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Kyoto University
On Instability in Geometric Evolution Equations

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ABSTRACT. A general parabolic evolution equation is considered for a closed hypersurface in Euclidean space. All stationary solutions are shown to be Lyapunov unstable if the normal velocity of a hypersurface depends only on its normal and second fundamental form and is independent of its position. Instability of time periodic solution is also discussed.

1. Introduction
This is a preliminary note. We consider the initial value problem of an evolution of a closed hypersurface $\Gamma_t$ in $\mathbb{R}^n$

$$V = f(n, -A).$$

(1)

Here $n$ is an inward unit normal vector field on $\Gamma_t$ and $V$ is normal velocity in the direction of $n$; $A = -\partial n$ denotes the second fundamental form. We shall prove that all stationary solutions $S$ of (1) is Lyapunov unstable provided that (1) is (nondegenerate) parabolic. This generalized a recent work of Ei and Yanagida [EY] where they assumed that $f$ depends on $A$ only through its mean curvature. Their method is completely different from ours. They linearized equation around stationary solution and appeal to spectral analysis. Their method applies to the equation depending on space variable but invariant under translation. We simply use a distance function of $S$ and appeals to the maximum principle. We believe our proof is simpler than theirs for this problem.

Our method also applies to instability of the periodic solution of

$$V = f(t, n, -A).$$

(2)

where $f$ is time periodic. We show that periodic solutions $S_t$ are unstable unless second fundamental form vanishes somewhere on $S_t$ for all $t$.

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2. Parabolic evolution equations

We formulate our equations as in [GG1]. Let $E$ be a bundle over the sphere $S^{n-1}$ of the form

$$E = \{ (\overline{p}, Q_{\overline{p}}(X)) \in S^{n-1} \times S_n; X \in S_n \}$$

with $Q_{\overline{p}}(X) = R_{\overline{p}}XR_{\overline{p}}$ and $R_{\overline{p}} = I - \overline{p} \otimes \overline{p}$; $R_{\overline{p}}$ is the projection orthogonal to $\overline{p}$. Here $S_n$ denotes the space of the $n \times n$ real symmetric matrices. By the standard Euclidean metric the bundle $E$ is identified with the tensor bundle $TS^{n-1} \otimes T^*S^{n-1}$ over $S^{n-1}$. Let $f$ be a function from $[0, \infty) \times E$ to $\mathbb{R}$. We shall always assume that $f$ is at least continuous.

Suppose that a hypersurface $\Gamma$ is given as a zero level set of $u$ in $\mathbb{R}^n$ such that the gradient $\nabla u \neq 0$ on $\Gamma$ and $n = \nabla u/|\nabla u|$. Then as in [GG1], the second fundamental form (in the direction of $n$) is of the form

$$A = -Q_{\overline{p}}(\nabla^2 u)/|\nabla u|$$

with $\overline{p} = \nabla u/|\nabla u|$, (3)

where $\nabla^2 u$ denotes the Hessian of $u$ in space variables.

We recall the notion of parabolicity of the equation

$$V = f(t, n, -A)$$

for evolving hypersurface $\Gamma_t$. It is convenient to introduce the level set equation

$$u_t + F_f(t, \nabla u, \nabla^2 u) = 0$$

with

$$F_f(t, p, X) = |p|f(t, p/|p|, Q_{\overline{p}}(X)/|p|).$$

This equation is uniquely determined if each level set of $u$ moves by (4) and a super level set $u > c$ is “inside” the level set $u = c$. Sign in (6) are different from those in [CGG] because our convention of $n$ is opposite.

We say (4) is strictly parabolic (uniformly in $t$) if for each $M > 0$ there is $\mu > 0$ such that

$$F_f(t, \overline{p}, X + Y) - F_f(t, \overline{p}, X) \leq -\mu \text{ trace}(Q_{\overline{p}}(Y))$$

(7)
for all $Y \geq 0$, $|X| \leq M$, $|p| = 1$, $t \in [0, \infty)$, where $|X|$ is the operator norm of $X$ as a selfadjoint operator. If (7) holds for $\mu = 0$ we say (4) is (degenerate) parabolic. A level set method [CGG], [ES] provides a unique global generalized solution. The following version is taken from [GG1].

2.1. Unique global existence. Suppose that (4) is parabolic. Let $\Gamma_0$ be the boundary of a bounded open set in $\mathbb{R}^n$. Then there is a unique generalized solution $\{\Gamma_t\}_{t \geq 0}$ of (4) starting from $\Gamma_0$.

If $f$ and $\Gamma_0$ are smooth enough and (4) is parabolic, there is a local-in-time classical solution $\Sigma_t$ (see e.g. [GG2]). Moreover $\Sigma_t$ agrees with $\Gamma_t$ as far as the former exists [GG2] (see also [ES] for the mean curvature flow). So our generalized solution is a natural extension of classical solution.

3. Instability of stationary solution

We say that $C^2$ hypersurface $S$ is stationary for

$$V = f(n, -A)$$ (8)

if $f(n, -A) = 0$ on $S$. Let $U(\alpha)$ denote a tubular neighborhood of $S$ of the form

$$U(\alpha) = \{x \in \mathbb{R}^n; \text{dist}(x, S) < \alpha\},$$

where dist denotes the distance. We say that $S$ is Lyapunov stable for (8) if for each $\epsilon > 0$ there is $\delta > 0$ such that a (generalized) solution $\Gamma_t$ with initial data $\Gamma_0$ stays in $U(\epsilon)$ for all $t > 0$ provided that $\Gamma_0$ is contained in $U(\epsilon)$. If not, $S$ is called unstable. If $\alpha$ is a supremum of $\alpha'$ such that every point of $U(\alpha')$ has a unique nearest point on $S$, $\alpha$ is called the reach of $S$ and dented by reach $S$.

3.1. Instability Theorem. Suppose that (8) is strictly parabolic. Let $S$ be a stationary $C^2$ hypersurface of (8) such that $S = \partial D$ for some open set $D$ in $\mathbb{R}^n$. Suppose that
reach $S = \alpha_0 > 0$, $\inf_S |A| = \sigma > 0$ and that $|A|$ is bounded on $S$ (if $S$ is not compact).

For $0 < \alpha < \alpha_0$ let

$$\Gamma^\alpha = \{x \in D; \text{dist}(x, S) = \alpha\}. \quad (9)$$

Let $\Gamma^\alpha_t$ be a (generalized) solution of (8) starting from $\Gamma^\alpha$. Then there are $\alpha_1$ ($0 < \alpha_1 < \alpha_0$) and $c_0 = c_0(f, \sigma, S) > 0$ such that

$$\text{dist}(\Gamma^\alpha_t, S) \geq (\alpha + c_0 t) \wedge \alpha_1 \quad \text{for all } t > 0 \quad (10)$$

where $a \wedge b = \min(a, b)$. The same inequality holds if $D$ in (9) is replaced by its complement.

3.2. Corollary. Suppose that $S$ is a stationary $C^2$ closed hypersurface for (8) with non-vanishing second fundamental form. Then $S$ is (Lyapunov) unstable provided that (8) is strictly parabolic.

Of course Lyapunov instability follows from (10).

3.3. Remark on noncompact surface. Following formula for distance function is a key for the proof of Theorem 3.1. Let $v$ be the signed distance function of $S$, i.e.,

$$v(x) = \begin{cases} 
\text{dist}(x, S) & \text{for } x \in D \\
-\text{dist}(x, S) & \text{otherwise.}
\end{cases}$$

If $S$ is $C^2$, then so is $d$ which is proved in [GT; §14].

Even for a non bounded open set generalized solution can be constructed by levelset method; see Ilmanen [I] and Ishiiand Souganidis [IS].
3.4. Lemma. For a general $C^2$ hypersurface $S = \partial D$ with an open set $D$

$$\nabla^2 v(y)(I - v(y)\nabla^2 v(y))^{-1} = \nabla^2 v(x), \quad y = x + vn, \quad x \in S$$

for $|v| < $ reach $S$.

This is also a key in [GG2], where the local existence of classical solution is proved for (2) by a level set method.

3.5. Proof of Theorem 3.1. We set

$$w(t, x) = v(x) - \rho(t) \quad \text{with} \quad \rho(t) = \alpha + c_0 t.$$ 

Our goal is to take $c_0 > 0$ so that $w$ is a supersolution of the level set equation of (8):

$$u_t + F(\nabla u, \nabla^2 u) = 0, \quad F = F_f$$

in a set $U_+^{\alpha_1} \setminus U_+^{\alpha/2}$, where

$$U_+^{\alpha} = \{x \in D; \text{dist}(x, S) < \alpha\}.$$ 

If such a $c_0$ exists, comparison principle ([CGG], [GG1]) implies that $\Gamma_t^\alpha$ is contained in \{w \geq 0\}. This yields (10).

Since $S$ is stationary, we see

$$F(\nabla v, \nabla^2 v) = 0 \quad \text{on} \quad S.$$ 

By Lemma 3.4 this yields

$$F(\nabla v, \nabla^2 v(I - v \nabla^2 v)^{-1}) = 0 \quad \text{in} \quad U(\alpha_0).$$

This implies

$$w_t + F(\nabla w, \nabla^2 w) = -c_0 + F(\bar{p}, X) - F(\bar{p}, X(I - vX)^{-1}) \quad \text{in} \quad U_+^{\alpha_0}. \quad (11)$$
with $\overline{p} = \nabla v$, $X = \nabla^2 v$.

We take $\alpha_1 > 0$ small so that

$$|\nabla^2 v| \geq \sigma/2 \quad \text{in } U_+(\alpha_1) \quad \text{(by } \inf |A|),$$

$$(I - v\nabla^2 v)^{-1} \geq I/2 \quad \text{in } U_+(\alpha_1).$$

Since $X\overline{p} \otimes \overline{p} = 0$ by $|\nabla v| = 1$, we see

$$Q_{\overline{p}}(X^2(I - rX))^{-1} = X^2(I - rX)^{-1} = -\frac{1}{v}(X - X(I - vX)^{-1})$$

By parabolicity

$$F(\overline{p}, X) - F(\overline{p}, X(I - vX)^{-1}) \geq \mu v \text{trace } Q_{\overline{p}}[X^2(I - rX)^{-1}]$$

$$= \mu v \text{trace } X^2(I - rX)^{-1} \quad \text{on } U_+(\alpha_1).$$

with $M = \sup_{U_+(\alpha_1)} |\nabla^2 v|$. Using (12), (13) we see

$$\text{trace } X^2(I - rX)^{-1} \geq \frac{1}{2} \left( \frac{\sigma}{2} \right)^2 = c_1$$

which yields

$$F(\overline{p}, X) - F(\overline{p}, X(I - vX)^{-1}) \geq \mu c_1 v \quad \text{on } U_+(\alpha_1).$$

If we set $c_0 = \mu c_1 \alpha/2$, from (11) it follows that $w$ is a classical supersolution of the level set equation of (8) in $U_+(\alpha_1) \setminus \overline{U_+(\alpha/2)}$.

The proof for the last statement is parallel so is omitted.

3.6. General instability Theorem. For (8) there is no stable stationary $C^2$ closed hypersurface provided that (8) is strictly parabolic and that $f$ is $C^1$.

Proof. By Corollary 2.2 we may assume that there is a point on at which

$$(n, A) = (\overline{p}_0, O), f(n, A) = (\overline{p}, O)$$

for some $\overline{p}_0 \in S^{n-1}$. Since $S$ is stationary,

$$f(\overline{p}_0, O) = 0.$$
3.7. Nonexistence Lemma. Suppose that \((8)\) is strictly parabolic and \(f\) is \(C^1\). Suppose that \(f(\overline{p}_0, O) = 0\) for some \(\overline{p}_0 \in S^{n-1}\). Then there is no stationary \(C^2\) closed hypersurface for \((8)\).

**Proof.** Let \(S\) be a stationary \(C^2\) closed hypersurface. Since \(S\) is compact, there is a half space \(H\) such that

\[
H = \{ x + c \in \mathbb{R}^n; \quad x \cdot \overline{p}_0 \geq 0 \}, \quad S \subset H \text{ with } c \in S.
\]

Note that \(\partial H\) is a stationary solution of \((8)\) since \(f(\overline{p}_0, O) = 0\). Since \((8)\) is strictly parabolic and \(f\) is \(C^1\) we may apply the strong maximum principle and conclude \(S\) cannot touch \(\partial H\) for \(t > 0\). This contradicts the existence of stationary closed hypersurface \(S\).

3.8. Remark. If \(f\) is \(C^1\) in \(A\), the parabolicity is equivalent to say that \(\partial f / \partial A\) is positive definite.

4. Instability of periodic solutions

We consider

\[
V = f(t, n, -A),
\]

where \(f : [0, T] \times E \to \mathbb{R}\) is continuous and \(T\)-periodic, i.e. \(f(t, \overline{p}, -A) = f(t + T, \overline{p}, -A)\).

We say \(S_t (-\infty < t < \infty)\) is a \(T\)-periodic \(C^{2,1}\) solution of \((14)\) such that \(S_t = S_{t+T}\) where \(C^{2,1}\) implies that \(C^2\) in space and \(C^1\) in time. Note that the signed distance function \(v\) of \(S_t\) is now a \(C^{2,1}\) function.

Let \(U(\alpha, t)\) denote the \(\alpha\)-tubular neighborhood of \(S_t\). We say \(S_t\) is *Lyapunov stable* for \((14)\) if for each \(\epsilon > 0\) there is \(\delta > 0\) such that a generalized solution \(\Gamma_t\) with \(\Gamma_t|_{t=t_0} = \Gamma_0\) always stays in \(U(\epsilon, t)\) for all \(t > t_0\) provided that \(\Gamma_{t_0} \subset U(\delta, t_0)\).

In some cases \(S_t\) is called \(S_t\) is called \(T\)-periodic even if \(S_t\) is \(T\)-periodic up to \(T\)-periodic translation \(a(t)\), i.e., \(S_{t+T} = S_t + a(t)\). Here \(a(t)\) is assumed to have the form

\[
a(t) = b(t + T) - b(t)
\]
for some $C^1$ function $b(t)$. If we set

$$\Sigma_t = S_t - b(t)$$

then $\Sigma_t = \Sigma_{t+T}$ and $\Sigma_t$ solves

$$V = f + n \cdot b(t).$$

Thus by $T$-periodic solution we shall always mean $T$-periodic with no ambiguity of translation.

4.1. Instability Theorem. Suppose that (14) is strictly parabolic. Let $S_t$ be a $T$-periodic $C^2$ solution of (8) of closed hypersurfaces surrounding an bounded open set $D$ in $\mathbb{R}^n$. Suppose that $\inf S_t |A| = a(t) \neq 0$. Let $\alpha_0 > 0$ denote the minimum of reach $S_t$ in $t$. For $0 < \alpha < \alpha_0$, $t_0 \in \mathbb{R}$

$$\Gamma^\alpha = \{ x \in D ; \text{dist}(x, S_{t_0}) = \alpha \} \text{ for } t > t_0.$$

Let $\Gamma_t^\alpha$ be a (generalized) solution of (14) with $\Gamma_t^\alpha = \Gamma^\alpha$ at $t = t_0$. Then there are $\alpha_1(S_t) (0 < \alpha_1 < \alpha_0)$ and nonnegative $T$-periodic function $c_0(t)(\neq 0)$ depending only on $f, a, S_t$ such that

$$\text{dist}(\Gamma_t^\alpha, S_t) \geq (\alpha + \int_0^t c_0(\tau) \, d\tau) \wedge \alpha_1 \text{ for all } t > 0.$$

The same inequality holds if $D$ in the definition of $\Gamma^\alpha$ replaced by $\mathbb{R}^n \setminus D$.

Proof. As in the proof of Theorem 3.1 we set

$$w(x, t) = v(x, t) - \rho(t), \quad \rho(t) = c'_0 \int_0^t a(\tau)^2 \, d\tau,$$

and choose $c'_0 > 0$ in suitable way so that $w$ is a supersolution of the level set equation of (14). Here $v$ denotes the signed distance function of $S_t$. 
4.2. Corollary. Suppose that (14) is strictly parabolic. If $S_t$ is a $T$-periodic $C^{2,1}$ solution of (14) consisting of closed hypersurfaces, then $S_t$ is Lyapunov unstable if $\inf_{S_t} |A| \not\equiv 0$ as a function of time.

This is an immediate from Theorem 4.1.

References


[IS] H. Ishii and P. Souganidis, forthcoming