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The Carleman type estimates and non-well-posed problems.

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1 Introduction

Let Ω be a connected open set in $\mathbb{R}^n$, and let $P = P(x, D)$ be a differential operator of order m in Ω with principal symbol p. Let $\phi : \overline{\Omega} \to \mathbb{R}$ be a C^∞ function, with $\nabla\phi(x) \neq 0$, $x \in \Omega$ and which is strongly pseudo convex (this is a convexity property relatively to p.) We say that the Carleman type estimate holds for P if there exists a constant $K > 0$ such that

$$\sum_{|\alpha|<m} \tau^{2(m-|\alpha|)-1} \int_\Omega |D^\alpha u|^2 e^{2\tau\phi} dx \leq K \int_\Omega |P(x, D)u|^2 e^{2\tau\phi} dx \tag{1}$$

$$\forall u \in C_0^\infty(\Omega), \quad \tau > 0 \text{ large enough.}$$

Estimates of this form were first used by Carleman in work on unique continuation property for second order elliptic operators in $\mathbb{R}^2$. Here P is said to have the unique continuation property if the following holds: Suppose u solves $P(x, D)u = 0$ on Ω and $u = 0$ on a empty open set in Ω. Then, u vanishes identically in Ω.

This property is equivalent to uniqueness in the Cauchy problem for any smooth hypersurface.

The Carleman type estimates are established under various assumptions on $P(x, D)$ and have a large field of applications:

1. Unique continuation property and uniqueness of Cauchy problem. (see [3], [4], [6], [9], [10])

2. Spectral properties of Schrödinger operator.(see [12])
3. Generic properties of nonlinear elliptic equations. (see [13]).
4. Stability of (non-well-posed) Cauchy problem (see [1]).
5. Identifiability of spatially-varying coefficients in partial differential equations. (see [1], [2])

The aim of this paper is to present new results concerning the last two subjects. In section 2 we establish an abstract analogue of Carleman estimates, which is an extension of Bukhgeim's result ([1]). In section 3 we apply it to the uniqueness question and identifiability of coefficients for the initial-boundary value problems for some (nonlinear) partial differential equations.

2 Stability estimates

Let H be a complex (or real) Hilbert space, the scalar product and the norm in H being denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $M(t)$ and $A(t)$ be linear operators whose domains are dense subspaces in H and are possibly changeable in t for $t \in [0, T]$.

The subscript t denotes differentiation with respect to t.

In Theorem 1 stated below, we assume the following.

(A1) For every $t \in [0,T]$ $M(t)$ is a selfadjoint operator.

(A2) $M(t)$ and $A(t)$ are strongly continuous and weakly differentiable with respect to t.

(A3) Let

$$\begin{aligned} D(P) = \Big\{ u : [0,T] \to H \mid\ & u(t), u_t(t) \in D(M(t)), \\ & u(t) \in D(A(t)) \text{ for each } \ t \in [0,T], \\ & M(\cdot)u(\cdot) \in C^1([0,T]; H) \text{ and } A(\cdot)u(\cdot) \in C([0,T]; H) \Big\}. \end{aligned}$$

and

$$\begin{aligned} Z = \Big\{ u : [0,T] \to H \mid\ & u(t) \in D(A(t) + A^*(t)), \text{ for each } t \in [0,T] \\ & \{A(\cdot) + A^*(\cdot)\} u(\cdot) \in C([0,T]; H) \Big\} \end{aligned}$$

There exists a linear subspace $\mathbf{D}$ dense in $D(P) \cap Z$ such that, setting $D(t) = \mathbf{D} \cap (\{t\} \times H) \subset ([0,T] \times H)$,

(a) There exists a positive constant C_1 such that

$$\|M_t(t)v\| \leq C_1 \|M(t)v\|, \qquad \forall v \in D(t).$$

(b) $M(t)$ and $A(t) + A^*(t)$ commute each other on $D(t)$, that is, for $v \in D(t)$ $(A(t) + A^*(t))v \in D(M(t))$ and $M(t)v \in D(A(t) + A^*(t))$, we have

$$M(t)(A(t) + A^*(t))v = (A(t) + A^*(t))M(t)v.$$

(c) There exist positive constants C_j $(j = 2, 3, 4)$ such that

$$\|(A(t) - A^*(t))v\| \leq C_2 \|M(t)v\| \qquad \forall v \in D(t),$$

$$\|A^*(t)v\|^2 - \|A(t)v\|^2 \leq C_3 \|M(t)v\|^2 \qquad \forall v \in D(t),$$

and

$$\| \ (A_t(t) + A_t^*(t))v \ \| \leq C_4 \|Mv\| \qquad \forall v \in D(t).$$

We define the operator

$$P(t)u(t) = M(t)u_t - A(t)u(t) \qquad \text{for } \forall u \in D(P).$$

For brevity we write

$$\|u\|_T = \|u\|_{L^2(0,T;H)} \qquad \|u\|_{s,T} = \|e^{s\phi}u\|_T$$

where $\phi = \phi(t)$ is a real-valued continuous function defined on $[0,T]$ and s is an arbitrary nonnegative number.

The following theorem is an extension of abstract versions of Carleman's estimates for the Cauchy problems. (see Nirenberg [11], Bukhgeim [1]).

Theorem 1 *Suppose that the assumptions* (A1)-(A3) hold. Suppose that $\phi \in C^2([0,T])$ satisfies

$$\phi_t(t) \leq 0 \qquad \forall t \in [0,T],$$

and

$$\phi_{tt}(t) + (C_1 + C_2)\phi_t(t) \geq \delta > 0 \qquad \forall t \in [0,T].$$

Then, there exist positive constants s_0 and C_5 such that for all $s \geq s_0$ and $u \in D(P) \cap Z$

$$
\begin{aligned}
s\|Mu\|^2_{s,T} &+ \frac{1}{1+s|\phi_t(0)|^2}\Big(\|(A+A^*)v\|^2_{s,T} + \|Mu_t\|^2_{s,T}\Big) \\
&\leq C_5\Big(\|Pu\|^2_{s,T} + \Big[s\phi_t(t)e^{2s\phi(t)}\|M(t)u(t)\|^2 \\
&\quad + e^{2s\phi(t)}\langle (A(t)+A^*(t))u(t), M(t)u(t)\rangle\Big]\Big|_0^T\Big) \qquad (2)
\end{aligned}
$$

Using Theorem 1, we can establish stability estimates as follows.

Theorem 2 *Suppose that all the assumptions stated in Theorem* 1 *are satisfied. Let* $f \in C([0,T];H)$*. Suppose that there exists a subset* $U \subset (D(P) \cap Z)$ *such that* $\forall u \in U$

$$
\begin{aligned}
&\|P(t)u(t)\| \\
&\quad \leq C_6 \int_0^t (\|(A(\tau)+A^*(\tau)u(\tau)\| + \|M(\tau)u_t(\tau)\| + \|M(\tau)u(\tau)\|)\, d\tau \\
&\qquad + C_7\|M(t)u(t)\| + C_8\|f(t)\| \qquad (3)
\end{aligned}
$$

where C_j $(j = 6,7,8)$ *are positive constants independent of* t*. Then, there exists positive constants* s_0*,* C_9 *and* C_{10}*, independent of* u*,* f *and* t*, such that for* $\forall u \in U$ *and* $\forall s \geq s_0$

$$
\begin{aligned}
\|Mu\|_T \leq C_9\Big[&\frac{1}{\sqrt{s}}\|(A(T)+A^*(T))u(T)\| \\
&+ \exp(sC_{10})\,(\|M(0)u(0)\| + \|(A(0)+A^*(0))u(0)\| + \|f\|_T)\Big]. \qquad (4)
\end{aligned}
$$

Furthermore, if $\langle M(T)u(T), (A(T)+A^*(T))u(T)\rangle \leq C\|M(T)u(T)\|^2$*, then*

$$
\|Mu\|_T \leq C_9\frac{\exp(sC_8)}{\sqrt{s}}\,(\|M(0)u(0)\| + \|(A(0)+A^*(0))u(0)\| + \|f\|_T). \qquad (5)
$$

Proof of Theorem 1. Let $u \in \mathbf{D}$, $v = e^{s\phi}u$ and

$$
\begin{aligned}
\mathsf{P}_\phi(\mathsf{t})\mathsf{v} &= e^{s\phi(t)}P(t)(e^{-s\phi(t)}v) \\
&= -s\phi_t(t)M(t)v + M(t)v_t - A(t)v.
\end{aligned}
$$

Then we have

$$
\mathsf{P}^*_\phi(t)v = -s\phi_t(t)M(t)v - M_t(t)v - M(t)v_t - A^*v.
$$

Define P^s_ϕ and P^a_ϕ by

$$\begin{aligned}\mathsf{P}^s_\phi &= \frac{1}{2}\left(\mathsf{P}_\phi + \mathsf{P}^*_\phi\right) v \\ &= -s\phi_t M v - \frac{1}{2}M_t v - \frac{1}{2}(A + A^*)v \end{aligned} \tag{6}$$

and

$$\begin{aligned}\mathsf{P}^a_\phi &= \frac{1}{2}\left(\mathsf{P}_\phi - \mathsf{P}^*_\phi\right) v \\ &= \frac{1}{2}M_t v + M v_t - \frac{1}{2}(A - A^*)v, \end{aligned} \tag{7}$$

respectively.

We see that

$$\begin{aligned}\|Pu\|_{s,T} = \int_0^T &\left\{\|\mathsf{P}^s_\phi(\tau)v(\tau)\|^2 + \|\mathsf{P}^a_\phi(\tau)v(\tau)\|^2\right. \\ &\left.+ 2\mathrm{Re}\langle \mathsf{P}^s_\phi(\tau)v(\tau), \mathsf{P}^a_\phi(\tau)v(\tau)\rangle\right\} d\tau. \end{aligned} \tag{8}$$

Making use of the assumptions (A1)-(A3), we have

$$\begin{aligned} 2\mathrm{Re}\langle \mathsf{P}^s_\phi \mathsf{v}, \mathsf{P}^a_\phi \mathsf{v}\rangle &= -s\phi_t\left\{\mathrm{Re}\langle Mv, M_t v\rangle + 2\mathrm{Re}\langle Mv, Mv_t\rangle\right\} \\ &\quad +s\phi_t \mathrm{Re}\langle Mv, (A - A^*)v\rangle \\ &\quad -\left\{\frac{1}{2}\|M_t v\|^2 + \mathrm{Re}\langle M_t v, M v_t\rangle\right\} \\ &\quad +\frac{1}{2}\mathrm{Re}\langle M_t v, (A - A^*)v\rangle - \frac{1}{2}\left\{\mathrm{Re}\langle (A + A^*)v, M_t v\rangle\right. \\ &\quad \left.+ 2\mathrm{Re}\langle (A + A^*)v, M v_t\rangle + (\|Av\|^2 - \|A^*\|^2)\right\} \\ &\geq -\frac{d}{dt}\left\{s\phi_t\|Mv\|^2 + \frac{1}{2}\langle (A + A^*)v, Mv\rangle\right\} \\ &\quad +s\phi_{tt}\|Mv\|^2 + s\phi_t(C_1 + C_2)\|Mv\|^2 \\ &\quad -\frac{1}{2}\{3C_1^2 + C_1C_2 + 2C_3 + C_4\}\|Mv\|^2 - \frac{1}{4}\|Mv_t\|^2. \end{aligned}$$

We also have

$$\begin{aligned}\|\mathsf{P}^a_\phi v\|^2 &\geq \frac{1}{2}\left(\|Mv_t\|^2 - \|M_t v\|^2 - \|(A - A^*)v\|^2\right) \\ &\geq \frac{1}{2}\|Mv_t\|^2 - \frac{1}{2}(C_1^2 + C_2^2)\|Mv\|^2. \end{aligned}$$

Hence, if we take s so large that

$$s \geq \frac{1}{\delta}\left(4C_1^2 + C_1C_2 + C_2^2 + 2C_3 + C_4\right),$$

we obtain

$$\begin{aligned}
&\frac{s\delta}{2}\|Mv\|_T^2 + \|\mathsf{P}_\phi^s v\|_T^2 + \frac{1}{2}\|Mv_t\|_T^2 \\
&\quad \leq \|Pu\|_{s,T}^2 + \left\{s\phi_t\|Mv\|^2 + \frac{1}{2}\langle (A+A^*)v, Mv\rangle\right\}\Big|_0^T \equiv I.
\end{aligned} \tag{9}$$

We have

$$\|\mathsf{P}_\phi^s v\|^2 \geq \frac{1}{2}s\phi_t\mathrm{Re}\langle (A+A^*)v, Mv\rangle + \frac{1}{8}\|(A*A^*)v\|^2 - \frac{1}{4}C_2^2\|Mv\|^2,$$

from which it follows that

$$\begin{aligned}
&\frac{1}{8}\|(A+A^*)v\|_T^2 \\
&\quad \leq \|\mathsf{P}_\phi^s v\|_T^2 + \frac{s}{2}|\phi_t(0)|\int_0^T \|Mv\|\ \|(A+A^*)v\|dt + \frac{1}{4}C_2^2\|Mv\|_T^2
\end{aligned}$$

Making use of (9), we have

$$\begin{aligned}
&\frac{1}{8}\|(A+A^*)v\|_T^2 \\
&\quad \leq I + \left(\frac{s}{2\delta}\right)^{1/2}|\phi_t(0)|\ I^{1/2}\left(\int_0^T \|(A+A^*)v\|^2 dt\right)^{1/2} + \frac{1}{2s\delta}C_2^2 I \\
&\quad \leq \left(1 + \frac{s}{\delta}|\phi_t(0)|^2 + \frac{2}{s\delta}C_2^2\right) I + \frac{1}{16}\|(A+A^*)v\|_T^2.
\end{aligned}$$

from which we deduce

$$\int_0^T \|(A+A^*)v\|^2 dt \leq C(1 + s|\phi_t(0)|^2)\ I \tag{10}$$

where and in the sequel by C we denote various positive constants which do not depend on t and u and are changeable from line to line. From (9) and (10), we have

$$\frac{s\delta}{2}\|Mv\|_T^2 + \frac{1}{1+s|\phi_t(0)|^2}\|(A+A^*)v\|_T^2 + \frac{1}{2}\|Mv_t\|_T^2 \leq C\ I.$$

Noting that $Mv_t = s\phi_t e^{s\phi} Mu + e^{s\phi} Mu_t$, we get

$$\begin{aligned}\|Mu_t\|_{s,T}^2 &\leq \|Mv_t\|_T^2 + 2s|\phi_t(0)|\|Mv\|_T\|Mu_t\|_{s,T} \\ &\leq I + 2\sqrt{2}s^{1/2}|\phi_t(0)|I^{1/2}\|Mu_t\|_{s,T} \\ &\leq I + 4s|\phi_t(0)|^2 I + \frac{1}{2}\|Mu_t\|_{s,T}^2\end{aligned}$$

from which it follows that

$$\|Mu_t\|_{s,T}^2 \leq 2(1+4s|\phi_t(0)|^2)I.$$

Hence, we finally obtain

$$\frac{s\delta}{2}\|Mu\|_{s,T}^2 + \frac{1}{1+s|\phi_t(0)|^2}\left(\|(A+A^*)u\|_{s,T}^2 + \|Mu_t\|_{s,T}^2\right) \leq C_5 I. \tag{11}$$

Since $\mathbf{D}$ is dense in $D(P)\cap Z$, the estimate (11 holds for any $u \in D(P)\cap Z$. This completes the proof of Theorem 1.

In order to establish Theorem 2, we need

Lemma 1 *Suppose that $\phi(t)$ is a real-valued C^1-function defined on $[0,T]$ satisfying $\phi_t < 0$ $\forall t \in [0,T]$. Then we have for any $f \in C([0,T];H)$*

$$s\left\|\int_0^t f(\tau)d\tau\right\|_{s,T} \leq \frac{1}{\min_{t\in[0,T]}|\phi_t(t)|}\|f\|_{s,T} \tag{12}$$

Proof. Note that

$$\phi(t) - \phi(\tau) = \phi_t(\xi)(t-\tau) \leq L(t-\tau)$$

where $L = \max_{t\in[0,T]}\phi_t(t)$. Set $g = e^{s\phi}f$ and $F = e^{s\phi}\int_0^t f(\tau)d\tau$. Then

$$\begin{aligned}\|F(t)\| &\leq \int_0^t e^{s(\phi(t)-\phi(\tau))}\|g(\tau)\|d\tau \\ &\leq \int_0^t e^{sL(t-\tau)}\|g(\tau)\|d\tau.\end{aligned}$$

Hence, we have

$$\|F(t)\|_T \leq \|e^{sLt}\|_{L^1(0,T)}\|g\|_T \leq -\frac{1}{L}\|g\|_T$$

which implies (12). ∎

Proof of Theorem 2. From the assumptions and Lemma 1, we see that

$$\begin{aligned}\|Pu\|_{s,T}^2 &\le \frac{2C_6^2}{s^2|\phi_t(T)|^2}\left\{\|(A+A^*)u\|_{s,T}^2+\|Mu_t\|_{s,T}^2+\|Mu\|_{s,T}^2\right\}\\ &\qquad +2C_7^2\|Mu\|_{s,T}^2+2C_8^2\|f\|_{s,T}^2.\end{aligned}$$

We take s_0 so large that for any $s \ge s_0$

$$|\phi_t(T)|^2 \ge \frac{2C_5C_6^2}{s^2}(1+s|\phi_t(0)|^2)$$

and

$$\frac{s}{2} \ge \frac{2C_5C_6^2}{s^2}+2C_7.$$

Then, Theorem 1 yields that

$$\begin{aligned}s\|Mu\|_{s,T}^2 &\le 2C_5\Big\{s\phi_t(t)e^{2s\phi(t)}\|M(t)u(t)\|^2\\ &\qquad +e^{2s\phi(t)}\langle (A(t)+A^*(t))u(t), M(t)u(t)\rangle\Big\}\Big|_0^T+2C_5C_8^2\|f\|_{s,T}^2 \end{aligned}\tag{13}$$

from which it follows that

$$\begin{aligned}&se^{2s\phi(T)}\|Mu\|_T^2-2C_5s\phi_t(T)e^{2s\phi(T)}\|M(T)u(T)\|^2\\ &\le -2C_5s\phi_t(0)e^{2s\phi(0)}\|M(0)u(0)\|^2+\frac{1}{2}e^{2s\phi(T)}\|(A(T)+A^*(T))u(T)\|^2\\ &\qquad +\frac{1}{2}e^{2s\phi(T)}\|M(T)u(T)\|^2+\frac{1}{2}e^{2s\phi(0)}\|(A(0)+A^*(0))u(0)\|^2\\ &\qquad +\frac{1}{2}e^{2s\phi(0)}\|M(0)u(0)\|^2+2C_5C_8^2\|f\|_{s,T}^2.\end{aligned}$$

Hence, taking s_0 so large that

$$s_0 \ge -\frac{1}{4C_5\phi_t(T)},$$

we conclude that (4) holds.

If $\langle M(t)u, (A(t)+A^*(t))u\rangle \le C\|M(t)u\|^2$ for all $t \in [0,T]$, from (13) we see that (5) follows. This completes the proof of Theorem 2.

3 Applications

In this section we discuss the uniqueness of Cauchy problems for semilinear evolution equations and identifiability of coefficients of evolution equations.

3.1 Uniqueness

Let $M(t)$ and $A(t)$ be the same as in section 2. We consider the Cauchy problem for semilinear evolution equation of the form

$$\begin{aligned} M(t)u_t &= A(t)u + \int_0^t f(t,s,u(s))ds + g_1(t,u) + g_2(t), \qquad t \in [0,T], \quad (14)\\ u(0) &= u_0. \quad (15)\end{aligned}$$

For brevity we introduce

$$|||u(t)|||_t = \|(A(t) + A^*(t))u(t)\| + \|M(t)u_t(t)\| + \|M(t)u(t)\|.$$

Theorem 3 *Suppose that $M(t)$ and $A(t)$ satisfy* (A1)-(A3). *Moreover we assume that $M(t)$ or $A(t) + A^*(t)$ is injective for each $t \in [0,T]$. Suppose that*

$$\begin{aligned} &\| \int_0^t (f(\tau,x,u(\tau)) - f(\tau,x,v(\tau)))\, d\tau \| \\ &\qquad \le C \int_0^t K_1(|||u(\tau)|||_\tau + |||v(\tau)|||_\tau)\, (|||u(\tau) - v(\tau)|||_\tau d\tau \qquad (16)\end{aligned}$$

and

$$\begin{aligned} &\|g_1(t,u(t)) - g_2(t,u(t))\| \\ &\qquad \le K_2(\|M(t)u\| + \|M(t)v\|)\ \|M(t)(u-v)\| \qquad (17)\end{aligned}$$

where K_1 and K_2 are non decreasing continuous functions defined on $[0,\infty)$. Then, for every $u_0 \in D(M(0)) \cap D(A(0) + A^*(0))$ and $g_2 \in C([0,T];H)$, the problem (14)-(15) *has at most one solution.*

Proof. We take the set U in Theorem 2 as

$$U = \left\{\, u \in (D(P) \cap Z) \mid \sup_{0 \le t \le T} (|||u|||_t + \|M(t)u(t)\|) \le R \right\}$$

for some $R > 0$. Let $u(t)$ and $v(t)$ be two solutions of (14)-(15). Put $w = u - v$. Then

$$\begin{aligned} M(t)w_t &= A(t)w + \int_0^t (f(t,\tau,u(\tau)) - f(t,\tau,v(\tau))\, d\tau \\ &\quad + (g_1(t,u) - g_1(t,v)), \qquad t \in [0,T], \qquad (18) \\ u(0) &= 0. \qquad (19) \end{aligned}$$

The assumptions yield that

$$\| \int_0^t (f(\tau,x,u(\tau)) - f(\tau,x,v(\tau)))\, d\tau \| \leq C \int_0^t K_1(2R) \| w(\tau) \|_\tau d\tau \qquad (20)$$

and

$$\|g_1(t,u(t)) - g_2(t,v(t))\| \leq K_2(2R)\|M(t)w\|. \qquad (21)$$

Hence, from Theorem 2 we see that

$$\|Mw\|_T \leq \frac{C_9}{\sqrt{s}} \|(A(T) + A^*(T))u(T)\|$$

By letting $s \to \infty$, we conclude that

$$\|Mw\|_T = 0$$

which implies

$$M(t)w(t) = 0 \qquad \forall t \in [0,T]. \qquad (22)$$

If $M(t)$ is injective for each t, then

$$w(t) = 0 \qquad \forall t \in [0,T]. \qquad (23)$$

Assume that $A(t) + A^*$ is injective. In much the same way as in the proof of Theorem 2, using (22), we see that

$$\|(A + A^*)w\|_{s,T} \leq 0$$

provided that s is taken large enough. Hence we easily see that (23) holds for this case. This completes the proof.

Remark 1 *Since our assumptions does not require positivity or accretiveness of the operators $M(t)$, $A(t)$, Theorem 3 covers very wide class of uniqueness question for the Cauchy problem. For instance we can show the backward uniqueness for the heat equation and for pseudo-parabolic equations (see below).*

Examples

Let Ω be a domain in $\mathbb{R}^N$. Let

$$M(t,x,D)u = \sum_{0\leq|\alpha|,|\beta|\leq p} (-1)^{\alpha} D^{\alpha}(m_{\alpha\beta}(t,x)D^{\beta}u),$$

and

$$A(t,x,D)u = \sum_{0\leq|\alpha|,|\beta|\leq q} (-1)^{\alpha} D^{\alpha}(a_{\alpha\beta}(t,x)D^{\beta}u)$$

be linear differential operators of order $2p$ and $2q$, respectively with complex-valued smooth coefficients defined on $[0,T]\times\Omega$. Let $H = L^2(\Omega)$ and define

$$D(M(t)) = \{u : \Omega \to \mathbb{C} \mid u \in H^{2p}(\Omega) \cap H_0^p(\Omega)\}$$

and for any $u \in D(M(t))$

$$(M(t)u)(x) = M(t,x,D)u(t,x), \qquad (t,x) \in [0,T]\times\Omega.$$

We assume that $M(t,x,D)$ is formally symmetric, that is,

$$m_{\alpha\beta} = \overline{m}_{\beta\alpha}, \quad \forall\alpha,\beta.$$

Then, under some suitable assumptions on the coefficients $m_{\alpha\beta}$, we can see that for each t $M(t)$ is selfadjoint in H and $D(t) = C_0^{\infty}(\Omega)$ is the core of $M(t)$.

Let

$$D(A(t)) = \{u : \Omega \to \mathbb{C} \mid u \in H^{2q}(\Omega) \cap H_0^q(\Omega)\}$$

and define for any $u \in D(A(t))$

$$(A(t)u)(x) = A(t,x,D)u(t,x), \qquad (t,x) \in [0,T]\times\Omega.$$

In this case the Cauchy problem (14)-(15) is as follows:

$$\begin{aligned} M(t,x,D)u_t &= A(t,x,D)u + \int_0^t f(t,s,x,u(s))ds \\ &\quad + g_1(t,x,u) + g_2(t,x) \qquad (t,x) \in [0,T]\times\Omega, \end{aligned} \tag{24}$$

with

$$
\begin{aligned}
u(0,x) &= u_0(x), \qquad x \in \Omega, && (25)\\
D^\alpha u(t,x) &= 0, \qquad (t,x) \in [0,T] \times \partial\Omega, \qquad |\alpha| \le q, && (26)\\
D^\alpha u_t(t,x) &= 0, \qquad (t,x) \in [0,T] \times \partial\Omega, \qquad |\alpha| \le p. && (27)
\end{aligned}
$$

If the coefficients $m_{\alpha\beta}(t,x)$ and $a_{\alpha\beta}(t,x)$ are many-times boundedly differentiable in (t,x) on $(0,T) \times \Omega$, we easily see that the assumption holds valid.

We can impose additional conditions on $M(t)$, $A(t)$ so as to satisfy (A3). We list up below some of them:

(Ex.1) $M(t,x,D)$ and $A(t,x,D)$ are of constant coefficients and $A(t,x,D)$ is formally symmetric.

(Ex.2) $M(t,x,D)$ or $-M(t,x,D)$ is a uniformly elliptic operator for each t, and $m_{\alpha\beta}(t,x)$ and $a_{\alpha\beta}(t,x)$ are independent of x, and $p \ge q$.

(Ex.3) $M(t,x,D) = m(t) \neq 0$ for $t \in [0,T]$ and $A(t,x,D)$ is independent of t and formally symmetric or anti-symmetric with many-times boundedly differentiable coefficients.

Remark 2 *The form of Eq.* (24) *contains pseudo-parabolic equations. Concerning the well-posedness of the initial-boundary value problem for them we refer to the book of Carroll and Showalter* [5].

3.2 Identifiability

Consider the initial-periodic boundary value problem

$$
\begin{aligned}
&u_t = u_{xx} + a(t)f(x,u), \qquad 0 < x < 1, \quad t > 0 && (28)\\
&u(0,t) = u(1,t) \qquad t > 0 && (29)\\
&u_x(0,t) = u_x(1,t) \qquad t > 0 && (30)\\
&u(x,0) = u_0(x), \qquad 0 < x < 1, && (31)
\end{aligned}
$$

where $f(x,u)$ is a known function of u and u_0 is a given function.

Our problem is to recover the coefficient $a(t)$ when we know some observation of the state. Here we are interested in the case when our observation is given by

$$
u(x_0,t) = u_{ob}(t) \qquad 0 < t < T \tag{32}
$$

for some point $x_0 \in [0,1]$. We establish identifiability of the coefficients for the problem, that is, to show that the coefficient $a(t)$ is uniquely determined by the data and the observation (32).

Theorem 4 *Suppose that $a(t), u_{ob} \in C(0,T]$ and $u_0 \in C([0,1])$. Assume that for given $a(t)$ and u_0 there exists a unique solution $u \in C^2([0,1] \times [0,T])$ of* (28)-(31), *which satisfies*

$$u_{xx}(0,t) = u_{xx}(1,t)$$

and

$$f(x,u(x,t)) > 0 \qquad \forall t \in [0,1] \times [0,T]. \tag{33}$$

Then, (u,a) is uniquely determined by the initial condition (31) *and the observation* (32).

Remark 3 *The assumption* (33) *is satisfied by, for example,*

$$f(x,u) = q(x)e^u$$

where $q(x)$ is a known positive function. or, if we consider positive solutions, it is satisfied by

$$f(x,u) = q(x)|u|^{p-1}u.$$

Proof. Let (u_1, a_1) and (u_2, a_2) be two solutions. Then, putting $w = u_1 - u_2$ and $a = a_1 - a_2$, we have

$$\begin{aligned} w_t = w_{xx} + a_1(t) \int_0^1 f'(\theta u_1 + (1-\theta)u_2) d\theta w & \\ + a(t) f(u_2), \qquad 0 < x < 1, \quad t > 0 & \qquad (34) \\ w(x_0, t) = 0, \qquad t > 0, & \qquad (35) \\ \frac{\partial^k}{\partial x^k} w(0,t) = \frac{\partial^k}{\partial x^k} w(1,t) \qquad (k = 0,1,2) \qquad t > 0, & \qquad (36) \\ w(x,0) = 0, \qquad 0 < x < 1. & \qquad (37) \end{aligned}$$

Define

$$Q = \partial_x - (\log f(u_2))_x$$

and

$$\tilde{P}(t) = \partial_t - \partial_{xx} - a_1(t) G(x,t),$$

where

$$G(x,t) = \int_0^1 f'(\theta u_1 + (1-\theta)u_2)d\theta.$$

We easily see that

$$Q(a(t)f(u_2)) = 0 \qquad \forall (x,t) \in [0,1] \times [0,T].$$

Applying Q to (34), we have

$$Q\tilde{P}w = 0 \qquad \forall (x,t) \in [0,1] \times [0,T].$$

Hence, we have

$$\begin{aligned} \tilde{P}Qw &= [\tilde{P}, Q]w \\ &= H(x,t)w + 2(\log(f(u_2))_{xx}w_x \end{aligned} \tag{38}$$

where

$$H(x,t) = -(\log f(u_2))_{xt} + (\log f(u_2))_{xxx} + a_1(t)G_x. \tag{39}$$

Put $v = Qw$. Since $w(x_0, t) = 0$, we get

$$Q^{-1}v = \int_{x_0}^{x} \frac{f(u_2(x,t))}{f(u_2(\xi,t))} v(\xi,t)d\xi \tag{40}$$

Hence, we can rewrite (38) as

$$\tilde{P}v = [\tilde{P}, Q]Q^{-1}v \tag{41}$$

with

$$v(0,t) = v(1,t), \qquad v_x(0,t) = v_x(1,t), \qquad \forall t > 0 \tag{42}$$

and

$$v(x,0) = 0. \tag{43}$$

In view of (38)-(40) we easily see that

$$\|[\tilde{P}, Q]Q^{-1}v\|_{L^2([0,1])} \leq C\|u\|_{L^2([0,1])}. \tag{44}$$

Let $H = L^2([0,1])$ and $A : H \to H$ defined by

$$Au = u_{xx} \qquad u \in D(A)$$

with

$$\begin{aligned} D(A) &= \Big\{u : [0,1] \to \mathbb{R} \mid u \in H^2([0,1]) \\ &\qquad \text{u satisfies (29), (30)}\Big\} \end{aligned}$$

Then, we can apply Theorem 2 to obtain

$$\|v\|_T \leq \frac{C}{\sqrt{s}}\|A(T)u(T)\|$$

for any $s \geq s_0$ where C and s_0 are positive constants independent of v. Letting s tend to infinity, we get

$$\|v\|_T \equiv 0$$

from which it follows that $v \equiv 0$ on $[0, T^*]$. Then, we conclude that

$$w(t) \equiv 0 \qquad t \in [0, T^*]$$

from which we deduce

$$0 = \tilde{P}(t)w = af(u_2) \qquad t \in [0, T^*]$$

Noting (33), we see that

$$a = a_1 - a_2 = 0.$$

This completes the proof.

Remark 4 *In much the same manner we can obtain analogous results for the initial periodic-boundary value problems in many space dimensions (or equivalently, initial value problems on multi-dimensional torus) not only for (nonlinear) heat equations like* (28) *but also for the Schrödinger-type or Korteweg-de Vries type equations.(see* [15]*) In [1] Bukhgeim considered initial-Dirichelet or Neumann boundary value problems with point observations at the boundary.*

Remark 5 *Another approach showing identifiability of coefficients relies on the inverse Sturm-Liouville problems (see* [7], [14] *).*

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