<table>
<thead>
<tr>
<th>Title</th>
<th>The Carleman type estimates and non-well-posed problems (Nonlinear Evolution Equations and Their Applications)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Tsutsumi, M.</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1995), 898: 27-43</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1995-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/84492">http://hdl.handle.net/2433/84492</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
The Carleman type estimates and non-well-posed problems.

M. Tsutsumi
Department of Applied Physics
Waseda University
Tokyo 169, Japan

1 Introduction

Let $\Omega$ be a connected open set in $\mathbb{R}^n$, and let $P = P(x, D)$ be a differential operator of order $m$ in $\Omega$ with principal symbol $p$. Let $\phi : \Omega \rightarrow \mathbb{R}$ be a $C^\infty$ function, with $\nabla \phi(x) \neq 0$, $x \in \Omega$ and which is strongly pseudo convex (this is a convexity property relatively to $p$.) We say that the Carleman type estimate holds for $P$ if there exists a constant $K > 0$ such that

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int_{\Omega} |D^\alpha u|^2 e^{2\tau \phi} dx \leq K \int_{\Omega} |P(x, D)u|^2 e^{2\tau \phi} dx$$  \hspace{1cm} (1)

\forall u \in C_0^\infty(\Omega), \quad \tau > 0 \quad \text{large enough.}

Estimates of this form were first used by Carleman in work on unique continuation property for second order elliptic operators in $\mathbb{R}^2$. Here $P$ is said to have the unique continuation property if the following holds: Suppose $u$ solves $P(x, D)u = 0$ on $\Omega$ and $u = 0$ on a empty open set in $\Omega$. Then, $u$ vanishes identically in $\Omega$.

This property is equivalent to uniqueness in the Cauchy problem for any smooth hypersurface.

The Carleman type estimates are established under various assumptions on $P(x, D)$ and have a large field of applications:

1. Unique continuation property and uniqueness of Cauchy problem. (see [3], [4], [6], [9], [10])
2. Spectral properties of Schrödinger operator. (see [12])

3. Generic properties of nonlinear elliptic equations. (see [13]).

4. Stability of (non-well-posed) Cauchy problem (see [1]).

5. Identifiability of spatially-varying coefficients in partial differential equations. (see [1], [2])

The aim of this paper is to present new results concerning the last two subjects. In section 2 we establish an abstract analogue of Carleman estimates, which is an extension of Bukhgeim's result ([11]). In section 3 we apply it to the uniqueness question and identifiability of coefficients for the initial-boundary value problems for some (nonlinear) partial differential equations.

2 Stability estimates

Let $H$ be a complex (or real) Hilbert space, the scalar product and the norm in $H$ being denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $M(t)$ and $A(t)$ be linear operators whose domains are dense subspaces in $H$ and are possibly changeable in $t$ for $t \in [0, T]$.

The subscript $t$ denotes differentiation with respect to $t$.

In Theorem 1 stated below, we assume the following.

(A1) For every $t \in [0, T]$ $M(t)$ is a selfadjoint operator.

(A2) $M(t)$ and $A(t)$ are strongly continuous and weakly differentiable with respect to $t$.

(A3) Let

$$D(P) = \{ u : [0, T] \rightarrow H \mid u(t), u_t(t) \in D(M(t)), u(t) \in D(A(t)) \text{ for each } t \in [0, T],$$

$$M(\cdot)u(\cdot) \in C^1([0, T]; H) \text{ and } A(\cdot)u(\cdot) \in C([0, T]; H) \}.$$ 

and

$$Z = \{ u : [0, T] \rightarrow H \mid u(t) \in D(A(t) + A^*(t)), \text{ for each } t \in [0, T]$$

$$\{ A(\cdot) + A^*(\cdot) \} u(\cdot) \in C([0, T]; H) \}$$
There exists a linear subspace $D$ dense in $D(P) \cap Z$ such that, setting $D(t) = D \cap (\{t\} \times H) \subset ([0, T] \times H),

(a) There exists a positive constant $C_1$ such that

$$\|M_t(t)v\| \leq C_1\|M(t)v\|, \quad \forall v \in D(t).$$

(b) $M(t)$ and $A(t) + A^*(t)$ commute each other on $D(t)$, that is, for $v \in D(t)$

$$(A(t) + A^*(t))v \in D(M(t)) \text{ and } M(t)v \in D(A(t) + A^*(t)), \quad \text{we have}$$

$$M(t)(A(t) + A^*(t))v = (A(t) + A^*(t))M(t)v.$$

(c) There exist positive constants $C_j$ ($j = 2, 3, 4$) such that

$$\|(A(t) - A^*(t))v\| \leq C_2\|M(t)v\| \quad \forall v \in D(t),$$

$$\|A^*(t)v\|^2 - \|A(t)v\|^2 \leq C_3\|M(t)v\|^2 \quad \forall v \in D(t),$$

and

$$\|(A(t) + A^*_t(t))v\| \leq C_4\|Mv\| \quad \forall v \in D(t).$$

We define the operator

$$P(t)u(t) = M(t)u_t - A(t)u(t) \quad \text{for } \forall u \in D(P).$$

For brevity we write

$$\|u\|_T = \|u\|_{L^2(0,T;H)} \quad \|u\|_{s,T} = \|e^{st}u\|_T$$

where $\phi = \phi(t)$ is a real-valued continuous function defined on $[0,T]$ and $s$ is an arbitrary nonnegative number.

The following theorem is an extension of abstract versions of Carleman’s estimates for the Cauchy problems. (see Nirenberg [11], Bukhgeim [1]).

**Theorem 1** Suppose that the assumptions (A1)-(A3) hold. Suppose that $\phi \in C^2([0,T])$ satisfies

$$\phi_t(t) \leq 0 \quad \forall t \in [0,T],$$

and

$$\phi_{tt}(t) + (C_1 + C_2)\phi_t(t) \geq \delta > 0 \quad \forall t \in [0,T].$$
Then, there exist positive constants $s_0$ and $C_5$ such that for all $s \geq s_0$ and $u \in D(P) \cap Z$

\[ s\|Mu\|_{s,T}^2 + \frac{1}{1 + s|\phi_t(0)|^2} \left( \| (A + A^*)v \|_{s,T}^2 + \|Mu_t\|_{s,T}^2 \right) \]
\[ \leq C_5 \left( \|Pu\|_{s,T}^2 + \left[ s\phi_t(t)e^{2s\phi(t)}\|M(t)u(t)\|^2 \right. \right. \]
\[ + \left. \left. e^{2s\phi(t)}\langle (A(t) + A^*(t))u(t), M(t)u(t)\rangle \right]_0^T \right) \]

(2)

Using Theorem 1, we can establish stability estimates as follows.

**Theorem 2** Suppose that all the assumptions stated in Theorem 1 are satisfied. Let $f \in C([0,T]; H)$. Suppose that there exists a subset $U \subset (D(P) \cap Z)$ such that $\forall u \in U$

\[ \|P(t)u(t)\| \]
\[ \leq C_6 \int_0^t \left( \| (A(\tau) + A^*(\tau))u(\tau)\| + \|M(\tau)u_t(\tau)\| + \|M(\tau)u(t)\| \right) d\tau \]
\[ + C_7\|M(t)u(t)\| + C_8\|f(t)\| \]

(3)

where $C_j (j = 6, 7, 8)$ are positive constants independent of $t$. Then, there exists positive constants $s_0$, $C_9$ and $C_{10}$, independent of $u$, $f$ and $t$, such that for $\forall u \in U$ and $\forall s \geq s_0$

\[ \|Mu\|_T \leq C_9 \left[ \frac{1}{\sqrt{s}}\| (A(T) + A^*(T))u(T)\| \right. \]
\[ + \exp(sC_{10}) \left( \|M(0)u(0)\| + \|(A(0) + A^*(0))u(0)\| + \|f\|_T \right) \].

(4)

Furthermore, if $\langle M(T)u(T), (A(T) + A^*(T))u(T)\rangle \leq C\|M(T)u(T)\|^2$, then

\[ \|Mu\|_T \leq C_9 \frac{\exp(sC_8)}{\sqrt{s}} \left( \|M(0)u(0)\| + \|(A(0) + A^*(0))u(0)\| + \|f\|_T \right). \]

(5)

**Proof of Theorem 1.** Let $u \in D$, $v = e^{s\phi}u$ and

\[ P_\phi(t)v = e^{s\phi(t)}P(t)(e^{-s\phi(t)}v) \]
\[ = -s\phi_t(t)M(t)v + M(t)v_t - A(t)v. \]

Then we have

\[ P_\phi^*(t)v = -s\phi_t(t)M(t)v - M_t(t)v - M(t)v_t - A^*v. \]
Define $P_{\phi}^{s}$ and $P_{\phi}^{a}$ by

$$
P_{\phi}^{s} = \frac{1}{2} (P_{\phi} + P_{\phi}^*) v
= -s\phi_t M v - \frac{1}{2} M_t v - \frac{1}{2} (A + A^*) v$$
\hspace{1cm} (6)

and

$$
P_{\phi}^{a} = \frac{1}{2} (P_{\phi} - P_{\phi}^*) v
= \frac{1}{2} M_t v + M v_t - \frac{1}{2} (A - A^*) v,$$
\hspace{1cm} (7)

respectively.

We see that

$$
\|P u\|_{s,T} = \int_0^T \{ \|P_{\phi}^{s}(\tau)v(\tau)\|^2 + \|P_{\phi}^{a}(\tau)v(\tau)\|^2
+ 2 \text{Re}\langle P_{\phi}^{s}(\tau)v(\tau), P_{\phi}^{a}(\tau)v(\tau)\rangle \} d\tau.
\hspace{1cm} (8)
$$

Making use of the assumptions (A1)-(A3), we have

$$
2\text{Re}\langle P_{\phi}^{s} v, P_{\phi}^{a} v \rangle = -s\phi_t \{ \text{Re}\langle M v, M v_t \rangle + 2\text{Re}\langle M v, M v_t \rangle \}
+ s\phi_t \text{Re}\langle M v, (A - A^*) v \rangle
- \left\{ \frac{1}{2} \| M v \|^2 + \text{Re}\langle M v, M v_t \rangle \right\}
+ \frac{1}{2} \text{Re}\langle M v, (A - A^*) v \rangle - \frac{1}{2} \{ \text{Re}\langle (A + A^*) v, M v \rangle
+ 2\text{Re}\langle (A + A^*) v, M v_t \rangle + (\| Av \|^2 - \| A^* \|^2) \} \geq
- \frac{d}{dt} \left\{ s\phi_t \| M v \|^2 + \frac{1}{2} \langle (A + A^*) v, M v \rangle \right\}
+ s\phi_t \| M v \|^2 + s\phi_t (C_1 + C_2) \| M v \|^2
- \frac{1}{2} \{ 3C_1^2 + C_1 C_2 + 2C_3 + C_4 \} \| M v \|^2
- \frac{1}{4} \| M v_t \|^2.
$$

We also have

$$
\|P_{\phi}^{a} v\|^2 \geq \frac{1}{2} \left( \| M v_t \|^2 - \| M v \|^2 - \| (A - A^*) v \|^2 \right)
\geq \frac{1}{2} \| M v_t \|^2 - \frac{1}{2} (C_1^2 + C_2^2) \| M v \|^2.
$$
Hence, if we take $s$ so large that

$$s \geq \frac{1}{\delta} \left( 4C_1^2 + C_1C_2 + C_2^2 + 2C_3 + C_4 \right),$$

we obtain

$$\frac{s\delta}{2} \| Mv \|_T^2 + \| P_{\phi}^s v \|_T^2 + \frac{1}{2} \| Mv_t \|_T^2$$

$$ \leq \| Pu \|_{\epsilon,T}^2 + \left\{ s\phi_t \| Mv \|^2 + \frac{1}{2} \langle (A + A^*)v, Mv \rangle \right\}_0^T \equiv I. \quad (9)$$

We have

$$\| P_{\phi}^s v \|^2 \geq \frac{1}{2} s\phi_t \text{Re} \langle (A + A^*)v, Mv \rangle + \frac{1}{8} \| (A + A^*)v \|^2 - \frac{1}{4} C_2^2 \| Mv \|^2,$$

from which it follows that

$$\frac{1}{8} \| (A + A^*)v \|_T^2$$

$$ \leq \| P_{\phi}^s v \|_T^2 + \frac{s}{2} |\phi_t(0)| \int_0^T \| Mv \| \| (A + A^*)v \| dt + \frac{1}{4} C_2^2 \| Mv \|_T^2$$

Making use of (9), we have

$$\frac{1}{8} \| (A + A^*)v \|_T^2$$

$$ \leq I + \left( \frac{s}{2\delta} \right)^{1/2} |\phi_t(0)| \left\{ \int_0^T \| (A + A^*)v \|^2 dt \right\}^{1/2} + \frac{1}{2s\delta} C_2^2 I$$

$$ \leq \left( 1 + \frac{s}{\delta} |\phi_t(0)|^2 + \frac{2}{s\delta^2} C_2^2 \right) I + \frac{1}{16} \| (A + A^*)v \|_T^2.$$

from which we deduce

$$\int_0^T \| (A + A^*)v \|^2 dt \leq C(1 + s|\phi_t(0)|^2) \ I \quad (10)$$

where and in the sequel by $C$ we denote various positive constants which do not depend on $t$ and $u$ and are changeable from line to line. From (9) and (10), we have

$$\frac{s\delta}{2} \| Mv \|_T^2 + \frac{1}{1 + s|\phi_t(0)|^2} \| (A + A^*)v \|_T^2 + \frac{1}{2} \| Mv_t \|_T^2 \leq C \ I.$$
Noting that $Mv_{t} = s\phi_{t} e^{s\phi} Mu + e^{s\phi} Mu_{t}$, we get
\[
\|Mu_{t}\|_{s,T}^{2} \leq \|Mv_{t}\|_{s,T}^{2} + 2s|\phi(0)|\|Mu\|_{s,T} \|Mu_{t}\|_{s,T}
\leq I + 2\sqrt{2}s^{1/2}|\phi(0)|^{1/2} \|Mu_{t}\|_{s,T}
\leq I + 4s|\phi(0)|^{2} I + \frac{1}{2} \|Mu_{t}\|_{s,T}^{2},
\]
from which it follows that
\[
\|Mu_{t}\|_{s,T}^{2} \leq 2(1 + 4s|\phi(0)|^{2}) I.
\]
Hence, we finally obtain
\[
\frac{s\delta}{2} \|Mu\|_{s,T}^{2} + \frac{1}{1 + s|\phi(0)|^{2}} \left( \|(A + A^{*})u\|_{s,T}^{2} + \|Mu_{t}\|_{s,T}^{2} \right) \leq C_{5} I.
\]
(11)

Since $D$ is dense in $D(P) \cap Z$, the estimate (11 holds for any $u \in D(P) \cap Z$. This completes the proof of Theorem 1.

In order to establish Theorem 2, we need

**Lemma 1** Suppose that $\phi(t)$ is a real-valued $C^{1}$-function defined on $[0, T]$ satisfying $\phi_{t} < 0 \forall t \in [0, T]$. Then we have for any $f \in C([0, T]; H)$
\[
s \left\| \int_{0}^{t} f(\tau) d\tau \right\|_{s,T} \leq \frac{1}{\min_{t \in [0,T]} |\phi_{t}(t)|} \|f\|_{s,T}
\]
(12)

**Proof.** Note that
\[
\phi(t) - \phi(\tau) = \phi_{t}(\xi)(t - \tau) \leq L(t - \tau)
\]
where $L = \max_{t \in [0,T]} \phi_{t}(t)$. Set $g = e^{s\phi} f$ and $F = e^{s\phi} \int_{0}^{t} f(\tau) d\tau$. Then
\[
\|F(t)\| \leq \int_{0}^{t} e^{s(\phi(t) - \phi(\tau))} \|g(\tau)\| d\tau
\leq \int_{0}^{t} e^{sL(t - \tau)} \|g(\tau)\| d\tau.
\]
Hence, we have
\[
\|F(t)\|_{T} \leq \|e^{sLt}\|_{L^{1}(0,T)} \|g\|_{T} \leq \frac{1}{L} \|g\|_{T}
\]
which implies (12).

Proof of Theorem 2. From the assumptions and Lemma 1, we see that

\[
\|Pu\|_{s,T}^2 \leq \frac{2C_6^2}{s^2|\phi_t(T)|^2} \left\{ \| (A + A^*)u \|_{s,T}^2 + \| Mu \|_{s,T}^2 + \| Mu_f \|_{s_2,T}^2 + \| Mu \|_{s,T}^2 \right\} + 2C_7^2 \| Mu \|_{s,T}^2 + 2C_8^2 \| f \|_{s,T}^2.
\]

We take \( s_0 \) so large that for any \( s \geq s_0 \)

\[
|\phi_t(T)|^2 \geq \frac{2C_5C_6^2}{s^2} (1 + s|\phi_t(0)|^2)
\]

and

\[
\frac{s}{2} \geq \frac{2C_5C_6^2}{s^2} + 2C_7.
\]

Then, Theorem 1 yields that

\[
s \| Mu \|_{s,T}^2 \leq 2C_5 \left\{ s\phi_t(t)e^{2s\phi(t)} \| M(t)u(t) \|^2 + e^{2s\phi(t)} \langle (A(t) + A^*(t))u(t), M(t)u(t) \rangle \right\}_{0}^{T} + 2C_5C_8^2 \| f \|_{s,T}^2, \tag{13}
\]

from which it follows that

\[
s e^{2s\phi(T)} \| Mu \|_{T}^2 - 2C_5 s\phi_t(T) e^{2s\phi(T)} \| M(T)u(T) \|^2 \leq -2C_5 s\phi_t(0) e^{2s\phi(0)} \| M(0)u(0) \|^2 + \frac{1}{2} e^{2s\phi(T)} \| (A(T) + A^*(T))u(T) \|^2

+ \frac{1}{2} e^{2s\phi(T)} \| M(T)u(T) \|^2 + \frac{1}{2} e^{2s\phi(0)} \| (A(0) + A^*(0))u(0) \|^2 + 2C_5C_8^2 \| f \|_{s,T}^2.
\]

Hence, taking \( s_0 \) so large that

\[
s_0 \geq -\frac{1}{4C_5\phi_t(T)},
\]

we conclude that (4) holds.

If \( \langle M(t)u, (A(t) + A^*(t))u \rangle \leq C \| M(t)u \|^2 \) for all \( t \in [0,T] \), from (13) we see that (5) follows. This completes the proof of Theorem 2.
3 Applications

In this section we discuss the uniqueness of Cauchy problems for semilinear evolution equations and identifiability of coefficients of evolution equations.

3.1 Uniqueness

Let $M(t)$ and $A(t)$ be the same as in section 2. We consider the Cauchy problem for semilinear evolution equation of the form

\begin{align}
M(t)u_t &= A(t)u + \int_0^t f(t, s, u(s))ds + g_1(t, u) + g_2(t), \quad t \in [0, T], \quad (14) \\
u(0) &= u_0. \quad (15)
\end{align}

For brevity we introduce

\[ \|u(t)\|_t = \|(A(t) + A^*(t))u(t)\| + \|M(t)u_t(t)\| + \|M(t)u(t)\|. \]

**Theorem 3** Suppose that $M(t)$ and $A(t)$ satisfy (A1)-(A3). Moreover we assume that $M(t)$ or $A(t) + A^*(t)$ is injective for each $t \in [0, T]$. Suppose that

\[ \|\int_0^t (f(\tau, x, u(\tau)) - f(\tau, x, v(\tau)))d\tau\| \leq C \int_0^t K_1(\|u(\tau)\|_\tau + \|v(\tau)\|_\tau) (\|u(\tau) - v(\tau)\|_\tau) d\tau \]  

(16)

and

\[ \|g_1(t, u(t)) - g_2(t, u(t))\| \leq K_2(\|M(t)u\| + \|M(t)v\|) \|M(t)(u - v)\| \]  

(17)

where $K_1$ and $K_2$ are non decreasing continuous functions defined on $[0, \infty)$. Then, for every $u_0 \in D(M(0)) \cap D(A(0) + A^*(0))$ and $g_2 \in C([0, T]; H)$, the problem (14)-(15) has at most one solution.

**Proof.** We take the set $U$ in Theorem 2 as

\[ U = \{ u \in (D(P) \cap Z) \mid \sup_{0 \leq t \leq T} (\|u\|_t + \|M(t)u(t)\|) \leq R \} \]
for some $R > 0$. Let $u(t)$ and $v(t)$ be two solutions of (14)-(15). Put $w = u - v$. Then
\[
M(t)w_t = A(t)w + \int_0^t (f(t, \tau, u(\tau)) - f(t, \tau, v(\tau)) \, d\tau + (g_1(t, u) - g_1(t, v)), \quad t \in [0, T],
\]

(18)
\[
u(0) = 0.
\]
(19)

The assumptions yield that
\[
\|\int_0^t (f(\tau, x, u(\tau)) - f(\tau, x, v(\tau))) \, d\tau\| \leq C \int_0^t K_1(2R) \|w(\tau)\|_\tau \, d\tau
\]
and
\[
\|g_1(t, u(t)) - g_2(t, v(t))\| \leq K_2(2R) \|M(t)w\|.
\]

Hence, from Theorem 2 we see that
\[
\|Mw\|_T \leq \frac{C}{\sqrt{s}} \|(A(T) + A^*(T))u(T)\|
\]
By letting $s \to \infty$, we conclude that
\[
\|Mw\|_T = 0
\]
which implies
\[
M(t)w(t) = 0 \quad \forall t \in [0, T].
\]
(22)

If $M(t)$ is injective for each $t$, then
\[
w(t) = 0 \quad \forall t \in [0, T].
\]
(23)

Assume that $A(t) + A^*$ is injective. In much the same way as in the proof of Theorem 2, using (22), we see that
\[
\|(A + A^*)w\|_s,T \leq 0
\]
provided that $s$ is taken large enough. Hence we easily see that (23) holds for this case. This completes the proof.
Remark 1 Since our assumptions does not require positivity or accretiveness of the operators $M(t)$, $A(t)$, Theorem 3 covers very wide class of uniqueness question for the Cauchy problem. For instance we can show the backward uniqueness for the heat equation and for pseudo-parabolic equations (see below).

Examples

Let $\Omega$ be a domain in $\mathbb{R}^N$. Let

$$M(t, x, D)u = \sum_{0 \leq |\alpha|, |\beta| \leq p} (-1)^{\alpha} D^\alpha (m_{\alpha\beta}(t, x) D^\beta u),$$

and

$$A(t, x, D)u = \sum_{0 \leq |\alpha|, |\beta| \leq q} (-1)^{\alpha} D^\alpha (a_{\alpha\beta}(t, x) D^\beta u)$$

be linear differential operators of order $2p$ and $2q$, respectively with complex-valued smooth coefficients defined on $[0, T] \times \Omega$. Let $H = L^2(\Omega)$ and define

$$D(M(t)) = \{ u : \Omega \rightarrow \mathbb{C} | u \in H^{2p}(\Omega) \cap H^p_0(\Omega) \}$$

and for any $u \in D(M(t))$

$$(M(t)u)(x) = M(t, x, D)u(t, x), \quad (t, x) \in [0, T] \times \Omega.$$ We assume that $M(t, x, D)$ is formally symmetric, that is,

$$m_{\alpha\beta} = \overline{m}_{\beta\alpha}, \quad \forall \alpha, \beta.$$ Then, under some suitable assumptions on the coefficients $m_{\alpha\beta}$, we can see that for each $t$ $M(t)$ is selfadjoint in $H$ and $D(t) = C_0^\infty(\Omega)$ is the core of $M(t)$.

Let

$$D(A(t)) = \{ u : \Omega \rightarrow \mathbb{C} | u \in H^{2q}(\Omega) \cap H^q_0(\Omega) \}$$

and define for any $u \in D(A(t))$

$$(A(t)u)(x) = A(t, x, D)u(t, x), \quad (t, x) \in [0, T] \times \Omega.$$ In this case the Cauchy problem (14)-(15) is as follows:

$$M(t, x, D)u_t = A(t, x, D)u + \int_0^t f(t, s, x, u(s))ds$$

$$+ g_1(t, x, u) + g_2(t, x) \quad (t, x) \in [0, T] \times \Omega, \quad (24)$$
with
\[ u(0, x) = u_0(x), \quad x \in \Omega, \quad (25) \]
\[ D^\alpha u(t, x) = 0, \quad (t, x) \in [0, T] \times \partial \Omega, \quad |\alpha| \leq q, \quad (26) \]
\[ D^\alpha u_t(t, x) = 0, \quad (t, x) \in [0, T] \times \partial \Omega, \quad |\alpha| \leq p. \quad (27) \]

If the coefficients \( m_{\alpha\beta}(t, x) \) and \( a_{\alpha\beta}(t, x) \) are many-times boundedly differentiable in \((t, x)\) on \((0, T) \times \Omega\), we easily see that the assumption holds valid.

We can impose additional conditions on \( M(t) \), \( A(t) \) so as to satisfy (A3). We list up below some of them:

(Ex.1) \( M(t, x, D) \) and \( A(t, x, D) \) are of constant coefficients and \( A(t, x, D) \) is formally symmetric.

(Ex.2) \( M(t, x, D) \) or \(-M(t, x, D)\) is a uniformly elliptic operator for each \( t \), and \( m_{\alpha\beta}(t, x) \) and \( a_{\alpha\beta}(t, x) \) are independent of \( x \), and \( p \geq q \).

(Ex.3) \( M(t, x, D) = m(t) \neq 0 \) for \( t \in [0, T] \) and \( A(t, x, D) \) is independent of \( t \) and formally symmetric or anti-symmetric with many-times boundedly differentiable coefficients.

Remark 2 The form of Eq. (24) contains pseudo-parabolic equations. Concerning the well-posedness of the initial-boundary value problem for them we refer to the book of Carroll and Showalter [5].

3.2 Identifiability

Consider the initial-periodic boundary value problem
\[ u_t = u_{xx} + a(t)f(x, u), \quad 0 < x < 1, \quad t > 0 \quad (28) \]
\[ u(0, t) = u(1, t) \quad t > 0 \quad (29) \]
\[ u_x(0, t) = u_x(1, t) \quad t > 0 \quad (30) \]
\[ u(x, 0) = u_0(x), \quad 0 < x < 1, \quad (31) \]

where \( f(x, u) \) is a known function of \( u \) and \( u_0 \) is a given function.

Our problem is to recover the coefficient \( a(t) \) when we know some observation of the state. Here we are interested in the case when our observation is given by
\[ u(x_0, t) = u_{ob}(t) \quad 0 < t < T \quad (32) \]
for some point \( x_0 \in [0,1] \). We establish identifiability of the coefficients for the problem, that is, to show that the coefficient \( a(t) \) is uniquely determined by the data and the observation (32).

**Theorem 4** Suppose that \( a(t), u_{ob} \in C(0,T) \) and \( u_0 \in C([0,1]) \). Assume that for given \( a(t) \) and \( u_0 \) there exists a unique solution \( u \in C^2([0,1] \times [0,T]) \) of (28)-(31), which satisfies

\[
 u_{xx}(0,t) = u_{xx}(1,t)
\]

and

\[
f(x, u(x,t)) > 0 \quad \forall t \in [0,1] \times [0,T]. \tag{33}
\]

Then, \((u,a)\) is uniquely determined by the initial condition (31) and the observation (32).

**Remark 3** The assumption (33) is satisfied by, for example,

\[
f(x,u) = q(x)e^u
\]

where \( q(x) \) is a known positive function. or, if we consider positive solutions, it is satisfied by

\[
f(x,u) = q(x)|u|^{p-1}u.
\]

**Proof.** Let \((u_1, a_1)\) and \((u_2, a_2)\) be two solutions. Then, putting \( w = u_1 - u_2 \) and \( a = a_1 - a_2 \), we have

\[
w_t = w_{xx} + a_1(t) \int_0^1 f'(\theta u_1 + (1-\theta)u_2) d\theta w + a(t)f(u_2), \quad 0 < x < 1, \quad t > 0 \tag{34}
\]

\[
w(x_0, t) = 0, \quad t > 0, \tag{35}
\]

\[
\frac{\partial^k}{\partial x^k} w(0, t) = \frac{\partial^k}{\partial x^k} w(1, t), \quad (k = 0, 1, 2) \quad t > 0, \tag{36}
\]

\[
w(x, 0) = 0, \quad 0 < x < 1. \tag{37}
\]

Define

\[
Q = \partial_x - (\log f(u_2))_x
\]

and

\[
\tilde{P}(t) = \partial_t - \partial_{xx} - a_1(t)G(x,t),
\]
where
\[ G(x, t) = \int_0^1 f'(\theta u_1 + (1 - \theta)u_2) d\theta. \]

We easily see that
\[ Q(a(t)f(u_2)) = 0 \quad \forall (x, t) \in [0, 1] \times [0, T]. \]

Applying \( Q \) to (34), we have
\[ Q\tilde{P}w = 0 \quad \forall (x, t) \in [0, 1] \times [0, T]. \]

Hence, we have
\[
\tilde{P}Qw = [\tilde{P}, Q]w = H(x, t)w + 2(\log f(u_2))_{xx}w_x
\]  \hspace{1cm} (38)

where
\[ H(x, t) = -((\log f(u_2))_{xt} + (\log f(u_2))_{xxx} + a_1(t)G_x). \]  \hspace{1cm} (39)

Put \( v = Qw \). Since \( w(x_0, t) = 0 \), we get
\[
Q^{-1}v = \int_{x_0}^x \frac{f(u_2(x, t))}{f(u_2(\xi, t))}v(\xi, t) d\xi
\]  \hspace{1cm} (40)

Hence, we can rewrite (38) as
\[ \tilde{P}v = [\tilde{P}, Q]Q^{-1}v \]  \hspace{1cm} (41)

with
\[
v(0, t) = v(1, t), \quad v_x(0, t) = v_x(1, t), \quad \forall t > 0
\]  \hspace{1cm} (42)

and
\[ v(x, 0) = 0. \]  \hspace{1cm} (43)

In view of (38)-(40) we easily see that
\[ ||[\tilde{P}, Q]Q^{-1}v||_{L^2([0,1])} \leq C||u||_{L^2([0,1])}. \]  \hspace{1cm} (44)
Let $H = L^2([0,1])$ and $A : H \to H$ defined by

$$Au = u_{xx} \quad u \in D(A)$$

with

$$D(A) = \{ u : [0,1] \to \mathbb{R} \mid u \in H^2([0,1]) \text{ satisfies (29), (30)} \}$$

Then, we can apply Theorem 2 to obtain

$$\|v\|_T \leq \frac{C}{\sqrt{s}} \|A(T)u(T)\|$$

for any $s \geq s_0$ where $C$ and $s_0$ are positive constants independent of $v$. Letting $s$ tend to infinity, we get

$$\|v\|_T \equiv 0$$

from which it follows that $v \equiv 0$ on $[0,T^*]$. Then, we conclude that

$$w(t) \equiv 0 \quad t \in [0,T^*]$$

from which we deduce

$$0 = \tilde{P}(t)w = af(u_2) \quad t \in [0,T^*]$$

Noting (33), we see that

$$a = a_1 - a_2 = 0.$$ 

This completes the proof.

**Remark 4** In much the same manner we can obtain analogous results for the initial periodic-boundary value problems in many space dimensions (or equivalently, initial value problems on multi-dimensional torus) not only for (nonlinear) heat equations like (28) but also for the Schrödinger-type or Korteweg-de Vries type equations. (see [15]) In [1] Bukhgeim considered initial-Dirichelet or Neumann boundary value problems with point observations at the boundary.

**Remark 5** Another approach showing identifiability of coefficients relies on the inverse Sturm-Liouville problems (see [7], [14]).
References


