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Balanced Families in Compact Spaces

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1 Introduction

We shall denote by $N$ the set $\{1, \ldots, n\}$ and by $\mathcal{N}$ the family of the nonempty subsets of $N$. A subfamily $\{S_i\}_{i=1}^{p}$ of $\mathcal{N}$ is said to be balanced if there is a corresponding family $\{\lambda_i\}_{i=1}^{p}$ of nonnegative numbers such that $\sum_i \lambda_i \chi_{S_i} = \chi_N$, where $\chi_A$ denotes the characteristic vector of the set $A$, i.e., $\chi_A$ is an $n$-vector whose $i$-the coordinate is 1 if $i \in A$ and 0 if $i \notin A$.

The balancedness plays a crucial role in covering theorems of simplexes which are basic tools to prove the nonemptiness of the core of nontransferable utility games. (cf. [2], [3]) We shall examine the balancedness of a subfamily of $\mathcal{N}$ profoundly and extend the study to the case that a compact Hausdorff space is the substitute of the finite set $N$. The research would be expected to be a basis of the study of infinite dimensional game theory, that is, the game theory with infinitely many players.

We prepare mathematical background necessary for the arguments hereafter. Let $Q$ be a compact Hausdorff space and let $C(Q)$ be the Banach space of all continuous real valued functions on $Q$ with the supremum norm $\|\xi\| = \max_{q \in Q} |\xi(q)|$. Let $M(Q)$ be the Banach space of all regular signed Borel measures on $Q$ with the norm $\|\mathcal{M}\| = |\mathcal{M}|(Q)$, where $|\mathcal{M}|$ denotes the total variation of the regular signed Borel measure $\mathcal{M}$ on $Q$. Then we can regard $M(Q)$ as the dual Banach space $C(Q)'$ of $C(Q)$ by the bijection $x \mapsto \tilde{x}$ from $M(Q)$ onto $C(Q)'$ defined by

$$\tilde{x}(\xi) = \int \xi dx, \quad \xi \in C(Q).$$

The space $M(Q)$ is equipped with the weak star topology throughout this note. We shall write $x(\xi)$ in place of $\int \xi dx$ when no confusion is likely to arise. We denote by $\Sigma$ the $\sigma$-field of the Borel sets in $Q$. The support
supp$(x)$ of an element $x$ of $M(Q)$ is defined by
\[ \text{supp}(x) = Q \setminus \bigcup \{ G : x(G) = 0, \ G \text{ is open} \} . \]

We introduce two binary relations $\geq$ and $\gg$ in $M(Q)$ by
\[ x \geq y \text{ if } x(A) \geq y(A) \text{ for all } A \in \Sigma, \]
\[ x \gg y \text{ if } x \geq y \text{ and } \text{supp}(x - y) = Q, \]
respectively. We shall use the symbol $\Delta$ to denote the convex subset
\[ \{ x \in M(Q) : \| x \| = x(1) = 1 \} \]
of $M_+(Q) = \{ x \in M(Q) : x \geq 0 \}$, and the symbol $\Delta_{++}$ to denote the set
\[ \{ x \in \Delta : x \gg 0 \} . \] It may happen that the set $\Delta_{++}$ is empty. Consider a discrete uncountably infinite space $Q$ and its one-point compactification $Q^*$. Let $x \in M(Q^*)$ and $x \geq 0$. Put $Q_n = \{ q \in Q : x(\{ q \}) \geq 1/n \}$. Since $|Q_n| \leq n\| x \|$, $\cup_{n=1}^\infty Q_n$ is countable and there is a point $q_0 \in Q \setminus \cup Q_n$. Thus, $x(\{ q_0 \}) = 0$ and $\{ q_0 \}$ is open. Therefore, $\Delta_{++}$ is empty.

Recall that $\Delta$ is compact and $M_+(Q)$ is closed. Moreover, if we correspond a point $q$ in $Q$ to the mass measure $\hat{q}$ at $q$ on $Q$, then the correspondence is into-homeomorphism. For any nonempty subsets $A$ of $Q$, let $\Delta^A$ be the closed convex hull of $\{ \hat{q} : q \in A \}$. We shall use the same symbols as in the finite dimensional case, but no confusion may occur.

2 Balanced families in compact spaces

We start with an examination of balanced subfamilies of $\mathcal{N}$. It is well known that a subfamily $\{ S_i \}_{i=1}^p$ of $\mathcal{N}$ is balanced if and only if the vector $\chi_{S_i}/|S_i|$ is a convex combination of the vectors $\chi_{S_i}/|S_i|$. Geometrically this means the barycenter of the simplex $\Delta^N$ is contained in the polytope spanned by the barycenters of the faces $\Delta^S$.

The concept of balancedness has been characterized in terms of the specific vectors such as $\chi_{N}$ or $\chi_{N}/n$, but balancedness is free from the specification as shown in Proposition 1 below.

Let $r$ be a point of $\Delta^N$ such that $r \gg 0$. Define a vector $r^S$ for $S \in \mathcal{N}$ by
\[ r^S = \begin{cases} r_i/\sum_{j \in S} r_j & \text{for } i \in S \\ 0 & \text{otherwise.} \end{cases} \]
Proposition 1 For any vector $r$ of $\Delta^N$ such that $r \gg 0$, a subfamily $\{S_i\}_{i=1}^p$ of $\mathcal{N}$ is balanced if and only if $r$ is a convex combination of the points $\{r^{S_i}\}_{i=1}^p$.

Proof. Suppose that the family $\{S_i\}_{i=1}^p$ is balanced. Then there is a corresponding family $\{\lambda_i\}_{i=1}^p$ of nonnegative numbers such that $\chi_N = \sum_{i=1}^p \lambda_i \chi_{S_i}$. Multiply the diagonal matrix $(a_{ij})_{i,j=1}^n$, where $a_{ij} = r_i$ if $i = j$ and $a_{ij} = 0$ otherwise, to both sides of the equality above. Then we have

$$r = \sum_{i=1}^p \lambda_i \left( \sum_{j \in S_i} r_j \right) r^{S_i}$$

and $\sum_{i=1}^p \lambda_i \left( \sum_{j \in S_i} r_j \right) = \sum_{k=1}^n r_k = 1$.

Conversely if $r$ is represented as a convex combination of $\{r^{S_i}\}_{i=1}^p$ such as $r = \sum_{i=1}^p \mu_i r^{S_i}$, then we have the equation

$$\chi_N = \sum_{i=1}^p (\mu_i / \sum_{j \in S_i} r_j) \chi_{S_i}$$

by multiplying the diagonal matrix $(b_{ij})_{i,j=1}^n$, where $b_{ij} = r_i^{-1}$ if $i = j$ and $b_{ij} = 0$ otherwise, to both sides of the equality above. Therefore the family $\{S_i\}_{i=1}^p$ is balanced. $\square$

Similar to the definition of $r^S$, we can define an element $\bar{x}^S$ of $\Delta$ for any $\bar{x} \in \Delta_{++}$ and any Borel subset $S$ of $Q$ with $\bar{x}(S) > 0$ by

$$\bar{x}^S(A) = \bar{x}(A \cap S) / \bar{x}(S), \quad A \in \Sigma.$$  

Note that $\bar{x}^S$ belongs to $\Delta^S$ and $\bar{x}^S(\xi) = \int_S \xi d\bar{x} / \bar{x}(S)$ for any $\xi \in C(Q)$.

According to Proposition 1, we can define the balancedness of subfamilies of $\mathcal{N}$ by means of any vector $r$ with $r \gg 0$. However, we cannot expect such uniformity in the infinite dimensional spaces. See the following example.

Example 1 Let $m$ be the Lebesgue measure on $[0,1]$, and consider the two elements $\bar{x} = m$ and $\bar{y} = m/2 + 1/2$ of $\Delta \subset M([0,1])$. Let $S = [0,1)$, and consider the family $\{S\}$. Then we have $\bar{x} = m = \bar{x}^S$ and $\bar{y} \neq m = \bar{y}^S$ in spite of the fact $\bar{x} \gg 0$ and $\bar{y} \gg 0$.

Inspired by Proposition 1 and Example 1, we define balancedness in compact Hausdorff spaces as follows:
Definition 1  Let $Q$ be a compact Hausdorff space such that $\Delta_{++}$ is not empty, and let $\Sigma$ be a Borel $\sigma$-field of $Q$. For an element $\bar{x}$ of $\Delta_{++}$ in $M(Q)$, let $\Sigma_{\bar{x}} = \{ S \in \Sigma : \bar{x}(S) > 0 \}$. A subfamily $B$ of $\Sigma$ is said to be $\bar{x}$-balanced if $\bar{x}$ belongs to the closed convex hull of the set $\{ \bar{x}^S : S \in B \cap \Sigma_{\bar{x}} \}$.

We probe the balancedness just defined hereafter. The following is the infinite dimensional version of the proposition obtained in Ichishi[2].

Proposition 2  Let $\bar{x}$ be an element of $\Delta_{++}$ and $B = \{ S_1, \ldots, S_p \}$ be a finite subfamily of $\Sigma$ such that $0 < \bar{x}(S_i) < 1$ for all $i = 1, \ldots, p$. Then $B$ is $\bar{x}$-balanced if and only if the family $B' = \{ Q \setminus S_1, \ldots, Q \setminus S_p \}$ is $\bar{x}$-balanced.

Proof. We need to prove only the "only if" part because of the symmetry of the statement. There are nonnegative numbers $\lambda_1, \ldots, \lambda_p$ such that

$$\bar{x} = \sum_{i=1}^{p} \lambda_i \bar{x}^{S_i}$$

and

$$\sum_{i=1}^{p} \lambda_i = 1$$

by the hypothesis. Then we have $\sum_{i=1}^{p} \lambda_i (\bar{x} - \bar{x}^{S_i}) = 0$. On the other hand, we have $\bar{x} = \bar{x}(S_i) \bar{x}^{S_i} + \bar{x}(Q \setminus S_i) \bar{x}(Q \setminus S_i)$; hence we have

$$\bar{x} - \bar{x}^{S_i} = - \frac{\bar{x}(Q \setminus S_i)}{\bar{x}(S_i)} (\bar{x} - \bar{x}(Q \setminus S_i)).$$

Therefore we have

$$\sum_{i=1}^{p} \lambda_i \frac{\bar{x}(Q \setminus S_i)}{\bar{x}(S_i)} (\bar{x} - \bar{x}(Q \setminus S_i)) = 0.$$ 

If we put $\mu = \sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\bar{x}(S_i)}$ and $\mu_i = \sum_{i=1}^{p} \frac{\lambda_i \bar{x}(Q \setminus S_i)}{\mu \bar{x}(S_i)}$, then we have the desired result $\bar{x} = \sum_{i=1}^{p} \mu_i \bar{x}(Q \setminus S_i)$. □

We cannot expect the corresponding result for infinite families as shown in the following examples.

Example 2  Let $N^*$ be the one-point compactification of the positive integers and $\bar{x}$ the Borel measure on $N^*$ defined by $\bar{x}(n) = 1/2^{(n+1)}$ for $n = 1, 2, \ldots$, and $\bar{x}(\infty) = 1/2$. Let $S_n = N^* \setminus \{ n \}$ and consider the family $B = \{ S_n : n = 2, 3, \ldots \}$. Then $B$ is $\bar{x}$-balanced because $\bar{x}^{S_n}$ converges to $\bar{x}$. On the other hand, it is trivial that the family $B' = \{ \{ 2 \}, \{ 3 \}, \ldots \}$ is not $\bar{x}$-balanced.
We need the following lemma to present the next example and we shall also use it later.

**Lemma 1** Let \( \{x_\alpha\} \) be a net in \( \Delta \) and \( x \) an element of \( \Delta \). Then \( x_\alpha(A) \to x(A) \) for every \( A \in \Sigma \) implies \( x_\alpha \to x \).

**Proof.** Let \( \xi \) be an element of \( C(Q) \). Since \( \xi \) is bounded, for any \( \epsilon > 0 \), there is a measurable simple function \( \sigma \) on \( Q \) such that \( \|\xi - \sigma\| < \epsilon/3 \). Since \( x_\alpha(\sigma) \to x(\sigma) \) by the hypothesis, there is \( \alpha_0 \) such that \( |x_\alpha(\sigma) - x(\sigma)| < \epsilon/3 \) for \( \alpha \geq \alpha_0 \). Therefore, for any \( \alpha \geq \alpha_0 \), we have

\[
|\xi - \sigma| < \epsilon/3 + \|\sigma - \xi\|
\]

\[
< \epsilon.
\]

\( \square \)

**Example 3** Consider the compact Hausdorff space \( Q = \{0,1\}^N \) with the product topology, where \( N = \{1,2,\ldots\} \) and \( \{0,1\} \) has the usual topological group structure, and let \( \bar{x} \) be the Haar measure on \( Q \). For any two disjoint finite subsets \( A \) and \( B \) of \( N \), define the subset \( H^{A,B} \) of \( Q \) by

\[
H^{A,B} = \{q \in Q : q(n) = 0 \text{ for } n \in A, q(n) = 1 \text{ for } n \in B\}.
\]

Then it is easily seen that \( \bar{x}(H^{A,B}) = 1/2^{|A|+|B|} \). Define a sequence \( S_n \) by

\[
S_1 = H^{(1),\emptyset}, \quad S_{n+1} = H^{(n+1),\{1,\ldots,n\}} \cup S_n.
\]

Then we have \( \bar{x}(S_n) = 1 - 1/2^n \) and \( S_n \nearrow Q \setminus \{(1,1,\ldots,1,\ldots)\} \). Therefore, we have

\[
\bar{x}^{S_n}(A) = \frac{\bar{x}(A \cap S_n)}{\bar{x}(S_n)} \to \bar{x}(A) \quad \text{for all } A \in \Sigma;
\]

and hence, \( \bar{x}^{S_n} \) converges to \( \bar{x} \) by Lemma 1. Therefore the family \( \{S_n\} \) is \( \bar{x} \)-balanced. On the other hand, since \( Q \setminus S_n = H^{\emptyset,\{1,\ldots,n\}} \subset Q \setminus S_1 \subset H^{\emptyset,\{1\}} \), \( \bar{x}Q\setminus S_n \) belongs to \( \Delta^{H^{\emptyset,\{1\}}} \), i.e. \( \supp(\bar{x}Q\setminus S_n) \subset H^{\emptyset,\{1\}} \) for all \( n = 1,2,\ldots \). Therefore, every point of \( \overline{\supp(\bar{x}Q\setminus S_n)} : n = 1,2,\ldots \) has the support in \( H^{\emptyset,\{1\}} \). However, since \( \supp(\bar{x}) = Q \), we have \( \bar{x} \notin \overline{\supp(\bar{x}Q\setminus S_n)} : n = 1,2,\ldots \) and \( B' = \{Q \setminus S_n : n = 1,2,\ldots\} \) is not \( \bar{x} \)-balanced.

We expect that suitable partitions of \( Q \) satisfy the balancedness we have defined. The following proposition assures us our definition of balancedness is appropriate.
Proposition 3 Let $\overline{x}$ be an element of $\Delta_{++}$. Let $\{A_i\}$ be a countable covering of a compact Hausdorff space $Q$ such that $A_i \in \Sigma$ for all $i$ and $\overline{x}(A_i \cap A_j) = 0$ for $i \neq j$. Then $\{A_i\}$ is $\overline{x}$-balanced. In particular, any countable partition of $Q$ consisting of Borel sets is $\overline{x}$-balanced for any $\overline{x} \in \Delta_{++}$.

Proof. Define a disjoint countable covering $\{B_j\}$ of $Q$ by $B_j = A_j \setminus \bigcup_{i>j} A_i$. Then it is easily seen that $\overline{x}(B_j) = \overline{x}(A_j)$ and $\overline{x}^{B_j} = \overline{x}^{A_j}$. Therefore, for any $A \in \Sigma$,

$$\overline{x}(A) = \sum \overline{x}(A \cap B_j) = \sum \overline{x}(B_j) \overline{x}^{B_j}(A) = \sum \overline{x}(B_j) \overline{x}^{A_j}(A).$$

Since $\{B_j\}$ is a disjoint covering of $Q$, we have $\sum \overline{x}(B_j) = 1$. If the sum is essentially finite, then the proof is completed. Suppose the sum has infinite terms essentially. We can assume $\overline{x}(B_1) \neq 0$ without loss of generality. For any $n = 1, 2, \ldots$, define an element $x_n$ of $\text{co}\{\overline{x}^{A_j} : j = 1, 2, \ldots\}$ by $x_n = \sum_{j=1}^{n} (\overline{x}(B_j)/\lambda_n) \overline{x}^{A_j}$, where $\lambda_n = \sum_{j=1}^{n} \overline{x}(B_j)$. Then we have the equations

$$\overline{x}(A) = (\lambda_n x_n)(A) + \sum_{j>n} \overline{x}(B_j) \overline{x}^{A_j}(A)$$

$$= x_n(A) + (\lambda_n - 1)x_n(A) + \sum_{j>n} \overline{x}(B_j) \overline{x}^{A_j}(A).$$

Therefore we have

$$|\overline{x}(A) - x_n(A)| \leq (1 - \lambda_n)x_n(A) + \sum_{j>n} \overline{x}(B_j) \leq 2(1 - \lambda_n).$$

We can conclude $x_n \to \overline{x}$ from Lemma 1 since $\lambda_n \to 1$. Therefore we have $\overline{x} \in \text{co}\{\overline{x}^{A_j} : j = 1, 2, \ldots\}$. □

We give another example of a balanced family such that any two sets of the family have a nonempty intersection.

Example 4 Let $N^*$ be the one point compactification of the positive integers, and $\overline{x}$ the element defined in Example 2 above. Consider the family
\{A, B, C\} of the subsets of \(N^*\) defined by \(A = \{1, 2\}\), \(B = \{2, 3, \ldots, \infty\}\), and \(C = \{3, 4, \ldots, \infty, 1\}\). Then the family \(\{A, B, C\}\) is \(\bar{x}\)-balanced.

In fact, we have

\[
\bar{x}^A(n) = \begin{cases} 
2/3 & \text{for } n = 1 \\
1/3 & \text{for } n = 2 \\
0 & \text{otherwise}
\end{cases}, \quad
\bar{x}^B(n) = \begin{cases} 
0 & \text{for } n = 1 \\
2/3 & \text{for } n = \infty \\
1/(3 \times 2^{(n-1)}) & \text{otherwise}
\end{cases}, \\
\bar{x}^C(n) = \begin{cases} 
2/7 & \text{for } n = 1 \\
4/7 & \text{for } n = \infty \\
1/(7 \times 2^{(n-2)}) & \text{otherwise}
\end{cases}
\]

and

\[
\bar{x} = \frac{3}{16} \bar{x}^A + \frac{3}{8} \bar{x}^B + \frac{7}{16} \bar{x}^C.
\]

References

