (Use Proposition 3 for Lemma 5.)

**Lemma 4.** If \((G, \chi, l)\) is sharp, then \( \alpha(l) \) divides \(|G|\).

**Lemma 5.** Let \( \chi \) be a character of degree \( n \). If \((G, \chi, l)\) is sharp for all \( l \in L-\{n\} \), then \((G, \chi)\) is a sharp pair.

**Question 6.** If \((G, \chi, l)\) is sharp with \( \chi \) a faithful character, then is it true that the set \( \{x \in G \mid \chi(x) = l\} \) is a single conjugacy class of \( G \)?

**Problem 7.** Determine all finite groups \( G \) such that \((G, \chi, l)\) is sharp for every non-trivial irreducible character \( \chi \) and for every \( l \in \text{Im} \chi \). Note that abelian groups and dihedral groups of twice odd prime order are such examples.

3. Classification of sharp triples for given \( L \)

From now on we assume \( \chi \) is a faithful character of \( G \) of degree \( n \). Set
$L = \text{Im}\chi$ and $L^{*} = L - \{n\}$. Cameron and Kiyota [CK] posed the problem of determining all the sharp pairs $(G, \chi)$ for a given set $L^{*}$. There are many papers on this subject; see the references of [AKN]. In particular Alvis and Nozawa [AN] have given a complete classification of sharp pairs when $L^{*}$ contains an irrational number.

Now we will consider the analogous problem for sharp triples $(G, \chi, l)$. The results known to me are very few. The first one is the simplest case and easy to prove.

**Result 1.** Let $L^{*} = \{\alpha_{1}, \ldots, \alpha_{t}\}$ with all $\alpha_{i}$ are algebraically conjugate. If $(G, \chi, \alpha_{1})$ is sharp, then $G$ is cyclic of prime order.

**Proof.** Since all $\alpha_{i}$ are conjugate, $(G, \chi, \alpha_{1})$ are all sharp, and so $(G, \chi)$ is sharp by Lemma 5. If $t \geq 2$, then the result follows from Theorem 4.1 in [CK]. Now assume $t = 1$. Then by Lemma 4, $a(\alpha_{1})$ divides $|G| = 1 + a(\alpha_{1})$. Thus $a(\alpha_{1}) = 1$, and so $G$ is cyclic of order two. This completes the proof.

We will state the other known results without proofs.

**Result 2.** Let $L^{*} = \{0, \alpha_{1}, \ldots, \alpha_{t}\}$ with all $\alpha_{i}$ are algebraically conjugate and $t \geq 2$. If $(\chi, 1_{G}) = 0$ and $(G, \chi, 0)$ is sharp, then $G$ is cyclic of order 4, dihedral of twice odd prime order, or $E_{2^{v}} \times \mathbb{Z}_{p}$, where $p = 2^{v} - 1$ is a Mersenne prime.

**Result 3.** (Matsuhisa and Yamaki [MY]) Let $L^{*} = \{0, \varepsilon_{1}, \ldots, \varepsilon_{t}\}$ with all $\varepsilon_{i}$ are roots of unity. If $(G, \chi, 0)$ is sharp, then $G$ is a sharply 3-transitive group or a 2-transitive Frobenius group.

**Result 4.** Let $L^{*} = \{l, k\}$ with integers $l, k$. If $(\chi, 1_{G}) = 0$ and $(G, \chi, l)$ is sharp, then one of the following holds:

(i) $k = 0$ and $(G, \chi)$ is sharp of type $(0, l)$. 

(ii) $k=-1, l=0$ and $G$ is the symmetric group of degree 3.

(iii) $k=-1, l=1$ and $G$ is quaternion or dihedral of order 8.

Problem 8. Determine all sharp triples $(G, \chi, l)$ when $L^*$ contains an irrational number.

References


[B] H.F.Blichfeldt, A theorem concerning the invariants of linear homogeneous groups, with some applications to substitution groups, Trans. Amer. Math. Soc. 5 (1904), 461-466.
