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ON A LINEAR THERMOELASTIC PLATE EQUATION

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In this note, let me consider the linear thermoelastic plate equation:

$$(1) \qquad u_{tt} - h\Delta u_{tt} + \Delta^2 u + \alpha\Delta\theta = 0 \qquad \text{in } (0,\infty)\times\Omega,$$

$$(2) \qquad \theta_t - \beta\Delta\theta - \alpha\Delta u_t = 0 \qquad \text{in } (0,\infty)\times\Omega,$$

$$(3) \qquad u(0,x) = u_0(x),\ \ u_t(0,x) = u_1(x),\ \ \theta(0,x) = \theta_0(x) \ \text{ in } \Omega,$$

where $\alpha \neq 0$, $\beta > 0$ and $h \geq 0$ are real constants. Ω is a bounded domain in $\mathbf{R}^n$ with C^∞ boundary $\partial\Omega$, which is identified with a thin plate of height h. u and θ denote vertical deflection of the plate and temperature, respectively. The derivation of (1) and (2) can be found in J. Lagnese's book,[1] where Lagnese discussed stability of various plate models and showed that the energy of a linear thermoelastic plate decays exponentially fast with a certain dissipative boundary condition. In this note, I would like to consider the following two questions under suitable boundary conditions.

(Q.1) Does the first energy decay exponentially fast ?
(Q.2) Do solutions become smooth enough even if the initial data belong to a first energy class only ?

From a physical point of view, the energy of motion changes to the temperature, so that even if the total energy is conserved, the motion will stop at time infinity. The exponential decay of solutions represents this physical aspect. Namely, (Q.1) should have an affirmative answer. The second question is concerning the fact that the dissipation from temperature smoothen the motion. Thus, (Q.2) has an affirmative answer if the dissipation from temperature is strong enough. From a mathematical point of view, if $h = 0$, then both (1) and (2) seem to be parabolic, so that (Q.2) has an affirmative answer. But, if $h > 0$, the first equation is a hyperbolic equation with respect to u, so that (Q.2) must have a negative answer.

Now, let us try to answer two questions under the following boundary conditon:

$$(4) \qquad u = \Delta u = \theta = 0 \quad \text{on } (0,\infty)\times\partial\Omega.$$

Roughly speaking, I shall prove that

(A.1) the first energy of solutions to (1)–(4) decays exponentially fast:

[1] *Boundary stabilization of thin plate,* SIAM Studies in Appl. Math. **10**, Philadelphia, 1989.

(A.2) when $h = 0$, solutions to (1)–(4) become smooth for $t > 0$ even if the initial data u_0, u_1 and θ_0 belong to the first energy class only:

(A.3) when $h > 0$, each time section of solutions to (1)– (4) belongs to the same class for all $t \geq 0$.

Namely, (A.2) is the affirmative answer of (Q.2) and (A.3) is the negative answer of (Q.2).

Now, let me give you a sketch of proofs of the assertions (A.1)–(A.3). The key idea is to use an orthonormal system $\{\phi_n\}$ of $L^2(\Omega)$, where each ϕ_n is an eigenfunction of $-\Delta$ with zero Dirichlet boundary conditon corresponding to the eigenvalue λ_n, i.e.

$$-\Delta\phi_n = \lambda_n\phi_n \text{ in } \Omega \text{ and } \phi_n = 0 \text{ on } \partial\Omega;$$
$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \quad (\lambda_n \to \infty \text{ as } n \to \infty).$$

Using the fact that $\Delta^2\phi_n = \lambda_n^2\phi_n$ in Ω and $\phi_n = \Delta\phi_n = 0$ on $\partial\Omega$, you can reduce the problem (1)– (4) to the ordinary differential equations:

$$\tag{5} \begin{cases} (1+h\lambda_n)u_n'' + \lambda_n^2 u_n - \alpha\lambda_n\theta_n = 0, & t > 0, \\ \theta_n' + \beta\lambda_n\theta_n + \alpha\lambda_n u_n' = 0, & t > 0, \\ u_n(0) = u_n^0, \ \ u_n'(0) = u_n^1, \ \ \theta_n(0) = \theta_n^0, \end{cases}$$

where

$$\tag{6} u_i(x) = \sum_{n=1}^{\infty} u_n^i\phi_n(x) \quad (i = 0, 1), \quad \theta_0(x) = \sum_{n=1}^{\infty} \theta_n^0\phi_n(x).$$

And then, solutions $u(t, x)$ and $\theta(t, x)$ to (1)–(4) are represented by the relations:

$$\tag{7} u(t, x) = \sum_{n=1}^{\infty} u_n(t)\phi_n(x) \text{ and } \theta(t, x) = \sum_{n=1}^{\infty} \theta_n(t)\phi_n(x).$$

To investigate the properties of u and θ, in view of (5), you have to know the asymptotic behaviour of the characteristic roots. In fact, the equations in (5) are written in the following matrix form:

$$U_n' = A_n U_n \ \ t > 0 \text{ and } U_n(0) = \begin{bmatrix} u_n^0 \\ u_n^1 \\ \theta_n^0 \end{bmatrix},$$

where

$$U_n(t) = \begin{bmatrix} u_n(t) \\ u_n'(t) \\ \theta_n(t) \end{bmatrix} \text{ and } A_n = \begin{pmatrix} 0 & 1 & 0 \\ \frac{-\lambda_n^2}{1+h\lambda_n} & 0 & \frac{\alpha\lambda_n}{1+h\lambda_n} \\ 0 & -\alpha\lambda_n & -\beta\lambda_n \end{pmatrix}$$

Put

$$f_n(k) = det(kI - A) = k^3 + \beta\lambda_n k^2 + \frac{(\alpha^2+1)\lambda_n^2}{1+h\lambda_n}k + \frac{\beta\lambda_n^3}{1+h\lambda_n}.$$

Let me denote three roots of the algebraic equation: $f_n(k) = 0$ by $k_0(\lambda_n)$ and $k_\pm(\lambda_n)$ where $k_0(\lambda_n)$ is real and $\pm Im \ \ k_\pm(\lambda_n) > 0$. To know the property of the roots we use the following fact.

Corollary of Hurwitz's theorem. *Let a, b, $c \in \mathbf{R}$. In order that all the roots of the algebraic equation: $x^3 + ax^2 + bx + c = 0$, have negative imaginary part, it is necessary and sufficient that a, b, $c > 0$ and $ab - c > 0$.*

And then, we know that $k_0(\lambda_n) < 0$ and Re $k_\pm(\lambda_n) < 0$ for all $n \geq 0$. You have to know the asymptotic behaviour of the roots as $n \to \infty$.

First I consider the case that $h = 0$. The argument below follows the paper due to Racke and Rivera,[2] where they handled with thermoelastic bar and plate equations having the Kirchhoff type nonlocal nonlinearity with the boundary condition (4) in the rather abstract setting and they show the exponential decay and smoothing property when $h = 0$. Put $k = \lambda_n l$, and then

$$f_n(k) = \lambda_n^3(l^3 + \beta l^2 + (\alpha^2 + 1)l + \beta) = 0.$$

Denoting three roots of the equation: $l^3 + \beta l^2 + (\alpha^2 + 1)l + \beta = 0$, by l_0 and $l_\pm$ where $l_0 < 0$, Re $l_\pm < 0$ and $\pm$Im $l_\pm > 0$ (cf. Hurwitz's theorem), we have

$$k_0(\lambda_n) = l_0\lambda_n \text{ and } k_\pm(\lambda_n) = l_\pm\lambda_n \qquad \text{when } h = 0. \tag{8}$$

Put

$$U(t,x) = \begin{bmatrix} u(t,x) \\ u_t(t,x) \\ \theta(t,x) \end{bmatrix},$$

and then

$$U(t,x) = \sum_{n=1}^{\infty} e^{tA_n} U_n(0)\phi_n(x).$$

By (8) we can see that

$$|e^{tA_n} U_n(0)| \leq Ce^{-c_0\lambda_n t}|U_n(0)| \tag{9}$$

where $c_0 = -min\ (l_0, \text{Re } l_+, \text{Re } l_-)$, which immediately implies that

$$\|\Delta u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2 + \|\theta(t,\cdot)\|^2 \leq Ce^{-c_0\lambda_1 t}\left\{\|\Delta u_0\|^2 + \|u_1\|^2 + \|\theta_0\|^2\right\},$$

where $\|\cdot\|$ denotes the usual L^2-norm on Ω. This is the exponential result, the affirmative answer to (Q.1), when $h = 0$. To show (A.2), you observe that

$$\left(\frac{\partial}{\partial t}\right)^K (-\Delta)^M U(t,x) = \sum_{n=1}^{\infty} e^{tA_n}(A_n)^K \lambda_n^M U_n(0)\phi_n(x). \tag{10}$$

When $t > 0$, by (9) you have

$$\left|e^{tA_n} U_n(0)\right| \leq \frac{N!}{(C\lambda_n t)^N}|U_n(0)| \quad \text{for any } N \geq 1,$$

[2] *Smoothing properties, decay and global existence of solutions to nonlinear coupled systems of thermoelastic type,* Preprint in 1993

which together with (10) implies that

$$\sharp\left(\frac{\partial}{\partial t}\right)^K(-\Delta)^M U(t,\cdot)\sharp \le C_N t^{-N}\{\|\Delta u_0\|^2+\|u_1\|^2+\|\theta_0\|^2\}$$

for a large N depending on K and M, where

$$\sharp U\sharp^2 = \|\Delta u\|^2+\|v\|^2+\|\theta\|^2 \quad \text{for } U=\begin{bmatrix} u\\ v\\ \theta\end{bmatrix}.$$

This shows that the solutions u and θ become C^∞ for $t>0$ when $h=0$, so that (A.2) is proved.

Now, let us consider the case that $h>0$. The roots $k_0(\lambda_n)$ and $k_\pm(\lambda_n)$ have the following asymptotic behaviours:

$$\begin{aligned} k_0(\lambda_n) &= -\beta\lambda_n+\sum_{j=0}^{\infty} d_0^j\lambda_n^{-j},\\ k_\pm(\lambda_n) &= \pm\frac{\sqrt{-1}}{\sqrt{h}}\lambda_n^{1/2}-\frac{\alpha^2}{2\beta\sqrt{h}}+\sum_{j=1}^{\infty} d_\pm^j\lambda_n^{-j/2} \end{aligned} \tag{12}$$

as $n\to\infty$. Since $k_0(\lambda_n)<0$ and Re $k_\pm(\lambda_n)<0$ as follows from Hurwitz's theorem, by (12) we see that there exists a $c_1>0$ such that

$$k_0(\lambda_n),\ \text{Re } k_\pm(\lambda_n)\le -c_1 \quad \text{for all } n\ge 1,$$

so that we can also prove that

$$\begin{aligned} &\|\Delta u(t,\cdot)\|^2+\|u_t(t,\cdot)\|^2+h\|\nabla u_t(t,\cdot)\|^2+\|\theta(t,\cdot)\|^2\\ &\quad\le Ce^{-c_1t}\left\{\|\Delta u_0\|^2+\|u_1\|^2+h\|\nabla u_1\|^2+\|\theta_0\|^2\right\} \end{aligned}$$

for a suitable $C>0$, where u and θ are solutions to (1)–(4) for $h>0$ and $\nabla v = (\partial v/\partial x_1,\ldots,\partial v/\partial x_n)$. This means that the first energy of solutions to (1)–(4) decays exponentially fast when $h>0$, i.e., (A.1) is proved.

Finally, let me discuss about (A.3). For simplicity, I consider the case that $u_1=\theta_0=0$. And then, by representing solutions to (5), you can show that

$$\lambda_n^2|u_n(t)|^2+(1+h\lambda_n)|v_n(t)|^2+|\theta_n(t)|^2\ge C_2e^{-c_3t}\lambda_n^2|u_n^0|^2$$

for large n with suitable positive constants C_2 and c_3[3] which implies (A.3). In fact, for example if we assume that

$$\sum_{n=1}^{\infty}\lambda_n^4|u_n^0|^2=\infty \ \ (\text{i.e., } \Delta^2u\notin L^2),$$

[3] I shall give a proof elsewhere in future.

then

$$\|\Delta^2 u(t,\cdot)\|^2 + \|\Delta u_t(t,\cdot)\|^2 + h\|\Delta\nabla u_t(t,\cdot)\|^2 + \|\Delta\theta(t,\cdot)\|^2 = \infty \quad \text{for } t \geq 0.$$

I think that it is very interesting in considering the same problem under other boundary conditions, for example,

$$\text{(D)} \qquad u = \frac{\partial u}{\partial \nu} = \theta = 0 \qquad \text{on } \partial\Omega,$$

$$\text{(N)} \qquad \Delta u + \alpha\theta = \frac{\partial}{\partial \nu}(\Delta u + \alpha\theta) = \frac{\partial \theta}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where $\partial/\partial\nu$ denotes the outward normal derivatives on $\partial\Omega$. When $h = 0$, the exponential decay result is known. Namely, J.U.Kim[4] proved the following theorem.

Theorem. *There exist C and $\gamma > 0$ such that*

$$\begin{aligned} &\|u(t,\cdot)\|_2^2 + \|u_t(t,\cdot)\|^2 + \|\theta(t,\cdot)\|^2 \\ &\qquad \leq Ce^{-\alpha t}\left\{\|u_0\|_2^2 + \|u_1\|^2 + \|\theta_0\|^2\right\} \end{aligned}$$

where $\|v\|_2^2 = \sum_{|\alpha|\leq 2}\|\partial_x^\alpha v\|^2$, provided that u and θ solve the problem (1), (2), (3) *and* (D).

Recently, the author[5] proved the exponential decay result when $h = 0$ and the boundary condition is (N) case. To state the theorem more precisely, I have to introduce some functional spaces

$$\begin{aligned} H_\Delta^2 &= \{u \in L^2 \mid \Delta u \in L^2\}, & Y &= \{u \in L^2 \mid \Delta u = 0 \ \text{ in } \Omega\}, \\ X_0 &= \{u \in L^2 \mid (u,v) = 0 \ \ \forall v \in Y\}, & X_1 &= \{u \in H_\Delta^2 \mid (u,v)_\Delta = 0 \ \ \forall v \in Y\}, \end{aligned}$$

where $(\cdot,\cdot)$ is the usual L^2-innerproduct and $(u,v)_\Delta = (\Delta u, \Delta v) + (u,v)$, which is the innerproduct of H_Δ^2.

Theorem. *Let H_X be the set of all (u,v,θ) satisfying the condition:*

$$u \in X_1, \ \ v \in X_0, \ \ \theta \in L^2, \ \ \int_\Omega (\theta - \alpha\Delta u)dx = 0.$$

Then, there exist positive constants C and γ such that

$$\begin{aligned} &\|\Delta u(t,\cdot)\|^2 + \|u(t,\cdot)\| + \|u_t(t,\cdot)\|^2 + \|\theta(t,\cdot)\|^2 \\ &\qquad \leq Ce^{-\gamma t}\left\{\|\Delta u_0\| + \|u_0\|^2 + \|u_1\|^2 + \|\theta_0\|^2\right\} \end{aligned} \tag{13}$$

[4] *On the energy decay of a linear thermoelastic bar and plate,* SIAM J. Math. Anal., **23** (1992), 889-899.

[5] *On the exponential decay of the energy of a linear thermoelastic plate,* Preprint in 1993.

provided that $(u_0, u_1, \theta_0) \in H_X$ *and* u *and* θ *solve the problem* (1), (2), (3) *and* (N).

Moreover, for general initial data $u_0 \in H^2_\Delta$, $u_1 \in L^2$ and $\theta_0 \in L^2$, we can represent solutions by the relation

$$u(t,x) = u_E(t,x) + u_S(t,x), \quad \theta(t,x) = \theta_E(t,x) + \theta_S(t,x)$$

where u_E and θ_E satisfy the estimate of type (13) and

$$\begin{aligned} u_S(t,x) &= ty_{1Y}(x) + w_0(x) + u_{0y}(x), \quad \theta_S(t,x) = \theta_0(x) - \theta_1; \\ \theta_1 &= \frac{1}{(1+\alpha^2)|\Omega|} \int_\Omega (\theta_0(x) - \alpha \Delta u_0(x)) dx; \\ w_0(x) &\in X_1, \quad \Delta w_0 = -\alpha \theta_1 \text{ in } \Omega; \\ u_0(x) &= u_{0X}(x) + u_{0Y}(x) \in X_1 \oplus Y = H^2_\Delta; \\ u_1(x) &= u_{1X}(x) + u_{1Y}(x) \in X^0 \oplus Y = L^2. \end{aligned}$$

When $h = 0$, to show that (Q.2) has an affirmative answer is an open problem for (D) and (N). Moreover, when $h > 0$, (Q.1) and (Q.2) have so far no answers at all for (D) and (N). This is, I think, very interesting problem.