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Periodic behavior of solutions to a continuous casting problem

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1. Introduction

In this paper we consider a continuous casting problem

\[
(P)\nu
\begin{cases}
\partial_t \eta + \nu \partial_z \eta - \Delta \theta = 0 & \text{in } Q_\infty := ]0, \infty[ \times \Omega, \\
\eta \in \beta(\theta) & \text{in } Q_\infty, \\
\frac{\partial \theta}{\partial n} + g(t, x, \theta) = 0 & \text{on } \Sigma_N := ]0, \infty[ \times \Gamma_N, \\
\theta = M & \text{on } \Sigma_0 := ]0, \infty[ \times \Gamma_0, \\
\theta = -m & \text{on } \Sigma_L := ]0, \infty[ \times \Gamma_L,
\end{cases}
\]

under periodic (in time) boundary condition

\[
g(t + T, x, \theta) = g(t, x, \theta) \quad \text{on } \Sigma_\infty \times \mathbb{R},
\]

for a given period \( T > 0 \). Here \( \Omega = ]-l, l[ \times [0, L], \Gamma_N = \{l, -l\} \times [0, L], \Gamma_0 = ]-l, l[ \times \{0\}, \Gamma_L = ]-l, l[ \times \{L\}, L, l > 0, x = (y, z); \nu, m \) and \( M \) are given constants with \( \nu \geq 0 \) and \( m, M > 0 \); \( \beta \) is a maximal monotone graph of the form

\[
\beta(r) = \begin{cases}
\lambda + \int_0^r b(\tau) d\tau & \text{if } r > 0, \\
[0, \lambda] & \text{if } r = 0, \\
\int_0^r b(\tau) d\tau & \text{if } r < 0,
\end{cases}
\]

for a given constant \( \lambda > 0 \) and a locally bounded measurable function \( b \) such that

\[
(1.1) \quad b(r) \geq b_\ast > 0 \quad \text{for a.e. } r \in \mathbb{R}.
\]

Furthermore \( g = g(t, x, \cdot) \) is a given function on \( \mathbb{R}_+ \times \Gamma_N \times \mathbb{R} \) such that

\[
(g1) \ g(t, x, \cdot) \text{ is a nondecreasing function for a.e. } (t, x) \in \mathbb{R}_+ \times \Gamma_N;
\]
(g2) \( g(\cdot, \cdot, \theta) \in L_{loc}^{2}(\mathbb{R}_+; L^{2}(\Gamma_N)) \) for all \( \theta \in \mathbb{R} \);

(g3) For any \( K > 0 \) there is a constant \( C_g(K) > 0 \) such that

\[
|g(t, x, \theta_1) - g(t, x, \theta_2)| \leq C_g(K) |\theta_1 - \theta_2|
\]

for all \( \theta_1, \theta_2 \in [-K, K] \) and a.e. \( (t, x) \in \mathbb{R}_+ \times \Gamma_N \);

(g4) There exist constants \( K_1, K_2 > 0 \) such that

\[
g(t, x, -K_1) \leq 0, \quad g(t, x, K_2) \geq 0 \quad \text{for a.e.} \quad (t, x) \in \mathbb{R}_+ \times \Gamma_N.
\]

For details of continuous casting problems, see Rodrigues [5], Rodrigues-Yi [6], Yi [9] and the literatures in their references. We remark here that problem \((P)^0\) is a Stefan problem. For results to periodic solutions of Stefan problems we refer to Aiki et al. [1], Damlamian-Kenmochi [2] and Haraux-Kenmochi [3]. In the following chapters, we shall discuss problem \((P)^\nu\) due to Shinoda [7, 8].

2. Main results

Throughout this paper we denote \( Q_S =]0, S[\times\Omega, \Sigma_S^N =]0, S[\times\Gamma_N, \text{etc. for } S \in ]0, +\infty[. \)

Now let us give a notion of a weak solution on an interval of the form \([0, S]\) or \([0, +\infty[. \)

**Definition 2.1.** Let \( S \) be a positive number. Then a couple \((\theta, \eta) \in L^2(0, S; H^1(\Omega)) \times L^\infty(Q_S)\) is called a weak solution of \((P)^\nu\) on \([0, S]\) when the following four conditions are satisfied:

(w1) \( \eta \in C_w([0, S]; L^2(\Omega)), \) that is, \( \eta \) is a weakly continuous function from \([0, S]\) to \( L^2(\Omega) \);

(w2) \( \theta = M \) a.e. on \( \Sigma_S^0 \) and \( \theta = -m \) a.e. on \( \Sigma_S^L \);

(w3) \( \eta \in \beta(\theta) \) a.e. in \( Q_S \);
(w4) for any $\varphi \in W_S := \{\varphi \in H^1(Q_S) ; \varphi(S, \cdot) = 0$ a.e. in $\Omega$, $\varphi = 0$ a.e. on $\Sigma_S^D\},$

$$-\int_{Q_S} \eta(\partial_t \varphi + \nu \partial_z \varphi) dx dt + \int_{Q_S} \nabla \theta \nabla \varphi dx dt + \int_{\Sigma_S^N} g(\cdot, \cdot, \theta) \varphi d\Gamma dt = \int_{\Omega} \eta(0, \cdot) \varphi(0, \cdot) dx,$$

where $\Sigma_S^D = ]0, S[\times \Gamma_D$, $\Gamma_D = \Gamma_0 \cup \Gamma_L$. In the case when $S = +\infty$, $(\theta, \eta)$ is called a weak solution of $(P)$ on $\mathbb{R}_+$ if $(\theta, \eta)$ is a weak solution of $(P)$ on $[0, S]$ for any finite $S > 0$.

**Definition 2.2.** Let $0 < S \leq +\infty$ and let $(\theta_0, \eta_0)$ be a pair of functions in $L^\infty(\Omega)$ satisfying $\eta_0 \in \beta(\theta_0)$ a.e. in $\Omega$. Then we call a pair $(\theta, \eta)$ a weak solution for $CP(\theta_0, \eta_0)^\nu$ on $[0, S]$ ($\mathbb{R}_+$ if $S = +\infty$) if $(\theta, \eta)$ is a weak solution of $(P)$ on $[0, S]$ and the initial conditions $\theta(0, \cdot) = \theta_0$ and $\eta(0, \cdot) = \eta_0$ are satisfied, respectively.

Concerning the existence and the uniqueness results for $CP(\theta_0, \eta_0)^\nu$, we quote them from Rodrigues-Yi [6]. The first proposition assures the existence of a weak solution for $CP(\theta_0, \eta_0)^\nu$.

**Proposition 2.1.** (cf. [6; theorem 1]) Let $(\theta_0, \eta_0) \in (L^\infty(\Omega))^2$ be any pair of functions such that $\eta_0 \in \beta(\theta_0)$ a.e. in $\Omega$. Choose two positive constants $\tilde{K}_1$ and $\tilde{K}_2$ so that $\tilde{K}_i \geq \max\{m, M, K_i\}, i = 1, 2$, and that

$$\beta(-\tilde{K}_1) \leq \eta_0(x) \leq \beta(\tilde{K}_2) \quad \text{for a.e. } x \in \Omega.$$

Then, there exists at least one weak solution $(\theta, \eta)$ for $CP(\theta_0, \eta_0)^\nu$ on $\mathbb{R}_+$ such that

$$\beta(-\tilde{K}_1) \leq \eta(t, x) \leq \beta(\tilde{K}_2) \quad \text{for a.e. } (t, x) \in Q_\infty,$$

hence

$$-\tilde{K}_1 \leq \theta(t, x) \leq \tilde{K}_2 \quad \text{for a.e. } (t, x) \in Q_\infty.$$
Remark 2.1. In view of the proof of [6; theorem 1] we may assume that the solution $(\theta, \eta)$ obtained in proposition 2.1 is constructed as a limit of an approximate solution $(\theta_\epsilon, \beta_\epsilon(\theta_\epsilon))$ of

$$
\begin{align*}
\partial_t \beta_\epsilon(\theta_\epsilon) + \nu \partial_x \beta_\epsilon(\theta_\epsilon) - \Delta \theta_\epsilon &= 0 &\text{in } Q_\infty, \\
\frac{\partial \theta_\epsilon}{\partial n} + g_\epsilon(t, x, \theta_\epsilon) &= 0 &\text{on } \Sigma_N, \\
\theta_\epsilon &= M &\text{on } \Sigma_0, \\
\theta_\epsilon &= -m &\text{on } \Sigma_L, \\
\theta_\epsilon(0, \cdot) &= \theta_{0\epsilon} &\text{in } \Omega,
\end{align*}
$$

in the sense that for some subsequence $\{\epsilon_n\}$ of $\{\epsilon\}$

(2.1) \quad \beta_{\epsilon_n}(\theta_{\epsilon_n}) \rightharpoonup \eta \quad \text{weakly* in } L^\infty_{loc}(\mathbb{R}_+; L^\infty(\Omega));

(2.2) \quad \theta_{\epsilon_n} \rightharpoonup \theta \quad \text{weakly in } L^2_{loc}(\mathbb{R}_+; H^1(\Omega)) \cap H^1_{loc}(Q_\infty);

(2.3) \quad g_{\epsilon_n}(\cdot, \cdot, \theta_{\epsilon_n}) \rightharpoonup g(\cdot, \cdot, \theta) \quad \text{in } L^2_{loc}(\mathbb{R}_+; L^2(\Gamma_N)).

Here $\{\beta_\epsilon\}$, $\{g_\epsilon\}$ and $\{\theta_{0\epsilon}\}$ are smooth approximations to $\beta$, $g$ and $\theta_0$, respectively. Furthermore, $\{\beta_\epsilon\}$ satisfies (1.1) with $b_\epsilon = \beta'_\epsilon$, $\beta_\epsilon(0) = 0$, $\beta'_\epsilon \leq 1/\epsilon$ and

$$
\beta_\epsilon(r) \rightharpoonup \beta(r) \quad \text{for any compact interval in } \mathbb{R} \setminus \{0\} \text{ as } \epsilon \to 0;
$$

$\{g_\epsilon\}$ satisfies (g1)\sim(g4) and

$$
g_\epsilon(\cdot, \cdot, \theta) \rightharpoonup g(\cdot, \cdot, \theta) \quad \text{in } L^2_{loc}(\mathbb{R}_+; L^2(\Gamma_N))
$$

uniformly with respect to $\theta$ on any compact set in $\mathbb{R}$ as $\epsilon \to 0$;

$\{\theta_{0\epsilon}\}$ satisfies the compatibility conditions

(2.4) \quad \theta_{0\epsilon} = M \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta_{0\epsilon} = -m \quad \text{on } \Gamma_L.
$\beta_\epsilon(\theta_{0\epsilon}) \rightarrow \eta_0$ in $L^2(\Omega)$ as $\epsilon \rightarrow 0$.

The second proposition is the continuous dependence of the weak solutions. This requires the following condition to a weak solution $(\theta, \eta)$ of $(P)^{\nu}$:

For some positive constants $\delta, \rho > 0$,

$$\theta(t, y, z) \geq \rho > 0 \quad \text{a.e. in } Q_\infty^\delta := \{(t, y, z) \in Q_\infty; 0 < z < \delta\}.$$ 

**Proposition 2.2.** (cf. [6; theorem 2]) Fix $\nu > 0$. Let $(\theta_1, \eta_1)$ and $(\theta_2, \eta_2)$ be two weak solutions for $CP(\theta_{10}, \eta_{10})^{\nu}$ and $CP(\theta_{20}, \eta_{20})^{\nu}$, respectively. If at least one of $(\theta_i, \eta_i)$ satisfies (2.5), then the following is valid:

$$\int_{Q_\infty} |\eta_1 - \eta_2| dx dt \leq \frac{L}{\nu} \int_{\Omega} |\eta_{10} - \eta_{20}| dx.$$ 

As a direct corollary we have:

**Corollary 2.1.** If at least one of the weak solution $(\theta, \eta)$ for $CP(\theta_0, \eta_0)^{\nu}$ on $\mathbb{R}^+$ satisfies (2.5), then $(\theta, \eta)$ is the only weak solution for $CP(\theta_0, \eta_0)^{\nu}$ on $\mathbb{R}^+$.

Using well-known $L^1$-space technique, we have in the manner similar to that of [1]:

**Proposition 2.3.** Let $\nu > 0$, and let $(\theta_1, \eta_1)$, $(\theta_2, \eta_2)$ be two weak solutions for $CP(\theta_{10}, \eta_{10})^{\nu}$ and $CP(\theta_{20}, \eta_{20})^{\nu}$ on $\mathbb{R}^+$ satisfying (2.5), respectively. Then we have

$$[[\eta_1(t, \cdot) - \eta_2(t, \cdot)]^+]_{L^1(\Omega)} \leq [[\eta_1(s, \cdot) - \eta_2(s, \cdot)]^+]_{L^1(\Omega)} \quad \text{for any } s, t \in \mathbb{R}^+ \text{ with } s \leq t,$$

and

$$|\eta_1(t, \cdot) - \eta_2(t, \cdot)|_{L^1(\Omega)} \leq |\eta_1(s, \cdot) - \eta_2(s, \cdot)|_{L^1(\Omega)} \quad \text{for any } s, t \in \mathbb{R}^+ \text{ with } s \leq t.$$
In particular, if $\eta_{10} \leq \eta_{20}$ a.e. in $\Omega$ then

$$\eta_{1} \leq \eta_{2} \text{ hence } \theta_{1} \leq \theta_{2} \text{ a.e. in } Q_{\infty}.$$ 

**Remark 2.2.** Propositions 2.1, 2.3 and corollary 2.1 are also valid for $\nu = 0$. We can prove them by using similar techniques to those in the proofs of [6; theorem 1,4; theorem 4.2,1; lemma 2.1], respectively.

Next we state a definition of a $T$-periodic weak solution of $(P)^{\nu}$ on $\mathbb{R}_{+}$.

**Definition 2.3.** Let $T$ be a given positive number (period). Then $(\theta, \eta)$ is called a $T$-periodic weak solution of $(P)^{\nu}$ on $\mathbb{R}_{+}$ provided that $(\theta, \eta)$ is a weak solution of $(P)^{\nu}$ on $\mathbb{R}_{+}$ and satisfies the periodic conditions $\theta(t + T, \cdot) = \theta(t, \cdot)$ and $\eta(t + T, \cdot) = \eta(t, \cdot)$ for all $t \in \mathbb{R}_{+}$.

Finally we mention the main results for the $T$-periodic weak solution of $(P)^{\nu}$ on $\mathbb{R}_{+}$.

**Theorem 2.1.** Let $\nu > 0$. Assume that the periodicity condition

$$g(t + T, x, \theta) = g(t, x, \theta) \text{ for all } \theta \in \mathbb{R}_{+} \text{ and a.e. } (t, x) \in \mathbb{R}_{+} \times \Gamma_{N}$$

holds. Then there exists one and only one $T$-periodic weak solution $(\theta^{\nu}, \eta^{\nu})$ of $(P)^{\nu}$ on $\mathbb{R}_{+}$.

**Theorem 2.2.** Assume that the same conditions as in theorem 2.1 hold. Then for any weak solution $(\theta, \eta)$ satisfying (2.5) for some positive constants $\delta, \rho > 0$, we have

$$\eta^{\nu}_{p}(t, \cdot) - \eta(t, \cdot) \rightarrow 0 \text{ and } \theta^{\nu}_{p}(t, \cdot) - \theta(t, \cdot) \rightarrow 0 \text{ in } L^{q}(\Omega) \text{ for all } q \geq 1 \text{ as } t \rightarrow +\infty.$$
Remark 2.3. Yi [9] treated the periodic solutions under the Dirichlet boundary condition. He proved there the existence of periodic solutions using Schauder fixed point theorem.

Remark 2.4. There exists a $T$-periodic weak solution $(\theta_p^0, \eta_p^0)$ of $(P)^0$ on $\mathbb{R}_+$ under the periodicity condition (2.7). But for the uniqueness of $T$-periodic weak solutions of $(P)^0$ on $\mathbb{R}_+$, we can only prove that of $g(\cdot, \cdot, \theta_p^0)$ on $\Sigma_{\infty}^N$ and moreover that of $\theta_p^0$ in $Q_{\infty}$ (see [7,8] and also [2]).

3. Lemmas

In this chapter we prepare some lemmas to prove theorems 2.1 and 2.2.

Firstly we define a function $g_* = g_*(\theta)$ by $g_*(\theta) = C_g(K_3)[\theta + K_3]^+$ for $\theta \in \mathbb{R}$, where $K_3 = \max\{M, m, K_1, K_2\}$. Then the following is valid.

Lemma 3.1. $g_*$ defined as above is nondecreasing and satisfies

$$g(t, x, \theta) \leq g_*(\theta) \quad \text{for all } \theta \leq K_3 \text{ and a.e. } (t, x) \in \mathbb{R}_+ \times \Gamma_N,$$

Next we construct a smooth function $\theta_* = \theta_*(x)$ satisfying for any $\epsilon > 0$ the following system

$$\begin{cases}
\nu \partial_z \beta_\epsilon(\theta_*) - \Delta \theta_* \leq 0 \quad \text{in } \Omega, \\
\frac{\partial \theta_*}{\partial n} + g_*(\theta_*) \leq 0 \quad \text{on } \Gamma_N, \\
\theta_* \leq M \quad \text{in } \overline{\Omega}, \\
\theta_* \leq -K_3 \quad \text{on } \Gamma_L.
\end{cases}
$$

Choose a function $\chi = \chi(y) \in C^\infty([-l, l])$ such that

$$0 \leq \chi \leq \frac{M}{2} \quad \text{in } ]-l, l[,$$

and

$$\pm \frac{\partial \chi}{\partial y}(\pm l) + g_*(M) \leq 0.$$
For a positive parameter \( \mu \), let us define \( \theta_* \) by

\[
\theta_*(y, z) = -\mu z + \chi(y) + \frac{M}{2}.
\]

Then we see that \( \theta_* \) satisfies for some constants \( \delta, \rho > 0 \)

\[
\theta_*(y, z) \geq \rho \quad \text{in} \quad \Omega_\delta := \{(y, z) \in \Omega; 0 < z < \delta\}.
\]

Moreover it is readily seen that (3.1) is fulfilled for sufficiently large \( \mu \) dependent upon \( \nu \). Thus we have the following lemma.

**Lemma 3.2.** There is a smooth function \( \theta_* = \theta_*(x) \) on \( \Omega \) which is independent of \( \varepsilon \) and satisfies (3.1) and (3.2) for some positive constants \( \delta, \rho \).

Put \( \eta_* = \beta(\theta_*) \). We remark that \( \eta_* \) is a.e. defined since the Lebesgue measure of the set \( \{x \in \Omega; \theta_*(x) = 0\} \) is zero. Then we have:

**Lemma 3.3.** The unique weak solution \( (? \theta, \eta) \) for \( CP(\theta_*, \eta_*)^\nu \) on \( \mathbb{R}^+ \) satisfies (2.5) for some \( \delta, \rho > 0 \).

**Proof.** Let \( \{\theta_{0\varepsilon}\} \subset C^\infty(\overline{\Omega}) \) such that \( \theta_* \leq \theta_{0\varepsilon} \) in \( \Omega \), \( \beta_{\varepsilon}(\theta_{0\varepsilon}) \to \eta_* \) in \( L^2(\Omega) \) as \( \varepsilon \to 0 \), and that (2.4) holds. Recalling proposition 2.1 and remark 2.1, we get a weak solution \( (? \theta, \eta) \) for \( CP(\theta_*, \eta_*)^\nu \) on \( \mathbb{R}^+ \) as a limit of an approximate solution \( \theta_{\varepsilon n} \) corresponding to initial value \( \theta_{0\varepsilon n} \) in the sense of (2.1)\( \sim \) (2.3) for some subsequence \( \{\varepsilon_n\} \) of \( \{\varepsilon\} \). We note that for any \( \varepsilon \in [0, 1] \).

\[
\partial_t(\beta_\varepsilon(\theta_* - \beta_\varepsilon(\theta_*))) + \nu \partial_z(\beta_\varepsilon(\theta_* - \beta_\varepsilon(\theta_*))) - \Delta(\theta_* - \theta_\varepsilon) \leq 0 \quad \text{in} \quad Q_\infty.
\]

Now let us denote by \( \{\sigma_m\} \) a sequence of smooth functions on \( \mathbb{R} \) such that \( \sigma_m(0) = 0 \), and for any \( r \in \mathbb{R}, \sigma_m'(r) \geq 0, -1 \leq \sigma_m(r) \leq 1 \) and

\[
\sigma_m(r) \to \sigma_0(r) := \begin{cases} 
1 & \text{for } r > 0, \\
0 & \text{for } r = 0, \\
-1 & \text{for } r < 0,
\end{cases}
\]

as \( m \to +\infty \).
Multiply (3.3) by \( \sigma_m([\theta_\ast - \theta_e]^+) \) and integrate it over \( Q_t \). By lemma 3.1 and 3.2,

\[
\int_{Q_t} \triangle(\theta_\ast - \theta_e)\sigma_m([\theta_\ast - \theta_e]^+)dxd\tau \\
\geq \int_{\Sigma_t} (g_\ast(\theta_\ast) - g_e(\cdot, \cdot, \theta_e))\sigma_m([\theta_\ast - \theta_e]^+)d\Gamma d\tau \\
\geq \int_{\Sigma_t} (g_\ast(\cdot, \cdot, \theta_e) - g_e(\cdot, \cdot, \theta_e))\sigma_m([\theta_\ast - \theta_e]^+)d\Gamma d\tau \\
\rightarrow \int_{\Sigma_t} (g_\ast(\cdot, \cdot, \theta_e) - g_e(\cdot, \cdot, \theta_e))\sigma_0([\theta_\ast - \theta_e]^+)d\Gamma d\tau \quad \text{as } m \rightarrow +\infty.
\]

By the strict monotonicity of \( \beta_e \),

\[
\int_{Q_t} \partial_z(\beta_e(\theta_\ast) - \beta_e(\theta_e))\sigma_m([\theta_\ast - \theta_e]^+)dxd\tau \\
\rightarrow \int_{Q_t} \partial_z(\beta_e(\theta_\ast) - \beta_e(\theta_e))\sigma_0([\theta_\ast - \theta_e]^+)dxd\tau \quad \text{as } m \rightarrow +\infty \\
= \int_{Q_t} \partial_z([\beta_e(\theta_\ast) - \beta_e(\theta_e)])\sigma_0([\beta_e(\theta_\ast) - \beta_e(\theta_e)]^+)dxd\tau \\
= \int_0^L \int_{-l}^l [\beta_e(\theta_\ast) - \beta_e(\theta_e)]^+dx'd\tau \bigg|_{z=0} = 0,
\]

and

\[
\int_{Q_t} \partial_t(\beta_e(\theta_\ast) - \beta_e(\theta_e))\sigma_m([\theta_\ast - \theta_e]^+)dxd\tau \\
\rightarrow \int_{Q_t} \partial_t(\beta_e(\theta_\ast) - \beta_e(\theta_e))\sigma_0([\theta_\ast - \theta_e]^+)dxd\tau \quad \text{as } m \rightarrow +\infty \\
= \int_{Q_t} \partial_t([\beta_e(\theta_\ast) - \beta_e(\theta_e)])\sigma_0([\beta_e(\theta_\ast) - \beta_e(\theta_e)]^+)dxd\tau \\
= \int_\Omega [\beta_e(\theta_\ast) - \beta_e(\theta_e(t, \cdot))]^+dx
\]

Therefore we have for all \( t \in \mathbb{R}_+ \)

\[
\int_\Omega [\beta_e(\theta_\ast) - \beta_e(\theta_e(t, \cdot))]^+dx + \int_{\Sigma_t} (g_\ast(\cdot, \cdot, \theta_e) - g_e(\cdot, \cdot, \theta_e))\sigma_0([\theta_\ast - \theta_e]^+)d\Gamma d\tau \leq 0.
\]

Taking \( \varepsilon = \varepsilon_n \) and letting \( n \rightarrow +\infty \) we have by lemma 3.1

\[
\int_\Omega [\eta_\ast - \eta(t, \cdot)]^+dx \leq 0 \quad \text{for all } t \in \mathbb{R}_+,
\]
which implies that

(3.4) \[ \eta_* \leq \eta \quad \text{hence} \quad \theta_* \leq \theta \quad \text{a.e. in } Q_\infty. \]

Because of lemma 3.2, we thus have

\[ \theta(t, y, z) \geq \rho \quad \text{a.e. in } Q_\infty^\delta \]

for the same constants \( \delta \) and \( \rho \) as in (3.2). By corollary 2.1 we see that \((\theta, \eta)\) is the unique weak solution for \( CP(\theta_*, \eta_*) \) on \( \mathbb{R}_+ \).

q.e.d.

4. Proof of main theorems

Let us prove theorems 2.1 and 2.2.

Proof of theorem 1.1. Firstly we construct a T-periodic weak solution of \((P)^\nu\) on \( \mathbb{R}_+ \).

Let \((\theta, \eta)\) be as in lemma 3.3, that is, the unique weak solution for \( CP(\theta_*, \eta_*)^\nu \) on \( \mathbb{R}_+ \).

For each \( m \in \mathbb{N} \) we denote by \((\theta_m, \eta_m)\) the weak solution for \( CP(\theta(mT, \cdot), \eta(mT, \cdot))^\nu \) on \([0, T]\). By proposition 2.1 and (3.4), we have

\[ \eta_* \leq \eta \leq \beta(K_3) \quad \text{a.e. in } Q_\infty. \]

In particular

\[ \eta_* \leq \eta(T, \cdot) \leq \beta(K_3) \quad \text{a.e. in } \Omega. \]

Applying proposition 2.3 to \( \eta \) and \( \eta_1 \),

\[ \eta_* \leq \eta \leq \eta_1 \leq \beta(K_3) \quad \text{a.e. in } Q_T. \]

Recursive use of this procedure derives that

\[ \eta_* \leq \eta \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_m \leq \cdots \leq \beta(K_3) \quad \text{a.e. in } Q_T, \]
hence

$$\theta_* \leq \theta \leq \theta_1 \leq \cdots \leq \theta_m \leq \cdots \leq K_3 \text{ a.e. in } Q_T.$$  

Then we can define $\eta_\infty(t, x) = \lim_{m \to +\infty} \eta_m(t, x)$ and $\theta_\infty(t, x) = \lim_{m \to +\infty} \theta_m(t, x)$ for (t, x) \in Q_T. It is easily verified that $\eta_\infty \in \beta(\theta_\infty)$ a.e. in $Q_T$, $\eta_\infty(0, \cdot) = \eta_\infty(T, \cdot)$ and $\theta_\infty(0, \cdot) = \theta_\infty(T, \cdot)$ a.e. in $\Omega$. Further we have estimates

$$\eta_* \leq \eta_m \leq \beta(K_3) \text{ hence } \theta_* \leq \theta_m \leq K_3 \text{ a.e. in } Q_T;$$

$$|\theta_m|_{L^2(0,T;H^1(\Omega))} \leq C_1,$$

and for any bounded subdomain $A$ with $\overline{A} \subset Q_T$,

$$|\theta_m|_{H^1(A)} \leq C_2 := C_2(A),$$

where $C_i, i = 1, 2$ are positive constants independent of $m$. Then we easily see that $(\theta_\infty, \eta_\infty)$ is a weak solution of $(P)^\nu$ on $[0, T]$. Consequently, $T$-periodic extension $(\theta^\nu_p, \eta^\nu_p)$ of $(\theta_\infty, \eta_\infty)$ onto $\mathbf{R}_+$ is a $T$-periodic weak solution of $(P)^\nu$ on $\mathbf{R}_+$.

Next we prove the uniqueness of $T$-periodic weak solutions. To do this, we shall show that any $T$-periodic weak solution $(\theta, \eta)$ is equal to $(\theta^\nu_p, \eta^\nu_p)$ constructed as above. Since $\theta^\nu_p$ satisfies (2.5), (2.6) holds for $\theta_1 = \theta^\nu_p$ and $\theta_2 = \theta$, from which it follows that

$$(4.1) \quad \int_{mT}^{(m+1)T} \int_\Omega |\eta^\nu_p - \eta| \, dx \, dt \to 0 \quad \text{as } m \to +\infty.$$  

On the other hand, by $T$-periodicity of $\eta^\nu_p$ and $\eta$,

$$\int_0^T \int_\Omega |\eta^\nu_p - \eta| \, dx \, dt = \int_{mT}^{(m+1)T} \int_\Omega |\eta^\nu_p - \eta| \, dx \, dt.$$  

So we must have $\int_0^T \int_\Omega |\eta^\nu_p - \eta| \, dx \, dt = 0$. Therefore $\eta^\nu_p = \eta$ a.e. in $Q_T$. Again, by $T$-periodicity of $\eta^\nu_p$ and $\eta$, $\eta^\nu_p = \eta$ a.e. in $Q_\infty$. Hence $\theta^\nu_p = \theta$ a.e. in $Q_\infty$. Thus the proof has been completed.  

q.e.d.
**Proof of theorem 2.** Let \((\theta, \eta)\) be an arbitrary weak solution of \((P)^\nu\) on \(\mathbb{R}_+\) satisfying (2.5). From proposition 2.3 we find that

\[
d := \lim_{t \to +\infty} |\eta_p^\nu(t, \cdot) - \eta(t, \cdot)|_{L^1(\Omega)}
\]
exists. Further as \(m \to +\infty\) we have

\[
\int_{mT}^{(m+1)T} \int_{\Omega} |\eta_p^\nu - \eta| \, dx \, dt \geq T |\eta_p^\nu((m+1)T, \cdot) - \eta((m+1)T, \cdot)|_{L^1(\Omega)} \to dT.
\]

Note that (4.1) also holds for \(\eta_p^\nu\) and \(\eta\), hence we deduce \(d = 0\). That is \(\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \to 0\) in \(L^1(\Omega)\). On account of the boundedness of \(\eta_p^\nu\) and \(\eta\) in \(Q_\infty\), we obtain

\[
\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \to 0 \quad \text{in } L^q(\Omega) \quad \text{for all } q \geq 1 \text{ as } t \to +\infty.
\]

From (1.1), it results that

\[
b_* |\theta_p^\nu(t, x) - \theta(t, x)| \leq |\eta_p^\nu(t, x) - \eta(t, x)| \quad \text{for a.e. } (t, x) \in Q_\infty,
\]
consequently

\[
\theta_p^\nu(t, \cdot) - \theta(t, \cdot) \to 0 \quad \text{in } L^q(\Omega) \quad \text{for all } q \geq 1 \text{ as } t \to +\infty.
\]

q.e.d.

In the rest of this chapter we study the convergence of the \(T\)-periodic weak solution of \((P)^\nu\) on \(\mathbb{R}_+\) to that of the Stefan problem when \(\nu \to 0\). The result is as follows.

**Theorem 4.3.** Assume that (2.7) holds. When \(\nu \to 0\), \((\theta_p^\nu, \eta_p^\nu)\) converges to some periodic solution \((\theta_p^0, \eta_p^0)\) of the Stefan problem \((P)^0\) in the following sense:

\[
\theta_p^\nu \to \theta_p^0 \quad \text{weakly in } L^2(0,T;H^1(\Omega)) \text{ and strongly in } L^q(Q_T) \quad \text{for all } q \geq 1,
\]
\[ g(\cdot, \cdot, \theta_p^\nu) \rightarrow g(\cdot, \cdot, \theta_p^0) \quad \text{in} \quad L^2(\Sigma_T^N), \]

and there exists a subsequence \( \{\nu_k\} \) of \( \{\nu\} \) such that

\[ \eta_p^{\nu_k} \rightarrow \eta_p^0 \quad \text{weakly in} \quad L^\infty(Q_T). \]

We claim that the following estimates hold for \( \{(\theta_p^\nu, \eta_p^\nu)\} \):

\[
\beta(-K_3) \leq \eta_p^\nu(t, x) \leq \beta(K_3) \quad \text{hence} \quad -K_3 \leq \theta_p^\nu(t, x) \leq K_3 \quad \text{a.e. in} \quad Q_T, 
\]

\[
|\theta_p^\nu|_{L^2(0, T; H^1(\Omega))} \leq C_3, 
\]

and for any bounded subdomain \( A \) with \( \overline{A} \subset Q_T, \)

\[
|\theta_p^\nu|_{H^1(A)} \leq C_4, 
\]

where \( C_i > 0, \ i = 3, 4, \) are constants independent of \( \nu \in [0, 1]. \) Hence there exist a subsequence \( \{\nu_k\} \) of \( \{\nu\} \) and \( (\theta, \eta) \in L^2(0, T; H^1(\Omega)) \times L^\infty(Q_T) \) such that

\[ \eta_p^{\nu_k} \rightarrow \eta \quad \text{weakly}^* \text{ in} \quad L^\infty(Q_T), \]

(4.2) \quad \theta_p^{\nu_k} \rightarrow \theta \quad \text{weakly in} \quad L^2(0, T; H^1(\Omega)) \quad \text{and strongly in} \quad L^q(Q_T) \quad \text{for all} \quad q \geq 1, \]

(4.3) \quad g(\cdot, \cdot, \theta_p^{\nu_k}) \rightarrow g(\cdot, \cdot, \theta) \quad \text{in} \quad L^2(\Sigma_T^N). \]

We easily see that \( (\theta, \eta) \) is a weak solution of \( (P)^0 \) on \([0, T]. \) Moreover, since \( (\theta_p^\nu, \eta_p^\nu) \) is \( T \)-periodic, \( (\theta, \eta) \) is also \( T \)-periodic. On account of remark 2.4 we can replace \( \{\nu_k\} \) with \( \{\nu\} \) in (4.2) and (4.3). Therefore \( T \)-periodic extension of \( (\theta, \eta) \) onto \( \mathbb{R}_+ \) is a desired one.
References


