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Nonlinear Ergodic Theorems for Semigroups of Nonexpansive Mappings and Left Ideals

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1 Introduction

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $s \to a \cdot s$ and $s \to s \cdot a$ from S to S are continuous. Let E be a uniformly convex Banach space and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings on a closed convex subset C of E into C, i.e., $T_{ab}x = T_aT_bx$ for every $a, b \in S$ and $x \in C$ and the mapping $(s, x) \to T_s(x)$ from $S \times C$ into C is continuous when $S \times C$ has the product topology. Let $F(\mathcal{S})$ denote the set $\{x \in C : T_sx = x \text{ for all } s \in S\}$ of common fixed points of $\mathcal{S}$ in C. Then as well known, $F(\mathcal{S})$ (possibly empty) is a closed convex subset of C (see [5]).
In this paper, we shall study the distance between left ideal orbits and elements in the fixed point set $F(\mathcal{S})$. We shall prove (Theorem 3.11) among other things that if E has a Fréchet differentiable norm, then for any semitopological semigroup S and $x \in C$, the set $Q(x) = \bigcap \overline{co}\{T_tx : t \in L\}$, with the intersection taking over all closed left ideals L of S, contains at most one common fixed point of $\mathcal{S}$ (where $\overline{co}A$ denotes the closed convex hull of A). This result is then applied to show (Theorem 4.1) that if $F(\mathcal{S}) \cap Q(x) \neq \emptyset$ for any $x \in C$, then there exists a retraction P from C onto $F(\mathcal{S})$ such that $T_tP = PT_t = P$ for every $t \in S$ and $P(x) \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Both Theorem 3.11 and Theorem 4.1 were established by Lau and Takahashi in [18] when S has finite intersection property for closed left ideals.
The first nonlinear ergodic theorem for nonexpansive mappings was established in 1975 by Baillon [1]: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and $Px \in \overline{co}\{T^nx : n = 1, 2, \cdots\}$ for each $x \in C$. In [24], Takahashi proved the existence of such a retraction for an amenable semigroup. This result is further extended to certain Banach spaces by Hirano and Takahashi in [12].

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Our paper is organized as follows : In section 2 we define some terminologies that we use ; in section 3 we study the distance between ideals determined by left orbits and the fixed point set ; in section 4 we apply our results in section 3 to establish our main nonlinear ergodic theorems ; finally in section 5 we study an almost fixed point property determined by the minimal left ideals in the enveloping semigroup of a semigroup of nonexpansive mappings on a weak compact convex set and obtain a generalization of De Marr's fixed point theorem [6].

2 Preliminaries

Throughout this paper, we assume that a Banach (or Hilbert) space is real.
Let E be a Banach space and let E^* be its dual. Then, the value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$ or $f(x)$. The duality mapping J of E is a multivalued operator $J : E \to E^*$ where $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$ (which is nonempty by simple application of the Hahn-Banach theorem). Let $B = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the norm of E is said to be Fréchet differentiable if for each $x \in B$, the limit

$$\lim_{\lambda\to 0} \frac{\|x+\lambda y\| - \|x\|}{\lambda}$$

is attained uniformly for $y \in B$. In this case, J is a single-valued and norm to norm continuous mapping from E into E^* (see [5] or [8] for more details).
Let S be a nonempty set and let X be a subspace of $l^\infty(S)$ (bounded real-valued functions on S) containing constants. By a submean on X we shall mean a real-valued function μ on X satisfying the following properties:

(1) $\mu(f+g) \le \mu(f) + \mu(g)$ for every $f, g \in X$;

(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \ge 0$;

(3) For $f, g \in X, f \le g$ implies $\mu(f) \le \mu(g)$;

(4) $\mu(c) = c$ for every constant function c.

A semitopological semigroup S is called left reversible (resp. right reversible) if S has finite intersection property for right (resp. left) ideals. S is called reversible if S is both left and right reversible.
Let S be a semitopological semigroup and let $C(S)$ denote the closed subalgebra of $l^\infty(S)$ consisting of bounded continuous functions. For each $f \in C(S)$ and $a \in S$, let $(l_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$. Let $RUC(S)$ denote all $f \in C(S)$ such that the mapping $S \to C(S)$ defined by $s \to r_s f$ is continuous when $C(S)$ has the norm topology. Then $RUC(S)$ is a translation invariant subalgebra of $C(S)$ containing constants. Further, $RUC(S)$ is precisely the space of bounded left uniformly continuous functions on S when S is a group (see [11]).
A submean μ on $RUC(S)$ is called invariant if $\mu(l_a f) = \mu(r_a f) = \mu(f)$ for every $f \in RUC(S)$ and $a \in S$. If S is a discrete semigroup, then $RUC(S)$ has an invariant submean if and only if S is reversible. Also if S is normal and $C(S)$ has an invariant submean, then S is reversible. However S need not be reversible when $C(S)$ has an invariant submean in general (see [19] for details).

3 Left ideal orbits and the fixed point set

Unless otherwise specified, S denotes a semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ a continuous representation of S as nonexpansive mappings from a nonempty closed convex subset C of a Banach space E into C.
Let $\mathcal{L}(S)$ denote the collection of closed left ideals in S. Assume that $F(\mathcal{S}) \neq \emptyset$. For each $x \in C$ and $L \in \mathcal{L}(S)$, define the real-valued function $q_{x,L}$ on $F(\mathcal{S})$ by

$$q_{x,L}(f) = \inf\{\|T_t x - f\|^2 : t \in L\}$$

and let

$$q_x(f) = \sup\{q_{x,L} : L \in \mathcal{L}(S)\}.$$

Then

$$q_x(f) = \sup_s \inf_t \|T_{ts}x - f\|^2$$

as readily checked.

LEMMA 3.1 *Let C be a nonempty closed convex subset of a Banach space E. If $F(\mathcal{S}) \neq \emptyset$, then for each $x \in C$, q_x is a continuous real-valued function on $F(\mathcal{S})$ such that $0 \leq q_x(f) \leq \|x - f\|^2$ for each $f \in F(\mathcal{S})$ and $q_x(f_n) \to \infty$ if $\|f_n\| \to \infty$. Further, if $F(\mathcal{S})$ is convex, then q_x is a convex function on $F(\mathcal{S})$.*

Proof. Since $0 \leq \|T_t x - f\|^2 = \|T_t x - T_t f\|^2 \leq \|x - f\|^2$ for every $f \in F(\mathcal{S})$ and $t \in S$, it follows readily that $0 \leq q_x(f) \leq \|x - f\|^2$. Also if $f \in F(\mathcal{S})$ and $t \in S$, then $\|T_t x - f\| \leq \|x - f\|$. Hence $\|T_t x\| \leq \|T_t x - f\| + \|f\| \leq \|x - f\| + \|f\|$, i.e., $M = \sup\{\|T_t x\| : t \in S\} < \infty$. Let $\{f_n\}$ be a sequence in $F(\mathcal{S})$ such that $\|f_n\| \to \infty$. Then we have for each $t \in S$,

$$\begin{aligned} \|T_t x - f_n\|^2 &\geq (\|T_t x\| - \|f_n\|)^2 \\ &= \|f_n\|^2 - 2\|T_t x\|\|f_n\| + \|T_t x\|^2 \\ &\geq \|f_n\|^2 - 2M\|f_n\| \\ &= \|f_n\|^2 \left(1 - \frac{2M}{\|f_n\|}\right) \end{aligned}$$

and hence for each $L \in \mathcal{L}(S)$,

$$q_{x,L}(f_n) \geq \|f_n\|^2 \left(1 - \frac{2M}{\|f_n\|}\right) \to \infty.$$

So we have $q_x(f_n) \to \infty$.
To see that q_x is continuous, let $\{f_n\}$ be a sequence in $F(\mathcal{S})$ converging to some $f \in F(\mathcal{S})$ and

$$M' = \sup\{\|T_t x - f_n\| + \|T_t x - f\| : n = 1, 2, \cdots \text{ and } t \in S\}.$$

Then since

$$\begin{aligned} \|T_t x - f_n\|^2 - \|T_t x - f\|^2 &\leq (\|T_t x - f_n\| + \|T_t x - f\|)\,|\,\|T_t x - f_n\| - \|T_t x - f\|\,| \\ &\leq M'\|f_n - f\|, \end{aligned}$$

we have for each $L \in \mathcal{L}(S)$,

$$q_{x,L}(f_n) \le q_{x,L}(f) + M'\|f_n - f\|.$$

Similarly, we have

$$q_{x,L}(f) \le q_{x,L}(f_n) + M'\|f_n - f\|.$$

So we obtain

$$|q_x(f_n) - q_x(f)| \le M'\|f_n - f\|.$$

This implies that q_x is continuous on $F(\mathcal{S})$.
If $F(\mathcal{S})$ is convex, for each $f, g \in F(\mathcal{S})$ and $\alpha, \beta \ge 0$ with $\alpha + \beta = 1, \alpha f + \beta g \in F(\mathcal{S})$.
Let $\varepsilon > 0$. Then there exists $L_0 \in \mathcal{L}(S)$ such that

$$\sup_{L\in\mathcal{L}(S)} \inf_{t\in L}(\alpha\|T_t x - f\|^2 + \beta\|T_t x - g\|^2) < \inf_{t\in L_0}(\alpha\|T_t x - f\|^2 + \beta\|T_t x - g\|^2) + \frac{\varepsilon}{2}.$$

Let $u \in L_0$. Then $Su \subseteq L_0$ and hence

$$\sup_{L\in\mathcal{L}(S)} \inf_{t\in L}(\alpha\|T_t x - f\|^2 + \beta\|T_t x - g\|^2) < \inf_{t\in S}(\alpha\|T_{tu} x - f\|^2 + \beta\|T_{tu} x - g\|^2) + \frac{\varepsilon}{2}.$$

Moreover, there exist $v, w \in S$ such that

$$\|T_{vu}x - f\|^2 < \inf_{t\in S}\|T_{tu}x - f\|^2 + \frac{\varepsilon}{2}$$

and

$$\|T_{wvu}x - f\|^2 < \inf_{t\in S}\|T_{tvu}x - f\|^2 + \frac{\varepsilon}{2}.$$

Therefore we obtain

$$\begin{aligned}
q_x(\alpha f + \beta g) &= \sup_{L\in\mathcal{L}(S)} \inf_{t\in L}\|T_t x - (\alpha f + \beta g)\|^2 \\
&\le \sup_{L\in\mathcal{L}(S)} \inf_{t\in L}(\alpha\|T_t x - f\|^2 + \beta\|T_t x - g\|^2) \\
&< \inf_{t\in S}(\alpha\|T_{tu} x - f\|^2 + \beta\|T_{tu} x - g\|^2) + \frac{\varepsilon}{2} \\
&\le \alpha\|T_{wvu}x - f\|^2 + \beta\|T_{wvu}x - g\|^2 + \frac{\varepsilon}{2} \\
&\le \alpha\|T_{vu}x - f\|^2 + \beta\|T_{wvu}x - g\|^2 + \frac{\varepsilon}{2} \\
&< \alpha \inf_{t\in S}\|T_{tu}x - f\|^2 + \beta \inf_{t\in S}\|T_{tvu}x - g\|^2 + \frac{\alpha\varepsilon}{2} + \frac{\beta\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \alpha \inf_{t\in L_1}\|T_t x - f\|^2 + \beta \inf_{t\in L_2}\|T_t x - g\|^2 + \varepsilon \\
&\quad (\text{ where } L_1 = \overline{Su} \text{ and } L_2 = \overline{Svu}) \\
&\le \alpha q_x(f) + \beta q_x(g) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$q_x(\alpha f + \beta g) \le \alpha q_x(f) + \beta q_x(g).\square$$

THEOREM 3.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Assume that $F(\mathcal{S}) \neq \emptyset$. Then for any $x \in C$, there exists a unique element $h \in F(\mathcal{S})$ such that*

$$q_x(h) = \inf\{q_x(f) : f \in F(\mathcal{S})\}.$$

Proof. Since E is uniformly convex, the fixed point set $F(\mathcal{S})$ in C is closed and convex (see [5]). Hence it follows from Lemma 3.1 and [2] that there exists $h \in F(\mathcal{S})$ such that

$$q_x(h) = \inf\{q_x(f) : f \in F(\mathcal{S})\}.$$

To see that h is unique, let $k \in F(\mathcal{S})$. Then by [27], there exists a strictly increasing and convex function (depending on h and k) $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\begin{aligned} \|T_t x - (\lambda h + (1-\lambda)k)\|^2 &= \|\lambda(T_t x - h) + (1-\lambda)(T_t x - k)\|^2 \\ &\leq \lambda\|T_t x - h\|^2 + (1-\lambda)\|T_t x - k\|^2 - \lambda(1-\lambda)g(\|h-k\|) \end{aligned}$$

for each $t \in S$ and λ with $0 \leq \lambda \leq 1$. So we have for each λ with $0 \leq \lambda \leq 1$,

$$\begin{aligned} q_x(h) &\leq q_x(\lambda h + (1-\lambda)k) \\ &\leq \lambda q_x(h) + (1-\lambda)q_x(k) - \lambda(1-\lambda)g(\|h-k\|) \end{aligned}$$

and hence

$$q_x(h) \leq q_x(k) - \lambda g(\|h-k\|).$$

It follows that

$$q_x(h) \leq q_x(k) - g(\|h-k\|) \text{ as } \lambda \to 1.$$

Since g is strictly increasing, it follows that if $q_x(h) = q_x(k)$, then $h = k$.□

We call the unique element $h \in F(\mathcal{S})$ in Theorem 3.2 the minimizer of q_x in $F(\mathcal{S})$. For each $x \in C$, let

$$Q(x) = \bigcap_{L \in \mathcal{L}(S)} \overline{co}\{T_t x : t \in L\} (= \bigcap_{s \in S} \overline{co}\{T_{ts} x : t \in S\}).$$

THEOREM 3.3 *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings from C into C. Then for any $x \in C$, any element in $Q(x) \cap F(\mathcal{S})$ is the unique minimizer of q_x in $F(\mathcal{S})$. In particular, $Q(x) \cap F(\mathcal{S})$ contains at most one point.*

Proof. Let $z \in F(\mathcal{S})$ be the minimizer of q_x in $F(\mathcal{S})$ and $y \in Q(x) \cap F(\mathcal{S})$. Then for some $\varepsilon > 0$, there exists $u \in S$ such that

$$\begin{aligned} &\sup_s \inf_t (\|T_{ts}x - z\|^2 + 2\langle T_{ts}x - z, z - y\rangle + \|z-y\|^2) \\ &\quad < \inf_t(\|T_{tu}x - z\|^2 + 2\langle T_{tu}x - z, z - y\rangle + \|z-y\|^2) + \frac{\varepsilon}{4}. \end{aligned}$$

Moreover there exist $v, w \in S$ such that

$$\|T_{vu}x - z\|^2 < \inf_t \|T_{tu}x - z\|^2 + \frac{\varepsilon}{4}$$

and

$$\langle T_{wvu}x - z, z - y\rangle < \inf_t \langle T_{tvu}x - z, z - y\rangle + \frac{\varepsilon}{4}.$$

Therefore we obtain

$$\begin{aligned}
q_x(y) &= \sup_s \inf_t \|T_{ts}x - y\|^2 \\
&= \sup_s \inf_t \left(\|T_{ts}x - z\|^2 + 2\langle T_{ts}x - z, z - y\rangle + \|z - y\|^2\right) \\
&< \inf_t\left(\|T_{tu}x - z\|^2 + 2\langle T_{tu}x - z, z - y\rangle + \|z - y\|^2\right) + \frac{\varepsilon}{4} \\
&\leq \|T_{wvu}x - z\|^2 + 2\langle T_{wvu}x - z, z - y\rangle + \|z - y\|^2 + \frac{\varepsilon}{4} \\
&\leq \|T_{vu}x - z\|^2 + 2\langle T_{wvu}x - z, z - y\rangle + \|z - y\|^2 + \frac{\varepsilon}{4} \\
&< \inf_t \|T_{tu}x - z\|^2 + 2\inf_t \langle T_{tvu}x - z, z - y\rangle \\
&\quad + \|z - y\|^2 + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\
&\leq \sup_s \inf_t \|T_{ts}x - z\|^2 + 2\sup_s \inf_t \langle T_{ts}x - z, z - y\rangle \\
&\quad + \|z - y\|^2 + \varepsilon \\
&= q_x(z) + 2\sup_s \inf_t \langle T_{ts}x - z, z - y\rangle + \|z - y\|^2 + \varepsilon.
\end{aligned}$$

This implies

$$\begin{aligned}
2\sup_s \inf_t \langle T_{ts}x - z, z - y\rangle &> q_x(y) - q_x(z) - \|z - y\|^2 - \varepsilon \\
&\geq -\|z - y\|^2 - \varepsilon.
\end{aligned}$$

So, there exists $a \in S$ such that

$$2\langle T_{ta}x - z, z - y\rangle > -\|z - y\|^2 - \varepsilon$$

for every $t \in S$. From $y \in \overline{co}\{T_{ta}x : t \in S\}$, we have

$$2\langle y - z, z - y\rangle \geq -\|z - y\|^2 - \varepsilon.$$

This inequality implies $\|z - y\|^2 \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $z = y$. □

REMARK 3.4 *From Theorem 3.3, it is natural to ask the following:*
Problem 1. If E is a uniformly convex Banach space, $x \in C$ and $y \in Q(x) \cap F(\mathcal{S})$, is y always the minimizer of q_x in $F(\mathcal{S})$?
Problem 2. If E is a uniformly convex Banach space, does $Q(x) \cap F(\mathcal{S})$ contain at most one point for each $x \in C$?

Clearly, by Theorem 3.2, an affirmative answer for Problem 1 gives an affirmative answer to Problem 2. We now proceed to give an affirmative answer for Problem 2 when E has a Fréchet differentiable norm.

LEMMA 3.5 *Let C be a nonempty closed convex subset of a Banach space E. Let $x \in C$ and $f \in F(\mathcal{S})$. Then*

$$\inf_s \|T_s x - f\| = \inf_s \sup_t \|T_{ts} x - f\|.$$

Proof. Let $r = \inf_s \|T_s x - f\|$ and $\varepsilon > 0$. Then there exists $a \in S$ such that

$$\|T_a x - f\| < r + \varepsilon.$$

So, for each $t \in S$, we have

$$\|T_{ta} x - f\| \leq \|T_a x - f\| < r + \varepsilon$$

and hence

$$\begin{aligned} \inf_s \sup_t \|T_{ts} x - f\| &\leq \sup_t \|T_{ta} x - f\| \\ &\leq r + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\inf_s \sup_t \|T_{ts} x - f\| \leq r.$$

It is clear that $\inf_s \sup_t \|T_{ts} x - f\| \geq r$. So we have

$$\inf_s \sup_t \|T_{ts} x - f\| = r. \Box$$

LEMMA 3.6 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $x \in C, f \in F(\mathcal{S})$ and $0 < \alpha \leq \beta < 1$. Then for any $\varepsilon > 0$, there exists a closed left ideal L of S such that*

$$\|T_s(\lambda T_t x + (1-\lambda) f) - (\lambda T_s T_t x + (1-\lambda) f)\| < \varepsilon$$

for every $s \in S, t \in L$ and $\alpha \leq \lambda \leq \beta$.

Proof. Let $r = \inf_s \|T_s x - f\|$. By Lemma 3.5, for any $d > 0$, there exists $t_0 \in S$ such that

$$\sup_t \|T_{tt_0} x - f\| \leq r + d.$$

Apply now Lemma 1 in [18] and let $L = \overline{St_0}$.$\Box$
Let E be a Banach space and let S be a semigroup. Let $\{x_\alpha : \alpha \in S\}$ be a subset of E and $x, y \in E$. Then we write $x_\alpha \to x \, (\alpha \to \infty_R)$ if for any $\varepsilon > 0$, there exists $\alpha_0 \in S$ such that $\|x_{\alpha\alpha_0} - x\| < \varepsilon$ for every $\alpha \in S$ (see [23]). We also denote by $[x, y]$ the set $\{\lambda x + (1-\lambda) y : 0 \leq \lambda \leq 1\}$.

LEMMA 3.7 *Let C be a nonempty closed convex subset of a Banach space E with a Fréchet differentiable norm and let S be a semigroup. Let $\{x_\alpha : \alpha \in S\}$ be a bounded subset of C. Let $z \in \bigcap_\beta \overline{co}\{x_{\alpha\beta} : \alpha \in S\}$, $y \in C$ and $\{y_\alpha : \alpha \in S\}$ be a subset of C with $y_\alpha \in [y, x_\alpha]$ and*

$$\|y_\alpha - z\| = \min\{\|u - z\| : u \in [y, x_\alpha]\}.$$

If $y_\alpha \to y \, (\alpha \to \infty_R)$, then $y = z$.

Proof. Since the duality mapping J of E is single-valued, for each $\alpha \in S$, it follows from [7] that

$$\langle u - y_\alpha, J(y_\alpha - z)\rangle \geq 0$$

for every $u \in [y, x_\alpha]$. Putting $u = x_\alpha$, we have

$$\langle x_\alpha - y_\alpha, J(y_\alpha - z)\rangle \geq 0$$

for every $\alpha \in S$. Since $\{x_\alpha : \alpha \in S\}$ is bounded, there exists $K > 0$ such that $\|x_\alpha - y\| \leq K$ and $\|y_\alpha - z\| \leq K$ for every $\alpha \in S$. Let $\varepsilon > 0$ and choose $\delta > 0$ so small that $2\delta K < \varepsilon$. Then since the norm of E is Fréchet differentiable, there exists $\delta_0 > 0$ such that $\delta_0 < \delta$ and

$$\|J(u) - J(y - z)\| < \delta$$

for every $u \in E$ with $\|u - (y - z)\| < \delta_0$. Since $y_\alpha \to y\,(\alpha \to \infty_R)$, there exists $\alpha_0 \in S$ such that

$$\|y_{\alpha\alpha_0} - y\| < \delta_0$$

for every $\alpha \in S$. So, for each $\alpha \in S$, we have

$$\begin{aligned}
&|\langle x_{\alpha\alpha_0} - y_{\alpha\alpha_0}, J(y_{\alpha\alpha_0} - z)\rangle - \langle x_{\alpha\alpha_0} - y, J(y - z)\rangle| \\
&\leq\ |\langle x_{\alpha\alpha_0} - y_{\alpha\alpha_0}, J(y_{\alpha\alpha_0} - z)\rangle - \langle x_{\alpha\alpha_0} - y, J(y_{\alpha\alpha_0} - z)\rangle| \\
&\quad + |\langle x_{\alpha\alpha_0} - y, J(y_{\alpha\alpha_0} - z)\rangle - \langle x_{\alpha\alpha_0} - y, J(y - z)\rangle| \\
&=\ |\langle y - y_{\alpha\alpha_0}, J(y_{\alpha\alpha_0} - z)\rangle| + |\langle x_{\alpha\alpha_0} - y, J(y_{\alpha\alpha_0} - z) - J(y - z)\rangle| \\
&\leq\ \|y - y_{\alpha\alpha_0}\|\|y_{\alpha\alpha_0} - z\| + \|x_{\alpha\alpha_0} - y\|\|J(y_{\alpha\alpha_0} - z) - J(y - z)\| \\
&<\ \delta_0 K + \delta K < \varepsilon
\end{aligned}$$

and hence

$$\langle x_{\alpha\alpha_0} - y, J(y - z)\rangle > \langle x_{\alpha\alpha_0} - y_{\alpha\alpha_0}, J(y_{\alpha\alpha_0} - z)\rangle - \varepsilon \geq -\varepsilon.$$

From $z \in \overline{co}\{x_{\alpha\alpha_0} : \alpha \in S\}$, we have

$$\langle z - y, J(y - z)\rangle \geq -\varepsilon,$$

that is

$$\|y - z\|^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $y = z$. □

LEMMA 3.8 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $x \in C$. Assume that $F(\mathcal{S}) \neq \emptyset$. Then for $y \in F(\mathcal{S})$ and $y \notin Q(x)$,*

$$k = \inf_s \|T_s x - y\| > 0.$$

Proof. Supposing that $k = 0$, by Lemma 3.5,

$$\inf_s \sup_t \|T_{ts} x - y\| = k = 0.$$

Let $z \in Q(x)$. For each $t \in S$, let y_t be the unique element in $[y, T_t x]$ such that

$$\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.$$

So, for any $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$\sup_t \|T_{ts_0} x - y\| < \frac{\varepsilon}{2}$$

and hence we have

$$\begin{aligned} \|y_{ts_0} - y\| &\leq \|y_{ts_0} - T_{ts_0} x\| + \|T_{ts_0} x - y\| \\ &\leq 2\|T_{ts_0} x - y\| < \varepsilon \end{aligned}$$

for every $t \in S$, that is, $y_t \to y\,(t \to \infty_R)$. So by Lemma 3.7, we have $y = z$. This is a contradiction. So we have $k > 0$. □

LEMMA 3.9 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $x \in C$. Then for any $y \in F(\mathcal{S})$ and $z \in Q(x)$, there exists a closed left ideal L of S such that*

$$\langle T_t x - y, J(y - z)\rangle \leq 0$$

for every $t \in L$.

Proof. If $x = y$ or $y = z$, Lemma 3.9 is obvious. So, let $x \neq y$ and $y \neq z$. For any $t \in S$, define a unique element y_t such that $y_t \in [y, T_t x]$ and

$$\|y_t - z\| = \min\{\|u - z\| : u \in [y, T_t x]\}.$$

Then since $y \neq z$, by Lemma 3.7 we have $y_t \not\to y\,(t \to \infty_R)$. So we obtain $c > 0$ such that for any $t \in S$, there exists $t' \in S$ with $\|y_{t't} - y\| \geq c$. Setting

$$y_{t't} = a_{t't} T_{t't} x + (1 - a_{t't}) y, \quad a_{t't} \in [0, 1],$$

we also obtain $c_0 > 0$ so small that $a_{t't} \geq c_0$. In fact, since $T_{t't}$ is nonexpansive and $y \in F(\mathcal{S})$, we have

$$c \leq \|y_{t't} - y\| = a_{t't}\|T_{t't} x - y\| \leq a_{t't}\|x - y\|.$$

So, put $c_0 = c/\|x - y\|$. Let $k = \inf_s \|T_s x - y\|$. By Lemma 3.5 and $y_t \not\to y\,(t \to \infty_R)$, we have $k > 0$.
Now, choose $\varepsilon > 0$ so small that

$$(R + \varepsilon)\left(1 - \delta\left(\frac{c_0 k}{R + \varepsilon}\right)\right) < R,$$

where δ is the modulus of convexity of E and $R = \|z - y\|$. Then by Lemma 3.6, there exists $t_0 \in S$ such that

$$\|T_s(c_0 T_{tt_0} x + (1 - c_0) y) - (c_0 T_s T_{tt_0} x + (1 - c_0) y)\| < \varepsilon \qquad (*)$$

for every $s, t \in S$. Fix $t_1 \in S$ with $\|y_{t_1t_0} - y\| \geq c$. Then since $a_{t_1t_0} \geq c_0$, we have

$$\begin{aligned} c_0T_{t_1t_0}x + (1-c_0)y &= \left(1 - \frac{c_0}{a_{t_1t_0}}\right)y + \frac{c_0}{a_{t_1t_0}}(a_{t_1t_0}T_{t_1t_0}x + (1-a_{t_1t_0})y) \\ &= \left(1 - \frac{c_0}{a_{t_1t_0}}\right)y + \frac{c_0}{a_{t_1t_0}}y_{t_1t_0} \in [y, y_{t_1t_0}] \end{aligned}$$

and hence

$$\begin{aligned} \|c_0T_{t_1t_0}x + (1-c_0)y - z\| &\leq \max\{\|y-z\|, \|y_{t_1t_0} - z\|\} \\ &\leq \|y-z\| = R. \end{aligned}$$

By using (*), we obtain

$$\begin{aligned} \|c_0T_sT_{t_1t_0}x + (1-c_0)y - z\| &< \|T_s(c_0T_{t_1t_0}x + (1-c_0)y) - z\| + \varepsilon \\ &\leq \|c_0T_{t_1t_0}x + (1-c_0)y - z\| + \varepsilon \\ &\leq R + \varepsilon \end{aligned}$$

for every $s \in S$. On the other hand, since $\|y - z\| = R < R + \varepsilon$ and

$$\|c_0T_sT_{t_1t_0}x + (1-c_0)y - y\| = c_0\|T_{st_1t_0}x - y\| \geq c_0k$$

for every $s \in S$, we have, by uniform convexity,

$$\begin{aligned} &\left\|\frac{1}{2}\left((c_0T_sT_{t_1t_0}x + (1-c_0)y - z) + (y - z)\right)\right\| \\ &\leq (R+\varepsilon)\left(1 - \delta\left(\frac{c_0k}{R+\varepsilon}\right)\right) < R, \end{aligned}$$

that is

$$\left\|\frac{c_0}{2}T_sT_{t_1t_0}x + \left(1 - \frac{c_0}{2}\right)y - z\right\| < R$$

for every $s \in S$. Putting

$$u_s = \frac{c_0}{2}T_sT_{t_1t_0}x + \left(1 - \frac{c_0}{2}\right)y,$$

we have

$$\begin{aligned} \|u_s + \alpha(y - u_s) - z\| &= \|\alpha(y-z) - (\alpha-1)(u_s - z)\| \\ &\geq \alpha\|y-z\| - (\alpha-1)\|u_s - z\| \\ &\geq \alpha\|y-z\| - (\alpha-1)\|y-z\| = \|y-z\| \end{aligned}$$

for every $s \in S$ and $\alpha \geq 1$. So, by Theorem 2.5 in [7], we have

$$\langle u_s + \alpha(y - u_s) - y, J(y-z)\rangle \geq 0$$

for every $s \in S$ and $\alpha \geq 1$ and hence

$$\langle u_s - y, J(y-z)\rangle \leq 0$$

for every $s \in S$. Therefore we obtain

$$\begin{aligned}\langle T_sT_{t_1t_0}x - y, J(y-z)\rangle &= \frac{2}{c_0}\left\langle \frac{c_0}{2}T_sT_{t_1t_0}x - \frac{c_0}{2}y, J(y-z)\right\rangle \\ &= \frac{2}{c_0}\langle u_s - y, J(y-z)\rangle \le 0\end{aligned}$$

for every $s \in S$. Let $L = \overline{St_1t_0}$. □

LEMMA 3.10 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $x \in C$. If for any $y, z \in Q(x) \cap F(\mathcal{S})$,*

$$\inf_{L\in\mathcal{L}(S)} \inf_{\phi\in J(y-z)} \sup_{t\in L}\langle T_tx - y, \phi\rangle \le 0,$$

then $Q(x) \cap F(\mathcal{S})$ has at most one point.

Proof. Let $y, z \in Q(x)\cap F(\mathcal{S})$. Then by convexity of $Q(x)\cap F(\mathcal{S})$, we have $(y+z)/2 \in Q(x) \cap F(\mathcal{S})$. Let $\varepsilon > 0$. By assumption, there exist $L \in \mathcal{L}(S)$ and $\phi \in J((y+z)/2 - z)$ such that

$$\left\langle T_tx - \frac{y+z}{2}, \phi\right\rangle \le \varepsilon$$

for every $t \in L$. Since $y \in \overline{co}\{T_tx : t \in L\}$, it follows

$$\left\langle y - \frac{y+z}{2}, \phi\right\rangle \le \varepsilon$$

and hence

$$\frac{1}{2}\langle y - z, \phi\rangle = \frac{1}{2}\|y - z\|^2 \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $y = z$. □

Combining Lemma 3.9 and Lemma 3.10, we have the following result.

THEOREM 3.11 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $x \in C$. Then $Q(x) \cap F(\mathcal{S})$ contains at most one point.*

4 Ergodic theorems

We are now ready to prove our main nonlinear ergodic theorems.

THEOREM 4.1 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings from C into C. Assume that $F(\mathcal{S}) \neq \emptyset$. Then the following are equivalent:*

(1) For each $x \in C$, the set $Q(x) \cap F(\mathcal{S})$ is nonempty.

(2) There exists a retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$.

Proof. (1) $\Rightarrow$ (2). If for each $x \in C$, the set $Q(x) \cap F(\mathcal{S}) \neq \emptyset$, then by Theorem 3.11, $Q(x) \cap F(\mathcal{S})$ contains exactly one point Px. Then clearly P is a retraction of C onto $F(\mathcal{S})$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. Clearly $T_tP = P$ for every $t \in S$. Also if $u \in S$ and $x \in C$, we have

$$\bigcap_{s\in S} \overline{co}\{T_{ts}x : t \in S\} \subset \bigcap_{s\in S} \overline{co}\{T_{tsu}x : t \in S\}$$

and hence

$$Q(x) \cap F(\mathcal{S}) = Q(T_ux) \cap F(\mathcal{S}).$$

This implies $PT_t = P$ for every $t \in S$.
(2) $\Rightarrow$ (1). Let $x \in C$. Then it is obvious that $Px \in F(\mathcal{S})$. Since

$$Px = PT_sx \in \overline{co}\{T_tT_sx : t \in S\} = \overline{co}\{T_{ts} : t \in S\}$$

for every $s \in S$, we have

$$Px \in \bigcap_{s\in S} \overline{co}\{T_{ts}x : t \in S\} = Q(x).\Box$$

THEOREM 4.2 *Let C be a nonempty closed convex subset of a Hilbert space H and let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of a semitopological semigroup S as nonexpansive mappings from C into C. If for each $x \in C$, the set $Q(x) \cap F(\mathcal{S})$ is nonempty, then there exists a nonexpansive retraction P of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$.*

Proof. For each $x \in C$, let Px be the unique element in $Q(x) \cap F(\mathcal{S})$. Then, as in the proof of Theorem 4.1 (1) $\Rightarrow$ (2), P is a retraction of C onto $F(\mathcal{S})$ such that $PT_t = T_tP = P$ for every $t \in S$ and $Px \in \overline{co}\{T_tx : t \in S\}$ for every $x \in C$. It remains to show that P is nonexpansive. Let $y \in C$ and $0 < \lambda < 1$. Then as in the proof of Theorem 3.3 we have for any $\varepsilon > 0$,

$$\begin{aligned}
&q_x((1-\lambda)Px + \lambda Py) \\
&= \sup_s \inf_t \|T_{ts}x - ((1-\lambda)Px + \lambda Py)\|^2 \\
&= \sup_s \inf_t \|T_{ts}x - Px + \lambda(Px - Py)\|^2 \\
&= \sup_s \inf_t (\|T_{ts}x - Px\|^2 + 2\lambda\langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2) \\
&< q_x(Px) + 2\lambda \sup_s \inf_t \langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2 + \varepsilon.
\end{aligned}$$

Since Px is the minimizer of q_x, we have

$$\begin{aligned}
&2\lambda \sup_s \inf_t \langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2 + \varepsilon \\
&> q_x((1-\lambda)Px + \lambda Py) - q_x(Px) \geq 0.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$2\lambda \sup_s \inf_t \langle T_{ts}x - Px, Px - Py\rangle + \lambda^2\|Px - Py\|^2 \geq 0$$

and hence

$$2 \sup_s \inf_t \langle T_{ts}x - Px, Px - Py\rangle \geq -\lambda\|Px - Py\|^2.$$

Now, if $\lambda \to 0$, then

$$\sup_s \inf_t \langle T_{ts}x - Px, Px - Py\rangle \geq 0.$$

Let $\varepsilon > 0$. Then there exists $u \in S$ such that

$$\langle T_{tu}x - Px, Px - Py\rangle > -\varepsilon$$

for every $t \in S$. For such an element $u \in S$, we also have

$$\sup_s \inf_t \langle T_{ts}T_u y - PT_u y, PT_u y - Px\rangle \geq 0$$

and hence there exists $v \in S$ such that

$$\langle T_{tvu}y - PT_u y, PT_u y - Px\rangle > -\varepsilon$$

for every $t \in S$. Then, from $PT_u y = Py$, we have

$$\langle T_{tvu}y - Py, Py - Px\rangle > -\varepsilon$$

for every $t \in S$. Therefore we have

$$\begin{aligned} -2\varepsilon &< \langle T_{uvu}x - Px, Px - Py\rangle + \langle T_{uvu}y - Py, Py - Px\rangle \\ &= \langle T_{uvu}x - T_{uvu}y - (Px - Py), Px - Py\rangle \\ &= \langle T_{uvu}x - T_{uvu}y, Px - Py\rangle - \|Px - Py\|^2 \\ &\leq \|T_{uvu}x - T_{uvu}y\|\|Px - Py\| - \|Px - Py\|^2 \\ &\leq \|x - y\|\|Px - Py\| - \|Px - Py\|^2. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies $\|Px - Py\| \leq \|x - y\|$. □

We now proceed to find conditions on S and E such that $Q(x) \cap F(\mathcal{S}) \neq \emptyset$ for every $x \in C$.

LEMMA 4.3 *[20] Let C be a nonempty closed convex subset of a Hilbert space H, let S be an index set, and let $\{x_t : t \in S\}$ be a bounded set of H. Let X be a subspace of $l^\infty(S)$ containing constants, and let μ be a submean on X. Suppose that for each $x \in C$, the real-valued function f on S defined by*

$$f(t) = \|x_t - x\|^2 \textit{ for all } t \in S$$

belongs to X. If

$$r(x) = \mu_t\|x_t - x\|^2 \textit{ for all } x \in C$$

and $r = \inf\{r(x) : x \in C\}$, then there exists a unique element $z \in C$ such that $r(z) = r$. Further the following inequality holds :

$$r + \|z - x\|^2 \leq r(x) \textit{ for every } x \in C.$$

THEOREM 4.4 *Let C be a nonempty closed convex subset of a Hilbert space H and let S be a semitopological semigroup such that $RUC(S)$ has an invariant submean. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as nonexpansive mappings from C into C. Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Then the set $Q(x) \cap F(\mathcal{S})$ is nonempty.*

Proof. First we observe that for any $y \in H$, the function $f(t) = \|T_t x - y\|^2$ is in $RUC(S)$ (see [16]). Let μ be an invariant submean and define a real-valued function g on H by

$$g(y) = \mu_t \|T_t x - y\|^2 \text{ for each } y \in H.$$

If $r = \inf\{g(y) : y \in H\}$, then by Lemma 4.3 there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leq g(y) \text{ for every } y \in H.$$

For each $s \in S$, let Q_s be the metric projection of H onto $\overline{co}\{T_{ts}x : t \in S\}$. Then by Phelps [22], Q_s is nonexpansive and for each $t \in S$,

$$\|T_{ts}x - Q_s z\|^2 = \|Q_s T_{ts} x - Q_s z\|^2 \leq \|T_{ts}x - z\|^2.$$

So, we have

$$\begin{aligned} \mu_t \|T_t x - Q_s z\|^2 &= \mu_t \|T_{ts} x - Q_s z\|^2 \\ &\leq \mu_t \|T_{ts} x - z\|^2 \\ &= \mu_t \|T_t x - z\|^2 \end{aligned}$$

and thus $Q_s z = z$. This implies

$$z \in \overline{co}\{T_{ts}x : t \in S\} \text{ for all } s \in S$$

and hence

$$z \in \bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\}.$$

On the other hand, by Lemma 4.3

$$\|z - y\|^2 \leq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2 \text{ for every } y \in H.$$

So, putting $y = T_s z$ for each $s \in S$, we have

$$\begin{aligned} \|z - T_s z\|^2 &\leq \mu_t \|T_t x - T_s z\|^2 - \mu_t \|T_t x - z\|^2 \\ &= \mu_t \|T_{st} x - T_s z\|^2 - \mu_t \|T_t x - z\|^2 \\ &\leq \mu_t \|T_t x - z\|^2 - \mu_t \|T_t x - z\|^2 = 0. \end{aligned}$$

Therefore, we have $T_s z = z$ for every $s \in S$. □

PROPOSITION 4.5 *Let S be a discrete reversible semigroup and let C be a nonempty weakly compact convex subset of a Banach space E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as affine nonexpansive mappings from C into C. Then for each $x \in C$, the set $Q(x) \cap F(\mathcal{S})$ is nonempty.*

Proof. Let $x \in C$. Clearly $Q(x) = \bigcap_{s \in S} \overline{co}\{T_{ts}x : t \in S\}$ is compact and convex. Also $Q(x)$ is nonempty. Indeed, for each $s \in S$, let $K_s = \overline{co}\{T_{ts}x : t \in S\}$. Then $\{K_s : s \in S\}$ are weakly closed subsets of C with finite intersection property: If $s_1, \ldots, s_n \in S$, choose $t_0 \in \bigcap_{i=1}^n Ss_i$. Then $T_{t_0}x \in \bigcap_{i=1}^n K_{s_i}$. Consequently $\bigcap_{s \in S} K_s = Q(x)$ is nonempty. Let $a, s \in S, y \in Q(x)$ and let $\{y_k\}$ be a sequence in K_s such that $\|y_k - y\| \to 0$. Then $T_a y_k \in K_s$ (by affiness and continuity of T_a) and $\|T_a y_k - T_a y\| \leq \|y_k - y\| \to 0$. Hence $T_a y \in K_s$. Consequently $T_a y \in \bigcap_{s \in S} K_s = Q(x)$. Hence $Q(x)$ is $\mathcal{S}$-invariant.
Now since S is left reversible, by [14] the space $WAP(S)$ of weakly almost periodic functions on S has a left invariant mean (see also [17]). Hence by [15], there exists a common fixed point in $Q(x)$, i.e., $Q(x) \cap F(\mathcal{S}) \neq \emptyset$. □

5 Minimal left ideals

Let $(\Sigma, \circ)$ be a compact right topological semigroup, i.e., a smigroup and a compact Hausdorff topological space such that for each $\tau \in \Sigma$ the mapping $\gamma \to \gamma \circ \tau$ from Σ into Σ is continuous. In this case, Σ must contain minimal left ideals. Any minimal left ideal in Σ is closed and any two minimal left ideals of Σ are homeomorphic and algebraically isomorphic (see [3]).

LEMMA 5.1 *Let X be a nonempty weakly compact convex subset of a Banach space E. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup S as nonexpansive and weak-weak continuous mappings from X into X. Let Σ be the closure of $\mathcal{S}$ in the product space $(X, weak)^X$. Then Σ is a compact right topological semigroup consisting of nonexpansive mappings from X into X. Further, for any $T \in \Sigma$, there exists a sequence $\{T_n\}$ of convex combination of operators from $\mathcal{S}$ such that $\|T_n x - Tx\| \to 0$ for every $x \in X$.*

Proof. It is easy to see that Σ is a compact right topological semigroup. We now prove the last statement (which implies that each $T \in \Sigma$ is nonexpansive). Consider $\mathcal{S} \subseteq (E, \|\cdot\|)^X$ with the product topology. Let $\Phi = co\mathcal{S}$. Then each $T \in \Phi$ is nonexpansive. Hence each $T \in \overline{\Phi}$ is also nonexpansive. Since the weak topology of the locally convex space $(E, \|\cdot\|)^X$ is the product space $(E, \text{weak})^X$, it follows that $\Sigma \subseteq \overline{\Phi}^{\text{weak}} = \overline{\Phi}$, and hence the last statement holds. □.
Σ is called the enveloping semigroup of $\mathcal{S}$.
A subset X of a Banach space E is said to have normal structure if for any bounded (closed) convex subset W of X which contains more than one point, there exists $x \in W$ such that $\sup\{\|x - y\| : y \in W\} < \text{diam}(W)$, where $\text{diam}(W) = \sup\{\|x - y\| : x, y \in W\}$ (see [10] for more details).

THEOREM 5.2 *Let X be a nonempty weakly compact convex subset of a Banach space E and X has normal structure. Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a semigroup* *as norm-nonexpansive and weakly continuous mappings from X into X and let Σ be the enveloping of $\mathcal{S}$. Let I be a minimal left ideal of Σ and let Y be a minimal $\mathcal{S}$-invariant closed convex subset of X. Then there exists a nonempty weakly closed subset C of Y such that I is constant on C.*

Proof. Since I is a minimal left ideal of Σ and Σ is a compact right topological semigroup (Lemma 5.1), $I = \Sigma e$ for a minimal idempotent e of Σ and $G = e\Sigma e$ is a maximal subgroup contained in I (see [3]). Since each $T \in G$ is a nonexpansive mapping from Y into Y (Lemma 5.1), by Broskii-Milman Theorem [4], there exists $x \in Y$ such that $Tx = x$ for every $T \in G$. Now put $C = Ix$. Then C is weakly closed and $\mathcal{S}$-invariant. Also if $y_1, y_2 \in C, y_1 = T_1ex, y_2 = T_2ex, T_1, T_2 \in \Sigma$, then, since $eT_1e \in G$, we have

$$(Te)y_1 = Te(T_1ex) = Tx$$

for every $T \in \Sigma$ and similarly

$$(Te)y_2 = Tx$$

for every $T \in \Sigma$. The assertion is proved. □
The following improves the main theorem in [13] for Banach spaces (see also [21]).

COROLLARY 5.3 *Let Σ and X be as in Theorem 5.2. Then there exist $T_0 \in \Sigma$ and $x \in X$ such that $T_0Tx = T_0x$ for every $T \in \Sigma$.*

Proof. Pick $x \in C$ and $T_0 \in I$ of the above theorem. □

REMARK 5.4 *If S is commutative, then for any $T \in \Sigma$ and $s \in S, T_s \circ T = T \circ T_s$, i.e., $z = T_0x$ is in fact a common fixed point for Σ (and hence for $\mathcal{S}$). Note that if X is norm compact, the weak and norm topology agree on X. Hence every nonexpansive mapping from X into X must be weakly continuous. Therefore Corollary 5.3 improves the well known fixed point theorem of De Marr [6] for commuting semigroups of nonexpansive mappings on compact convex sets.*

References

[1] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert,* C.R. Acad. Sci. Paris Sér. A-B, **280**(1975), 1511-1514.

[2] V. Barbu and Th. Precupanu, *Convexity and Optimization in Banach spaces,* Editura Academiei R. S. R.,Bucuresti, 1978.

[3] J. F. Berglund, H. Q. Junghenn and P. Milnes, *Analysis on semigroups,* John Wiley 2 Sons, New York, 1989.

[4] M. S. Broskii and D. P. Milman, *On the center of a convex set,* Dokl. Akad. Nauk SSSR, **59**, 837-840.

[5] F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* Proc. Sympos. Pure Math., **18**, no. 2, Amer. Math. Soc., Providence, R. I., 1976.

[6] R. De Marr, *Common fixed points for commuting contraction mappings,* Pacific J. Math., **13**(1963), 1139-1141.

[7] F. R. Deutsch and P. H. Maserick, *Applications of the Hahn-Banach theorem in approximation theory,* SIAM Rev., **9**(1967), 516-530.

[8] J. Diestel, *Geometry of Banach spaces, Selected Topics,* Lecture notes in mathematics, #485, Springer Verlag, Berlin-Heidelberg, New York, 1975.

[9] N. Dunford and J. T. Schwartz, *Linear Operators I,* Interscience, New York, 1958.

[10] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory,* Cambridge Shedies in Advanced Mathematics, **28**(1990).

[11] E. Hewitt and K. Ross, *Abstract harmonic analysis I,* Springer Verlag, Berlin-Gottingen-Heidelberg, 1963.

[12] N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for an amenable semigroup of nonexpansive mappings in a Banach space,* Pacific J. Math., **112**(1984), 333-346.

[13] R. D. Holmes and A. T. Lau, *Almost fixed points of semigroups of nonexpansive mappings,* Shedia Math., **XL111**(1972), 217-218.

[14] R. Hsu, *Topics on weakly almost periodic functions,* Ph. D. Thesis, SUNY, Buffalo, 1985.

[15] A. T. Lau, *Some fixed point theorems and their applications to W^*-algebras* in *Fixed Point Theory and its Applications,* edited by S. Swamirathan, Academic Press(1976), 121-129.

[16] A. T. Lau, *Semigroup of nonexpansive mappings on a Hilbert space,* J. Math. Anal. Appl., **105**(1985), 514-522.

[17] A. T. Lau, *Amenability and fixed point property for semigroup of nonexpansive mappings* in *Fixed Point Theory and Applcations,* edited by J. B. Baillon and M. A. Théra, Pitman Research Notes in Mathematics Series, **252**(1991), Longman Scientific and Technology, Essex, UK.

[18] A. T. Lau and W. Takahashi, *Weak convergence and non-linear ergodic theorems for reversible semigroups of nonexpansive mappings,* Pacific J. Math., **126**(1987), 277-294.

[19] A. T. Lau and W. Takahashi, *Invariant means and fixed point properties for nonexpansive representations of topological semigroups,* Topological Methods in Nonlinear Analysis (to appear).

[20] N. Mizoguchi and W. Takahashi, *On the existence of fixed points and ergodic retractions for Lipschitzian semigroups in Hilbert spaces,* Nonlinear Analysis, Theory, Methods and Applications, **14**(1990), 69-80.

[21] B. J. Pettis, *On Nikaidô's proof of the invariant mean-valued theorem,* Shedia Math., **33**(1969), 193-196.

[22] R. R. Phelps, *Convex sets and nearest points,* Proc. Am. Math. Soc., **8**(1957), 790-797.

[23] G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert Space,* J. Math. Anal. Appl., **85**(1982), 172-178.

[24] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space,* Proc. Am. Math. Soc., **81**(1981), 253-256.

[25] W. Takahashi, *A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space,* Proc. Am. Math. Soc., **97**(1986), 55-58.

[26] W. Takahashi, *Nonlinear Functional Analysis(Japanese),* Kindaikagakusya, Japan, 1988.

[27] H. K. Xu, *Inequalities in Banach spaces with applications,* Nonlinear Analysis, Theory, Methods and Applications, **16**(1991), 1127-1138.

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