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Nonlinear Analysis and Convex Analysis

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Citation
数理解析研究所講究録 (1995), 897: 133-142

Issue Date
1995-02

URL
http://hdl.handle.net/2433/84466

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
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Nonexpansive Mappings in Banach spaces

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Abstract

Let $C$ be a nonempty weakly compact convex subset of a Banach space
which has normal structure and let $S$ be a semitopological semigroup such
that $RUC(S)$ has a left invariant mean. Then we prove a fixed point theorem
for a continuous representation of $S$ as nonexpansive mappings on $C$.

1991 Mathematics Subject Classification. Primary 47H10.
Key words and phrases. Fixed point, nonexpansive mapping, invariant mean.
1 Introduction.

Let $S$ be a semitopological semigroup, i.e., $S$ is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \rightarrow sa$ and $s \rightarrow as$ from $S$ into $S$ are continuous and let $RUC(S)$ be the space of bounded right uniformly continuous functions on $S$. Let $C$ be a nonempty subset of a Banach space and let $S = \{T_t : t \in S\}$ be a family of self-maps of $C$. $S$ is said to be a continuous representation of $S$ as nonexpansive mappings on $C$ if the following conditions are satisfied:

1. $T_{st}x = T_sT_tx$ for all $t, s \in S$ and $x \in C$;

2. for each $x \in C$, the mapping $s \rightarrow T_sx$ from $S$ into $C$ is continuous.

Let $F(S)$ denote the set of common fixed points of $T_s$, $s \in S$. Fixed point theorems for noncommutative families of nonexpansive mappings on $C$ have been investigated by several authors; see, for example, Bartoszek[1], Holmes-Lau[2,3], Lau[4,5,6], Lau-Takahashi[7,8], Lim[9,10], Mitchell[11,12], Takahashi[13,14,15,16], Takahashi-Jeong[17] and others. Among these, Lim[9] proved that if $S$ is left reversible (i.e., any two closed right ideals in $S$ have non-void intersection) and $C$ is weakly compact, convex, and has normal structure, then $S$ has a common fixed point in $C$.

In this paper, we prove a fixed point theorem for a continuous representation of $S$ as nonexpansive mappings on $C$ in the case of which $RUC(S)$
has a left invariant mean and $C$ is weakly compact, convex, and has normal structure. It is well known that left reversibility and existence of a left invariant mean on $RUC(S)$ do not imply each other.

2 Fixed point theorem.

Let $S$ be a set and $m(S)$ be the Banach space of all bounded real-valued functions on $S$ with the supremum norm. Let $X$ be a subspace of $m(S)$ containing constants. Then $\mu \in X^*$ is called a mean on $X$ if $\|\mu\| = \mu(1) = 1$. Let $\mu \in X^*$ be a mean on $X$ and $f \in X$. Then we denote by $\mu(f)$ the value of $\mu$ at the function $f$. According to time and circumstances, we write $\mu_t(f(t))$ the value $\mu(f)$. As is well known, $\mu \in X^*$ is a mean on $X$ if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for every $f \in X$. If $S$ is a semigroup, $a \in S$, and $f \in m(S)$, define $(\ell_a f)(t) = f(at)$ and $(r_a f)(t) = f(ta)$, $t \in S$. If $\ell_a(X) \subseteq X$ for all $a \in S$, then a mean $\mu$ on $X$ is left invariant if $\mu(\ell_a f) = \mu(f)$ for all $a \in S$ and $f \in X$. Let $S$ be a semitopological semigroup. Let $C(S)$ be the Banach space of bounded continuous real-valued functions on $S$. Let $RUC(S)$ denote the space of bounded right uniformly continuous functions on $S$, i.e., all $f \in C(S)$ such that the mapping $s \to r_s f$ of $S$ into $C(S)$ is continuous. Then $RUC(S)$ is a closed subalgebra of $C(S)$ containing constants and invariant under left and right translations (see [12] for details). A closed convex subset $C$ of a Banach
space is said to have normal structure if for each closed bounded convex subset $K$ of $C$, which contains at least two points, there exists an element of $K$ which is not a diametral point of $K$. It is well known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure. Lim[9] also proved the following.

Lemma[9]. A closed convex subset $C$ of a Banach space has normal structure if and only if it does not contain a sequence $\{x_n\}$ such that for some $c > 0$, $\|x_n - x_m\| \leq c$, $\|x_{n+1} - \overline{x_n}\| \geq c - \frac{1}{n^2}$ for all $n \geq 1$, $m \geq 1$, where $\overline{x_n} = \frac{1}{n} \sum_{i=1}^{n} x_i$.

Now we can prove a fixed point theorem for noncommutative families of nonexpansive mappings in Banach spaces.

Theorem. Let $S$ be a semitopological semigroup, let $D$ be a weakly compact subset of a Banach space $B$ which has normal structure and let $S = \{T_s : s \in S\}$ be a continuous representation of $S$ as nonexpansive mappings on $D$. Suppose $RUC(S)$ has a left invariant mean. Then $S$ has a common fixed point in $D$.

Proof. We first prove that for any $x \in D$ and $y \in B$, a function $h$ defined by $h(t) = \|T_t x - y\|$ for all $t \in S$ is in $RUC(S)$. In fact, we have, for
$s, u \in S,$

$$\|r_s h - r_u h\| = \sup_{t \in S} |(r_s h)(t) - (r_u h)(t)| = \sup_{t \in S} |h(ts) - h(tu)|$$

$$= \sup_{t \in S} \|T_{ts} x - y\| - \|T_{tu} x - y\| \leq \sup_{t \in S} \|T_{ts} x - T_{tu} x\|$$

$$\leq \|T_s x - T_u x\|.$$ 

Let

$$E = \{ K \subset D : K \text{ is nonempty, closed, convex, and } T_s\text{-invariant } \}.$$ 

Then by Zorn’s Lemma, there exists a minimal element $C$ of $E$. Let $\delta(C) > 0$ and let $\mu$ be a left invariant mean. Then, for any $x \in C$, 

$$A_x = \{ z \in C : \mu_t T_t x - z\| = \min_{y \in C} \mu_t T_t x - y\| \}$$

is nonempty, closed, convex, and $T_s$-invariant (see [8,13] for details). So, we have $A_x = C$ from minimality of $C$. Since $\mu$ is a mean, there exists a net of finite means $\lambda_\alpha$ such that $\lambda_\alpha \rightarrow^{w^*} \mu$. Let $x_0 \in C$, $\epsilon > 0$, and $x_1, x_2, \cdots, x_n \in C$. Since $A_{x_0} = C$, there exists $\alpha_0$ such that

$$(\mu_{\alpha_0})_t T_t x_0 - x_i\| \leq r + \epsilon, \ \forall i = 1, 2, \cdots, n,$$

where $r = \min_{y \in C} \mu_t T_t x_0 - y\|$. That is, there exists $z = \sum_{j=1}^{n_{\alpha_0}} \lambda_j T_{S_j} x_0$ with $\lambda_1, \cdots, \lambda_{n_{\alpha_0}} \geq 0$ and $\sum_{j=1}^{n_{\alpha_0}} \lambda_j = 1$ such that

$$\|z - x_i\| \leq \sum_{j=1}^{n_{\alpha_0}} \lambda_j \|T_{S_j} x_0 - x_i\| \leq r + \epsilon, \ \forall i = 1, 2, \cdots, n. \quad (1)$$
Let $C_{y, \epsilon} = \{z \in C : \|z - y\| \leq r + \epsilon\}$ for each $y \in C$. Then by (1),

$$\{C_{y, \epsilon} : y \in C\}$$

has finite intersection property. Since $C$ is weakly compact, there is $z_0 \in C$ such that $\|z_0 - y\| \leq r + \epsilon$ for every $y \in C$. Since $\{T_t x_0\} \subset C$, we have

$$\sup_{t \in S} \|z_0 - T_t x_0\| \leq \sup_{y \in C} \|z_0 - y\| \leq r + \epsilon.$$

Since

$$r = \mu_t \|T_t x_0 - z_0\| \leq \sup_{t} \|T_t x_0 - z_0\| \leq \sup_{y \in C} \|z_0 - y\| \leq r + \epsilon,$$

we have

$$r \leq \inf_{x \in C} \sup_{t} \|T_t x_0 - x\| \leq \inf_{x \in C} \sup_{y \in C} \|y - x\| \leq r + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$r = \mu_t \|T_t x_0 - x\| = \inf_{z \in C} \sup_{y \in C} \|y - z\|, \quad \forall x \in C. \quad (2)$$

Since $x_0 \in C$ is arbitrary, for any $x, z \in C$, we have

$$r = \mu_t \|T_t x - z\| = \inf_{u \in C} \sup_{t \in S} \|T_t x - u\| = \inf_{u \in C} \sup_{y \in C} \|y - u\|.$$

So, let

$$A_0 = \{z \in C : \sup_{t \in S} \|z - T_t x\| \leq r, \quad \forall x \in C\}.$$
By (2), since there exists $z_0 \in C$ such that
\[
\sup_{y \in C} \| y - z_0 \| = r,
\]
we have that $A_0$ is nonempty. Let $z_0 \in A_0$ and $s \in S$. Then putting
\[
A_s = \{ z \in C : \sup_{t \in S} \| T_{st} x - z \| \leq r, \ \forall x \in C \},
\]
we have $z_0, T_s z_0 \in A_s$. Further, for any $x \in C$,
\[
\begin{align*}
\mu_t \| T_t x - z_0 \| &= \mu_t \| T_{st} x - z_0 \| \leq \sup_{t \in S} \| T_{st} x - z_0 \| \\
&\leq \sup_{t \in S} \| T_t x - z_0 \| \leq r.
\end{align*}
\]
and
\[
\begin{align*}
\mu_t \| T_t x - T_s z_0 \| &= \mu_t \| T_{st} x - T_s z_0 \| \leq \sup_{t \in S} \| T_{st} x - T_s z_0 \| \\
&\leq \sup_{t \in S} \| T_t x - z_0 \| \leq r.
\end{align*}
\]
For using Lim's Lemma, fix $z_0 \in A_0$. Then since $r = \mu_t \| T_t z_0 - z_0 \|$, there exists $s_1 \in S$ such that $\| T_{s_1} z_0 - z_0 \| \geq r - 1$. Since $z_0, T_s z_0 \in A_{s_1}$ and $A_{s_1}$ is convex,
\[
\overline{x_2} = \frac{1}{2} z_0 + \frac{1}{2} T_{s_1} z_0 \in A_{s_1}.
\]
Let $x_1 = z_0$ and $x_2 = T_{s_1} z_0$. Since $r = \mu_t \| T_t z_0 - \overline{x_2} \| = \mu_t \| T_{s_1} z_0 - \overline{x_2} \|$, there exists $s_2 \in S$ such that $\| T_{s_1 s_2} z_0 - \overline{x_2} \| \geq r - \frac{1}{2^2}$. So, let $x_3 = T_{s_1 s_2} z_0$. Then, we have
\[
\| x_1 - x_2 \| = \| z_0 - T_{s_1} z_0 \| \leq \sup_{t \in S} \| z_0 - T_t z_0 \| = r,
\]
\[\|x_2 - x_3\| = \|T_{s_1}z_0 - T_{s_1s_2}z_0\| \leq \|z_0 - T_{s_1}z_0\| \leq r,\]

and

\[\|x_3 - x_1\| = \|T_{s_1s_2}z_0 - z_0\| \leq \sup_{t \in S} \|T_t z_0 - z_0\| = r.\]

Similarly, let

\[\overline{x}_3 = \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3.\]

Then, \(r = \mu_t \|T_t z_0 - \overline{x}_3\| = \mu_t \|T_{s_1s_2s_3}z_0 - \overline{x}_3\|\), there exists \(s_3 \in S\) such that

\[\|T_{s_1s_2s_3}z_0 - \overline{x}_3\| \geq r - \frac{1}{3^2}.\]

So, let \(x_4 = T_{s_1s_2s_3}z_0\). Then, we have

\[\|x_4 - x_1\| = \|T_{s_1s_2s_3}z_0 - z_0\| \leq \sup_{t \in S} \|T_t z_0 - z_0\| = r,\]

\[\|x_4 - x_2\| = \|T_{s_1s_2s_3}z_0 - T_{s_1}z_0\| \leq \|T_{s_2s_3}z_0 - z_0\| \leq \sup_{t \in S} \|T_t z_0 - z_0\| = r,\]

and

\[\|x_4 - x_3\| = \|T_{s_1s_2s_3}z_0 - T_{s_1s_2}z_0\| \leq \|T_{s_3}z_0 - z_0\| \leq \sup_{t \in S} \|T_t z_0 - z_0\| = r.\]

By mathematical induction, let \(x_5 = T_{s_1s_2s_3s_4}z_0, x_6 = T_{s_1s_2s_3s_4s_5}z_0, \ldots\). Then we have

\[\|x_n - x_m\| \leq r, \ \forall n, m \ \text{and} \ \|x_{n+1} - \overline{x}_n\| \geq r - \frac{1}{n^2}.\]

Using Lim's Lemma, \(C\) has not normal structure. This is a contradiction.
References


