Nonlinear Ergodic Theorems for Semigroups of Nonexpansive Mappings and Left Ideals

Anthony T. M. Lau*, Koji Nishiura and Wataru Takahashi

1 Introduction

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from $S$ to $S$ are continuous. Let $E$ be a uniformly convex Banach space and let $S = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of $E$ into $C$, i.e., $T_{ab}x = T_a T_b x$ for every $a, b \in S$ and $x \in C$ and the mapping $(s, x) \to T_s(x)$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology. Let $F(S)$ denote the set $\{x \in C : T_s x = x$ for all $s \in S\}$ of common fixed points of $S$ in $C$. Then as well known, $F(S)$ (possibly empty) is a closed convex subset of $C$ (see [5]).

In this paper, we shall study the distance between left ideal orbits and elements in the fixed point set $F(S)$. We shall prove (Theorem 3.11) among other things that if $E$ has a Fréchet differentiable norm, then for any semitopological semigroup $S$ and $x \in C$, the set $Q(x) = \bigcap \overline{co}\{T_t x : t \in L\}$, with the intersection taking over all closed left ideals $L$ of $S$, contains at most one common fixed point of $S$ (where $\overline{co}A$ denotes the closed convex hull of $A$). This result is then applied to show (Theorem 4.1) that if $F(S) \cap Q(x) \neq \emptyset$ for any $x \in C$, then there exists a retraction $P$ from $C$ onto $F(S)$ such that $T_t P = P T_t = P$ for every $t \in S$ and $P(x) \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. Both Theorem 3.11 and Theorem 4.1 were established by Lau and Takahashi in [18] when $S$ has finite intersection property for closed left ideals.

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let $C$ be a closed convex subset of a Hilbert space and let $T$ be a nonexpansive mapping of $C$ into itself. If the set $F(T)$ of fixed points of $T$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = P x$ for each $x \in C$, $P$ is a nonexpansive retraction of $C$ onto $F(T)$ such that $PT = TP = P$ and $P x \in \overline{co}\{T^n x : n = 1, 2, \cdots\}$ for each $x \in C$. In [24], Takahashi proved the existence of such a retraction for an amenable semigroup. This result is further extended to certain Banach spaces by Hirano and Takahashi in [12].

*This research is supported by NSERC-grant A7679
Our paper is organized as follows: In section 2 we define some terminologies that we use; in section 3 we study the distance between ideals determined by left orbits and the fixed point set; in section 4 we apply our results in section 3 to establish our main nonlinear ergodic theorems; finally in section 5 we study an almost fixed point property determined by the minimal left ideals in the enveloping semigroup of a semigroup of nonexpansive mappings on a weak compact convex set and obtain a generalization of De Marr's fixed point theorem [6].

2 Preliminaries

Throughout this paper, we assume that a Banach (or Hilbert) space is real. Let $E$ be a Banach space and let $E^*$ be its dual. Then, the value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$ or $f(x)$. The duality mapping $J$ of $E$ is a multivalued operator $J : E \to E^*$ where $J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \|^2 = \| f \|^2 \}$ (which is nonempty by simple application of the Hahn-Banach theorem). Let $B = \{ x \in E : \| x \| = 1 \}$ be the unit sphere of $E$. Then the norm of $E$ is said to be Fréchet differentiable if for each $x \in B$, the limit

$$\lim_{\lambda \to 0} \frac{\| x + \lambda y \| - \| x \|}{\lambda}$$

is attained uniformly for $y \in B$. In this case, $J$ is a single-valued and norm to norm continuous mapping from $E$ into $E^*$ (see [5] or [8] for more details).

Let $S$ be a nonempty set and let $X$ be a subspace of $l^\infty(S)$ (bounded real-valued functions on $S$) containing constants. By a submean on $X$ we shall mean a real-valued function $\mu$ on $X$ satisfying the following properties:

1. $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
2. $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
3. For $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
4. $\mu(c) = c$ for every constant function $c$.

A semitopological semigroup $S$ is called left reversible (resp. right reversible) if $S$ has finite intersection property for right (resp. left) ideals. $S$ is called reversible if $S$ is both left and right reversible.

Let $S$ be a semitopological semigroup and let $C(S)$ denote the closed subalgebra of $l^\infty(S)$ consisting of bounded continuous functions. For each $f \in C(S)$ and $a \in S$, let $(l_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$. Let $RUC(S)$ denote all $f \in C(S)$ such that the mapping $S \to C(S)$ defined by $s \to r_s f$ is continuous when $C(S)$ has the norm topology. Then $RUC(S)$ is a translation invariant subalgebra of $C(S)$ containing constants. Further, $RUC(S)$ is precisely the space of bounded left uniformly continuous functions on $S$ when $S$ is a group (see [11]).

A submean $\mu$ on $RUC(S)$ is called invariant if $\mu(l_a f) = \mu(r_a f) = \mu(f)$ for every $f \in RUC(S)$ and $a \in S$. If $S$ is a discrete semigroup, then $RUC(S)$ has an invariant submean if and only if $S$ is reversible. Also if $S$ is normal and $C(S)$ has an invariant submean, then $S$ is reversible. However $S$ need not be reversible when $C(S)$ has an invariant submean in general (see [19] for details).
3 Left ideal orbits and the fixed point set

Unless otherwise specified, $S$ denotes a semitopological semigroup and $S = \{T_s : s \in S\}$ a continuous representation of $S$ as nonexpansive mappings from a nonempty closed convex subset $C$ of a Banach space $E$ into $C$.

Let $\mathcal{L}(S)$ denote the collection of closed left ideals in $S$. Assume that $F(S) \neq \emptyset$. For each $x \in C$ and $L \in \mathcal{L}(S)$, define the real-valued function $q_{x,L}$ on $F(S)$ by

$$q_{x,L}(f) = \inf\{\|T_t x - f\|^2 : t \in L\}$$

and let

$$q_x(f) = \sup\{q_{x,L} : L \in \mathcal{L}(S)\}.$$

Then

$$q_x(f) = \sup_s \inf_t \|T_{ts} x - f\|^2$$

as readily checked.

**Lemma 3.1** Let $C$ be a nonempty closed convex subset of a Banach space $E$. If $F(S) \neq \emptyset$, then for each $x \in C$, $q_x$ is a continuous real-valued function on $F(S)$ such that $0 \leq q_x(f) \leq \|x - f\|^2$ for each $f \in F(S)$ and $q_x(f_n) \to \infty$ if $\|f_n\| \to \infty$. Further, if $F(S)$ is convex, then $q_x$ is a convex function on $F(S)$.

**Proof.** Since $0 \leq \|T_t x - f\|^2 = \|T_t x - T_t f\|^2 \leq \|x - f\|^2$ for every $f \in F(S)$ and $t \in S$, it follows readily that $0 \leq q_x(f) \leq \|x - f\|^2$. Also if $f \in F(S)$ and $t \in S$, then $\|T_t x - f\| \leq \|x - f\|$. Hence $\|T_t x\| \leq \|T_t x - f\| + \|f\| \leq \|x - f\| + \|f\|$, i.e., $M = \sup\{\|T_t x\| : t \in S\} < \infty$. Let $\{f_n\}$ be a sequence in $F(S)$ such that $\|f_n\| \to \infty$. Then we have for each $t \in S$,

$$\|T_t x - f_n\|^2 \geq \left(\|T_t x\| - \|f_n\|\right)^2$$

$$= \|f_n\|^2 - 2\|T_t x\|\|f_n\| + \|T_t x\|^2$$

$$\geq \|f_n\|^2 - 2M\|f_n\|$$

$$= \|f_n\|^2 \left(1 - \frac{2M}{\|f_n\|}\right)$$

and hence for each $L \in \mathcal{L}(S)$,

$$q_{x,L}(f_n) \geq \|f_n\|^2 \left(1 - \frac{2M}{\|f_n\|}\right) \to \infty.$$

So we have $q_x(f_n) \to \infty$.

To see that $q_x$ is continuous, let $\{f_n\}$ be a sequence in $F(S)$ converging to some $f \in F(S)$ and

$$M' = \sup\{\|T_t x - f_n\| + \|T_t x - f\| : n = 1, 2, \ldots \text{ and } t \in S\}.$$

Then since

$$\|T_t x - f_n\|^2 - \|T_t x - f\|^2 \leq (\|T_t x - f_n\| + \|T_t x - f\|) \|T_t x - f_n\| - \|T_t x - f\|$$

$$\leq M'\|f_n - f\|,$$
we have for each $L \in \mathcal{L}(S)$,
\[ q_{x,L}(f_{n}) \leq q_{x,L}(f) + M'\|f_{n} - f\|. \]

Similarly, we have
\[ q_{x,L}(f) \leq q_{x,L}(f_{n}) + M'\|f_{n} - f\|. \]

So we obtain
\[ |q_{x}(f_{n}) - q_{x}(f)| \leq M'\|f_{n} - f\|. \]

This implies that $q_{x}$ is continuous on $F(S)$.

If $F(S)$ is convex, for each $f, g \in F(S)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $\alpha f + \beta g \in F(S)$.

Let $\epsilon > 0$. Then there exists $L_{0} \in \mathcal{L}(S)$ such that
\[ \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} ( \alpha \|T_{t}x - f\|^{2} + \beta \|T_{t}x - g\|^{2}) < \inf_{t \in L_{0}} ( \alpha \|T_{t}x - f\|^{2} + \beta \|T_{t}x - g\|^{2}) + \frac{\epsilon}{2}. \]

Let $u \in L_{0}$. Then $Su \subseteq L_{0}$ and hence
\[ \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} ( \alpha \|T_{t}x - f\|^{2} + \beta \|T_{t}x - g\|^{2}) < \inf_{t \in S} ( \alpha \|T_{tu}x - f\|^{2} + \beta \|T_{tu}x - g\|^{2}) + \frac{\epsilon}{2}. \]

Moreover, there exist $v, w \in S$ such that
\[ \|T_{vu}x - f\|^{2} < \inf_{t \in S} \|T_{tu}x - f\|^{2} + \frac{\epsilon}{2} \]
\[ \text{and} \]
\[ \|T_{wvu}x - f\|^{2} < \inf_{t \in S} \|T_{tvu}x - f\|^{2} + \frac{\epsilon}{2}. \]

Therefore we obtain
\[ q_{x}(\alpha f + \beta g) = \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} \|T_{t}x - (\alpha f + \beta g)\|^{2} \]
\[ \leq \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_{t}x - f\|^{2} + \beta \|T_{t}x - g\|^{2}) \]
\[ < \inf_{t \in S} (\alpha \|T_{tu}x - f\|^{2} + \beta \|T_{tu}x - g\|^{2}) + \frac{\epsilon}{2} \]
\[ \leq \alpha \|T_{vu}x - f\|^{2} + \beta \|T_{wvu}x - f\|^{2} + \frac{\epsilon}{2} \]
\[ < \alpha \inf_{t \in S} \|T_{t}x - f\|^{2} + \beta \inf_{t \in L_{1}} \|T_{t}x - g\|^{2} + \frac{\alpha \epsilon}{2} + \frac{\beta \epsilon}{2} + \epsilon \]
\[ = \alpha \inf_{t \in L_{1}} \|T_{t}x - f\|^{2} + \beta \inf_{t \in L_{2}} \|T_{t}x - g\|^{2} + \epsilon \]
\[ ( \text{where } L_{1} = \overline{Su} \text{ and } L_{2} = \overline{Svu} ) \]
\[ \leq \alpha q_{x}(f) + \beta q_{x}(g) + \epsilon. \]

Since $\epsilon > 0$ is arbitrary, we have
\[ q_{x}(\alpha f + \beta g) \leq \alpha q_{x}(f) + \beta q_{x}(g). \]
Theorem 3.2 Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Assume that $F(S) \neq \emptyset$. Then for any $x \in C$, there exists a unique element $h \in F(S)$ such that

$$q_x(h) = \inf \{q_x(f) : f \in F(S) \}.$$

Proof. Since $E$ is uniformly convex, the fixed point set $F(S)$ in $C$ is closed and convex (see [5]). Hence it follows from Lemma 3.1 and [2] that there exists $h \in F(S)$ such that

$$q_x(h) = \inf \{q_x(f) : f \in F(S) \}.$$

To see that $h$ is unique, let $k \in F(S)$. Then by [27], there exists a strictly increasing and convex function (depending on $h$ and $k$) $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|T_t x - (\lambda h + (1 - \lambda)k)\|^2 = \|\lambda(T_t x - h) + (1 - \lambda)(T_t x - k)\|^2 \leq \lambda\|T_t x - h\|^2 + (1 - \lambda)\|T_t x - k\|^2 - \lambda(1 - \lambda)g(\|h - k\|)$$

for each $t \in S$ and $\lambda$ with $0 \leq \lambda \leq 1$. So we have for each $\lambda$ with $0 \leq \lambda \leq 1$,

$$q_x(h) \leq q_x(\lambda h + (1 - \lambda)k) \leq \lambda q_x(h) + (1 - \lambda)q_x(k) - \lambda(1 - \lambda)g(\|h - k\|)$$

and hence

$$q_x(h) \leq q_x(k) - \lambda g(\|h - k\|).$$

It follows that

$$q_x(h) \leq q_x(k) - g(\|h - k\|) \text{ as } \lambda \to 1.$$

Since $g$ is strictly increasing, it follows that if $q_x(h) = q_x(k)$, then $h = k$. \qed

We call the unique element $h \in F(S)$ in Theorem 3.2 the minimizer of $q_x$ in $F(S)$. For each $x \in C$, let

$$Q(x) = \bigcap_{L \in \mathcal{L}(S)} \overline{co}\{T_t x : t \in L\} = \bigcap_{s \in S} \overline{co}\{T_{ts} x : t \in S\}.$$

Theorem 3.3 Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $S = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings from $C$ into $C$. Then for any $x \in C$, any element in $Q(x) \cap F(S)$ is the unique minimizer of $q_x$ in $F(S)$. In particular, $Q(x) \cap F(S)$ contains at most one point.

Proof. Let $z \in F(S)$ be the minimizer of $q_x$ in $F(S)$ and $y \in Q(x) \cap F(S)$. Then for some $\varepsilon > 0$, there exists $u \in S$ such that

$$\sup_{t} \inf_{s} \left(\|T_{ts} x - z\|^2 + 2\langle T_{ts} x - z, z - y \rangle + \|z - y\|^2\right)$$

$$< \inf_{t} \left(\|T_{tu} x - z\|^2 + 2\langle T_{tu} x - z, z - y \rangle + \|z - y\|^2\right) + \frac{\varepsilon}{4}.$$

Moreover there exist $v, w \in S$ such that

$$\|T_{vu} x - z\|^2 < \inf_{t} \|T_{tu} x - z\|^2 + \frac{\varepsilon}{4}.$$
and
\[ \langle T_{wvu}x - z, z - y \rangle < \inf_{t} \langle T_{tvu}x - z, z - y \rangle + \frac{\epsilon}{4}. \]

Therefore we obtain
\[
q_{x}(y) = \sup_{s} \inf_{t} \| T_{ts}x - y \|^2 \\
= \sup_{s} \inf_{t} (\| T_{ts}x - z \|^2 + 2 \langle T_{ts}x - z, z - y \rangle + \| z - y \|^2) \\
< \inf_{t} (\| T_{tu}x - z \|^2 + 2 \langle T_{tu}x - z, z - y \rangle + \| z - y \|^2) + \frac{\epsilon}{4} \\
\leq \| T_{vu}x - z \|^2 + 2 \langle T_{vu}x - z, z - y \rangle + \| z - y \|^2 + \frac{\epsilon}{4} \\
< \inf_{t} (\| T_{tu}x - z \|^2 + 2 \langle T_{tvu}x - z, z - y \rangle + \| z - y \|^2 + \frac{\epsilon}{4}) \\
\leq \sup_{s} \inf_{t} (\| T_{ts}x - z \|^2 + 2 \langle T_{ts}x - z, z - y \rangle + \| z - y \|^2 + \frac{\epsilon}{4}) \\
\leq q_{x}(z) + 2 \sup_{s} \inf_{t} (\| T_{ts}x - z, z - y \| + \| z - y \|^2 + \frac{\epsilon}{4}).
\]

This implies
\[
2 \sup_{s} \inf_{t} (\| T_{ts}x - z, z - y \|) > q_{x}(y) - q_{x}(z) - \| z - y \|^2 - \epsilon \\
\geq -\| z - y \|^2 - \epsilon.
\]

So, there exists \( a \in S \) such that
\[
2 \langle T_{ta}x - z, z - y \rangle > -\| z - y \|^2 - \epsilon
\]
for every \( t \in S \). From \( y \in \overline{co}\{T_{ta}x : t \in S\} \), we have
\[
2\langle y - z, z - y \rangle \geq -\| z - y \|^2 - \epsilon.
\]

This inequality implies \( \| z - y \|^2 \leq \epsilon \). Since \( \epsilon > 0 \) is arbitrary, we have \( z = y \). \( \square \)

**Remark 3.4** From Theorem 3.3, it is natural to ask the following:

**Problem 1.** If \( E \) is a uniformly convex Banach space, \( x \in C \) and \( y \in Q(x) \cap F(S) \), is \( y \) always the minimizer of \( q_{x} \) in \( F(S) \) ?

**Problem 2.** If \( E \) is a uniformly convex Banach space, does \( Q(x) \cap F(S) \) contain at most one point for each \( x \in C \) ?

Clearly, by Theorem 3.2, an affirmative answer for Problem 1 gives an affirmative answer to Problem 2. We now proceed to give an affirmative answer for Problem 2 when \( E \) has a Fréchet differentiable norm.
Lemma 3.5 Let $C$ be a nonempty closed convex subset of a Banach space $E$. Let $x \in C$ and $f \in F(S)$. Then
\[ \inf_s \|T_s x - f\| = \inf_s \sup_t \|T_{ts} x - f\|. \]

Proof. Let $r = \inf_s \|T_s x - f\|$ and $\varepsilon > 0$. Then there exists $a \in S$ such that
\[ \|T_a x - f\| < r + \varepsilon. \]
So, for each $t \in S$, we have
\[ \|T_{ta} x - f\| \leq \|T_a x - f\| < r + \varepsilon \]
and hence
\[ \inf_s \sup_t \|T_{ts} x - f\| \leq r + \varepsilon. \]
Since $\varepsilon > 0$ is arbitrary, we have
\[ \inf_s \sup_t \|T_{ts} x - f\| \leq r. \]
It is clear that $\inf_s \sup_t \|T_{ts} x - f\| \geq r$. So we have
\[ \inf_s \sup_t \|T_{ts} x - f\| = r. \]

Lemma 3.6 Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $x \in C$, $f \in F(S)$ and $0 < \alpha \leq \beta < 1$. Then for any $\varepsilon > 0$, there exists a closed left ideal $L$ of $S$ such that
\[ \|T_s (\lambda T_t x + (1 - \lambda)f) - (\lambda T_s T_t x + (1 - \lambda)f)\| < \varepsilon \]
for every $s \in S, t \in L$ and $\alpha \leq \lambda \leq \beta$.

Proof. Let $r = \inf_s \|T_s x - f\|$. By Lemma 3.5, for any $d > 0$, there exists $t_0 \in S$ such that
\[ \sup_t \|T_{ts} x - f\| \leq r + d. \]
Apply now Lemma 1 in [18] and let $L = \overline{S \{t_0\}}$.

Let $E$ be a Banach space and let $S$ be a semigroup. Let $\{x_\alpha : \alpha \in S\}$ be a subset of $E$ and $x, y \in E$. Then we write $x_\alpha \to x (\alpha \to \infty_R)$ if for any $\varepsilon > 0$, there exists $\alpha_0 \in S$ such that $\|x_\alpha - x\| < \varepsilon$ for every $\alpha \in S$ (see [23]). We also denote by $[x, y]$ the set $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$.

Lemma 3.7 Let $C$ be a nonempty closed convex subset of a Banach space $E$ with a Fréchet differentiable norm and let $S$ be a semigroup. Let $\{x_\alpha : \alpha \in S\}$ be a bounded subset of $C$. Let $z \in \bigcap_{\alpha} \overline{co}\{x_\alpha x : \alpha \in S\}$, $y \in C$ and $\{y_\alpha : \alpha \in S\}$ be a subset of $C$ with $y_\alpha \in [y, x_\alpha]$ and
\[ \|y_\alpha - z\| = \min\{|u - z| : u \in [y, x_\alpha]\}. \]
If $y_\alpha \to y (\alpha \to \infty_R)$, then $y = z$. 
Proof. Since the duality mapping $J$ of $E$ is single-valued, for each $\alpha \in S$, it follows from [7] that

$$\langle u - y_\alpha, J(y_\alpha - z) \rangle \geq 0$$

for every $u \in [y, x_\alpha]$. Putting $u = x_\alpha$, we have

$$\langle x_\alpha - y_\alpha, J(y_\alpha - z) \rangle \geq 0$$

for every $\alpha \in S$. Since $\{x_\alpha : \alpha \in S\}$ is bounded, there exists $K > 0$ such that $\|x_\alpha - y\| \leq K$ and $\|y_\alpha - z\| \leq K$ for every $\alpha \in S$. Let $\epsilon > 0$ and choose $\delta > 0$ so small that $2\delta K < \epsilon$. Then since the norm of $E$ is Fréchet differentiable, there exists $\delta_0 > 0$ such that $\delta_0 < \delta$ and

$$\|J(u) - J(y - z)\| < \delta$$

for every $u \in E$ with $\|u - (y - z)\| < \delta_0$. Since $y_\alpha \to y (\alpha \to \infty_R)$, there exists $\alpha_0 \in S$ such that

$$\|y_{\alpha_0} - y\| < \delta_0$$

for every $\alpha \in S$. So, for each $\alpha \in S$, we have

$$\begin{align*}
\langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle &- \langle x_{\alpha_0} - y, J(y - z) \rangle \\
\leq &\ |\langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) \rangle| \\
&+ |\langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y - z) \rangle| \\
= &\ |\langle y - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle| + |\langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) - J(y - z) \rangle| \\
\leq &\ |\|y - y_{\alpha_0}\| \|y_{\alpha_0} - z\| + \|x_{\alpha_0} - y\| \|J(y_{\alpha_0} - z) - J(y - z)\| | \\
< &\ \delta_0 K + \delta K < \epsilon
\end{align*}$$

and hence

$$\langle x_{\alpha_0} - y, J(y - z) \rangle - \langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle \geq -\epsilon.$$ 

From $z \in \text{co}\{x_{\alpha_0} : \alpha \in S\}$, we have

$$\langle z - y, J(y - z) \rangle \geq -\epsilon,$$

that is

$$\|y - z\|^2 \leq \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, we have $y = z$. \qed

**Lemma 3.8** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $x \in C$. Assume that $F(S) \neq \emptyset$. Then for $y \in F(S)$ and $y \notin Q(x)$,

$$k = \inf_{x} \|T_s x - y\| > 0.$$ 

**Proof.** Supposing that $k = 0$, by Lemma 3.5,

$$\inf_{s} \sup_{t} \|T_{ts} x - y\| = k = 0.$$
Let $z \in Q(x)$. For each $t \in S$, let $y_t$ be the unique element in $[y, T_t x]$ such that

$$\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.$$ 

So, for any $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$\sup_t \|T_{ts_0} x - y\| < \frac{\varepsilon}{2}$$

and hence we have

$$\|y_{ts_0} - y\| \leq \|y_{ts_0} - T_{ts_0} x\| + \|T_{ts_0} x - y\| < \varepsilon$$

for every $t \in S$, that is, $y_t \to y (t \to \infty_R)$. So by Lemma 3.7, we have $y = z$. This is a contradiction. So we have $k > 0$. $\square$

**Lemma 3.9** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $x \in C$. Then for any $y \in F(S)$ and $z \in Q(x)$, there exists a closed left ideal $L$ of $S$ such that

$$\langle T_t x - y, J(y - z) \rangle \leq 0$$

for every $t \in L$.

**Proof.** If $x = y$ or $y = z$, Lemma 3.9 is obvious. So, let $x \neq y$ and $y \neq z$. For any $t \in S$, define a unique element $y_t$ such that $y_t \in [y, T_t x]$ and

$$\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.$$ 

Then since $y \neq z$, by Lemma 3.7 we have $y_t \not\to y (t \to \infty_R)$. So we obtain $c > 0$ such that for any $t \in S$, there exists $t' \in S$ with $\|y_{tt'} - y\| \geq c$. Setting

$$y_{tt'} = a_{tt'} T_{tt'} x + (1 - a_{tt'}) y, \quad a_{tt'} \in [0, 1],$$

we also obtain $c_0 > 0$ so small that $a_{tt'} \geq c_0$. In fact, since $T_{tt'}$ is nonexpansive and $y \in F(S)$, we have

$$c \leq \|y_{tt'} - y\| = a_{tt'} \|T_{tt'} x - y\| \leq a_{tt'} \|x - y\|.$$ 

So, put $c_0 = c/\|x - y\|$. Let $k = \inf_s \|T_s x - y\|$. By Lemma 3.5 and $y \not\to y (t \to \infty_R)$, we have $k > 0$.

Now, choose $\varepsilon > 0$ so small that

$$(R + \varepsilon) \left(1 - \delta \left(\frac{c_0 k}{R + \varepsilon}\right)\right) < R,$$

where $\delta$ is the modulus of convexity of $E$ and $R = \|z - y\|$. Then by Lemma 3.6, there exists $t_0 \in S$ such that

$$\|T_s (c_0 T_{tt_0} x + (1 - c_0) y) - (c_0 T_s T_{tt_0} x + (1 - c_0) y)\| < \varepsilon$$

$(\ast)$
for every $s, t \in S$. Fix $t_1 \in S$ with $\|y_{t_1t_0} - y\| \geq c$. Then since $a_{t_1t_0} \geq c_0$, we have
\[
c_0T_{t_1t_0}x + (1 - c_0)y = \left(1 - \frac{c_0}{a_{t_1t_0}}\right)y + \frac{c_0}{a_{t_1t_0}}(a_{t_1t_0}T_{t_1t_0}x + (1 - a_{t_1t_0})y)
\]
and hence
\[
\|c_0T_{t_1t_0}x + (1 - c_0)y - z\| \leq \max\{\|y - z\|, \|y_{t_1t_0} - z\|\}
\]
\[
\leq \|y - z\| = R.
\]
By using (*), we obtain
\[
\|c_0T_sT_{t_1t_0}x + (1 - c_0)y - z\| < \|T_s(c_0T_{t_1t_0}x + (1 - c_0)y) - z\| + \varepsilon
\]
\[
\leq \|c_0T_{t_1t_0}x + (1 - c_0)y - z\| + \varepsilon
\]
\[
\leq R + \varepsilon
\]
for every $s \in S$. On the other hand, since $\|y - z\| = R < R + \varepsilon$ and
\[
\|c_0T_sT_{t_1t_0}x + (1 - c_0)y - y\| = c_0\|T_{st_1t_0}x - y\| \geq c_0k
\]
for every $s \in S$, we have, by uniform convexity,
\[
\left\|\frac{1}{2}\left((c_0T_sT_{t_1t_0}x + (1 - c_0)y - z) + (y - z)\right)\right\|
\leq (R + \varepsilon)\left(1 - \delta\left(\frac{c_0k}{R + \varepsilon}\right)\right) < R,
\]
that is
\[
\left\|\frac{c_0}{2}T_sT_{t_1t_0}x + \left(1 - \frac{c_0}{2}\right)y - z\right\| < R
\]
for every $s \in S$. Putting
\[
u_s = \frac{c_0}{2}T_sT_{t_1t_0}x + \left(1 - \frac{c_0}{2}\right)y,
\]
we have
\[
\|u_s + \alpha(y - u_s) - z\| = \|\alpha(y - z) - (\alpha - 1)(u_s - z)\|
\geq \alpha\|y - z\| - (\alpha - 1)\|u_s - z\|
\geq \alpha\|y - z\| - (\alpha - 1)\|y - z\| = \|y - z\|
\]
for every $s \in S$ and $\alpha \geq 1$. So, by Theorem 2.5 in [7], we have
\[
\langle u_s + \alpha(y - u_s) - y, J(y - z)\rangle \geq 0
\]
for every $s \in S$ and $\alpha \geq 1$ and hence
\[
\langle u_s - y, J(y - z)\rangle \leq 0
\]
for every $s \in S$. Therefore we obtain
\[
(T_s T_{t_1 t_0} x - y, J(y - z)) = \frac{2}{c_0} \left( \frac{c_0}{2} T_s T_{t_1 t_0} x - \frac{c_0}{2} y, J(y - z) \right)
\]
\[
= \frac{2}{c_0} (u_s - y, J(y - z)) \leq 0
\]
for every $s \in S$. Let $L = \overline{St_1 t_0}$.

**Lemma 3.10** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $x \in C$. If for any $y, z \in Q(x) \cap F(S)$,
\[
\inf_{L \in \mathcal{L}(S)} \inf_{\phi \in J(y-z)} \sup_{t \in L} \langle T_t x - y, \phi \rangle \leq 0,
\]
then $Q(x) \cap F(S)$ has at most one point.

**Proof.** Let $y, z \in Q(x) \cap F(S)$. Then by convexity of $Q(x) \cap F(S)$, we have $(y + z)/2 \in Q(x) \cap F(S)$. Let $\varepsilon > 0$. By assumption, there exist $L \in \mathcal{L}(S)$ and $\phi \in J((y + z)/2 - z)$ such that
\[
\langle T_t x - \frac{y + z}{2}, \phi \rangle \leq \varepsilon
\]
for every $t \in L$. Since $y \in \overline{co}\{T_t x : t \in L\}$, it follows
\[
\langle y - \frac{y + z}{2}, \phi \rangle \leq \varepsilon
\]
and hence
\[
\frac{1}{2} \langle y - z, \phi \rangle = \frac{1}{2} \| y - z \|^2 \leq \varepsilon.
\]
Since $\varepsilon > 0$ is arbitrary, we have $y = z$. \qed

Combining Lemma 3.9 and Lemma 3.10, we have the following result.

**Theorem 3.11** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $x \in C$. Then $Q(x) \cap F(S)$ contains at most one point.

### 4 Ergodic theorems

We are now ready to prove our main nonlinear ergodic theorems.

**Theorem 4.1** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $S = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup $S$ as nonexpansive mappings from $C$ into $C$. Assume that $F(S) \neq \emptyset$. Then the following are equivalent:

1. For each $x \in C$, the set $Q(x) \cap F(S)$ is nonempty.
(2) There exists a retraction $P$ of $C$ onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. $(1) \Rightarrow (2)$. If for each $x \in C$, the set $Q(x) \cap F(S) \neq \emptyset$, then by Theorem 3.11, $Q(x) \cap F(S)$ contains exactly one point $Px$. Then clearly $P$ is a retraction of $C$ onto $F(S)$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Clearly $T_tP = P$ for every $t \in S$. Also if $u \in S$ and $x \in C$, we have

$$\bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\} \subset \bigcap_{s \in S} \overline{co}\{T_{tsu}x : t \in S\}$$

and hence

$$Q(x) \cap F(S) = Q(T_{u}x) \cap F(S).$$

This implies $PT_t = P$ for every $t \in S$.

$(2) \Rightarrow (1)$. Let $x \in C$. Then it is obvious that $Px \in F(S)$. Since

$$Px = PT_sx \in \overline{co}\{T_Trx : r \in S\} = \overline{co}\{T_{s}x : t \in S\}$$

for every $s \in S$, we have

$$Px \in \bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\} = Q(x). \square$$

**Theorem 4.2** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $S = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup $S$ as nonexpansive mappings from $C$ into $C$. If for each $x \in C$, the set $Q(x) \cap F(S)$ is nonempty, then there exists a nonexpansive retraction $P$ of $C$ onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$.

**Proof.** For each $x \in C$, let $Px$ be the unique element in $Q(x) \cap F(S)$. Then, as in the proof of Theorem 4.1 $(1) \Rightarrow (2)$, $P$ is a retraction of $C$ onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. It remains to show that $P$ is nonexpansive. Let $y \in C$ and $0 < \lambda < 1$. Then as in the proof of Theorem 3.3 we have for any $\epsilon > 0$,

$$q_x((1 - \lambda)Px + \lambda Py) = \sup_{s,t} \inf \|T_{ts}x - ((1 - \lambda)Px + \lambda Py)\|^2$$

$$= \sup_{s,t} \inf \|T_{ts}x - Px + \lambda(Px - Py)\|^2$$

$$= \sup_{s,t} \inf ((\|T_{ts}x - Px\|^2 + 2\lambda\langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2)$$

$$< q_x(Px) + 2\lambda \sup_{s,t} \inf \|T_{ts}x - Px, Px - Py\| + \lambda^2\|Px - Py\|^2 + \epsilon.$$

Since $Px$ is the minimizer of $q_x$, we have

$$2\lambda \sup_{s,t} \inf \|T_{ts}x - Px, Px - Py\| + \lambda^2\|Px - Py\|^2 + \epsilon > q_x((1 - \lambda)Px + \lambda Py) - q_x(Px) \geq 0.$$
Since $\varepsilon > 0$ is arbitrary, we have

$$2\lambda \sup_{s} \inf_{t} \langle T_{ts}x - Px, Px - Py \rangle + \lambda^2 \|Px - Py\|^2 \geq 0$$

and hence

$$2 \sup_{s} \inf_{t} \langle T_{ts}x - Px, Px - Py \rangle \geq -\lambda \|Px - Py\|^2.$$

Now, if $\lambda \to 0$, then

$$\sup_{s} \inf_{t} \langle T_{ts}x - Px, Px - Py \rangle \geq 0.$$

Let $\varepsilon > 0$. Then there exists $u \in S$ such that

$$\langle T_{tu}x - Px, Px - Py \rangle > -\varepsilon$$

for every $t \in S$. For such an element $u \in S$, we also have

$$\sup_{s} \inf_{t} \langle T_{ts}Tu y - PT_{u}y, PT_{u}y - Px \rangle \geq 0$$

and hence there exists $v \in S$ such that

$$\langle T_{tvu}y - PT_{u}y, PT_{u}y - Px \rangle > -\varepsilon$$

for every $t \in S$. Then, from $PT_{u}y = Py$, we have

$$\langle T_{tvu}y - Py, Py - Px \rangle > -\varepsilon$$

for every $t \in S$. Therefore we have

$$-2\varepsilon < \langle Tuuvx - Px, Px - Py \rangle + \langle Tuuvy - Py, Py - Px \rangle$$

$$= \langle Tuuvx - Tuuvy, Px - Py \rangle - \|Px - Py\|^2$$

$$\leq \|Tuuvx - Tuuvy\| \|Px - Py\| - \|Px - Py\|^2$$

$$\leq \|x - y\|\|Px - Py\| - \|Px - Py\|^2.$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$. □

We now proceed to find conditions on $S$ and $E$ such that $Q(x) \cap F(S) \neq \emptyset$ for every $x \in C$.

**Lemma 4.3** [20] Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $S$ be an index set, and let $\{x_t : t \in S\}$ be a bounded set of $H$. Let $X$ be a subspace of $l^\infty(S)$ containing constants, and let $\mu$ be a submean on $X$. Suppose that for each $x \in C$, the real-valued function $f$ on $S$ defined by

$$f(t) = \|x_t - x\|^2$$

for all $t \in S$ belongs to $X$. If

$$r(x) = \mu_t \|x_t - x\|^2$$

for all $x \in C$ and $r = \inf\{r(x) : x \in C\}$, then there exists a unique element $z \in C$ such that $r(z) = r$. Further the following inequality holds:

$$r + \|z - x\|^2 \leq r(x)$$

for every $x \in C$. 

\[155\]
Theorem 4.4 Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $S$ be a semitopological semigroup such that $\text{RUC}(S)$ has an invariant submean. Let $S = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings from $C$ into $C$. Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Then the set $Q(x) \cap F(S)$ is nonempty.

Proof. First we observe that for any $y \in H$, the function $f(t) = \|T_t x - y\|^2$ is in $\text{RUC}(S)$ (see [16]). Let $\mu$ be an invariant submean and define a real-valued function $g$ on $H$ by

$$g(y) = \mu_t \|T_t x - y\|^2$$

for each $y \in H$. If $r = \inf \{g(y) : y \in H\}$, then by Lemma 4.3 there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leq g(y)$$

for every $y \in H$.

For each $s \in S$, let $Q_s$ be the metric projection of $H$ onto $\overline{co}\{T_{ts} x : t \in S\}$. Then by Phelps [22], $Q_s$ is nonexpansive and for each $t \in S$,

$$\|T_{ts} x - Q_s z\|^2 = \|Q_s T_{ts} x - Q_s z\|^2 \leq \|T_{ts} x - z\|^2.$$

So, we have

$$\mu_t \|T_t x - Q_s z\|^2 = \mu_t \|T_{ts} x - Q_s z\|^2 \leq \mu_t \|T_{ts} x - z\|^2$$

and thus $Q_s z = z$. This implies

$$z \in \overline{co}\{T_{ts} x : t \in S\}$$

for all $s \in S$ and hence

$$z \in \bigcap_{s \in S} \overline{co}\{T_{ts} x : t \in S\}.$$

On the other hand, by Lemma 4.3

$$\|z - y\|^2 \leq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2$$

for every $y \in H$.

So, putting $y = T_s z$ for each $s \in S$, we have

$$\|z - T_s z\|^2 \leq \mu_t \|T_{ts} x - T_s z\|^2 - \mu_t \|T_t x - z\|^2$$

$$= \mu_t \|T_{ts} x - T_s z\|^2 - \mu_t \|T_t x - z\|^2$$

$$\leq \mu_t \|T_t x - z\|^2 - \mu_t \|T_t x - z\|^2 = 0.$$

Therefore, we have $T_s z = z$ for every $s \in S$. $\Box$
5 Minimal left ideals

Let $(\Sigma, o)$ be a compact right topological semigroup, i.e., a smigroup and a compact Hausdorff topological space such that for each $\tau \in \Sigma$ the mapping $\gamma \rightarrow \gamma o \tau$ from $\Sigma$ into $\Sigma$ is continuous. In this case, $\Sigma$ must contain minimal left ideals. Any minimal left ideal in $\Sigma$ is closed and any two minimal left ideals of $\Sigma$ are homeomorphic and algebraically isomorphic (see [3]).

**Lemma 5.1** Let $X$ be a nonempty weakly compact convex subset of a Banach space $E$. Let $S = \{T_s : s \in S\}$ be a representation of a semigroup $S$ as affine nonexpansive mappings from $C$ into $C$. Then $S$ is a compact right topological semigroup consisting of nonexpansive mappings from $X$ into $X$. Further, for any $T \in \Sigma$, there exists a sequence $\{T_n\}$ of convex combination of operators from $S$ such that $\|T_n x - T x\| \rightarrow 0$ for every $x \in X$.

**Proof.** It is easy to see that $\Sigma$ is a compact right topological semigroup. We now prove the last statement (which implies that each $T \in \Sigma$ is nonexpansive). Consider $S \subseteq (E, \|\cdot\|)^X$ with the product topology. Let $\Phi = co S$. Then each $T \in \Phi$ is nonexpansive. Hence each $T \in \Phi$ is also nonexpansive. Since the weak topology of the locally convex space $(E, \|\cdot\|)^X$ is the product space $(E, \text{weak})^X$, it follows that $\Sigma \subseteq \Phi^{\text{weak}} = \Phi$, and hence the last statement holds. $\square$

$\Sigma$ is called the enveloping semigroup of $S$.

A subset $X$ of a Banach space $E$ is said to have normal structure if for any bounded (closed) convex subset $W$ of $X$ which contains more than one point, there exists $x \in W$ such that $\sup\{\|x - y\| : y \in W\} < \text{diam}(W)$, where $\text{diam}(W) = \sup\{\|x - y\| : x, y \in W\}$ (see [10] for more details).

**Theorem 5.2** Let $X$ be a nonempty weakly compact convex subset of a Banach space $E$ and $X$ has normal structure. Let $S = \{T_s : s \in S\}$ be a representation of a semigroup
as norm-nonexpansive and weakly continuous mappings from $X$ into $X$ and let $\Sigma$ be the enveloping of $S$. Let $I$ be a minimal left ideal of $\Sigma$ and let $Y$ be a minimal $S$-invariant closed convex subset of $X$. Then there exists a nonempty weakly closed subset $C$ of $Y$ such that $I$ is constant on $C$.

**Proof.** Since $I$ is a minimal left ideal of $\Sigma$ and $\Sigma$ is a compact right topological semigroup (Lemma 5.1), $I = \Sigma e$ for a minimal idempotent $e$ of $\Sigma$ and $G = e\Sigma e$ is a maximal subgroup contained in $I$ (see [3]). Since each $T \in G$ is a nonexpansive mapping from $Y$ into $Y$ (Lemma 5.1), by Broskii-Milman Theorem [4], there exists $x \in Y$ such that $Tx = x$ for every $T \in G$. Now put $C = Ix$. Then $C$ is weakly closed and $S$-invariant. Also if $y_1, y_2 \in C, y_1 = T_1ex, y_2 = T_2ex, T_1, T_2 \in \Sigma$, then, since $eT_1e \in G$, we have

$$(Te)y_1 = Te(T_1ex) = Tx$$

for every $T \in \Sigma$ and similarly

$$(Te)y_2 = Tx$$

for every $T \in \Sigma$. The assertion is proved. □

The following improves the main theorem in [13] for Banach spaces (see also [21]).

**COROLLARY 5.3** Let $\Sigma$ and $X$ be as in Theorem 5.2. Then there exist $T_0 \in \Sigma$ and $x \in X$ such that $T_0Tx = T_0x$ for every $T \in \Sigma$.

**Proof.** Pick $x \in C$ and $T_0 \in I$ of the above theorem. □

**REMARK 5.4** If $S$ is commutative, then for any $T \in \Sigma$ and $s \in S, T_s \circ T = T \circ T_s$, i.e., $z = T_0x$ is in fact a common fixed point for $\Sigma$ (and hence for $S$). Note that if $X$ is norm compact, the weak and norm topology agree on $X$. Hence every nonexpansive mapping from $X$ into $X$ must be weakly continuous. Therefore Corollary 5.3 improves the well known fixed point theorem of De Marr [6] for commuting semigroups of nonexpansive mappings on compact convex sets.

**References**


Department of Mathematical Sciences
University of Alberta,
Edmonton, Alberta, Canada T6G-2G1
and
Department of Information Sciences
Tokyo Institute of Technology,
Oh-okayama, Meguro-ku,
Tokyo 152, Japan.