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THE CAUCHY PROBLEM FOR A WEAKLY CLOSED OPERATOR

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INTRODUCTION

We consider the Cauchy problem

$$(CP)\begin{cases}(d/dt)u(t)=Au(t) & \text{for } t\in[0,\infty),\\ u(0)=u_0,\end{cases}$$

in the largest space V^* of a triplet $\{V,H,V^*\}$ such that $V\subset H\subset V^*$, where the domain of A is the smallest space V and the initial value u_0 is an element of H. In [3], we gave an existence theorem of solutions to

$$(CP)_T\begin{cases}(d/dt)u(t)=Au(t) & \text{for } t\in[0,T),\ 0<T<\infty,\\ u(0)=u_0,\end{cases}$$

for a weakly closed operator A with range condition and "integrability" condition in a reflexive Banach space X. Moreover, in [4] we improved it and applied the result to the proof of existence of weak solutions of Navier-Stokes equations in a bounded domain in $\mathbb{R}^N (N=2 \text{ or } 3)$. The purpose of this report is twofold. First, we give two existence theorems of solutions to (CP). Second, we apply them to the proof of existence of weak solutions of Navier-Stokes equations in an unbounded domain in $\mathbb{R}^3$. We note that the existence of weak solutions of

Navier-Stokes equations is well known, see Leray [7], Hopf [2] and Temam [10], for example. The process of argument here is essentially along the same line as in [4]. We believe, however, that the applications to Navier-Stokes equations have become more elegant than in [4] because of two existence theorems.

1. Preliminaries

Let V be a reflexive Banach space with norm $\| \|_V$ and H a Hilbert space with inner product $(\, , \,)_H$ and norm $\| \|_H$, $V \subset H$, V dense in H with continuous injection. Let V^* be the dual of V with norm $\| \|_{V^*}$. Identifying H with its dual H^*, by the Riesz representation theorem we have

$$V \subset H \equiv H^* \subset V^*, \tag{1}$$

where each space is dense in the following one and the injections are continuous. Such a family $\{V, H, V^*\}$ is called a triplet. The scalar product between $u \in V^*$ and $v \in V$ is denoted by $\langle u, v \rangle_{V^*,V}$. We note that

$$\langle h, v \rangle_{V^*,V} = (h, v)_H \quad \text{for all } h \in H \text{ and } v \in V. \tag{2}$$

Definition. Let $\{V, H, V^*\}$ be a triplet. Let A be a single valued operator in V^* with domain V and let u_0 be an element of H. We say that $u : [0, \infty) \to V^*$ is a solution of (CP), if the following five conditions are satisfied.

(i) $u : [0, \infty) \to V^*$ is absolutely continuous;

(ii) $u(t) \in V$ for almost all $t \in [0, \infty)$;

(iii) $(d/dt)u(t) = Au(t)$ in V^* for almost all $t \in [0, \infty)$;

(iv) $u(0) = u_0$;

(v) $u(t) \in H$ for all $t \in [0, \infty)$.

In order to show the main theorems we use the following.

Theorem A [4, Corollary]. *Let $0 < T < \infty$. Let X be a reflexive Banach space with norm $\| \ \|$ and $u_k^n \in X$ for $n, k = 1, 2, 3, \cdots$. Let A be a single valued operator in X with domain $D(A)$ and range $R(A)$. Suppose the following three conditions hold.*

(H.1) there exists a subset $X_0 \subset X$ such that

$$D(A) \subset X_0 \subset \overline{D(A)} \quad \text{and} \quad R(1 - \lambda A) \supset X_0 \quad \text{for } \lambda > 0,$$

where $\overline{D(A)}$ denotes the closure of $D(A)$;

(H.2)

$$u_0^n \equiv u_0 \in X_0 \quad \text{and} \quad \left(1 - \frac{T}{n}A\right) u_k^n = u_{k-1}^n \quad \text{for } n, k = 1, 2, 3, \cdots,$$

and there exist a positive number $C(u_0)$ and a constant $p \in (1, \infty)$ such that

$$\text{(IA)} \qquad \frac{T}{n} \sum_{k=1}^{n} \|Au_k^n\|^p \leq C(u_0) \quad \text{for } n = 1, 2, 3, \cdots ;$$

(H.3) *A is a weakly closed operator, i.e., if* $x_n \in D(A)$, $x_n \to x$ *weakly and* $Ax_n \to y$ *weakly, then* $x \in D(A)$ *and* $Ax = y$.

Define the function $u^n : [0,T] \to X$, *setting*

$$u^n(t) = \begin{cases} u_k^n & \text{for } t \in \left(\frac{k-1}{n}T, \frac{k}{n}T\right], \\ u_0 & \text{for } t = 0. \end{cases}$$

Then there exist a subsequence $\{u^{n(j)}\}$ *of* $\{u^n\}$ *and an absolutely continuous function* $u : [0,T] \to X$ *which satisfy the following:*

(i) $\text{w-}\lim_{j\to\infty} u^{n(j)}(t) = u(t)$ *for all* $t \in [0,T]$;

(ii) $u(t) \in D(A)$ *for almost all* $t \in [0,T]$;

(iii) $(d/dt)u(t) = Au(t)$ *for almost all* $t \in [0,T)$;

(iv) $u(0) = u_0$;

(v) $Au \in L^p([0,T];X)$.

Remarks 1. (i) The symbol w-lim denotes weak limit.

(ii) See [3, Lemma 2] for Theorem A(v).

2. The main theorems

Theorem 1. *Let* $0 < T < \infty$. *Let* $\{V, H, V^*\}$ *be a triplet. Let* A *be a single valued operator in* V^* *with domain* V. *Suppose the following four conditions hold.*

(A.1)

$$R(1-\lambda A) \supset H \quad \text{for } \lambda > 0;$$

(A.2) for each $u_0 \in H$ *and a sequence* $\{u_k^n\}_{n,k\geq 1}$ *in* V *defined by*

$$u_0^n \equiv u_0 \quad \text{and} \quad \left(1 - \frac{T}{n}A\right) u_k^n = u_{k-1}^n \quad \text{for } n, k = 1, 2, 3, \cdots ,$$

there exist a positive number $C(u_0)$ *and a constant* $p \in (1, \infty)$ *such that*

(IA)
$$\frac{T}{n}\sum_{k=1}^{n} \|Au_k^n\|_{V^*}^p \leq C(u_0) \quad \text{for } n = 1, 2, 3, \cdots ;$$

(A.3) A *is a weakly closed operator in* V^*, *i.e., if* $x_n \in V$, $x_n \to x$ *weakly in* V^* *and* $Ax_n \to y$ *weakly in* V^*, *then* $x \in V$ *and* $Ax = y$;

(A.4) there exist two constants $\alpha \in \mathbb{R}$ *and* $\beta \geq 0$ *such that*

$$\langle Au, u\rangle_{V^*,V} \leq \alpha\|u\|_H^2 + \beta \quad \text{for all } u \in V.$$

Given $u_0 \in H$, *define the function* $u^n : [0, T] \to V^*$, *setting*

$$u^n(t) = \begin{cases} u_k^n & \text{for } t \in \left(\frac{k-1}{n}T, \frac{k}{n}T\right], \\ u_0 & \text{for } t = 0. \end{cases}$$

Then there exist a subsequence $\{u^{n(j)}\}$ *of* $\{u^n\}$ *and a solution* $u : [0, \infty) \to V^*$ *of (CP) which satisfy the following:*

(i) $u \in C_{\mathrm{w}}([0, \infty); H)$;

(ii) $\mathrm{w}\text{-}\lim_{j\to\infty} u^{n(j)}(t) = u(t)$ *in* H *for all* $t \in [0, T]$;

(iii) $Au \in L^p_{loc}([0,\infty); V^*)$;

(iv)

$$\begin{cases} \|u(t)\|_H^2 \le \|u_0\|_H^2 + 2\beta t & \text{for all } t \in [0,\infty), \text{ if } \alpha = 0; \\ \|u(t)\|_H^2 + \dfrac{\beta}{\alpha} \le e^{2\alpha t}\left(\|u_0\|_H^2 + \dfrac{\beta}{\alpha}\right) & \text{for all } t \in [0,\infty), \text{ if } \alpha \ne 0. \end{cases}$$

Remark 2. The symbol C_{w} in (i) denotes weak continuity.

Theorem 2. *Let V be a separable reflexive Banach space and let $\{V, H, V^*\}$ be a triplet. Let A be a single valued operator in V^* with domain V and let u_0 be an element of H. Suppose the following three conditions hold:*

(A.5) there exist $\alpha', \beta' \ge 0$ and $\gamma > 0$ such that

$$\langle Au, u\rangle_{V^*,V} \le \alpha'\|u\|_H^2 + \beta' - \gamma\|u\|_V^2 \quad \text{for all } u \in V;$$

(A.6) the operator $A : V \to V^*$ is weakly continuous, i.e.,

if $\text{w-lim}_{n\to\infty} u_n = u$ in V, then $\text{w-lim}_{n\to\infty} Au_n = Au$ in V^*;

(A.2)′ there exist an increasing function $\varphi : [0,\infty) \to [0,\infty)$ and a constant $p \in (1,\infty)$ such that

$$\text{(IP)} \qquad \|Au\|_{V^*} \le \varphi\left(\|u\|_H^2\right)\left(\|u\|_V^{2/p} + 1\right) \quad \text{for all } u \in V.$$

Then there exists a solution $u : [0,\infty) \to V^*$ of (CP) which satisfies the following:

(i) $u \in C_{\mathrm{w}}([0,\infty); H)$,

(ii) $u \in L^2_{loc}([0,\infty);V)$,

(iii) $Au \in L^p_{loc}([0,\infty);V^*)$.

Moreover, taking $\alpha \in \mathbb{R}$ *and* $\beta \geq 0$ *such that*

$$\langle Au, u\rangle_{V^*,V} \leq \alpha \|u\|_H^2 + \beta \quad \text{for all } u \in V,$$

we have the following:

(iv)

$$\begin{cases} \|u(t)\|_H^2 \leq \|u_0\|_H^2 + 2\beta t & \text{for all } t \in [0,\infty), \text{if } \alpha = 0, \\ \|u(t)\|_H^2 + \dfrac{\beta}{\alpha} \leq e^{2\alpha t}\left(\|u_0\|_H^2 + \dfrac{\beta}{\alpha}\right) & \text{for all } t \in [0,\infty), \text{if } \alpha \neq 0, \end{cases}$$

(v) if $\alpha = 0$, *then*

$$\|u(t)\|_H^2 + 2\gamma \int_0^t \|u(s)\|_V^2 ds \leq 2\alpha'\beta t^2 + 2(\alpha'\|u_0\|_H^2 + \beta')t + \|u_0\|_H^2$$

for all $t \in [0,\infty)$, *and if* $\alpha \neq 0$, *then*

$$\|u(t)\|_H^2 + 2\gamma \int_0^t \|u(s)\|_V^2 ds$$
$$\leq \frac{\alpha'}{\alpha}(e^{2\alpha t} - 1)\left(\|u_0\|_H^2 + \frac{\beta}{\alpha}\right) + 2\left(\beta' - \frac{\alpha'\beta}{\alpha}\right)t + \|u_0\|_H^2$$

for all $t \in [0,\infty)$.

Remark 3. If the injection $V \to H$ is compact, (v) above may be replaced by the following condition:

(v)$'$ if $\alpha = 0$, then

$$\|u(t)\|_H^2 + 2\gamma \int_s^t \|u(r)\|_V^2 dr$$
$$\leq 2\alpha'\beta(t-s)^2 + 2\left(\alpha'\|u(s)\|_H^2 + \beta'\right)(t-s) + \|u(s)\|_H^2$$

for $s = 0$, almost all $s > 0$, and all $t \geq s$; if $\alpha \neq 0$, then

$$\|u(t)\|_H^2 + 2\gamma \int_s^t \|u(r)\|_V^2 dr$$
$$\leq \frac{\alpha'}{\alpha}(e^{2\alpha(t-s)} - 1)\left(\|u(s)\|_H^2 + \frac{\beta}{\alpha}\right) + 2\left(\beta' - \frac{\alpha'\beta}{\alpha}\right)(t-s) + \|u(s)\|_H^2$$

for $s = 0$, almost all $s > 0$, and all $t \geq s$.

Inequalities in Theorem 2(v) and in (v)′ correspond to the energy inequalities in Navier-Stokes equations (see Ladyzhenskaya [6], and Shinbrot & Kaniel [9] for energy inequality).

Remarks 4.

(i) Condition (A.5) implies condition (A.4).

(ii) Condition (A.5) corresponds to the coerciveness on V, see Lions-Magenes [8, Definition 9.2, p.202].

Lemma 1. *Let $0 < T < \infty$. Let $\{V, H, V^*\}$ be a triplet. Let A be a single valued operator in V^* with domain V. Suppose that conditions (A.1) and (A.4) are satisfied. Let u_0 be in H. Set $u_0^n \equiv u_0$ and take a sequence $\{u_k^n\}_{n,k\geq 1}$ in V such that*

$$\left(1 - \frac{T}{n}A\right) u_k^n = u_{k-1}^n \quad \text{for } n, k = 1, 2, 3, \cdots. \tag{3}$$

Then the following hold:

$$\|u_k^n\|_H^2 \leq \|u_{k-1}^n\|_H^2 + \frac{2\beta T}{n} \tag{4}$$

for $\alpha = 0$ *and* $n, k = 1, 2, 3, \cdots$;

$$(5) \qquad \|u_k^n\|_H^2 + \frac{\beta}{\alpha} \le \left(1 - \frac{2\alpha T}{n}\right)^{-1} \left(\|u_{k-1}^n\|_H^2 + \frac{\beta}{\alpha}\right)$$

for $\alpha \neq 0, n > 2\alpha T$ *and* $k = 1, 2, 3, \cdots$.

Lemma 2. *Let* $\{V, H, V^*\}$ *be a triplet. Let* A *be a single valued operator in* V^* *with domain* V. *Suppose that conditions (A.5) and (A.6) hold.*

Then A *is a weakly closed operator in* V^*.

Lemma 3. *Let* $0 < T < \infty$. *Let* $\{V, H, V^*\}$ *be a triplet. Let* A *be a single valued operator in* V^* *with domain* V. *Suppose that conditions (A.1) and (A.5) hold. Let* u_0 *and* $\{u_k^n\}$ *be the same as in Lemma 1. Then the following hold:*

$$(6) \qquad \begin{aligned} &\|u_k^n\|_H^2 + 2\gamma \frac{T}{n} \sum_{i=l+1}^{k} \|u_i^n\|_V^2 \\ &\le 2\alpha' \frac{T}{n} \sum_{i=l+1}^{k} \|u_i^n\|_H^2 + 2\beta' \left(\frac{kT}{n} - \frac{lT}{n}\right) + \|u_l^n\|_H^2 \end{aligned}$$

for $n \ge 1$ *and* $k > l \ge 0$.

Combining Lemmas 1 and 3, we obtain the following.

Lemma 4. *Under the same assumptions as Lemma 3, taking* $\alpha \in \mathbb{R}$ *and* $\beta \ge 0$ *such that*

$$\langle Au, u \rangle_{V^*, V} \le \alpha \|u\|_H^2 + \beta \quad \text{for all } u \in V,$$

we have the following.

If $\alpha = 0$, *then*

$$
\begin{aligned}
&\|u_k^n\|_H^2 + 2\gamma\frac{T}{n}\sum_{i=l+1}^{k}\|u_i^n\|_V^2 \\
&\le 2\alpha'\beta\left(\frac{kT}{n}-\frac{lT}{n}\right)\left(\frac{(k+1)T}{n}-\frac{lT}{n}\right) \\
&\quad + 2\left(\alpha'\|u_l^n\|_H^2+\beta'\right)\left(\frac{kT}{n}-\frac{lT}{n}\right) + \|u_l^n\|_H^2
\end{aligned}
\tag{7}
$$

for $n \ge 1, k > l \ge 0$, *and if* $\alpha \ne 0$,*then*

$$
\begin{aligned}
&\|u_k^n\|_H^2 + 2\gamma\frac{T}{n}\sum_{i=l+1}^{k}\|u_i^n\|_V^2 \\
&\le \frac{\alpha'}{\alpha}\left(\left(1-\frac{2\alpha T}{n}\right)^{-(k-l)}-1\right)\left(\|u_l^n\|_H^2+\frac{\beta}{\alpha}\right) \\
&\quad + 2\left(\beta'-\frac{\alpha'\beta}{\alpha}\right)\left(\frac{kT}{n}-\frac{lT}{n}\right) + \|u_l^n\|_H^2
\end{aligned}
\tag{8}
$$

for $n > 2\alpha T$ *and* $k > l \ge 0$.

Lemma 5. *Let* $0 < T < \infty$. *Let* $\{V, H, V^*\}$ *be a triplet. Suppose that conditions (A.1), (A.2)′ and (A.5) are satisfied. Then the following hold.*

If $\alpha = 0$, *then*

$$
\begin{aligned}
&\frac{T}{n}\sum_{k=1}^{n}\|Au_k^n\|_{V^*}^p \le 2^{p-1}\gamma^{-1}\left(\varphi\left(\|u_0\|_H^2+2\beta T\right)\right)^p \\
&\times\left(\alpha'\beta\left(1+\frac{1}{n}\right)T^2+\left(\alpha'\|u_0\|_H^2+\beta'+\gamma\right)T+\frac{1}{2}\|u_0\|_H^2\right),
\end{aligned}
\tag{9}
$$

and if $\alpha \ne 0$, *then*

(10)

$$
\begin{aligned}
&\frac{T}{n}\sum_{k=1}^{n}\|Au_k^n\|_{V^*}^p \\
&\le 2^{p-1}\gamma^{-1}\left(\varphi\left(\left(1-\frac{2|\alpha|T}{n}\right)^{-n}\left(\|u_0\|_H^2+\frac{\beta}{|\alpha|}\right)+\frac{\beta}{|\alpha|}\right)\right)^p \\
&\times\left\{\frac{\alpha'}{2|\alpha|}\left(1-\frac{2|\alpha|T}{n}\right)^{-n}\left(\|u_0\|_H^2+\frac{\beta}{|\alpha|}\right)+\left(\frac{\alpha'\beta}{|\alpha|}+\beta'+\gamma\right)T+\frac{1}{2}\|u_0\|_H^2\right\}
\end{aligned}
$$

for $n > 2|\alpha|T$.

The following lemma is proved by the Galerkin method.

Lemma 6. *Let V be a separable reflexive Banach space and let $\{V, H, V^*\}$ be a triplet. Let A be a single valued operator in V^* with domain V. Suppose that conditions (A.5) and (A.6) hold.*
Then for any $f \in V^*$ and $\lambda > 0$ with $\alpha'\lambda \leq 1$, there exists an element $u \in V$ such that $(1 - \lambda A)u = f$.

Proof. Since V is a separable Banach space and $\{V, H, V^*\}$ is a triplet, there exists a subset $\{e_1, e_2, \cdots, e_n, \cdots\}$ of V satisfying the following two conditions:

$$\text{(O.1)} \qquad \text{if } \langle u, e_n \rangle_{V^*,V} = 0 \quad \text{for each } n, \text{ then } u = 0;$$

$$\text{(O.2)} \qquad (e_i, e_j)_H = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Let V_n be a linear space spanned by $e_1, e_2, \cdots, e_n$ and equipped with the inner product and the norm induced by H. We denote the inner product and the norm of V_n by $(\ ,\ )_{V_n}, \|\ \|_{V_n}$, respectively. Set

$$\text{(11)} \qquad P_n(u) = \sum_{j=1}^{n} \langle (1 - \lambda A)u - f, e_j \rangle_{V^*,V}\, e_j \quad \text{for all } u \in V.$$

Then by (A.6), P_n is a continuous mapping from V into V_n which satisfies

$$\text{(12)} \qquad (P_n(u), v)_{V_n} = \langle (1 - \lambda A)u - f, v \rangle_{V^*,V} \quad \text{for all } u \in V \text{ and } v \in V_n.$$

Furthermore, noting that on the space V_n all norms are equivalent, we also see that P_n is a continuous mapping from V_n into itself. Taking $v = u \in V_n$ in (12) and noting that $\lambda > 0$ and $1 - \lambda\alpha' \geq 0$, by (A.5) we have

$$\begin{aligned}(13)\qquad &(P_n(u), u)_{V_n} \\ &\geq \|u\|_H^2 - \lambda(\alpha'\|u\|_H^2 + \beta' - \gamma\|u\|_V^2) - \|f\|_{V^*}\|u\|_V \\ &= \lambda\gamma\|u\|_V^2 + (1 - \lambda\alpha')\|u\|_H^2 - \|f\|_{V^*}\|u\|_V - \lambda\beta' \\ &\geq \lambda\gamma\|u\|_V^2 - \|f\|_{V^*}\|u\|_V - \lambda\beta'.\end{aligned}$$

Thus there exists a positive number M_λ such that

$$(14)\qquad (P_n(u), u)_{V_n} > 0 \quad \text{for } u \in V_n \text{ with } \|u\|_V \geq M_\lambda.$$

In particular, we have

$$(15)\qquad (P_n(u), u)_{V_n} > 0 \quad \text{for } u \in V_n \text{ with } \|u\|_{V_n} \geq CM_\lambda,$$

where C is a positive constant such that

$$(16)\qquad \|u\|_H \leq C\|u\|_V \quad \text{for all } u \in V.$$

By [10, Lemma 1.4, p.164], (15) and (14), there exists $u_n \in V_n$ such that

$$(17)\qquad P_n(u_n) = 0 \quad \text{and} \quad \|u_n\|_V \leq M_\lambda.$$

Taking $u = u_n$ in (12), by (17) we get

$$(18)\qquad \langle (1 - \lambda A)u_n - f, v\rangle_{V^*,V} = 0 \quad \text{for all } v \in V_n.$$

Since V is a reflexive Banach space and the sequence $\{u_n\}$ is bounded in V, there exist a subsequence $\{u_{n(j)}\}$ of $\{u_n\}$ and an element $u \in V$ such that

$$(19)\qquad u_{n(j)} \to u \quad \text{weakly in } V.$$

By (18) we have

(20) $$\langle (1-\lambda A)u_{n(j)} - f, v\rangle_{V^*,V} = 0 \quad \text{for } n(j) \geq n \text{ and } v \in V_n.$$

Letting $j \to \infty$, by (19), (20) and (A.6) we obtain

(21) $$\langle (1-\lambda A)u - f, v\rangle_{V^*,V} = 0 \quad \text{for all } v \in V_n.$$

Taking $v = e_n$ in (21), we have

(22) $$\langle (1-\lambda A)u - f, e_n\rangle_{V^*,V} = 0 \quad \text{for } n = 1, 2, 3, \cdots.$$

It follows from (O.1) that

$$(1-\lambda A)u = f. \quad \square$$

Remark 5. By Lemmas 2, 5, and 6, if V is a separable reflexive Banach space and conditions (A.5), (A.6) and (A.2)$'$ hold, all the assumptions of Theorem 1 are satisfied essentially. In fact, we use condition (A.1) only for small $\lambda > 0$ in Theorem 1. Thus the assumptions of Therem 2 yield the conclusions of Theorem 1.

In order to prove Theorem 2(v) and Remark 3(v)$'$, we use the following lemma.

Lemma 7. *Make the assumptions of Theorem2. Let u be the solution of (CP) in Theorem 2 and let $\{u^{n(j)}\}$ be the sequence of functions in Theorem 1(ii). Then we have*

$$w\text{-}\lim_{j\to\infty} u^{n(j)} = u \quad \textit{in } L^2([0,T]; V).$$

To show Remark 3, we need two lemmas besides Lemma 7. In the following let u and $u^{n(j)}$ be the functions as in Lemma 7. We shall denote by S all the numbers $s \in [0, T]$ which satisfy the following:

(C) there exists a subsequence $\{n(j(k,s))\}$, depending on s, of $\{n(j)\}$ such that

$$\lim_{k\to\infty} u^{n(j(k,s))}(s) = u(s) \quad \text{in } H.$$

We note that $0 \in S$.

Lemma 8. *Make the assumptions of Theorem 2. If the injection $V \to H$ is compact, almost every $s \in [0, T]$ belongs to S.*

Lemma 9. *Make the assumptions of Theorem 2. Let α and β be the numbers as in (A.4). Then the following inequalities hold:*
if $\alpha = 0$, then

$$\|u(t)\|_H^2 \le \|u(s)\|_H^2 + 2\beta(t-s) \tag{23}$$

for $s \in S$ and $s \le t$; if $\alpha \neq 0$, then

$$\|u(t)\|_H^2 + \frac{\beta}{\alpha} \le e^{2\alpha(t-s)}\left(\|u(s)\|_H^2 + \frac{\beta}{\alpha}\right) \tag{24}$$

for $s \in S$ and $s \le t$.

Remark 6. From the construction of solution u and Lemma 8, if the injection $V \to H$ is compact, inequalities (23) and (24) hold for $s = 0$, almost all $s > 0$, and all $t \ge s$.

3. Applications to Navier-Stokes equations

We are concerned with the Cauchy problem for Navier-Stokes equations in an unbounded domain Ω in $\mathbb{R}^3$ with boundary $\partial\Omega$:

$$
\text{(NS)}\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - (u\cdot\nabla)u - \text{grad } p & \text{in } (0,\infty)\times\Omega, \\ \text{div } u = 0 \quad \text{in } (0,\infty)\times\Omega, \\ u = 0 \quad \text{on } [0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases}
$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ is the velocity field, $p = p(t,x)$ is the pressure, and $u_0 = u_0(x)$ is the initial velocity.

3.1 Notation

The Lebesgue space $L^p(\Omega)$ denotes the vector functions on Ω with finite norm:

$$
\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{1/p},
$$

where

$$
|u(x)| = \left(\sum_{i=1}^{3} |u_i(x)|^2\right)^{1/2}.
$$

Let $C_0^\infty(\Omega)$ be the space of infinitely differentiable functions on Ω with a compact support in Ω. Let

$$
C_{0,\sigma}^\infty(\Omega) = \{u \in C_0^\infty(\Omega);\ \text{div } u = 0\},
$$

$$
H \equiv L_\sigma^2(\Omega) = \text{the closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } L^2(\Omega).
$$

Then H is a Hilbert space with the inner product and the norm induced by $L^2(\Omega)$. Let

$$
H^1(\Omega) = \left\{u;\ u, \frac{\partial u}{\partial x_i} \in L^2(\Omega) \quad \text{for } i = 1,2,3\right\},
$$

$$(\nabla u, \nabla v)_{L^2(\Omega)} \equiv \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)} = \sum_{i,j=1}^{3} \int_\Omega \frac{\partial u_j}{\partial x_i} \frac{\partial v_j}{\partial x_i} dx,$$

$$\|\nabla u\|_{L^2(\Omega)} \equiv \left\{ (\nabla u, \nabla u)_{L^2(\Omega)} \right\}^{1/2}.$$

Then $H^1(\Omega)$ is a Hilbert space with inner product

$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)},$$

and the corresponding norm is given by

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Let

$$H_0^1(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in } H^1(\Omega),$$

$$V \equiv H^1_{0,\sigma}(\Omega) = \text{the closure of } C^\infty_{0,\sigma}(\Omega) \text{ in } H^1_0(\Omega).$$

Then V is a separable Hilbert space with the inner product and the norm induced by $H^1(\Omega)$. Moreover, if V^* denotes the dual of V, the family $\{V, H, V^*\}$ is a triplet. For each u in V, the form

$$v \in V \to -(\nabla u, \nabla v)_{L^2(\Omega)} \in \mathbb{R}$$

is linear and continuous on V; therefore, there exists an element of V^* which we denote by $\tilde{\Delta} u$ such that

$$\text{(25)} \qquad \langle \tilde{\Delta} u, v \rangle_{V^*,V} = -(\nabla u, \nabla v)_{L^2(\Omega)} \quad \text{for all } v \in V.$$

By the Sobolev imbedding theorem, for $u, v \in V$, there exists an element of V^* which we denote by $B(u,v)$ such that

$$\text{(26)} \qquad \langle B(u,v), w \rangle_{V^*,V} = \sum_{i,j=1}^{3} \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \text{for all } w \in V.$$

We set

$$Bu = B(u,u) \quad \text{for } u \in V,$$

and

$$\begin{cases} A = \tilde{\Delta} - B, \\ D(A) = V. \end{cases}$$

We consider the abstract Navier-Stokes equations

$$(\mathrm{NS})_\sigma \begin{cases} (d/dt)u(t) = Au(t) \quad \text{for } t \in [0,\infty), \\ u(0) = u_0, \end{cases}$$

in V^*, where u_0 is an element of H.

3.2 Existence of a solution of $(\mathrm{NS})_\sigma$

We use the following (see [10, Ch.II, §1; Ch. III, §3], [1] and [4]):

$$(27) \qquad \langle B(u,v), w\rangle_{V^*,V} = -\langle B(u,w), v\rangle_{V^*,V} \quad \text{for } u,v,w \in V,$$

in particular,

$$\langle Bu, u\rangle_{V^*,V} = 0 \quad \text{for } u \in V,$$

$$(28) \qquad \|B(u,v)\|_{V^*} \le \|u\|_{L^4(\Omega)} \cdot \|v\|_{L^4(\Omega)} \quad \text{for } u,v \in V,$$

$$(29) \qquad \|Bu - Bv\|_{V^*} \le (\|u\|_{L^4(\Omega)} + \|v\|_{L^4(\Omega)})\|u-v\|_{L^4(\Omega)} \quad \text{for } u,v \in V,$$

$$(30) \qquad \|u\|_{L^4(\Omega)} \le 3^{-3/8}\|\nabla u\|_{L^2(\Omega)}^{3/4}\|u\|_{L^2(\Omega)}^{1/4} \quad \text{for all } u \in H_0^1(\Omega),$$

$$(31) \qquad \|u\|_{L^4(\Omega)} \le 2^{-1}\|u\|_{H^1(\Omega)} \quad \text{for all } u \in H_0^1(\Omega),$$

(32) $$\|Au\|_{V^*} \leq \|u\|_V + \|u\|^2_{L^4(\Omega)} \quad \text{for all } u \in V.$$

In order to show the existence of a solution of $(\mathrm{NS})_\sigma$, we check that the following conditions (a), (b) and (c) hold.

(a) $\langle Au, u\rangle_{V^*,V} = \|u\|^2_H - \|u\|^2_V$ for all $u \in V$;

(b) the operators $\tilde{\Delta} : V \to V^*$ and $B : V \to V^*$ are weakly continuous, so that A is also weakly continuous;

(c) $\|Au\|_{V^*} \leq (1 + \|u\|^{1/2}_H)(\|u\|^{3/2}_V + 1)$ for all $u \in V$.

Proof of (a). Let $u \in V$. Then, by (25) and (27) we have

$$\langle Au, u\rangle_{V^*,V} = \langle \tilde{\Delta}u - Bu, u\rangle_{V^*,V}$$
$$= -\|\nabla u\|^2_{L^2(\Omega)} = \|u\|^2_H - \|u\|^2_V. \quad \square$$

We write down the proof of (b) for the sake of completeness, although it is seen essentially in [10].

Proof of (b). Let

(33) $$u^n, u \in V \quad \text{and } u^n \to u \quad \text{weakly in } V.$$

For any $v \in V$ we have

(34) $$\langle \tilde{\Delta}u^n - \tilde{\Delta}u, v\rangle_{V^*,V} = \langle \tilde{\Delta}v, u^n - u\rangle_{V^*,V} \to 0 \quad \text{as } n \to \infty.$$

Since V^* is a reflexive Banach space, it follows from (34) that $\tilde{\Delta} : V \to V^*$ is weakly continuous. We now prove that the operator $B : V \to V^*$ is weakly continuous. Let $f \in C_{0,\sigma}^{\infty}(\Omega)$ and let Ω_0 be a bounded open subset of Ω containing the support of f. Then, by the same argument as in [10, Lemma 1.7, Ch. II, §1] we have

$$\lim_{n\to\infty} \|u^n - u\|_{L^2(\Omega_0)} = 0. \tag{35}$$

Furthermore, by the Cauchy-Schwarz inequality we get

$$\begin{aligned} &|\langle B(u^n - u, f), u^n\rangle_{V^*,V}| \\ &\leq 3 \max_{1\leq i,j\leq 3} \left\| \frac{\partial f_j}{\partial x_i} \right\|_{L^\infty(\Omega)} \|u^n - u\|_{L^2(\Omega_0)} \|u^n\|_{L^2(\Omega_0)}. \end{aligned} \tag{36}$$

From (35) and (36) it follows that

$$\lim_{n\to\infty} \langle B(u^n - u, f), u^n\rangle_{V^*,V} = 0. \tag{37}$$

Combining (37) and (33) we get

$$\begin{aligned} &\langle Bu^n - Bu, f\rangle_{V^*,V} \\ &= \langle B(u - u^n, f), u^n\rangle_{V^*,V} + \langle B(u, f), u - u^n\rangle_{V^*,V} \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{38}$$

Thus, by (29) and (31), for any $v \in V$ we have

$$\begin{aligned} &|\langle Bu^n - Bu, v\rangle_{V^*,V}| \\ &\leq |\langle Bu^n - Bu, v - f\rangle_{V^*,V}| + |\langle Bu^n - Bu, f\rangle_{V^*,V}| \\ &\leq (\|u^n\|_{L^4(\Omega)} + \|u\|_{L^4(\Omega)})\|u^n - u\|_{L^4(\Omega)}\|v - f\|_V + |\langle Bu^n - Bu, f\rangle_{V^*,V}| \\ &\leq (\|u^n\|_V + \|u\|_V)\|u^n - u\|_V\|v - f\|_V + |\langle Bu^n - Bu, f\rangle_{V^*,V}|. \end{aligned} \tag{39}$$

From (38) and (39) we get

$$(40) \quad \overline{\lim_{n\to\infty}} |\langle Bu^n - Bu, v\rangle_{V^*,V}| \le \sup_n ((\|u^n\|_V + \|u\|_V)\|u^n - u\|_V) \|v - f\|_V.$$

Since the sequence $\{u^n\}$ is bounded in V and $C_{0,\sigma}^\infty(\Omega)$ is dense in V, it follows from (40) that

$$\lim_{n\to\infty} \langle Bu^n - Bu, v\rangle_{V^*,V} = 0. \quad \square$$

Proof of (c). For $u \in V$, we have

$$\begin{aligned} \|Au\|_{V^*} &\le \|u\|_V + \|u\|_{L^4(\Omega)}^2 \\ &\le \|u\|_V + \|\nabla u\|_{L^2(\Omega)}^{3/2} \|u\|_{L^2(\Omega)}^{1/2} \\ &\le 1 + \|u\|_V^{3/2} + \left(1 + \|u\|_V^{3/2}\right) \|u\|_H^{1/2} \\ &= \left(1 + \|u\|_H^{1/2}\right)\left(1 + \|u\|_V^{3/2}\right). \end{aligned}$$

This completes the proof of (c). $\square$

From (a), (b) and (c), applying Theorem 2 to the operator A we find that there exists a solution of $(\mathrm{NS})_\sigma$.

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