Nonlinear Ergodic Theorems for Semigroups of Nonexpansive Mappings and Left Ideals

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1 Introduction

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from $S$ to $S$ are continuous. Let $E$ be a uniformly convex Banach space and let $S = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings on a closed convex subset $C$ of $E$ into $C$, i.e., $T_{s+b}x = T_s T_b x$ for every $a, b \in S$ and $x \in C$ and the mapping $(s, x) \mapsto T_s(x)$ from $S \times C$ into $C$ is continuous when $S \times C$ has the product topology. Let $F(S)$ denote the set $\{x \in C : T_s x = x \text{ for all } s \in S\}$ of common fixed points of $S$ in $C$. Then as well known, $F(S)$ (possibly empty) is a closed convex subset of $C$ (see [5]).

In this paper, we shall study the distance between left ideal orbits and elements in the fixed point set $F(S)$. We shall prove (Theorem 3.11) among other things that if $E$ has a Fréchet differentiable norm, then for any semitopological semigroup $S$ and $x \in C$, the set $Q(x) = \bigcap \overline{co}\{T_t x : t \in L\}$, with the intersection taking over all closed left ideals $L$ of $S$, contains at most one common fixed point of $S$ (where $\overline{co}A$ denotes the closed convex hull of $A$). This result is then applied to show (Theorem 4.1) that if $F(S) \cap Q(x) \neq \emptyset$ for any $x \in C$, then there exists a retraction $P$ from $C$ onto $F(S)$ such that $T_t P = P T_t = P$ for every $t \in S$ and $P(x) \in \overline{co}\{T_t x : t \in S\}$ for every $x \in C$. Both Theorem 3.11 and Theorem 4.1 were established by Lau and Takahashi in [18] when $S$ has finite intersection property for closed left ideals.

The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let $C$ be a closed convex subset of a Hilbert space and let $T$ be a nonexpansive mapping of $C$ into itself. If the set $F(T)$ of fixed points of $T$ is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, $P$ is a nonexpansive retraction of $C$ onto $F(T)$ such that $P T = T P = P$ and $P x \in \overline{co}\{T^n x : n = 1, 2, \cdots\}$ for each $x \in C$. In [24], Takahashi proved the existence of such a retraction for an amenable semigroup. This result is further extended to certain Banach spaces by Hirano and Takahashi in [12].

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Our paper is organized as follows: In section 2 we define some terminologies that we use; in section 3 we study the distance between ideals determined by left orbits and the fixed point set; in section 4 we apply our results in section 3 to establish our main nonlinear ergodic theorems; finally in section 5 we study an almost fixed point property determined by the minimal left ideals in the enveloping semigroup of a semigroup of nonexpansive mappings on a weak compact convex set and obtain a generalization of De Marr's fixed point theorem [6].

2 Preliminaries

Throughout this paper, we assume that a Banach (or Hilbert) space is real.

Let $E$ be a Banach space and let $E^*$ be its dual. Then, the value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f \rangle$ or $f(x)$. The duality mapping $J$ of $E$ is a multivalued operator $J : E \rightarrow E^*$ where $J(x) = \{ f \in E^* : \langle x, f \rangle = \| x \| = \| f \| \}$ (which is nonempty by simple application of the Hahn-Banach theorem). Let $B = \{ x \in E : \| x \| = 1 \}$ be the unit sphere of $E$. Then the norm of $E$ is said to be Fréchet differentiable if for each $x \in B$, the limit

$$\lim_{\lambda \rightarrow 0} \frac{\| x + \lambda y \| - \| x \|}{\lambda}$$

is attained uniformly for $y \in B$. In this case, $J$ is a single-valued and norm to norm continuous mapping from $E$ into $E^*$ (see [5] or [8] for more details).

Let $S$ be a nonempty set and let $X$ be a subspace of $l^\infty(S)$ (bounded real-valued functions on $S$) containing constants. By a submean on $X$ we shall mean a real-valued function $\mu$ on $X$ satisfying the following properties:

1. $\mu(f + g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$;
2. $\mu(\alpha f) = \alpha \mu(f)$ for every $f \in X$ and $\alpha \geq 0$;
3. For $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$;
4. $\mu(c) = c$ for every constant function $c$.

A semitopological semigroup $S$ is called left reversible (resp. right reversible) if $S$ has finite intersection property for right (resp. left) ideals. $S$ is called reversible if $S$ is both left and right reversible.

Let $S$ be a semitopological semigroup and let $C(S)$ denote the closed subalgebra of $l^\infty(S)$ consisting of bounded continuous functions. For each $f \in C(S)$ and $a \in S$, let $(l_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$. Let $RUC(S)$ denote all $f \in C(S)$ such that the mapping $S \rightarrow C(S)$ defined by $s \rightarrow r_s f$ is continuous when $C(S)$ has the norm topology. Then $RUC(S)$ is a translation invariant subalgebra of $C(S)$ containing constants. Further, $RUC(S)$ is precisely the space of bounded left uniformly continuous functions on $S$ when $S$ is a group (see [11]).

A submean $\mu$ on $RUC(S)$ is called invariant if $\mu(l_a f) = \mu(r_a f) = \mu(f)$ for every $f \in RUC(S)$ and $a \in S$. If $S$ is a discrete semigroup, then $RUC(S)$ has an invariant submean if and only if $S$ is reversible. Also if $S$ is normal and $C(S)$ has an invariant submean, then $S$ is reversible. However $S$ need not be reversible when $C(S)$ has an invariant submean in general (see [19] for details).
3 Left ideal orbits and the fixed point set

Unless otherwise specified, \( S \) denotes a semitopological semigroup and \( S = \{ T_s : s \in S \} \) a continuous representation of \( S \) as nonexpansive mappings from a nonempty closed convex subset \( C \) of a Banach space \( E \) into \( C \).

Let \( \mathcal{L}(S) \) denote the collection of closed left ideals in \( S \). Assume that \( F(S) \neq \emptyset \). For each \( x \in C \) and \( L \in \mathcal{L}(S) \), define the real-valued function \( q_{x,L} \) on \( F(S) \) by

\[
q_{x,L}(f) = \inf \{ \| T_t x - f \|^2 : t \in L \}
\]

and let

\[
q_{x}(f) = \sup \{ q_{x,L} : L \in \mathcal{L}(S) \}.
\]

Then

\[
q_{x}(f) = \sup_{s} \inf_{t} \| T_{ts} x - f \|^2
\]

as readily checked.

**Lemma 3.1** Let \( C \) be a nonempty closed convex subset of a Banach space \( E \). If \( F(S) \neq \emptyset \), then for each \( x \in C \), \( q_{x} \) is a continuous real-valued function on \( F(S) \) such that \( 0 \leq q_{x}(f) \leq \| x - f \|^2 \) for each \( f \in F(S) \) and \( q_{x}(f_n) \to \infty \) if \( \| f_n \| \to \infty \). Further, if \( F(S) \) is convex, then \( q_{x} \) is a convex function on \( F(S) \).

**Proof.** Since \( 0 \leq \| T_t x - f \|^2 = \| T_t x - T_t f \|^2 \leq \| x - f \|^2 \) for every \( f \in F(S) \) and \( t \in S \), it follows readily that \( 0 \leq q_{x}(f) \leq \| x - f \|^2 \). Also if \( f \in F(S) \) and \( t \in S \), then \( \| T_t x - f \| \leq \| x - f \| \). Hence \( \| T_t x \| \leq \| T_t x - f \| + \| f \| \leq \| x - f \| + \| f \| \), i.e.,

\[
M = \sup \{ \| T_t x \| : t \in S \} < \infty.
\]

Let \( \{ f_n \} \) be a sequence in \( F(S) \) such that \( \| f_n \| \to \infty \). Then we have for each \( t \in S \),

\[
\| T_t x - f_n \|^2 \geq (\| T_t x \| - \| f_n \|)^2
= \| f_n \|^2 - 2 \| T_t x \| \| f_n \| + \| T_t x \|^2
\geq \| f_n \|^2 - 2M \| f_n \|
= \| f_n \|^2 \left( 1 - \frac{2M}{\| f_n \|} \right)
\]

and hence for each \( L \in \mathcal{L}(S) \),

\[
q_{x,L}(f_n) \geq \| f_n \|^2 \left( 1 - \frac{2M}{\| f_n \|} \right) \to \infty.
\]

So we have \( q_{x}(f_n) \to \infty \).

To see that \( q_{x} \) is continuous, let \( \{ f_n \} \) be a sequence in \( F(S) \) converging to some \( f \in F(S) \) and

\[
M' = \sup \{ \| T_t x - f_n \| + \| T_t x - f \| : n = 1, 2, \ldots \text{ and } t \in S \}.
\]

Then

\[
\| T_t x - f_n \|^2 - \| T_t x - f \|^2 \leq (\| T_t x - f_n \| + \| T_t x - f \|) \| T_t x - f_n \| - \| T_t x - f \|
\leq M' \| f_n - f \|,
\]

and

\[
q_{x,L}(f_n) \geq \| f_n \|^2 \left( 1 - \frac{2M}{\| f_n \|} \right) \to \infty.
\]

So we have \( q_{x}(f_n) \to \infty \).
we have for each $L \in \mathcal{L}(S)$,

$$q_{x,L}(f_n) \leq q_{x,L}(f) + M'\|f_n - f\|.$$ 

Similarly, we have

$$q_{x,L}(f) \leq q_{x,L}(f_n) + M'\|f_n - f\|.$$ 

So we obtain

$$|q_{x}(f_n) - q_{x}(f)| \leq M'\|f_n - f\|.$$ 

This implies that $q_{x}$ is continuous on $F(S)$.

If $F(S)$ is convex, for each $f, g \in F(S)$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1, \alpha f + \beta g \in F(S)$.

Let $\epsilon > 0$. Then there exists $L_0 \in \mathcal{L}(S)$ such that

$$\sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_{tx} - f\|^2 + \beta \|T_{tx} - g\|^2) < \inf_{t \in L_0} (\alpha \|T_{tx} - f\|^2 + \beta \|T_{tx} - g\|^2) + \frac{\epsilon}{2}.$$ 

Let $u \in L_0$. Then $Su \subseteq L_0$ and hence

$$\sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_{tx} - f\|^2 + \beta \|T_{tx} - g\|^2) < \inf_{t \in S} (\alpha \|T_{tx} - f\|^2 + \beta \|T_{tx} - g\|^2) + \frac{\epsilon}{2}.$$ 

Moreover, there exist $v, w \in S$ such that

$$\|T_{vu}x - f\|^2 < \inf_{t \in S} \|T_{tvu}x - f\|^2 + \frac{\epsilon}{2}$$

and

$$\|T_{wvu}x - f\|^2 < \inf_{t \in S} \|T_{tvu}x - f\|^2 + \frac{\epsilon}{2}.$$ 

Therefore we obtain

$$q_{x}(\alpha f + \beta g) = \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} \|T_{tx} - (\alpha f + \beta g)\|^2$$

$$\leq \sup_{L \in \mathcal{L}(S)} \inf_{t \in L} (\alpha \|T_{tx} - f\|^2 + \beta \|T_{tx} - g\|^2)$$

$$< \inf_{t \in S} (\alpha \|T_{vu}x - f\|^2 + \beta \|T_{vu}x - g\|^2) + \frac{\epsilon}{2}$$

$$\leq \alpha \|T_{vu}x - f\|^2 + \beta \|T_{vu}x - g\|^2 + \frac{\epsilon}{2}$$

$$< \alpha \inf_{t \in L_1} \|T_{tx} - f\|^2 + \beta \inf_{t \in L_2} \|T_{tx} - g\|^2 + \frac{\alpha \epsilon}{2} + \frac{\beta \epsilon}{2} + \frac{\epsilon}{2}$$

$$= \alpha q_{x}(f) + \beta q_{x}(g) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$q_{x}(\alpha f + \beta g) \leq \alpha q_{x}(f) + \beta q_{x}(g). \square$$
**Theorem 3.2** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Assume that $F(S) \neq \emptyset$. Then for any $x \in C$, there exists a unique element $h \in F(S)$ such that
\[
q_x(h) = \inf \{ q_x(f) : f \in F(S) \}.
\]

**Proof.** Since $E$ is uniformly convex, the fixed point set $F(S)$ in $C$ is closed and convex (see [5]). Hence it follows from Lemma 3.1 and [2] that there exists $h \in F(S)$ such that
\[
q_x(h) = \inf \{ q_x(f) : f \in F(S) \}.
\]
To see that $h$ is unique, let $k \in F(S)$. Then by [27], there exists a strictly increasing and convex function (depending on $h$ and $k$) $g : [0, \infty) \rightarrow [0, \infty)$ such that $g(0) = 0$ and
\[
\| T_t x - (\lambda h + (1 - \lambda)k) \|^2 = \| \lambda(T_t x - h) + (1 - \lambda)(T_t x - k) \|^2 \\
\leq \lambda\| T_t x - h \|^2 + (1 - \lambda)\| T_t x - k \|^2 - \lambda(1 - \lambda)g(\| h - k \|)
\]
for each $t \in S$ and $\lambda$ with $0 \leq \lambda \leq 1$. So we have for each $\lambda$ with $0 \leq \lambda \leq 1$,
\[
q_x(h) \leq q_x(\lambda h + (1 - \lambda)k) \\
\leq \lambda q_x(h) + (1 - \lambda)q_x(k) - \lambda(1 - \lambda)g(\| h - k \|)
\]
and hence
\[
q_x(h) \leq q_x(k) - \lambda g(\| h - k \|).
\]
It follows that
\[
q_x(h) \leq q_x(k) - g(\| h - k \|) \text{ as } \lambda \rightarrow 1.
\]
Since $g$ is strictly increasing, it follows that if $q_x(h) = q_x(k)$, then $h = k$.\(\square\)

We call the unique element $h \in F(S)$ in Theorem 3.2 the minimizer of $q_x$ in $F(S)$. For each $x \in C$, let
\[
Q(x) = \bigcap_{L \in \mathcal{L}(S)} \overline{co}\{T_t x : t \in L\}(= \bigcap_{s \in S} \overline{co}\{T_{ts} x : t \in S\}).
\]

**Theorem 3.3** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $S = \{ T_s : s \in S \}$ be a continuous representation of $S$ as nonexpansive mappings from $C$ into $C$. Then for any $x \in C$, any element in $Q(x) \cap F(S)$ is the unique minimizer of $q_x$ in $F(S)$. In particular, $Q(x) \cap F(S)$ contains at most one point.

**Proof.** Let $z \in F(S)$ be the minimizer of $q_x$ in $F(S)$ and $y \in Q(x) \cap F(S)$. Then for some $\varepsilon > 0$, there exists $u \in S$ such that
\[
\sup \inf_s (\| T_{ts} x - z \|^2 + 2(T_{ts} x - z, z - y) + \| z - y \|^2) \\
< \inf_t (\| T_{tu} x - z \|^2 + 2(T_{tu} x - z, z - y) + \| z - y \|^2) + \frac{\varepsilon}{4}.
\]
Moreover there exist $v, w \in S$ such that
\[
\| T_{vu} x - z \|^2 < \inf_t \| T_{tu} x - z \|^2 + \frac{\varepsilon}{4}
\]
and

\[ \langle T_{wvu}x - z, z - y \rangle \leq \inf_t \langle T_{tvu}x - z, z - y \rangle + \frac{\epsilon}{4}. \]

Therefore we obtain

\[
q_{x}(y) = \sup_s \inf_t \|T_{ts}x - y\|^2 \\
= \sup_s \inf_t (\|T_{ts}x - z\|^2 + 2\langle T_{ts}x - z, z - y \rangle + \|z - y\|^2) \\
< \inf_t (\|T_{tu}x - z\|^2 + 2\langle T_{tvu}x - z, z - y \rangle + \|z - y\|^2 + \frac{\epsilon}{4}) \\
\leq \|T_{vu}x - z\|^2 + 2\langle T_{wvu}x - z, z - y \rangle + \|z - y\|^2 + \frac{\epsilon}{4} \\
< \inf_t \|T_{tu}x - z\|^2 + 2\inf \langle T_{tvu}x - z, z - y \rangle \\
+ \|z - y\|^2 + \frac{\epsilon}{2} + \frac{\epsilon}{4} \\
\leq \sup_s \inf \|T_{ts}x - z\|^2 + 2\sup_s \inf \langle T_{ts}x - z, z - y \rangle \\
+ \|z - y\|^2 + \epsilon \\
= q_{x}(z) + 2\sup \inf \langle T_{ts}x - z, z - y \rangle + \|z - y\|^2 + \epsilon.
\]

This implies

\[
2\sup \inf \langle T_{ts}x - z, z - y \rangle > q_{x}(y) - q_{x}(z) - \|z - y\|^2 - \epsilon \\
\geq -\|z - y\|^2 - \epsilon.
\]

So, there exists \( a \in S \) such that

\[ 2\langle T_{ta}x - z, z - y \rangle > -\|z - y\|^2 - \epsilon \]

for every \( t \in S \). From \( y \in \overline{co}\{T_{ta}x : t \in S\} \), we have

\[ 2\langle y - z, z - y \rangle \geq -\|z - y\|^2 - \epsilon. \]

This inequality implies \( \|z - y\|^2 \leq \epsilon \). Since \( \epsilon > 0 \) is arbitrary, we have \( z = y \). \( \square \)

**Remark 3.4** From Theorem 3.3, it is natural to ask the following:

**Problem 1.** If \( E \) is a uniformly convex Banach space, \( x \in C \) and \( y \in Q(x) \cap F(S) \), is \( y \) always the minimizer of \( q_{x} \) in \( F(S) \)?

**Problem 2.** If \( E \) is a uniformly convex Banach space, does \( Q(x) \cap F(S) \) contain at most one point for each \( x \in C \)?

Clearly, by Theorem 3.2, an affirmative answer for Problem 1 gives an affirmative answer to Problem 2. We now proceed to give an affirmative answer for Problem 2 when \( E \) has a Fréchet differentiable norm.
Lemma 3.5 Let $C$ be a nonempty closed convex subset of a Banach space $E$. Let $x \in C$ and $f \in F(S)$. Then

$$\inf_{s} \| T_{s}x - f \| = \inf_{s} \sup_{t} \| T_{ts}x - f \|.$$ 

Proof. Let $r = \inf_{s} \| T_{s}x - f \|$ and $\epsilon > 0$. Then there exists $a \in S$ such that

$$\| T_{s}x - f \| < r + \epsilon.$$ 

So, for each $t \in S$, we have

$$\| T_{ta}x - f \| \leq \| T_{a}x - f \| < r + \epsilon$$ 

and hence

$$\inf_{s} \sup_{t} \| T_{ts}x - f \| \leq r + \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, we have

$$\inf_{s} \sup_{t} \| T_{ts}x - f \| \leq r.$$ 

It is clear that $\inf_{s} \sup_{t} \| T_{ts}x - f \| \geq r$. So we have

$$\inf_{s} \sup_{t} \| T_{ts}x - f \| = r. \square$$

Lemma 3.6 Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $x \in C$, $f \in F(S)$ and $0 < \alpha \leq \beta < 1$. Then for any $\epsilon > 0$, there exists a closed left ideal $L$ of $S$ such that

$$\| T_{s}(\lambda T_{t}x + (1 - \lambda)f) - (\lambda T_{s}T_{t}x + (1 - \lambda)f) \| < \epsilon$$

for every $s \in S$, $t \in L$ and $\alpha \leq \lambda \leq \beta$.

Proof. Let $r = \inf_{s} \| T_{s}x - f \|$. By Lemma 3.5, for any $d > 0$, there exists $t_{0} \in S$ such that

$$\sup_{t} \| T_{ts}x - f \| \leq r + d.$$ 

Apply now Lemma 1 in [18] and let $L = \overline{St_{0}}. \square$

Let $E$ be a Banach space and let $S$ be a semigroup. Let $\{x_{\alpha} : \alpha \in S\}$ be a subset of $E$ and $x, y \in E$. Then we write $x_{\alpha} \to x (\alpha \to \infty_{R})$ if for any $\epsilon > 0$, there exists $\alpha_{0} \in S$ such that $\| x_{\alpha_{0}} - x \| < \epsilon$ for every $\alpha \in S$ (see [23]). We also denote by $[x, y]$ the set $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$.

Lemma 3.7 Let $C$ be a nonempty closed convex subset of a Banach space $E$ with a Fréchet differentiable norm and let $S$ be a semigroup. Let $\{x_{\alpha} : \alpha \in S\}$ be a bounded subset of $C$. Let $z \in \bigcap_{\alpha} \overline{co}\{x_{\alpha} : \alpha \in S\}$, $y \in C$ and $\{y_{\alpha} : \alpha \in S\}$ be a subset of $C$ with $y_{\alpha} \in [y, x_{\alpha}]$ and

$$\| y_{\alpha} - z \| = \min\{\| u - z \| : u \in [y, x_{\alpha}]\}.$$ 

If $y_{\alpha} \to y (\alpha \to \infty_{R})$, then $y = z$. 


Proof. Since the duality mapping \( J \) of \( E \) is single-valued, for each \( \alpha \in S \), it follows from [7] that
\[
\langle u - y_\alpha, J(y_\alpha - z) \rangle \geq 0
\]
for every \( u \in [y, x_\alpha] \). Putting \( u = x_\alpha \), we have
\[
\langle x_\alpha - y_\alpha, J(y_\alpha - z) \rangle \geq 0
\]
for every \( \alpha \in S \). Since \( \{x_\alpha : \alpha \in S\} \) is bounded, there exists \( K > 0 \) such that \( \|x_\alpha - y\| \leq K \) and \( \|y_\alpha - z\| \leq K \) for every \( \alpha \in S \). Let \( \varepsilon > 0 \) and choose \( \delta > 0 \) so small that \( 2\delta K < \varepsilon \). Then since the norm of \( E \) is Fréchet differentiable, there exists \( \delta_0 > 0 \) such that \( \delta_0 < \delta \) and
\[
\|J(u) - J(y - z)\| < \delta
\]
for every \( u \in E \) with \( \|u - (y - z)\| < \delta_0 \). Since \( y_\alpha \to y (\alpha \to \infty_R) \), there exists \( \alpha_0 \in S \) such that
\[
\|y_{\alpha_0} - y\| < \delta_0
\]
for every \( \alpha \in S \). So, for each \( \alpha \in S \), we have
\[
\begin{align*}
\langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y - z) \rangle &
\leq \langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) \rangle \\
&\quad + \langle x_{\alpha_0} - y, J(y_{\alpha_0} - z) \rangle - \langle x_{\alpha_0} - y, J(y - z) \rangle \\
&\leq \|y - y_{\alpha_0}\| \|y_{\alpha_0} - z\| + \|x_{\alpha_0} - y\| \|J(y_{\alpha_0} - z) - J(y - z)\| \\
&\leq \delta_0 K + \delta K < \varepsilon
\end{align*}
\]
and hence
\[
\langle x_{\alpha_0} - y, J(y - z) \rangle > \langle x_{\alpha_0} - y_{\alpha_0}, J(y_{\alpha_0} - z) \rangle - \varepsilon \geq -\varepsilon.
\]
From \( z \in \overline{co}\{x_\alpha : \alpha \in S\} \), we have
\[
\langle z - y, J(y - z) \rangle \geq -\varepsilon,
\]
that is
\[
\|y - z\|^2 \leq \varepsilon.
\]
Since \( \varepsilon > 0 \) is arbitrary, we have \( y = z \). \( \Box \)

**Lemma 3.8** Let \( C \) be a nonempty closed convex subset of a uniformly convex Banach space \( E \) with a Fréchet differentiable norm. Let \( x \in C \). Assume that \( F(S) \neq \emptyset \). Then for \( y \in F(S) \) and \( y \notin Q(x) \),
\[
k = \inf_{x} \|T_s x - y\| > 0.
\]

**Proof.** Supposing that \( k = 0 \), by Lemma 3.5,
\[
\inf_{x} \sup_{t} \|T_{ts} x - y\| = k = 0.
\]
Let $z \in Q(x)$. For each $t \in S$, let $y_t$ be the unique element in $[y, T_t x]$ such that
\[\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.
\]So, for any $\epsilon > 0$, there exists $s_0 \in S$ such that
\[\sup_t \|T_{s_0} x - y\| < \frac{\epsilon}{2}
\]and hence we have
\[\|y_{t_0} - y\| \leq \|y_{t_0} - T_{t_0} x\| + \|T_{t_0} x - y\| < \epsilon
\]
for every $t \in S$, that is, $y_t \rightarrow y (t \rightarrow \infty_R)$. So by Lemma 3.7, we have $y = z$. This is a contradiction. So we have $k > 0$. □

**Lemma 3.9** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $x \in C$. Then for any $y \in F(S)$ and $z \in Q(x)$, there exists a closed left ideal $L$ of $S$ such that
\[\langle T_t x - y, J(y - z) \rangle \leq 0
\]
for every $t \in L$.

**Proof.** If $x = y$ or $y = z$, Lemma 3.9 is obvious. So, let $x \neq y$ and $y \neq z$. For any $t \in S$, define a unique element $y_t$ such that $y_t \in [y, T_t x]$ and
\[\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.
\]Then since $y \neq z$, by Lemma 3.7 we have $y_t \rightarrow y (t \rightarrow \infty_R)$. So we obtain $c > 0$ such that for any $t \in S$, there exists $t' \in S$ with $\|y_{t'} - y\| \geq c$. Setting
\[y_{t'} = a_{t'} T_{t'} x + (1 - a_{t'}) y, \quad a_{t'} \in [0, 1],
\]we also obtain $c_0 > 0$ so small that $a_{t'} \geq c_0$. In fact, since $T_{t'} x$ is nonexpansive and $y \in F(S)$, we have
\[c \leq \|y_{t'} - y\| = a_{t'} \|T_{t'} x - y\| \leq a_{t'} \|x - y\|
\]So, put $c_0 = c/\|x - y\|$. Let $k = \inf_s \|T_s x - y\|$. By Lemma 3.5 and $y \rightarrow y (t \rightarrow \infty_R)$, we have $k > 0$.

Now, choose $\epsilon > 0$ so small that
\[(R + \epsilon) \left(1 - \delta \left(\frac{c_0 k}{R + \epsilon}\right)\right) < R,
\]
where $\delta$ is the modulus of convexity of $E$ and $R = \|z - y\|$. Then by Lemma 3.6, there exists $t_0 \in S$ such that
\[\|T_s (c_0 T_{t_0} x + (1 - c_0) y) - (c_0 T_s T_{t_0} x + (1 - c_0) y)\| < \epsilon
\]
(*)
for every $s, t \in S$. Fix $t_1 \in S$ with $\|y_{t_1 t_0} - y\| \geq c$. Then since $a_{t_1 t_0} \geq c_0$, we have

$$c_0 T_{t_1 t_0} x + (1 - c_0) y = \left(1 - \frac{c_0}{a_{t_1 t_0}}\right) y + \frac{c_0}{a_{t_1 t_0}} (a_{t_1 t_0} T_{t_1 t_0} x + (1 - a_{t_1 t_0}) y)$$

$$= \left(1 - \frac{c_0}{a_{t_1 t_0}}\right) y + \frac{c_0}{a_{t_1 t_0}} y_{t_1 t_0} \in [y, y_{t_1 t_0}]$$

and hence

$$\|c_0 T_{t_1 t_0} x + (1 - c_0) y - z\| \leq \max\{\|y - z\|, \|y_{t_1 t_0} - z\|\} \leq \|y - z\| = R.$$}

By using $(*)$, we obtain

$$\|c_0 T_{s} T_{t_1 t_0} x + (1 - c_0) y - z\| < \|T_s (c_0 T_{t_1 t_0} x + (1 - c_0) y) - z\| + \varepsilon$$

$$\leq \|c_0 T_{t_1 t_0} x + (1 - c_0) y - z\| + \varepsilon$$

$$\leq R + \varepsilon$$

for every $s \in S$. On the other hand, since $\|y - z\| = R < R + \varepsilon$ and

$$\|c_0 T_{s} T_{t_1 t_0} x + (1 - c_0) y - y\| = c_0 \|T_{s t_1 t_0} x - y\| \geq c_0 k$$

for every $s \in S$, we have, by uniform convexity,

$$\left\| \frac{1}{2} ((c_0 T_{s} T_{t_1 t_0} x + (1 - c_0) y - z) + (y - z)) \right\|$$

$$\leq (R + \varepsilon) \left(1 - \delta \left(\frac{c_0 k}{R + \varepsilon}\right)\right) < R,$$

that is

$$\left\| \frac{c_0}{2} T_{s} T_{t_1 t_0} x + \left(1 - \frac{c_0}{2}\right) y - z \right\| < R$$

for every $s \in S$. Putting

$$u_s = \frac{c_0}{2} T_{s} T_{t_1 t_0} x + \left(1 - \frac{c_0}{2}\right) y,$$

we have

$$\|u_s + \alpha(y - u_s) - z\| \geq \alpha\|y - z\| - (\alpha - 1) \|u_s - z\|$$

for every $s \in S$ and $\alpha \geq 1$. So, by Theorem 2.5 in [7], we have

$$\langle u_s + \alpha(y - u_s) - y, J(y - z) \rangle \geq 0$$

for every $s \in S$ and $\alpha \geq 1$ and hence

$$\langle u_s - y, J(y - z) \rangle \leq 0$$
for every $s \in S$. Therefore we obtain
\[
\langle T_s T_{t_1 t_0} x - y, J(y - z) \rangle = \frac{2}{c_0} \left( \frac{c_0}{2} T_s T_{t_1 t_0} x - \frac{c_0}{2} y, J(y - z) \right) = \frac{2}{c_0} (u_s - y, J(y - z)) \leq 0
\]
for every $s \in S$. Let $L = \overline{St_1 t_0}$. \square

**Lemma 3.10** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $x \in C$. If for any $y, z \in Q(x) \cap F(S)$,
\[
\inf_{L \in L(S)} \inf_{\phi \in J(y - z)} \sup_{t \in L} \langle T_t x - y, \phi \rangle \leq 0,
\]
then $Q(x) \cap F(S)$ has at most one point.

**Proof.** Let $y, z \in Q(x) \cap F(S)$. Then by convexity of $Q(x) \cap F(S)$, we have $(y + z)/2 \in Q(x) \cap F(S)$. Let $\epsilon > 0$. By assumption, there exist $L \in L(S)$ and $\phi \in J((y + z)/2 - z)$ such that
\[
\langle T_t x - \frac{y + z}{2}, \phi \rangle \leq \epsilon
\]
for every $t \in L$. Since $y \in \overline{co}\{T_t x : t \in L\}$, it follows
\[
\langle y - \frac{y + z}{2}, \phi \rangle \leq \epsilon
\]
and hence
\[
\frac{1}{2} \langle y - z, \phi \rangle = \frac{1}{2} \|y - z\|^2 \leq \epsilon.
\]
Since $\epsilon > 0$ is arbitrary, we have $y = z$. \square
Combining Lemma 3.9 and Lemma 3.10, we have the following result.

**Theorem 3.11** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $x \in C$. Then $Q(x) \cap F(S)$ contains at most one point.

## 4 Ergodic theorems

We are now ready to prove our main nonlinear ergodic theorems.

**Theorem 4.1** Let $C$ be a nonempty closed convex subset of a uniformly convex Banach space $E$ with a Fréchet differentiable norm. Let $S = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup $S$ as nonexpansive mappings from $C$ into $C$. Assume that $F(S) \neq \emptyset$. Then the following are equivalent:

1. For each $x \in C$, the set $Q(x) \cap F(S)$ is nonempty.
(2) There exists a retraction $P$ of $C$ onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. (1) $\Rightarrow$ (2). If for each $x \in C$, the set $Q(x) \cap F(S) \neq \emptyset$, then by Theorem 3.11, $Q(x) \cap F(S)$ contains exactly one point $Px$. Then clearly $P$ is a retraction of $C$ onto $F(S)$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Clearly $T_tP = P$ for every $t \in S$. Also if $u \in S$ and $x \in C$, we have

\[
\bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\} \subset \bigcap_{s \in S} \overline{co}\{T_{tsu}x : t \in S\}
\]

and hence

\[
Q(x) \cap F(S) = Q(T_u x) \cap F(S).
\]

This implies $PT_t = P$ for every $t \in S$.

(2) $\Rightarrow$ (1). Let $x \in C$. Then it is obvious that $Px \in F(S)$. Since

\[
Px = PT_sx \in \overline{co}\{T_sT_tx : t \in S\} = \overline{co}\{T_{ts}x : t \in S\}
\]

for every $s \in S$, we have

\[
Px \in \bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\} = Q(x). \square
\]

THEOREM 4.2 Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $S = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup $S$ as nonexpansive mappings from $C$ into $C$. If for each $x \in C$, the set $Q(x) \cap F(S)$ is nonempty, then there exists a nonexpansive retraction $P$ of $C$ onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. For each $x \in C$, let $Px$ be the unique element in $Q(x) \cap F(S)$. Then, as in the proof of Theorem 4.1 (1) $\Rightarrow$ (2), $P$ is a retraction of $C$ onto $F(S)$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. It remains to show that $P$ is nonexpansive. Let $y \in C$ and $0 < \lambda < 1$. Then as in the proof of Theorem 3.3 we have for any $\varepsilon > 0$,

\[
q_x((1 - \lambda)Px + \lambda Py)
\]

\[
= \sup_{s} \inf_{t} \|T_{ts}x - ((1 - \lambda)Px + \lambda Py)\|^2
\]

\[
= \sup_{s} \inf_{t} \|T_{ts}x - Px + \lambda(Px - Py)\|^2
\]

\[
= \sup_{s} \inf_{t} (\|T_{ts}x - Px\|^2 + 2\lambda\langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2)
\]

\[
< q_x(Px) + 2\lambda \sup_{s} \inf_{t} \langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2 + \varepsilon.
\]

Since $Px$ is the minimizer of $q_x$, we have

\[
2\lambda \sup_{s} \inf_{t} \langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2 + \varepsilon
\]

\[
> q_x((1 - \lambda)Px + \lambda Py) - q_x(Px) \geq 0.
\]
Since $\varepsilon > 0$ is arbitrary, we have

$$2\lambda \sup_s \inf_t (T_{ts}x - Px, Px - Py) + \lambda^2 \|Px - Py\|^2 \geq 0$$

and hence

$$2 \sup_s \inf_t (T_{ts}x - Px, Px - Py) \geq -\lambda \|Px - Py\|^2.$$

Now, if $\lambda \to 0$, then

$$\sup_s \inf_t (T_{ts}x - Px, Px - Py) \geq 0.$$

Let $\varepsilon > 0$. Then there exists $u \in S$ such that

$$\langle T_{tu}x - Px, Px - Py \rangle > -\varepsilon$$

for every $t \in S$. For such an element $u \in S$, we also have

$$\sup_s \inf_t (T_{ts}Tu - PT_u y, PT_u y - Px) \geq 0$$

and hence there exists $v \in S$ such that

$$\langle T_{tvu}y - PT_u y, PT_u y - Px \rangle > -\varepsilon$$

for every $t \in S$. Then, from $PT_u y = Py$, we have

$$\langle T_{tvu}y - Py, Py - Px \rangle > -\varepsilon$$

for every $t \in S$. Therefore we have

$$-2\varepsilon < \langle Tuuvx - Px, Px - Py \rangle + \langle Tuuvy - Py, Py - Px \rangle$$

$$= \langle Tuuvx - Tuuvy - (Px - Py), Px - Py \rangle$$

$$= \langle Tuuvx - Tuuvy, Px - Py \rangle - \|Px - Py\|^2$$

$$\leq \|Tuuvx - Tuuvy\| \|Px - Py\| - \|Px - Py\|^2$$

$$\leq \|x - y\| \|Px - Py\| - \|Px - Py\|^2.$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$. □

We now proceed to find conditions on $S$ and $E$ such that $Q(x) \cap F(S) \neq \emptyset$ for every $x \in C$.

**Lemma 4.3 [20]** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $S$ be an index set, and let $\{x_t : t \in S\}$ be a bounded set of $H$. Let $X$ be a subspace of $l^{\infty}(S)$ containing constants, and let $\mu$ be a submean on $X$. Suppose that for each $x \in C$, the real-valued function $f$ on $S$ defined by

$$f(t) = \|x_t - x\|^2 \text{ for all } t \in S$$

belongs to $X$. If

$$r(x) = \mu_t \|x_t - x\|^2 \text{ for all } x \in C$$

and $r = \inf \{r(x) : x \in C\}$, then there exists a unique element $z \in C$ such that $r(z) = r$. Further the following inequality holds:

$$r + \|z - x\|^2 \leq r(x) \text{ for every } x \in C.$$
Theorem 4.4 Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $S$ be a semitopological semigroup such that $RUC(S)$ has an invariant submean. Let $S = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings from $C$ into $C$. Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Then the set $Q(x) \cap F(S)$ is nonempty.

Proof. First we observe that for any $y \in H$, the function $f(t) = \|T_t x - y\|^2$ is in $RUC(S)$ (see [16]). Let $\mu$ be an invariant submean and define a real-valued function $g$ on $H$ by

$$g(y) = \mu_t \|T_t x - y\|^2$$

for each $y \in H$.

If $r = \inf\{g(y) : y \in H\}$, then by Lemma 4.3 there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leq g(y) \text{ for every } y \in H.$$ 

For each $s \in S$, let $Q_s$ be the metric projection of $H$ onto $\overline{co}\{T_{ts} x : t \in S\}$. Then by Phelps [22], $Q_s$ is nonexpansive and for each $t \in S$,

$$\|T_{ts} x - Q_s z\|^2 = \|Q_s T_{ts} x - Q_s z\|^2 \leq \|T_{ts} x - z\|^2.$$ 

So, we have

$$\mu_t \|T_t x - Q_s z\|^2 = \mu_t \|T_t x - Q_s z\|^2$$

$$\leq \mu_t \|T_{ts} x - z\|^2$$

$$= \mu_t \|T_t x - z\|^2$$ 

and thus $Q_s z = z$. This implies

$$z \in \overline{co}\{T_{ts} x : t \in S\} \text{ for all } s \in S$$

and hence

$$z \in \bigcap_{s \in S} \overline{co}\{T_{ts} x : t \in S\}.$$ 

On the other hand, by Lemma 4.3

$$\|z - y\|^2 \leq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2 \text{ for every } y \in H.$$ 

So, putting $y = T_s z$ for each $s \in S$, we have

$$\|z - T_s z\|^2 \leq \mu_t \|T_t x - T_s z\|^2 - \mu_t \|T_t x - z\|^2$$

$$= \mu_t \|T_{ts} x - T_s z\|^2 - \mu_t \|T_t x - z\|^2$$

$$\leq \mu_t \|T_t x - z\|^2 - \mu_t \|T_t x - z\|^2 = 0.$$ 

Therefore, we have $T_s z = z$ for every $s \in S$. □
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as norm-nonexpansive and weakly continuous mappings from $X$ into $X$ and let $\Sigma$ be the enveloping of $S$. Let $I$ be a minimal left ideal of $\Sigma$ and let $Y$ be a minimal $S$-invariant closed convex subset of $X$. Then there exists a nonempty weakly closed subset $C$ of $Y$ such that $I$ is constant on $C$.

Proof. Since $I$ is a minimal left ideal of $\Sigma$ and $\Sigma$ is a compact right topological semigroup (Lemma 5.1), $I = \Sigma e$ for a minimal idempotent $e$ of $\Sigma$ and $G = e\Sigma e$ is a maximal subgroup contained in $I$ (see [3]). Since each $T \in G$ is a nonexpansive mapping from $Y$ into $Y$ (Lemma 5.1), by Broskii-Milman Theorem [4], there exists $x \in Y$ such that $Tx = x$ for every $T \in G$. Now put $C = Ix$. Then $C$ is weakly closed and $S$-invariant. Also if $y_1, y_2 \in C, y_1 = T_1 ex, y_2 = T_2 ex, T_1, T_2 \in \Sigma$, then, since $eT_1 e \in G$, we have

$$(Te)y_1 = Te(T_1 e x) = Tx$$

for every $T \in \Sigma$ and similarly

$$(Te)y_2 = T x$$

for every $T \in \Sigma$. The assertion is proved. □

The following improves the main theorem in [13] for Banach spaces (see also [21]).

**COROLLARY 5.3** Let $\Sigma$ and $X$ be as in Theorem 5.2. Then there exist $T_0 \in \Sigma$ and $x \in X$ such that $T_0 Tx = T_0 x$ for every $T \in \Sigma$.

Proof. Pick $x \in C$ and $T_0 \in I$ of the above theorem. □

**REMARK 5.4** If $S$ is commutative, then for any $T \in \Sigma$ and $s \in S, T_s \circ T = T \circ T_s$, i.e., $z = T_0 x$ is in fact a common fixed point for $\Sigma$ (and hence for $S$). Note that if $X$ is norm compact, the weak and norm topology agree on $X$. Hence every nonexpansive mapping from $X$ into $X$ must be weakly continuous. Therefore Corollary 5.3 improves the well known fixed point theorem of De Marr [6] for commuting semigroups of nonexpansive mappings on compact convex sets.

**References**


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