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<th>Sharp characters and their generalizations</th>
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Sharp characters and their generalizations

Masao KIYOTA

Department of General Education
Tokyo Medical and Dental University
2-8-30 Kohnodai, Ichikawa, Chiba, 272, Japan

1. Blichfeldt's Theorem

Let $G$ be a finite group and $\chi$ a virtual character of $G$. Let $L$ be the set of values of $\chi$. For $l \in L$, we define the number $B(l)$ as follows:

$$B(l) = \frac{a(l)}{|G|} \prod_{l' \in L - \{l\}} (l - l'),$$

where $a(l)$ denotes the number of elements $x$ in $G$ with $\chi(x) = l$.

Ninety years ago, Blichfeldt [B] proved that $B(l)$ is an algebraic integer for any $l \in L$. Our first aim is to extend this result. We will show that the numbers $B(l)$ ($l \in L$) are in fact the values of a virtual character $\tilde{\chi}$ of $G$, constructed from $\chi$ in a definite manner.

More precisely, we have the following

Theorem 1. Let $\tilde{\chi}$ be a class function on $G$ defined by $\tilde{\chi}(x) = B(\chi(x))$ for $x \in G$.

Then $\tilde{\chi}$ is a virtual character of $G$.

Since the value of a group character is a sum of roots of unity, it is clear that Theorem 1 implies Blichfeldt's Theorem mentioned above.

Proof of Theorem 1. (Outline) For $x \in G$, we let $f_x$ denote the monic polynomial of least degree whose set of roots is $L - \{\chi(x)\}$. Let $f$ be the average of $f_x$ over $G$:

$$f = \frac{1}{|G|} \sum_{x \in G} f_x.$$

Then we have the following
Claim. $f$ is a monic polynomial with integral coefficients of degree $|L| - 1$.

In fact, the coefficients of $f$ are expressed by integral linear combinations of $(\chi^i, 1_G)$ and symmetric functions of the elements in $L$. For example, if $L = \{n, l, k\}$ then we have $f(X) = X^2 - ((n+i+k) - (\chi, 1_G))X + ((ni+lk+kn) - (n+i+k)(\chi, 1_G) + (\chi^2, 1_G))$.

Now Theorem 1 follows easily from Claim since $\tilde{\chi} = f(\chi)$.

Remark. The above $f$ is the polynomial of least degree with $f(l) = B(l)$ for every $l \in L$, that is, the Lagrange interpolation polynomial through the points $((l, B(l)) \mid l \in L)$.

One of the typical properties of $\tilde{\chi}$ is that it does not take the value 0. So we can define the class function $1/\tilde{\chi}$. By direct calculation, we obtain

Proposition 2. $(\chi^i, 1/\tilde{\chi}) = 0$ for $i = 0, 1, \ldots, |L| - 2$.

Using Proposition 2 ($i=0$), we have the following divisibility conditions.

Proposition 3. For any $l \in L$, $B(l)$ divides $a(l) \prod_{l' \in L \setminus \{l\}} B(l')$ in the ring of algebraic integers. In particular, if $\chi$ is a character of degree $n$, then $B(n)$ divides $\prod_{l \in L \setminus \{n\}} B(l)$.

2. Sharp characters of finite groups

Under the same notation as in Section 1, we will define sharp triples for group characters.

Definitions. The triple $(G, \chi, l)$ is called a sharp triple if $B(l)$ is a unit in the ring of algebraic integers. The pair $(G, \chi)$ is called a sharp pair if $(G, \chi, \chi(l))$ is a sharp triple.
The concept of sharp pairs was first introduced by Cameron and Kiyota [CK], and
their definition of sharp pairs is slightly different from ours. But at least in case $\chi$ is
a faithful character of $G$, these two definitions are the same. So the concept of sharp
triples is a natural generalization of that of sharp pairs.

We will give some examples of sharp triples.

**Example 1.** Let $G$ be cyclic and $\chi$ be a faithful linear character of $G$. Then $(G,\chi,l)$ is
sharp for every $l \in \text{Im} \chi$.

**Example 2.** Let $G$ be a sharply $t$-transitive permutation group and $\pi$ be the associated
permutation character. Then $(G,\pi,t-2)$ is a sharp triple, and $(G,\pi)$ is a sharp pair.

The following Lemmas are easy to prove. (Use Proposition 3 for Lemma 5.)

**Lemma 4.** If $(G,\chi,l)$ is sharp, then $\alpha(l)$ divides $|G|$.

**Lemma 5.** Let $\chi$ be a character of degree $n$. If $(G,\chi,l)$ is sharp for all $l \in L-\{n\}$, then
$(G,\chi)$ is a sharp pair.

**Question 6.** If $(G,\chi,l)$ is sharp with $\chi$ a faithful character, then is it true that the set
$\{x \in G \mid \chi(x) = l\}$ is a single conjugacy class of $G$?

**Problem 7.** Determine all finite groups $G$ such that $(G,\chi,l)$ is sharp for every non-trivial
irreducible character $\chi$ and for every $l \in \text{Im} \chi$. Note that abelian groups and dihedral
groups of twice odd prime order are such examples.

3. Classification of sharp triples for given $L$

From now on we assume $\chi$ is a faithful character of $G$ of degree $n$. Set
Let $L = \text{Im} \chi$ and $L^* = L - \{n\}$. Cameron and Kiyota [CK] posed the problem of determining all the sharp pairs $(G, \chi)$ for a given set $L^*$. There are many papers on this subject; see the references of [AKN]. In particular Alvis and Nozawa [AN] have given a complete classification of sharp pairs when $L^*$ contains an irrational number.

Now we will consider the analogous problem for sharp triples $(G, \chi, l)$. The results known to me are very few. The first one is the simplest case and easy to prove.

Result 1. Let $L^* = \{\alpha_1, \ldots, \alpha_t\}$ with all $\alpha_i$ are algebraically conjugate. If $(G, \chi, \alpha_1)$ is sharp, then $G$ is cyclic of prime order.

Proof. Since all $\alpha_i$ are conjugate, $(G, \chi, \alpha_i)$ are all sharp, and so $(G, \chi)$ is sharp by Lemma 5. If $t \geq 2$, then the result follows from Theorem 4.1 in [CK]. Now assume $t = 1$. Then by Lemma 4, $a(\alpha_1)$ divides $|G| = 1 + a(\alpha_1)$. Thus $a(\alpha_1) = 1$, and so $G$ is cyclic of order two. This completes the proof.

We will state the other known results without proofs.

Result 2. Let $L^* = \{0, \alpha_1, \ldots, \alpha_t\}$ with all $\alpha_i$ are algebraically conjugate and $t \geq 2$. If $(\chi, 1_G) = 0$ and $(G, \chi, 0)$ is sharp, then $G$ is cyclic of order 4, dihedral of twice odd prime order, or $E_{2^v} \ltimes Z_p$, where $p = 2^v - 1$ is a Mersenne prime.

Result 3. (Matsuhisa and Yamaki [MY]) Let $L^* = \{0, \epsilon_1, \ldots, \epsilon_t\}$ with all $\epsilon_i$ are roots of unity. If $(G, \chi, 0)$ is sharp, then $G$ is a sharply 3-transitive group or a 2-transitive Frobenius group.

Result 4. Let $L^* = \{l, k\}$ with integers $l, k$. If $(\chi, 1_G) = 0$ and $(G, \chi, l)$ is sharp, then one of the following holds:

(i) $k = 0$ and $(G, \chi)$ is sharp of type $(0, l)$. 

(ii) \( k=-1, l=0 \) and \( G \) is the symmetric group of degree 3.

(iii) \( k=-1, l=1 \) and \( G \) is quaternion or dihedral of order 8.

**Problem 8.** Determine all sharp triples \((G, \chi, l)\) when \( L^* \) contains an irrational number.

**References**


[B] H.F.Blichfeldt, A theorem concerning the invariants of linear homogeneous groups, with some applications to substitution groups, Trans. Amer. Math. Soc. 5 (1904), 461-466.
