<table>
<thead>
<tr>
<th>Title</th>
<th>Periodic behavior of solutions to a continuous casting problem (Nonlinear Evolution Equations and Their Applications)</th>
</tr>
</thead>
<tbody>
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Kyoto University
Periodic behavior of solutions to a continuous casting problem

千葉大自然科学 篠田淳一 (Junichi Shinoda)

1. Introduction

In this paper we consider a continuous casting problem

\[(P)^\nu\begin{cases}
\partial_t \eta + \nu \partial_z \eta - \Delta \theta = 0 \quad \text{in } Q_{\infty} := ]0, \infty[ \times \Omega, \\
\eta \in \beta(\theta) \quad \text{in } Q_{\infty}, \\
\frac{\partial \theta}{\partial n} + g(t, x, \theta) = 0 \quad \text{on } \Sigma_{\infty}^N := ]0, \infty[ \times \Gamma_N, \\
\theta = M \quad \text{on } \Sigma_{\infty}^0 := ]0, \infty[ \times \Gamma_0, \\
\theta = -m \quad \text{on } \Sigma_{\infty}^L := ]0, \infty[ \times \Gamma_L,
\end{cases}\]

under periodic (in time) boundary condition

\[g(t + T, x, \theta) = g(t, x, \theta) \quad \text{on } \Sigma_{\infty}^N \times \mathbb{R},\]

for a given period \(T > 0\). Here \(\Omega = ]-l, l[ \times \mathbb{R}, L_N = \{l, -l\} \times \mathbb{R}, L, \Gamma_N = ]-l, l[ \times \{0\}, \Gamma_L = ]-l, l[ \times \{L\}, L, l > 0, x = (y, z); \nu, m \text{ and } M \text{ are given constants with } \nu \geq 0 \text{ and } m, M > 0; \beta \text{ is a maximal monotone graph of the form}

\[\beta(r) = \begin{cases}
\lambda + \int_0^r b(\tau) d\tau & \text{if } r > 0, \\
[0, \lambda] & \text{if } r = 0, \\
\int_0^r b(\tau) d\tau & \text{if } r < 0,
\end{cases}\]

for a given constant \(\lambda > 0\) and a locally bounded measurable function \(b\) such that

\[(1.1) \quad b(r) \geq b_* > 0 \quad \text{for a.e. } r \in \mathbb{R}.\]

Furthermore \(g = g(t, x, \theta)\) is a given function on \(\mathbb{R}_+ \times \Gamma_N \times \mathbb{R}\) such that

\[(g1) \quad g(t, x, \cdot) \text{ is a nondecreasing function for a.e. } (t, x) \in \mathbb{R}_+ \times \Gamma_N;\]
(g2) \(g(\cdot, \cdot, \theta) \in L_{loc}^{2}(R_{+};L^{2}(\Gamma_{N}))\) for all \(\theta \in R\);

(g3) For any \(K > 0\) there is a constant \(C_{g}(K) > 0\) such that

\[|g(t, x, \theta_{1}) - g(t, x, \theta_{2})| \leq C_{g}(K)|\theta_{1} - \theta_{2}|\]

for all \(\theta_{1}, \theta_{2} \in [-K, K]\) and a.e. \((t, x) \in R_{+} \times \Gamma_{N}\);

(g4) There exist constants \(K_{1}, K_{2} > 0\) such that

\[g(t, x, -K_{1}) \leq 0, \quad g(t, x, K_{2}) \geq 0\]

for a.e. \((t, x) \in R_{+} \times \Gamma_{N}\).

For details of continuous casting problems, see Rodrigues [5], Rodrigues-Yi [6], Yi [9] and the literatures in their references. We remark here that problem \((P)^{0}\) is a Stefan problem. For results to periodic solutions of Stefan problems we refer to Aiki et al. [1], Damlamian-Kenmochi [2] and Haraux-Kenmochi [3]. In the following chapters, we shall discuss problem \((P)^{\nu}\) due to Shinoda [7,8].

### 2. Main results

Throughout this paper we denote \(Q_{S} = ]0, S[ \times \Omega, \Sigma_{S}^{N} = ]0, S[ \times \Gamma_{N}\), etc. for \(S \in ]0, +\infty[\).

Now let us give a notion of a weak solution on an interval of the form \([0, S]\) or \([0, +\infty[\).

**Definition 2.1.** Let \(S\) be a positive number. Then a couple \((\theta, \eta) \in L^{2}(0, S; H^{1}(\Omega)) \times L^{\infty}(Q_{S})\) is called a weak solution of \((P)^{\nu}\) on \([0, S]\) when the following four conditions are satisfied:

1. \(\eta \in C_{w}([0, S]; L^{2}(\Omega)), \) that is, \(\eta\) is a weakly continuous function from \([0, S]\) to \(L^{2}(\Omega)\);
2. \(\theta = M\) a.e. on \(\Sigma_{S}^{0}\) and \(\theta = -m\) a.e. on \(\Sigma_{S}^{L}\);
3. \(\eta \in \beta(\theta)\) a.e. in \(Q_{S}\);
for any \( \varphi \in W_S := \{ \varphi \in H^1(Q_S); \varphi(S, \cdot) = 0 \text{ a.e. in } \Omega, \varphi = 0 \text{ a.e. on } \Sigma_S^D \}, \)

\[-\int_{Q_S} \eta(\partial_t \varphi + \nu \partial_z \varphi) dxdt + \int_{Q_S} \nabla \theta \nabla \varphi dxdt + \int_{\Sigma_S^N} g(\cdot, \cdot, \theta) \varphi d\Gamma dt = \int_{\Omega} \eta(0, \cdot) \varphi(0, \cdot) dx,\]

where \( \Sigma_S^D = ]0, S[ \times \Gamma_D, \Gamma_D = \Gamma_0 \cup \Gamma_L. \) In the case when \( S = +\infty, (\theta, \eta) \) is called a weak solution of \((P)^\nu\) on \( \mathbb{R}^+\), if \((\theta, \eta)\) is a weak solution of \((P)^\nu\) on \([0, S]\) for any finite \( S > 0 \).

**Definition 2.2.** Let \( 0 < S \leq +\infty \) and let \((\theta_0, \eta_0)\) be a pair of functions in \( L^\infty(\Omega) \) satisfying \( \eta_0 \in \beta(\theta_0) \text{ a.e. in } \Omega. \) Then we call a pair \((\theta, \eta)\) a weak solution for \( CP(\theta_0, \eta_0)^\nu\) on \([0, S]\) \((\mathbb{R}^+ \text{ if } S = +\infty)\) if \((\theta, \eta)\) is a weak solution of \((P)^\nu\) on \([0, S]\) and the initial conditions \( \theta(0, \cdot) = \theta_0 \) and \( \eta(0, \cdot) = \eta_0 \) are satisfied, respectively.

Concerning the existence and the uniqueness results for \( CP(\theta_0, \eta_0)^\nu\), we quote them from Rodrigues-Yi [6]. The first proposition assures the existence of a weak solution for \( CP(\theta_0, \eta_0)^\nu\).

**Proposition 2.1.** (cf. [6; theorem 1]) Let \((\theta_0, \eta_0) \in (L^\infty(\Omega))^2\) be any pair of functions such that \( \eta_0 \in \beta(\theta_0) \text{ a.e. in } \Omega. \) Choose two positive constants \( \tilde{K}_1 \) and \( \tilde{K}_2 \) so that \( \tilde{K}_i \geq \max\{m, M, K_i\}, i = 1, 2, \) and that

\[ \beta(-\tilde{K}_1) \leq \eta_0(x) \leq \beta(\tilde{K}_2) \text{ for a.e. } x \in \Omega. \]

Then, there exists at least one weak solution \((\theta, \eta)\) for \( CP(\theta_0, \eta_0)^\nu\) on \( \mathbb{R}^+\) such that

\[ \beta(-\tilde{K}_1) \leq \eta(t, x) \leq \beta(\tilde{K}_2) \text{ for a.e. } (t, x) \in Q_\infty, \]

hence

\[ -\tilde{K}_1 \leq \theta(t, x) \leq \tilde{K}_2 \text{ for a.e. } (t, x) \in Q_\infty. \]
Remark 2.1. In view of the proof of [6; theorem 1] we may assume that the solution $(\theta, \eta)$ obtained in proposition 2.1 is constructed as a limit of an approximate solution $(\theta_\epsilon, \beta_\epsilon(\theta_\epsilon))$ of
\[
\begin{cases}
\partial_t \beta_\epsilon(\theta_\epsilon) + \nu \partial_x \beta_\epsilon(\theta_\epsilon) - \Delta \theta_\epsilon = 0 & \text{in } Q_\infty, \\
\frac{\partial \theta_\epsilon}{\partial n} + g_\epsilon(t, x, \theta_\epsilon) = 0 & \text{on } \Sigma^N, \\
\theta_\epsilon = M & \text{on } \Sigma^0, \\
\theta_\epsilon = -m & \text{on } \Sigma^L, \\
\theta_\epsilon(0, \cdot) = \theta_{0\epsilon} & \text{in } \Omega,
\end{cases}
\]
in the sense that for some subsequence $\{\epsilon_n\}$ of $\{\epsilon\}$
\[
(2.1) \quad \beta_{\epsilon_n}(\theta_{\epsilon_n}) \rightharpoonup \eta \quad \text{weakly}^* \text{ in } L^\infty_{loc}(\mathbb{R}_+; L^\infty(\Omega));
\]
\[
(2.2) \quad \theta_{\epsilon_n} \rightharpoonup \theta \quad \text{weakly in } L^2_{loc}(\mathbb{R}_+; H^1(\Omega)) \cap H^1_{loc}(Q_\infty);
\]
\[
(2.3) \quad g_{\epsilon_n}(\cdot, \cdot, \theta_{\epsilon_n}) \rightharpoonup g(\cdot, \cdot, \theta) \quad \text{in } L^2_{loc}(\mathbb{R}_+; L^2(\Gamma_N)).
\]
Here $\{\beta_\epsilon\}$, $\{g_\epsilon\}$ and $\{\theta_{0\epsilon}\}$ are smooth approximations to $\beta$, $g$ and $\theta_0$, respectively. Furthermore, $\{\beta_\epsilon\}$ satisfies (1.1) with $b_\epsilon = \beta_\epsilon'$, $\beta_\epsilon(0) = 0$, $\beta_\epsilon' \leq 1/\epsilon$ and $\beta_\epsilon(r) \to \beta(r)$ for any compact interval in $\mathbb{R}\\backslash\{0\}$ as $\epsilon \to 0$;

$\{g_\epsilon\}$ satisfies (g1)$\sim$(g4) and
\[
g_\epsilon(\cdot, \cdot, \theta) \rightharpoonup g(\cdot, \cdot, \theta) \quad \text{in } L^2_{loc}(\mathbb{R}_+; L^2(\Gamma_N))
\]
uniformly with respect to $\theta$ on any compact set in $\mathbb{R}$ as $\epsilon \to 0$;

$\{\theta_{0\epsilon}\}$ satisfies the compatibility conditions
\[
(2.4) \quad \theta_{0\epsilon} = M \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta_{0\epsilon} = -m \quad \text{on } \Gamma_L.
and
\[ \beta_{\epsilon}(\theta_{0\epsilon}) \rightarrow \eta_{0} \quad \text{in} \quad L^{2}(\Omega) \quad \text{as} \quad \epsilon \rightarrow 0. \]

The second proposition is the continuous dependence of the weak solutions. This requires the following condition to a weak solution \((\theta, \eta)\) of \((P)^{\nu}\):

For some positive constants \(\delta, \rho > 0\),

\[ \theta(t, y, z) \geq \rho > 0 \quad \text{a.e. in} \quad Q_{\infty}^\delta := \{(t, y, z) \in Q_{\infty}; 0 < z < \delta\}. \]

**Proposition 2.2.** (cf. [6; theorem 2]) Fix \(\nu > 0\). Let \((\theta_{1}, \eta_{1})\) and \((\theta_{2}, \eta_{2})\) be two weak solutions for \(CP(\theta_{10}, \eta_{10})^{\nu}\) and \(CP(\theta_{20}, \eta_{20})^{\nu}\), respectively. If at least one of \((\theta_{i}, \eta_{i})\) satisfies (2.5), then the following is valid:

\[ \int_{Q_{\infty}} |\eta_{1} - \eta_{2}| \, dx \, dt \leq \frac{L}{\nu} \int_{\Omega} |\eta_{10} - \eta_{20}| \, dx. \]  

As a direct corollary we have:

**Corollary 2.1.** If at least one of the weak solution \((\theta, \eta)\) for \(CP(\theta_{0}, \eta_{0})^{\nu}\) on \(\mathbb{R}_{+}\) satisfies (2.5), then \((\theta, \eta)\) is the only weak solution for \(CP(\theta_{0}, \eta_{0})^{\nu}\) on \(\mathbb{R}_{+}\).

Using well-known \(L^{1}\)-space technique, we have in the manner similar to that of [1]:

**Proposition 2.3.** Let \(\nu > 0\), and let \((\theta_{1}, \eta_{1})\), \((\theta_{2}, \eta_{2})\) be two weak solutions for \(CP(\theta_{10}, \eta_{10})^{\nu}\) and \(CP(\theta_{20}, \eta_{20})^{\nu}\) on \(\mathbb{R}_{+}\) satisfying (2.5), respectively. Then we have

\[ |[\eta_{1}(t, \cdot) - \eta_{2}(t, \cdot)]^{+}|_{L^{1}(\Omega)} \leq |[\eta_{1}(s, \cdot) - \eta_{2}(s, \cdot)]^{+}|_{L^{1}(\Omega)} \quad \text{for any} \quad s, t \in \mathbb{R}_{+} \quad \text{with} \quad s \leq t, \]

and

\[ |\eta_{1}(t, \cdot) - \eta_{2}(t, \cdot)|_{L^{1}(\Omega)} \leq |\eta_{1}(s, \cdot) - \eta_{2}(s, \cdot)|_{L^{1}(\Omega)} \quad \text{for any} \quad s, t \in \mathbb{R}_{+} \quad \text{with} \quad s \leq t. \]
In particular, if $\eta_{10} \leq \eta_{20}$ a.e. in $\Omega$ then

$$\eta_1 \leq \eta_2 \quad \text{hence} \quad \theta_1 \leq \theta_2 \quad \text{a.e. in } Q_{\infty}.$$ 

**Remark 2.2.** Propositions 2.1, 2.3 and corollary 2.1 are also valid for $\nu = 0$. We can prove them by using similar techniques to those in the proofs of [6; theorem 1,4; theorem 4,1; lemma 2,1], respectively.

Next we state a definition of a $T$-periodic weak solution of $(P)^\nu$ on $\mathbb{R}_+$. 

**Definition 2.3.** Let $T$ be a given positive number (period). Then $(\theta, \eta)$ is called a $T$-periodic weak solution of $(P)^\nu$ on $\mathbb{R}_+$ provided that $(\theta, \eta)$ is a weak solution of $(P)^\nu$ on $\mathbb{R}_+$ and satisfies the periodic conditions $\theta(t + T, \cdot) = \theta(t, \cdot)$ and $\eta(t + T, \cdot) = \eta(t, \cdot)$ for all $t \in \mathbb{R}_+$.

Finally we mention the main results for the $T$-periodic weak solution of $(P)^\nu$ on $\mathbb{R}_+$.

**Theorem 2.1.** Let $\nu > 0$. Assume that the periodicity condition

$$(2.7) \quad g(t + T, x, \theta) = g(t, x, \theta) \quad \text{for all } \theta \in \mathbb{R}_+ \text{ and a.e. } (t, x) \in \mathbb{R}_+ \times \Gamma_N$$

holds. Then there exists one and only one $T$-periodic weak solution $(\theta_p^\nu, \eta_p^\nu)$ of $(P)^\nu$ on $\mathbb{R}_+$.

**Theorem 2.2.** Assume that the same conditions as in theorem 2.1 hold. Then for any weak solution $(\theta, \eta)$ satisfying (2.5) for some positive constants $\delta, \rho > 0$, we have

$$\eta_p^\nu(t, \cdot) - \eta(t, \cdot) \to 0 \quad \text{and} \quad \theta_p^\nu(t, \cdot) - \theta(t, \cdot) \to 0 \quad \text{in } L^q(\Omega) \quad \text{for all } q \geq 1 \text{ as } t \to +\infty.$$
Remark 2.3. Yi [9] treated the periodic solutions under the Dirichlet boundary condition. He proved there the existence of periodic solutions using Schauder fixed point theorem.

Remark 2.4. There exists a $T$-periodic weak solution $(\theta^0_p, \eta^0_p)$ of $(P)^0$ on $\mathbb{R}_+$ under the periodicity condition (2.7). But for the uniqueness of $T$-periodic weak solutions of $(P)^0$ on $\mathbb{R}_+$, we can only prove that of $g(\cdot, \cdot, \theta^0_p)$ on $\Sigma^N_{\infty}$ and moreover that of $\theta^0_p$ in $Q_{\infty}$ (see [7,8] and also [2]).

3. Lemmas

In this chapter we prepare some lemmas to prove theorems 2.1 and 2.2.

Firstly we define a function $g_* = g_*(\theta)$ by $g_*(\theta) = C_g(K_3)[\theta + K_3]^+$ for $\theta \in \mathbb{R}$, where $K_3 = \max\{M, m, K_1, K_2\}$. Then the following is valid.

Lemma 3.1. $g_*$ defined as above is nondecreasing and satisfies

$$g(t, x, \theta) \leq g_*(\theta) \quad \text{for all } \theta \leq K_3 \text{ and a.e. } (t, x) \in \mathbb{R}_+ \times \Gamma_N,$$

Next we construct a smooth function $\theta_* = \theta_*(x)$ satisfying for any $\epsilon > 0$ the following system

$$\begin{cases} 
\nu \partial_x \beta_\epsilon(\theta_*) - \Delta \theta_* \leq 0 & \text{in } \Omega, \\
\frac{\partial \theta_*}{\partial n} + g_*(\theta_*) \leq 0 & \text{on } \Gamma_N, \\
\theta_* \leq M & \text{in } \overline{\Omega}, \\
\theta_* \leq -K_3 & \text{on } \Gamma_L.
\end{cases}$$

(3.1)

Choose a function $\chi = \chi(y) \in C^\infty([-l, l])$ such that

$$0 \leq \chi \leq \frac{M}{2} \quad \text{in } ]-l, l[,$$

and

$$\pm \frac{\partial \chi}{\partial y}(\pm l) + g_*(M) \leq 0.$$
For a positive parameter $\mu$, let us define $\theta_*$ by

$$\theta_*(y, z) = -\mu z + \chi(y) + \frac{M}{2}.$$ 

Then we see that $\theta_*$ satisfies for some constants $\delta, \rho > 0$

(3.2) \hspace{1cm} \theta_*(y, z) \geq \rho \hspace{1cm} \text{in} \hspace{1cm} \Omega_\delta := \{(y, z) \in \Omega; 0 < z < \delta\}.

Moreover it is readily seen that (3.1) is fulfilled for sufficiently large $\mu$ dependent upon $\nu$. Thus we have the following lemma.

**Lemma 3.2.** There is a smooth function $\theta_* = \theta_*(x)$ on $\Omega$ which is independent of $\varepsilon$ and satisfies (3.1) and (3.2) for some positive constants $\delta, \rho$.

Put $\eta_* = \beta(\theta_*)$. We remark that $\eta_*$ is a.e. defined since the Lebesgue measure of the set $\{x \in \Omega; \theta_*(x) = 0\}$ is zero. Then we have:

**Lemma 3.3.** The unique weak solution $(\theta, \eta)$ for $CP(\theta_*, \eta_*)^\nu$ on $\mathbb{R}^+$ satisfies (2.5) for some $\delta, \rho > 0$.

**Proof.** Let \{\theta_{0\varepsilon}\} $\subset C^\infty(\overline{\Omega})$ such that $\theta_* \leq \theta_{0\varepsilon}$ in $\Omega$, $\beta_\varepsilon(\theta_{0\varepsilon}) \rightarrow \eta_*$ in $L^2(\Omega)$ as $\varepsilon \rightarrow 0$, and that (2.4) holds. Recalling proposition 2.1 and remark 2.1, we get a weak solution $(\theta, \eta)$ for $CP(\theta_*, \eta_*)^\nu$ on $\mathbb{R}^+$ as a limit of an approximate solution $\theta_{\varepsilon n}$ corresponding to initial value $\theta_{0\varepsilon n}$ in the sense of (2.1)\textendash(2.3) for some subsequence $\{\varepsilon_n\}$ of $\{\varepsilon\}$. We note that for any $\varepsilon \in ]0, 1]$.

(3.3) \hspace{1cm} \partial_t(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) + \nu \partial_z(\beta_\varepsilon(\theta_*) - \beta_\varepsilon(\theta_\varepsilon)) - \Delta(\theta_* - \theta_\varepsilon) \leq 0 \hspace{1cm} \text{in} \hspace{1cm} Q_\infty.

Now let us denote by \{\sigma_m\} a sequence of smooth functions on $\mathbb{R}$ such that $\sigma_m(0) = 0$, and for any $r \in \mathbb{R}$, $\sigma_m'(r) \geq 0$, $-1 \leq \sigma_m(r) \leq 1$ and

$$\sigma_m(r) \rightarrow \sigma_0(r) := \begin{cases} 1 & \text{for } r > 0, \\ 0 & \text{for } r = 0, \\ -1 & \text{for } r < 0, \end{cases} \quad \text{as } m \rightarrow +\infty.$$
Multiply (3.3) by \( \sigma_m([\theta_* - \theta_e]^+) \) and integrate it over \( Q_t \). By lemma 3.1 and 3.2,

\[
- \int_{Q_t} \Delta(\theta_* - \theta_e) \sigma_m([\theta_* - \theta_e]^+) dxd\tau \\
\geq \int_{\Sigma_t^N} (g_* - g_e(\cdot, \cdot, \theta_e)) \sigma_m([\theta_* - \theta_e]^+) d\Gamma d\tau \\
\geq \int_{\Sigma_t^N} (g_0(\cdot, \cdot, \theta_e) - g_e(\cdot, \cdot, \theta_e)) \sigma_m([\theta_* - \theta_e]^+) d\Gamma d\tau \\
\rightarrow \int_{\Sigma_t^N} (g_* - g_e(\cdot, \cdot, \theta_e)) \sigma_0([\theta_* - \theta_e]^+) d\Gamma d\tau \quad \text{as} \quad m \to +\infty.
\]

By the strict monotonicity of \( \beta_e \),

\[
\int_{Q_t} \partial_z(\beta_e(\theta_*) - \beta_e(\theta_e)) \sigma_m([\theta_* - \theta_e]^+) dxd\tau \\
\rightarrow \int_{Q_t} \partial_z(\beta_e(\theta_*) - \beta_e(\theta_e)) \sigma_0([\theta_* - \theta_e]^+) dxd\tau \quad \text{as} \quad m \to +\infty \\
= \int_{Q_t} \partial_z(\beta_e(\theta_*) - \beta_e(\theta_e)) \sigma_0([\beta_e(\theta_*) - \beta_e(\theta_e)]^+) dxd\tau \\
= \int_0^t \int_{-l}^l [\beta_e(\theta_*) - \beta_e(\theta_e)]^+ dx' d\tau \Big|_{x=0}^L = 0,
\]

and

\[
\int_{Q_t} \partial_t(\beta_e(\theta_*) - \beta_e(\theta_e)) \sigma_m([\theta_* - \theta_e]^+) dxd\tau \\
\rightarrow \int_{Q_t} \partial_t(\beta_e(\theta_*) - \beta_e(\theta_e)) \sigma_0([\theta_* - \theta_e]^+) dxd\tau \quad \text{as} \quad m \to +\infty \\
= \int_{Q_t} \partial_t(\beta_e(\theta_*) - \beta_e(\theta_e)) \sigma_0([\beta_e(\theta_*) - \beta_e(\theta_e)]^+) dxd\tau \\
= \int_\Omega [\beta_e(\theta_*) - \beta_e(\theta_e(t, \cdot))]^+ dx
\]

Therefore we have for all \( t \in \mathbb{R}_+ \)

\[
\int_\Omega [\beta_e(\theta_*) - \beta_e(\theta_e(t, \cdot))]^+ dx + \int_{\Sigma_t^N} (g_0 - g_e(\cdot, \cdot, \theta_e)) \sigma_0([\theta_* - \theta_e]^+) d\Gamma d\tau \leq 0.
\]

Taking \( \varepsilon = \varepsilon_n \) and letting \( n \to +\infty \) we have by lemma 3.1

\[
\int_\Omega [\eta_* - \eta(t, \cdot)]^+ dx \leq 0 \quad \text{for all} \quad t \in \mathbb{R}_+.
\]
which implies that

\[(3.4) \quad \eta_* \leq \eta \text{ hence } \theta_* \leq \theta \text{ a.e. in } Q_\infty.\]

Because of lemma 3.2, we thus have

\[\theta(t, y, z) \geq \rho \text{ a.e. in } Q_\infty^\delta\]

for the same constants \(\delta\) and \(\rho\) as in (3.2). By corollary 2.1 we see that \((\theta, \eta)\) is the unique weak solution for \(CP(\theta_*, \eta_*)\) on \(\mathbb{R}_+\).

\[\text{q.e.d.}\]

4. Proof of main theorems

Let us prove theorems 2.1 and 2.2.

*Proof of theorem 1.1.* Firstly we construct a \(T\)-periodic weak solution of \((P)^\nu\) on \(\mathbb{R}_+\).

Let \((\theta, \eta)\) be as in lemma 3.3, that is, the unique weak solution for \(CP(\theta_*, \eta_*)^\nu\) on \(\mathbb{R}_+\). For each \(m \in \mathbb{N}\) we denote by \((\theta_m, \eta_m)\) the weak solution for \(CP(\theta(mT, \cdot), \eta(mT, \cdot))^\nu\) on \([0, T]\). By proposition 2.1 and (3.4), we have

\[\eta_* \leq \eta \leq \beta(K_3) \text{ a.e. in } Q_\infty.\]

In particular

\[\eta_* \leq \eta(T, \cdot) \leq \beta(K_3) \text{ a.e. in } \Omega.\]

Applying proposition 2.3 to \(\eta\) and \(\eta_1\),

\[\eta_* \leq \eta \leq \eta_1 \leq \beta(K_3) \text{ a.e. in } Q_T.\]

Recursive use of this procedure derives that

\[\eta_* \leq \eta \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_m \leq \cdots \leq \beta(K_3) \text{ a.e. in } Q_T,\]
hence
\[ \theta_{*} \leq \theta \leq \theta_{1} \leq \theta_{2} \leq \cdots \leq \theta_{m} \leq \cdots \leq K_{3} \quad \text{a.e. in } Q_{T}. \]

Then we can define \( \eta_{\infty}(t, x) = \lim_{m \to +\infty} \eta_{m}(t, x) \) and \( \theta_{\infty}(t, x) = \lim_{m \to +\infty} \theta_{m}(t, x) \) for a.e. \((t, x) \in Q_{T}\). It is easily verified that \( \eta_{\infty} \in \beta(\theta_{\infty}) \) a.e. in \( Q_{T} \), \( \eta_{\infty}(0, \cdot) = \eta_{\infty}(T, \cdot) \) and \( \theta_{\infty}(0, \cdot) = \theta_{\infty}(T, \cdot) \) a.e. in \( \Omega \). Further we have estimates
\[ \eta_{*} \leq \eta_{m} \leq \beta(K_{3}) \quad \text{hence} \quad \theta_{*} \leq \theta_{m} \leq IC_{3} \quad \text{a.e. in } Q_{T}, \]

\[ |\theta_{m}|_{L^{2}(0,T;H^{1}(\Omega))} \leq C_{1}, \]

and for any bounded subdomain \( A \) with \( \overline{A} \subset Q_{T} \),
\[ |\theta_{m}|_{H^{1}(A)} \leq C_{2} := C_{2}(A), \]

where \( C_{i}, i = 1, 2 \) are positive constants independent of \( m \). Then we easily see that \((\theta_{\infty}, \eta_{\infty})\) is a weak solution of \((P)^{\nu}\) on \([0, T]\). Consequently, \( T \)-periodic extension \((\theta_{p}^{\nu}, \eta_{p}^{\nu})\) of \((\theta_{\infty}, \eta_{\infty})\) onto \( \mathbb{R}_{+} \) is a \( T \)-periodic weak solution of \((P)^{\nu}\) on \( \mathbb{R}_{+} \).

Next we prove the uniqueness of \( T \)-periodic weak solutions. To do this, we shall show that any \( T \)-periodic weak solution \((\theta, \eta)\) is equal to \((\theta_{p}^{\nu}, \eta_{p}^{\nu})\) constructed as above. Since \( \theta_{p}^{\nu} \) satisfies (2.5), (2.6) holds for \( \theta_{1} = \theta_{p}^{\nu} \) and \( \theta_{2} = \theta \), from which it follows that
\[ \int_{mT}^{(m+1)T} \int_{\Omega} |\eta_{p}^{\nu} - \eta| \, dxdt \to 0 \quad \text{as } m \to +\infty. \]

On the other hand, by \( T \)-periodicity of \( \eta_{p}^{\nu} \) and \( \eta \),
\[ \int_{0}^{T} \int_{\Omega} |\eta_{p}^{\nu} - \eta| \, dxdt = \int_{mT}^{(m+1)T} \int_{\Omega} |\eta_{p}^{\nu} - \eta| \, dxdt. \]

So we must have \( \int_{0}^{T} \int_{\Omega} |\eta_{p}^{\nu} - \eta| \, dxdt = 0 \). Therefore \( \eta_{p}^{\nu} = \eta \) a.e. in \( Q_{T} \). Again, by \( T \)-periodicity of \( \eta_{p}^{\nu} \) and \( \eta \), \( \eta_{p}^{\nu} = \eta \) a.e. in \( Q_{\infty} \). Hence \( \theta_{p}^{\nu} = \theta \) a.e. in \( Q_{\infty} \). Thus the proof has been completed.

q.e.d.
Proof of theorem 2. Let \((\theta, \eta)\) be an arbitrary weak solution of \((P)^\nu\) on \(\mathbb{R}_+\) satisfying (2.5). From proposition 2.3 we find that 

\[
d := \lim_{t \to +\infty} |\eta^\nu_p(t, \cdot) - \eta(t, \cdot)|_{L^1(\Omega)}
\]

exists. Further as \(m \to +\infty\) we have 

\[
\int_{mT}^{(m+1)T} \int_\Omega |\eta^\nu_p - \eta| \, dx \, dt \geq T |\eta^\nu_p((m+1)T, \cdot) - \eta((m+1)T, \cdot)|_{L^1(\Omega)} \to \infty.
\]

Note that (4.1) also holds for \(\eta^\nu_p\) and \(\eta\), hence we deduce \(d = 0\). That is \(\eta^\nu_p(t, \cdot) - \eta(t, \cdot) \to 0\) in \(L^1(\Omega)\). On account of the boundedness of \(\eta^\nu_p\) and \(\eta\) in \(Q_\infty\), we obtain 

\[
\eta^\nu_p(t, \cdot) - \eta(t, \cdot) \to 0 \quad \text{in} \quad L^q(\Omega) \quad \text{for all} \quad q \geq 1 \quad \text{as} \quad t \to +\infty.
\]

From (1.1), it results that 

\[
b_\ast |\theta^\nu_p(t, x) - \theta(t, x)| \leq |\eta^\nu_p(t, x) - \eta(t, x)| \quad \text{for a.e.} \quad (t, x) \in Q_\infty,
\]

consequently 

\[
\theta^\nu_p(t, \cdot) - \theta(t, \cdot) \to 0 \quad \text{in} \quad L^q(\Omega) \quad \text{for all} \quad q \geq 1 \quad \text{as} \quad t \to +\infty.
\]

q.e.d.

In the rest of this chapter we study the convergence of the \(T\)-periodic weak solution of \((P)^\nu\) on \(\mathbb{R}_+\) to that of the Stefan problem when \(\nu \to 0\). The result is as follows.

**Theorem 4.3.** Assume that (2.7) holds. When \(\nu \to 0\), \((\theta^\nu_p, \eta^\nu_p)\) converges to some periodic solution \((\theta^0_p, \eta^0_p)\) of the Stefan problem \((P)^0\) in the following sense:

\[
\theta^\nu_p \to \theta^0_p \quad \text{weakly in} \quad L^2(0, T; H^1(\Omega)) \quad \text{and strongly in} \quad L^q(Q_T) \quad \text{for all} \quad q \geq 1,
\]
\[ g(\cdot, \cdot, \theta_p^\nu) \to g(\cdot, \cdot, \theta_p^0) \quad \text{in } L^2(\Sigma_T^N), \]

and there exists a subsequence \( \{\nu_k\} \) of \( \{\nu\} \) such that

\[ \eta_p^{\nu_k} \to \eta_p^0 \quad \text{weakly in } L^\infty(Q_T). \]

We claim that the following estimates hold for \( \{(\theta_p^\nu, \eta_p^\nu)\} \):

\[ \beta(-K_3) \leq \eta_p^\nu(t, x) \leq \beta(K_3) \quad \text{hence} \quad -K_3 \leq \theta_p^\nu(t, x) \leq K_3 \quad \text{a.e. in } Q_T, \]

\[ \left| \theta_p^\nu \right|_{L^2(0, T; H^1(\Omega))} \leq C_3, \]

and for any bounded subdomain \( A \) with \( \overline{A} \subset Q_T \),

\[ \left| \theta_p^\nu \right|_{H^1(A)} \leq C_4, \]

where \( C_i > 0, i = 3, 4 \), are constants independent of \( \nu \in [0, 1] \). Hence there exist a subsequence \( \{\nu_k\} \) of \( \{\nu\} \) and \( (\theta, \eta) \in L^2(0, T; H^1(\Omega)) \times L^\infty(Q_T) \) such that

\[ \eta_p^{\nu_k} \to \eta \quad \text{weakly}\,* \text{ in } L^\infty(Q_T), \]

\[ (4.2) \quad \theta_p^{\nu_k} \to \theta \quad \text{weakly in } L^2(0, T; H^1(\Omega)) \text{ and strongly in } L^q(Q_T) \text{ for all } q \geq 1, \]

\[ (4.3) \quad g(\cdot, \cdot, \theta_p^{\nu_k}) \to g(\cdot, \cdot, \theta) \quad \text{in } L^2(\Sigma_T^N). \]

We easily see that \( (\theta, \eta) \) is a weak solution of \((P)^0\) on \([0, T]\). Moreover, since \((\theta_p^\nu, \eta_p^\nu)\) is \( T \)-periodic, \((\theta, \eta)\) is also \( T \)-periodic. On account of remark 2.4 we can replace \( \{\nu_k\} \) with \( \{\nu\} \) in (4.2) and (4.3). Therefore \( T \)-periodic extension of \((\theta, \eta)\) onto \( \mathbb{R}_+ \) is a desired one.
References


