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Kyoto University
Balanced Families in Compact Spaces

Hidetoshi Komiya

Faculty of Business and Commerce, Keio University
Kouhoku-ku, Yokohama 223, Japan

1 Introduction

We shall denote by $N$ the set $\{1, \ldots, n\}$ and by $\mathcal{N}$ the family of the nonempty subsets of $N$. A subfamily $\{S_i\}_{i=1}^{p}$ of $\mathcal{N}$ is said to be balanced if there is a corresponding family $\{\lambda_i\}_{i=1}^{p}$ of nonnegative numbers such that $\sum_{i} \lambda_i \chi_{S_i} = \chi_{N}$, where $\chi_{A}$ denotes the characteristic vector of the set $A$, i.e., $\chi_{A}$ is an $n$-vector whose $i$-the coordinate is 1 if $i \in A$ and 0 if $i \notin A$.

The balancedness plays a crucial role in covering theorems of simplexes which are basic tools to prove the nonemptiness of the core of nontransferable utility games. (cf. [2], [3]) We shall examine the balancedness of a subfamily of $\mathcal{N}$ profoundly and extend the study to the case that a compact Hausdorff space is the substitute of the finite set $N$. The research would be expected to be a basis of the study of infinite dimensional game theory, that is, the game theory with infinitely many players.

We prepare mathematical background necessary for the arguments hereafter. Let $Q$ be a compact Hausdorff space and let $C(Q)$ be the Banach space of all continuous real valued functions on $Q$ with the supremum norm $\|\xi\| = \max_{q \in Q} |\xi(q)|$. Let $M(Q)$ be the Banach space of all regular signed Borel measures on $Q$ with the norm $\|x\| = |x|(Q)$, where $|x|$ denotes the total variation of the regular signed Borel measure $x$ on $Q$. Then we can regard $M(Q)$ as the dual Banach space $C(Q)'$ of $C(Q)$ by the bijection $x \mapsto \tilde{x}$ from $M(Q)$ onto $C(Q)'$ defined by

$$\tilde{x}(\xi) = \int \xi dx, \quad \xi \in C(Q).$$

The space $M(Q)$ is equipped with the weak star topology throughout this note. We shall write $x(\xi)$ in place of $\int \xi dx$ when no confusion is likely to arise. We denote by $\Sigma$ the $\sigma$-field of the Borel sets in $Q$. The support
$\text{supp}(x)$ of an element $x$ of $M(Q)$ is defined by

$$\text{supp}(x) = Q \setminus \bigcup \{G : x(G) = 0, \ G \text{ is open}\}.$$  

We introduce two binary relations $\geq$ and $\gg$ in $M(Q)$ by

$$x \geq y \text{ if } x(A) \geq y(A) \text{ for all } A \in \Sigma,$$

$$x \gg y \text{ if } x \geq y \text{ and } \text{supp}(x - y) = Q,$$

respectively. We shall use the symbol $\Delta$ to denote the convex subset

$$\{x \in M(Q) : \|x\| = x(1) = 1\}$$

of $M_+(Q) = \{x \in M(Q) : x \geq 0\}$, and the symbol $\Delta_{++}$ to denote the set $\{x \in \Delta : x \gg 0\}$. It may happen that the set $\Delta_{++}$ is empty. Consider a discrete uncountably infinite space $Q$ and its one-point compactification $Q^*$. Let $x \in M(Q^*)$ and $x \geq 0$. Put $Q_n = \{q \in Q : x(\{q\}) \geq 1/n\}$. Since $|Q_n| \leq n\|x\|$, $\bigcup_{n=1}^\infty Q_n$ is countable and there is a point $q_0 \in Q \setminus \bigcup Q_n$. Thus, $x(\{q_0\}) = 0$ and $\{q_0\}$ is open. Therefore, $\Delta_{++}$ is empty.

Recall that $\Delta$ is compact and $M_+(Q)$ is closed. Moreover, if we correspond a point $q$ in $Q$ to the mass measure $\hat{q}$ at $q$ on $Q$, then the correspondence is into-homeomorphism. For any nonempty subsets $A$ of $Q$, let $\Delta^A$ be the closed convex hull of $\{\hat{q} : q \in A\}$. We shall use the same symbols as the finite dimensional case, but no confusion may occur.

## 2 Balanced families in compact spaces

We start with an examination of balanced subfamilies of $\mathcal{N}$. It is well known that a subfamily $\{S_i\}_{i=1}^p$ of $\mathcal{N}$ is balanced if and only if the vector $\chi_N/n$ is a convex combination of the vectors $\chi_{S_i}/|S_i|$. Geometrically this means the barycenter of the simplex $\Delta^N$ is contained in the polytope spanned by the barycenters of the faces $\Delta^{S_i}$.

The concept of balancedness has been characterized in terms of the specific vectors such as $\chi_N$ or $\chi_N/n$, but balancedness is free from the specification as shown in Proposition 1 below.

Let $r$ be a point of $\Delta^N$ such that $r \gg 0$. Define a vector $r^S$ for $S \in \mathcal{N}$ by

$$r^S = \begin{cases} r_i/\sum_{j \in S} r_j & \text{for } i \in S \\ 0 & \text{otherwise} \end{cases}$$
Proposition 1 For any vector $r$ of $\Delta^N$ such that $r \gg 0$, a subfamily $\{S_i\}_{i=1}^P$ of $\mathcal{N}$ is balanced if and only if $r$ is a convex combination of the points $\{r^{S_i}\}_{i=1}^P$.

Proof. Suppose that the family $\{S_i\}_{i=1}^P$ is balanced. Then there is a corresponding family $\{\lambda_i\}_{i=1}^P$ of nonnegative numbers such that $\chi_N = \sum_{i=1}^P \lambda_i \chi_{S_i}$. Multiply the diagonal matrix $(a_{ij})_{i,j=1}^n$, where $a_{ij} = r_j$ if $i = j$ and $a_{ij} = 0$ otherwise, to both sides of the equality above. Then we have

$$r = \sum_{i=1}^P \lambda_i \left( \sum_{j \in S_i} r_j \right) r^{S_i}$$

and $\sum_{i=1}^P \lambda_i \left( \sum_{j \in S_i} r_j \right) = \sum_{k=1}^n r_k = 1$.

Conversely if $r$ is represented as a convex combination of $\{r^{S_i}\}_{i=1}^P$ such as $r = \sum_{i=1}^P \mu_i r^{S_i}$, then we have the equation

$$\chi_N = \sum_{i=1}^P \left( \frac{\mu_i}{\sum_{j \in S_i} r_j} \right) \chi_{S_i}$$

by multiplying the diagonal matrix $(b_{ij})_{i,j=1}^n$, where $b_{ij} = r_i^{-1}$ if $i = j$ and $b_{ij} = 0$ otherwise, to both sides of the equality above. Therefore the family $\{S_i\}_{i=1}^P$ is balanced. \(\square\)

Similar to the definition of $r^S$, we can define an element $\bar{x}^S$ of $\Delta$ for any $\bar{x} \in \Delta_{++}$ and any Borel subset $S$ of $Q$ with $\bar{x}(S) > 0$ by

$$\bar{x}^S(A) = \frac{\bar{x}(A \cap S)}{\bar{x}(S)}, \quad A \in \Sigma.$$ 

Note that $\bar{x}^S$ belongs to $\Delta^S$ and $\bar{x}^S(\xi) = \int_S \xi d\bar{x}/\bar{x}(S)$ for any $\xi \in C(Q)$.

According to Proposition 1, we can define the balancedness of subfamilies of $\mathcal{N}$ by means of any vector $r$ with $r \gg 0$. However, we cannot expect such uniformity in the infinite dimensional spaces. See the following example.

Example 1 Let $m$ be the Lebesgue measure on $[0, 1]$, and consider the two elements $\bar{x} = m$ and $\bar{y} = m/2 + \hat{1}/2$ of $\Delta \subset M([0, 1])$. Let $S = [0, 1)$, and consider the family $\{S\}$. Then we have $\bar{x} = m = \bar{x}^S$ and $\bar{y} \neq m = \bar{y}^S$ in spite of the fact $\bar{x} \gg 0$ and $\bar{y} \gg 0$.

Inspired by Proposition 1 and Example 1, we define balancedness in compact Hausdorff spaces as follows:
Definition 1 Let $Q$ be a compact Hausdorff space such that $\Delta_{++}$ is not empty, and let $\Sigma$ be a Borel $\sigma$-field of $Q$. For an element $\bar{x}$ of $\Delta_{++}$ in $M(Q)$, let $\Sigma_{x} = \{ S \in \Sigma : \bar{x}(S) > 0 \}$. A subfamily $B$ of $\Sigma$ is said to be $\bar{x}$-balanced if $\bar{x}$ belongs to the closed convex hull of the set $\{ \bar{x}^S : S \in B \cap \Sigma_x \}$.

We probe the balancedness just defined hereafter. The following is the infinite dimensional version of the proposition obtained in Ichishi[2].

Proposition 2 Let $\bar{x}$ be an element of $\Delta_{++}$ and $B = \{ S_1, \ldots, S_p \}$ be a finite subfamily of $\Sigma$ such that $0 < \bar{x}(S_i) < 1$ for all $i = 1, \ldots, p$. Then $B$ is $\bar{x}$-balanced if and only if the family $B' = \{ Q \setminus S_1, \ldots, Q \setminus S_p \}$ is $\bar{x}$-balanced.

Proof. We need to prove only the "only if" part because of the symmetry of the statement. There are nonnegative numbers $\lambda_1, \ldots, \lambda_p$ such that

\[
\bar{x} = \sum_{i=1}^{p} \lambda_i \bar{x}^{S_i} \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i = 1
\]

by the hypothesis. Then we have $\sum_{i=1}^{p} \lambda_i (\bar{x} - \bar{x}^{S_i}) = 0$. On the other hand, we have $\bar{x} = \bar{x}(S_i) \bar{x}^{S_i} + \bar{x}(Q \setminus S_i) \bar{x}^{Q \setminus S_i}$; hence we have

\[
\bar{x} - \bar{x}^{S_i} = -\frac{\bar{x}(Q \setminus S_i)}{\bar{x}(S_i)} (\bar{x} - \bar{x}^{Q \setminus S_i}).
\]

Therefore we have

\[
\sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\bar{x}(S_i)} (\bar{x} - \bar{x}^{Q \setminus S_i}) = 0.
\]

If we put $\mu = \sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\bar{x}(S_i)}$ and $\mu_i = \sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\mu \bar{x}(S_i)}$, then we have the desired result $\bar{x} = \sum_{i=1}^{p} \mu_i \bar{x}^{Q \setminus S_i}$.

We cannot expect the corresponding result for infinite families as shown in the following examples.

Example 2 Let $N^*$ be the one-point compactification of the positive integers and $\bar{x}$ the Borel measure on $N^*$ defined by $\bar{x}(n) = 1/2^{(n+1)}$ for $n = 1, 2, \ldots$, and $\bar{x}(\infty) = 1/2$. Let $S_n = N^* \setminus \{ n \}$ and consider the family $B = \{ S_n : n = 2, 3, \ldots \}$. Then $B$ is $\bar{x}$-balanced because $\bar{x}^{S_n}$ converges to $\bar{x}$. On the other hand, it is trivial that the family $B' = \{ \{ 2 \}, \{ 3 \}, \ldots \}$ is not $\bar{x}$-balanced.
We need the following lemma to present the next example and we shall also use it later.

**Lemma 1** Let \( \{x_\alpha\} \) be a net in \( \Delta \) and \( x \) an element of \( \Delta \). Then \( x_\alpha(A) \to x(A) \) for every \( A \in \Sigma \) implies \( x_\alpha \to x \).

**Proof.** Let \( \xi \) be an element of \( C(Q) \). Since \( \xi \) is bounded, for any \( \varepsilon > 0 \), there is a measurable simple function \( \sigma \) on \( Q \) such that \( \|\xi - \sigma\| < \varepsilon/3 \). Since \( x_\alpha(\sigma) \to x(\sigma) \) by the hypothesis, there is \( \alpha_0 \) such that \( |x_\alpha(\sigma) - x(\sigma)| < \varepsilon/3 \) for \( \alpha \geq \alpha_0 \). Therefore, for any \( \alpha \geq \alpha_0 \), we have

\[
|x_\alpha(\xi) - x(\xi)| = |x_\alpha(\xi) - x_\alpha(\sigma)| + |x_\alpha(\sigma) - x(\sigma)| + |x(\sigma) - x(\xi)|
< \|\xi - \sigma\| + \varepsilon/3 + \|\sigma - \xi\|
< \varepsilon.
\]

\( \square \)

**Example 3** Consider the compact Hausdorff space \( Q = \{0,1\}^N \) with the product topology, where \( N = \{1,2,\ldots\} \) and \( \{0,1\} \) has the usual topological group structure, and let \( ^\times \) be the Haar measure on \( Q \). For any two disjoint finite subsets \( A \) and \( B \) of \( N \), define the subset \( H^{A,B} \) of \( Q \) by

\[
H^{A,B} = \{q \in Q: q(n) = 0 \text{ for } n \in A, q(n) = 1 \text{ for } n \in B\}.
\]

Then it is easily seen that \( ^\times(H^{A,B}) = 1/2^{|A|+|B|} \). Define a sequence \( S_n \) by

\[
S_1 = H^{(1),\emptyset}, \text{ and } S_{n+1} = H^{(n+1),(1,\ldots,n)} \cup S_n.
\]

Then we have \( ^\times(S_n) = 1 - 1/2^n \) and \( S_n \not\to Q \setminus \{(1,1,\ldots,1,\ldots)\} \). Therefore, we have

\[
^\times S_n(A) = \frac{^\times(A \cap S_n)}{^\times(S_n)} \to ^\times(A) \quad \text{for all } A \in \Sigma;
\]

and hence, \( ^\times S_n \) converges to \( ^\times \) by Lemma 1. Therefore the family \( \{S_n\} \) is \( ^\times \)-balanced. On the other hand, since \( Q \setminus S_n = H_{\emptyset,\{1,\ldots,n\}} \subset Q \setminus S_1 \subset H_{\emptyset,\{1\}} \), \( ^\times Q \setminus S_n \) belongs to \( \Delta^{H_{\emptyset,\{1\}}} \), i.e. \( \text{supp}(^\times Q \setminus S_n) \subset H_{\emptyset,\{1\}} \) for all \( n = 1,2,\ldots \). Therefore, every point of \( \text{co}\{^\times Q \setminus S_n : n = 1,2,\ldots\} \) has the support in \( H_{\emptyset,\{1\}} \). However, since \( \text{supp}(^\times) = Q \), we have \( ^\times \not\in \text{co}\{^\times Q \setminus S_n : n = 1,2,\ldots\} \) and \( B' = \{Q \setminus S_n : n = 1,2,\ldots\} \) is not \( ^\times \)-balanced.

We expect that suitable partitions of \( Q \) satisfy the balancedness we have defined. The following proposition assures us our definition of balancedness is appropriate.
Proposition 3 Let \( \bar{x} \) be an element of \( \Delta_{++} \). Let \( \{A_i\} \) be a countable covering of a compact Hausdorff space \( Q \) such that \( A_i \in \Sigma \) for all \( i \) and \( \bar{x}(A_i \cap A_j) = 0 \) for \( i \neq j \). Then \( \{A_i\} \) is \( \bar{x} \)-balanced. In particular, any countable partition of \( Q \) consisting of Borel sets is \( \bar{x} \)-balanced for any \( \bar{x} \in \Delta_{++} \).

Proof. Define a disjoint countable covering \( \{B_j\} \) of \( Q \) by \( B_j = A_j \setminus \bigcup_{i>j} A_i \). Then it is easily seen that \( \bar{x}(B_j) = \bar{x}(A_j) \) and \( \bar{x}^{B_j} = \bar{x}^{A_j} \). Therefore, for any \( A \in \Sigma \),

\[
\bar{x}(A) = \sum \bar{x}(A \cap B_j) = \sum \bar{x}(B_j)\bar{x}^{B_j}(A) = \sum \bar{x}(B_j)\bar{x}^{A_j}(A).
\]

Since \( \{B_j\} \) is a disjoint covering of \( Q \), we have \( \sum \bar{x}(B_j) = 1 \). If the sum is essentially finite, then the proof is completed. Suppose the sum has infinite terms essentially. We can assume \( \bar{x}(B_1) \neq 0 \) without loss of generality. For any \( n = 1, 2, \ldots \), define an element \( x_n \) of \( \text{co}\{\bar{x}^{A_j} : j = 1, 2, \ldots\} \) by \( x_n = \sum_{j=1}^{n} (\bar{x}(B_j)/\lambda_n)\bar{x}^{A_j} \), where \( \lambda_n = \sum_{j=1}^{n} \bar{x}(B_j) \). Then we have the equations

\[
\bar{x}(A) = (\lambda_n x_n)(A) + \sum_{j>n} \bar{x}(B_j)\bar{x}^{A_j}(A) = x_n(A) + (\lambda_n - 1)x_n(A) + \sum_{j>n} \bar{x}(B_j)\bar{x}^{A_j}(A).
\]

Therefore we have

\[
|\bar{x}(A) - x_n(A)| \leq (1 - \lambda_n)x_n(A) + \sum_{j>n} \bar{x}(B_j) \leq 2(1 - \lambda_n).
\]

We can conclude \( x_n \to \bar{x} \) from Lemma 1 since \( \lambda_n \to 1 \). Therefore we have \( \bar{x} \in \text{co}\{\bar{x}^{A_j} : j = 1, 2, \ldots\} \). \( \square \)

We give another example of a balanced family such that any two sets of the family have a nonempty intersection.

Example 4 Let \( N^* \) be the one point compactification of the positive integers, and \( \bar{x} \) the element defined in Example 2 above. Consider the family
\{A, B, C\} of the subsets of $N^*$ defined by $A = \{1, 2\}$, $B = \{2, 3, \ldots, \infty\}$, and $C = \{3, 4, \ldots, \infty, 1\}$. Then the family \{A, B, C\} is \(\bar{x}\)-balanced.

In fact, we have

\[
\bar{x}^A(n) = \begin{cases} 
2/3 & \text{for } n = 1 \\
1/3 & \text{for } n = 2 \\
0 & \text{otherwise}
\end{cases}, \quad \bar{x}^B(n) = \begin{cases} 
0 & \text{for } n = 1 \\
2/3 & \text{for } n = \infty \\
1/(3 \times 2^{(n-1)}) & \text{otherwise}
\end{cases},
\]

\[
\bar{x}^C(n) = \begin{cases} 
2/7 & \text{for } n = 1 \\
4/7 & \text{for } n = \infty \\
1/(7 \times 2^{(n-2)}) & \text{otherwise}
\end{cases}
\]

and

\[
\bar{x} = \frac{3}{16} \bar{x}^A + \frac{3}{8} \bar{x}^B + \frac{7}{16} \bar{x}^C.
\]

References

