数理解析研究所講究録
第 896 巻 1995 年 1-7

TRANSFERS AND THE STRUCTURE OF COHOMOLOGY RINGS

BY JON F. CARLSON[1]

Department of Mathematics
University of Georgia
Athens, GA 30602

Induction from a subgroup to a group is one of the oldest techniques in the representation theory of finite group. If H is a subgroup of finite group G and if N is a kH-module for some field k, then the induced kG-module from N is $N^{\uparrow G} \cong kG \otimes_{kH} N$ with left kG-action. Thus from the kH-module N we create a new kG-module. The formulas for the character of the induced module is classical and has been used many times in the construction of character tables. Moreover the induction theories of Brauer, Green, Dress, etc. have been very important in the development of representation theory [CR].

There is also a corresponding definition of induction of homomorphism which has been important in integral and modular representations. Suppose that M and N are kG-module and f is kH-homomorphism from M to N. Then the induced homomorphism $Tr_H^G(f) : M \longrightarrow N$ is defined by $Tr_H^G(m) = \sum_{i=1}^{n} x_i f(x_i^{-1} m)$ where $x_1, \ldots, x_n$ is a complete set of representatives of the left cosets of H in G. So we have a k-linear homomorphism

$$Tr_H^G : \mathrm{Hom}_{kH}(M, N) \longrightarrow \mathrm{Hom}_{kG}(M, N),$$

and a kG-homomorphism is in the image of this map if and only if it factors through a module which is induced from kH.

[1] Partially supported by a grant from NSF.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

One of the most useful properties of induction is Frobenius reciprocity. For modules it says the following. Let M be a kG-module and N a kH-module. Let $M \otimes_k N^{\uparrow G}$ be the tensor product over k, which becomes a G-module by the diagonal action ($g(a \otimes b) = ga \otimes gb$). Then

$$M \otimes_k N^{\uparrow G} \cong (M_H \otimes N)^{\uparrow G}$$

where M_H is the restriction of M to kH-module. For maps there is similar-looking property which is actually much easier to prove. It says that if L, M, N are kG-modules, $f : L \longrightarrow M$ a kG-homomorphism and $g : M \longrightarrow N$ a kH-homomorphism then

$$(*) \qquad Tr_H^G(g) \circ f = Tr_H^G(g \circ \mathrm{res}_{G,H}(f))$$

where $\mathrm{res}_{G,H} : \mathrm{Hom}_{kG}(L, M) \longrightarrow \mathrm{Hom}_{kH}(L, M)$ is the restriction.

In group cohomology we can apply the induction of maps to cocycles to define the transfer map. Suppose that M and N are kG-modules and that

$$\cdots \longrightarrow P_2 \xrightarrow{\partial} P_1 \xrightarrow{\partial} P_0 \longrightarrow M \longrightarrow 0$$

is a kG-projective resolution. Then, by restriction, (P_*, ∂) is also a kH-projective resolution. A cohomology class $\gamma \in \mathrm{Ext}^n_{kH}(M, N)$ is represent by a cocycle $f : P_n \longrightarrow N$. If we apply induction, we get a cohomology class, $Tr_H^G(\gamma)$, represented by $Tr_H^G(f)$. The k-linear map

$$Tr_H^G : \mathrm{Ext}^*_{kH}(M, N) \longrightarrow \mathrm{Ext}^*_{kG}(M, N)$$

is called the transfer or corestriction. Moreover it satisfies a Frobenius reciprocity type of relation as in (*).

For this paper we restrict our attention to the case in which k is a field of characteristic $p > 0$ and $M = N = k$ is the trival kG-module. Then $\mathrm{Ext}^*_{kG}(k, k) \cong H^*(G, k)$ is a ring by cup product. We thus have a transfer map

$$Tr_H^G : H^*(H, k) \longrightarrow H^*(G, k).$$

The Frobenius reciprocity type relation

$$Tr_H^G(\alpha)\cdot\beta = Tr_H^G(\alpha\cdot \mathrm{res}_{G,H}(\beta))$$

guarantees that the set $Tr_H^G(H^*(H,k))$ is an ideal in $H^*(G,k)$. However in many cases this ideal is rather small. For example if G is an abelian p-group and H is any proper subgroup then $Tr_H^G(H^*(H,k))$ consists only of nilpotent elements. This fact represents one extreme of the whole picture. The "size" of the image of the transfer map depends, to some extent, on the degree of noncommutativity of the group at the prime p. Specifically we have the following

Theorem 1. [C]. *Let I be the ideal*

$$I = \sum Tr_H^G(H^*(H,k))$$

where the sum is over all subgroup H of G with p dividing $|G:H|$. Let P be a Sylow p-subgroup of G and $Z = Z(P)$ its center. Let J be the kernel of the restriction map $res_{G,Z} : H^*(G,k) \longrightarrow H^*(Z,h)$. Then $\sqrt{I} = \sqrt{J}$.

In the last few years there has been a growing interest in the ring-theoretic properties of the mod-p cohomology, $H^*(G,k)$. The basic theorem is that of Quillen which says that the components of the maximal ideal spectrum $V_G(k)$ of $H^*(G,k)$ are precisely the images $\hat{V}_E$ of the map

$$\mathrm{res}^*_{G,E} : V_E(k) \longrightarrow V_G(k)$$

for E a maximal elementary abelian p-subgroup of G. That is, $\mathrm{res}^*_{G,E}$ is the map on spectra induced by the ring homomorphism $\mathrm{res}_{G,E} : H^*(G,k) \longrightarrow H^*(E,k)$ (see [B1] or [E]). Thus if $\mathbf{m} \in V_E(k)$ is a maximal ideal in $H^*(E,k)$ then $\mathrm{res}^*_{G,E}\mathbf{m}$ is the inverse image of $\mathbf{m}$ under $\mathrm{res}_{G,E}$. Quillen's theorem can be interpreted as saying that the minimal primes ideals in $H^*(G,E)$ have the form $\sqrt{\mathrm{Ker}_E}$ where Ker_E is the kernel of the restriction to a maximal elementary abelian p-subgroup E.

More recently several questions have focused on the depth of $H^*(G,k)$ or, more specifically, for which groups G is $H^*(G,k)$ Cohen-Macaulay. To understand the problem we need a few definition [M]. In a graded algebra such as $H^*(G,k)$, a regular sequence is a set of homogenous elements $\zeta_1, \ldots, \zeta_t$ of positive degree with the property that ζ_1 is not a divisor of zero on $H^*(G,k)$ and, for each $i = 2, \ldots, t, \zeta_i$ is not a divisor of zero on $H^*(G,k)/(\zeta_1, \ldots, \zeta_{i-1})H^*(G,k)$. The depth of $H^*(G,k)$ is the length of the longest regular sequence. The ring $H^*(G,k)$ is Cohen-Macauley (CM) if the depth is equal to the Krull dimension. In the case that $H^*(G,k)$ is CM, $H^*(G,k)/(\zeta_1, \ldots, \zeta_r)$. $H^*(G,k)$ is a finite dimensional algebra where $\zeta_1, \ldots, \zeta_r$ is a maximal regular sequence.

One of the consequences of Quillen's theorem is that the Krull dimension of $H^*(G,k)$ is the p-rank, r of an elementary abelian p-subgroup E of maximal rank in $G(|E| = p^r)$. A system of homogeneous parameters for $H^*(G,k)$ is a set $\zeta_1, \ldots, \zeta_r$ of (exactly r (!)) elements such that

$$H^*(G,k)/(\zeta_1, \ldots, \zeta_r)H^*(G,k)$$

is a finite dimensional k-algebra. Hence in the Cohen-Macaulay case any longest regular sequence must be a set of homogeneous parameters. But, it is a theorem that if $H^*(G,k)$ is CM then any system of homogeneous parameters must be a regular sequence.

Another theorem concerned with the Cohen-Maccaulay property states that if $H^*(G,k)$ is CM, then all components of $V_G(k)$, the maximal ideal spectrum, must have the same dimension. Now the dimension of a component $\hat{V}_E$ is precisely the p-rank of E. So for $H^*(G,k)$ to be CM, it is necessary for all maximal elementary abelian p-subgroups of G to have the same rank and hence also the same order. However there are several examples in which the maximal elementary abelian p-subgroups have the same rank, but the cohomology ring is not CM. These include the semidihedral group in characteristic 2 and the extra special group of order p^3 and exponent p^2 in the case that p is odd. The reason we know that the cohomology rings are not CM is that rings have actually been calculated (by Evens-Priddy [EP] in the semi-dihedral case, by Diethelm [Di] in the odd characteristic case). However there is a perfectly reasonable method for seeing this fact. It requires almost no direct calculations and can be summarized in the following steps.

Step 1. We assume, as with the above cited groups, that G is a p-group, of p-rank 2, with the property that all elements of order p are contained in some maximal subgroup H of G. Thus H also contains all elementary abelian p-subgroups of G.

Step 2. Recall the classical result that $H^1(G, \mathbb{Z}/p) \cong \mathrm{Hom}(G, \mathbb{Z}/p)$. So we get an element $\eta \in H^1(G,k) \cong k \otimes_{\mathbb{Z}} H^1(G, \mathbb{Z}/p)$ corresponding to the subgroup H. This means that, while $\eta \neq 0, \mathrm{res}_{G,H}(\eta) = 0$.

Step 3. Now we come to choosing a system of homogeneous parameters for $H^*(G,k)$. Because the group G has p-rank 2, the system consists of two homogeneous elements ζ_1 and ζ_2. Now ζ_1 can be chosen to have a nonnilpotent restriction to the center of G. By [D1] such an element is regular. By theorem 1, the second element can be chosen to be a transfer from the cohomology of H. (Actually we need an extension of Theorem 1 which is easily obtained from its proof [C1]). Thus we can assume that $\zeta_2 = Tr_H^G(\gamma)$ for some γ in $H^*(H,k)$.

Step 4. Now we notice that

$$\zeta_2 \cdot \eta = Tr_H^G(\gamma) \cdot \eta = Tr_H^G(\gamma \cdot \mathrm{res}_{G,H}(\eta)) = 0.$$

If $H^*(G,k)$ were Cohen-Maccaulay then ζ_1, ζ_2 would have to be a regular sequence. This is impossible.

The significant thing about the above argument is that it can be generalized. The first things required is a generalization of Theorem 1. This comes in the form of the following result of Benson.

Theorem 2. [B2]. *Let $\mathcal{H}$ be a nonempty collection of subgroups of G and let $\mathcal{K}$ denote the set of all subgroups K of G such that the Sylow p-subgroups of the centralizer $C_G(K)$ is not conjugate to a subgroup of any of the subgroups in $\mathcal{H}$. Let*

$$J = \sum_{H \in \mathcal{H}} Tr_H^G(H^*(H,k)), I = \bigcap_{K \in \mathcal{K}} Ker(res_{G,K}).$$

Then $\sqrt{I} = \sqrt{J}$.

Using Benson's result, the previous argument can be adapted to prove the following.

Theorem 3. [C2]. *Suppose that* $H^*(G,k)$ *has an element* ζ *such that* $\mathrm{res}_{G,C_G(E)}$ $(\zeta) = 0$ *for all elementary abelian p-subgroups E of rank s. Then depth* $(H^*(G,k)) < s$ *and also* $H^*(G,k)$ *has an associated prime ideal of dimension less than s.*

An associated prime is a prime ideal which annihilates some element of $H^*(G,k)$. In the above case, the element would be some element in the ideal generated by ζ. It is known that the associated primes of $H^*(G,k)$ are invariant under the Steenrod reduced power operations, and hence they are the radicals of the restrictions to certain elementary abelian p-subgroups. It follows that the associated primes are the radicals of ideals of transfers from certain collections of subgroups of G.

An open question is whether the associated primes and the images of transfers actually determine the depth and other ring-theoretic invariants of the mod-p cohomology. We might ask the following.

Question 1 [C2] Suppose that $H^*(G,k)$ has depth d. Is there an associated prime $\mathfrak{P}$ in $H^*(G,k)$ such that $V_G(\mathfrak{P})$ has dimension d?

The answer is yes in all of the examples that we have looked at. It can be shown to be true for the symmetric groups and their Sylow p-subgroup [CH1]. It holds for groups of p-rank 2 by an argument using results from [BC]. An equivalent question is the following.

Question 2 [CH2] Suppose that the mod-p cohomology of G is detected on the centralizers of elementary abelian p-subgroups of rank s. (That is, the intersection of the kernels of restrictions of $H^*(G,k)$ to the centralizers of the elementary abelian p-subgroups of rank s is zero). Is the depth of $H^*(G,k)$ at least s?

The equivalence of the two questions comes from the relationship of the detectability criterion to the existence of regular elements (see [HLS], [CH2]).

References

[B1] D. J. Benson, *Representations and Cohomology II: Cohomology of Groups and Modules*, Cambridge Studies in Advanced Mathematics **31** (1991), Cambridge University Press, Cambridge.

[B2] D. J. Benson, *The image of the transfer map*, Archiv Math. (to appear).

[BC] D. J. Benson and J. F. Carlson, *Products in negative cohomology*, J. Pure Appl. Algebra **82** (1992), 107–129.

[C1] J. F. Carlson, *Varieties and transfers*, J. Pure and Appl. Algebra **44** (1987), 99–105.

[C2] J. F. Carlson,, *Depth and transfer maps in the cohomology of groups*, Math Zeit. (to appear).

[CH1] J. F. Carlson and H.-W. Henn, *Wreath products and depths of cohomology rings* (to appear).

[CH2] J. F. Carlson and H.-W. Henn, *The support of regular elements of cohomology rings* (to appear).

[CR] C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Wiley, New York.

[Di] T. Diethelm, *The mod-p cohomology rings of the nonabelian split metacyclic p-groups*, Archiv Math. **44** (1985), 29–38.

[D1] J. Duflot, *Depth and equivariant cohomology*, Comm. Math. Helvetici **56** (1981), 627-637.

[D2] J. Duflot, *The associated primes of* $H^*_G(x)$, J. Pure and Appl. Algebra **30** (1983), 131–141.

[E] L. Evens, *The Cohomology of Groups*, Oxford University Press, New York.

[EP] L. Evens and S. Priddy, *The cohomology of the semi-dihedral group, Conference on Algebraic Topology in Honour of Peter Hilton*, Contemp. Math **37** (1985) (R. Piccinini and D. Sjerve, ed.).

[HLS] H.-W. Henn, J. Lannes and L. Schwartz, *Localization of unstable CA-modules and equivariant mod-p cohomology* (to appear).

[LS] P. S. Landweber and R. E. Stong, *The depth of rings of invariants over finite fields*, Number Theory, New York, 1984-1985Lecture Notes in Math , Springer-Verlag, Berlin.

[M] H. Matsumura, *Commutative Ring Theory*, Cambridge University Press, Cambridge.

[Q] D. Quillen, *The spectrum of an equivariant cohomology ring* I, II, Ann. Math. **94** (1971), 549-602.