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DISCONTINUITY OF SOLUTIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE (Nonlinear Evolution Equations and Their Applications)

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DISCONTINUITY OF SOLUTIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE

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0. Introduction and Theorem.

In this paper we consider the following integro-differential equation with time delay in a real Hilbert space $H$:

\[(0.1) \quad \frac{d}{dt} u(t) + Au(t) + A_1 u(t-h) + \int_{-h}^{0} a(-s) A_2 u(t+s) ds = f(t)\]

\[u(0) = x, \quad u(s) = y(s) \quad -h \leq s < 0.\]

Here, $A$ is a positive definite self-adjoint operator and $A_1$, $A_2$ are closed linear operators with domains containing that of $A$. The notations $h$ and $N$ denote a fixed positive number and a large natural number respectively. Let $a(\cdot)$ is a real valued function belonging to $C^3([0, h])$.

The equations of the type (0.1) were investigated by G.Di Blasio, K.Kunisch and E.Sinestrari [2], S.Nakagiri [4], H.Tanabe [6] and D.G.Park and S.Y.Kim [5]. Particularly, G.Di Blasio, K.Kunisch and E.Sinestrari [2] showed the existence and uniqueness of a solution for $f \in L^2(0, T; H)$, $Ay \in L^2(-h, 0; H)$ and $x \in (D(A), H)_{1/2, 2}$ where $(D(A), H)_{1/2, 2}$ is an interpolation space.

Since the equation (0.1) is of parabolic type, we want $x$ to be an arbitrary element of $H$. Then the integral in (0.1) exists only in the improper sense no
matter what nice functions $f$ and $Ay$ may be. Hence, it would be considered natural to investigate our problem under the following hypothesis:

$$f \in \cap_{\delta > 0} L^2(\delta, T; H) \quad \text{and} \quad Ay \in \cap_{\delta > 0} L^2(-h + \delta, 0; H),$$

$$f(t) \text{ and } Ay(t - h) \text{ are improperly integrable at } t = 0.$$ 

For the sake of simplicity we put

$$L^2_{loc}((0, T]; H) = \cap_{\delta > 0} L^2(\delta, T; H).$$

We first shall state the definition of a weak solution of (0.1).

**DEFINITION.** We say that a function $u$ defined on $[-h, T]$ is a weak solution of the equation (0.1) if the following four conditions satisfied:(see Definition 1.1 in [3])

1) $u \in L^2_{loc}((nh, (n+1)h]; D(A)) \cap W^{1,2}_{loc}((nh, (n+1)h]; H) \cap C([0, Nh]; D(A^{-\alpha}))$

for $n = 0, 1, 2, \ldots, N - 1$ and any $\alpha > 0$.

2) $\lim_{t \to 0} A^{-\alpha}u(t) = A^{-\alpha}x$

for any $\alpha > 0$ and $u(s) = y(s)$ for $-h \leq s < 0$.

3) $Au(\cdot + nh) \in L^2_{loc}((0, h]; H)$ and $A^{1-\alpha}u(\cdot + nh)$ is improper integrable at $t = 0$.

4) The function $u$ satisfies the equation (0.1) for a.e $t$.

In Theorem 1 in [3] we showed the existence and uniqueness of a weak solution for which $A^{-\alpha}u$ is continuous in $[0, T]$ for an arbitrary positive number $\alpha$ but this solution is not always in $C([0, T]; H)$.

As the notations we put

$$F_{-1} = \{ g \in L^2_{loc}((0, h]; H); \text{ there exists } \lim_{\epsilon \to 0} \int_\epsilon^0 g(s)ds \},$$
\[ F_m = \{ g \in F_{m-1}; \lim_{t \searrow 0} \int_{t/2}^{1} (t-s)^m A_1^m S(t-s)g(s) ds = 0 \} \]

where \( S(\cdot) \) is an analytic semigroup of the positive defined self-adjoint operator \( A \) and \( m = 1, 2, \ldots, N - 1 \).

In Proposition 6.9 of [3] we also showed the following resultant.

Let \( f \) belong to \( F_{-1} \cap L^2_{loc}((0, Nh): H) \) and \( m \) is a nonnegative integer such that \( 0 \leq m \leq N - 1 \). Then following two conditions are equivalent.

1) A weak solution of (0.1) is continuous on \([0, mh]\), but at \( t = mh \) this solution is discontinuous.

2) \( f - A_1 y(\cdot - h) \in F_{m-1} \), but \( f - A_1 y(\cdot - h) \notin F_m \).

In [3] we could not show that \( F_m \) is a proper subset in \( F_{m-1} \). The object in this paper is to show that \( F_m \) is a proper subset in \( F_{m-1} \) (i.e. there exists a inhomogeneous function \( f \) and an initial data function \( y \) such that the solution of (0.1) is continuous on \([0, mh]\), but at \( t = mh \) this solution is discontinuous on \( H \)).

Throughout this paper we assume

\( A - 1) \quad A = A_1 = A_2 \),

\( A - 2) \quad \) the operator \( A \) holds eigenvalues \( \{ \lambda_q \}_{q=1}^{\infty} \) such that

\[ \lambda_q = C q^\alpha + o(q^\alpha), \quad \lambda_q \leq \lambda_{q+1} \]

where \( \alpha \) and \( C \) are some positive numbers. We denote normal eigenfunctions of eigenvalues \( \lambda_q \) by \( \varphi_j \).

\textbf{THEOREM} Under the assumptions \( A-1) \) and \( A-2) \) there exist a inhomogeneous function \( f \) and the initial valued function \( y \) such that the weak solution of (0.1) is continuous on \([0, mh]\), but at \( t = mh \) it is discontinuous.

1. Properties of eigenvalues.

We denote \( 10^{-1} \) by \( \epsilon_0 \).
**Lemma 1.** Let $\epsilon_0$ be a small positive number and $t_0$ be sufficiently small positive number. Then there exists a eigenvalue $\lambda_q$ such that

$1 - \epsilon_0 < t \lambda_q < 1 + \epsilon_0$ for any $t: 0 < t < t_0$.

Proof. We suppose that there exists a small positive number $t_0$ such that

$t \lambda_q \leq 1 - \epsilon_0$ or $t \lambda_q \geq 1 + \epsilon_0$ for any natural number $q$.

We put $p = \max_q \{q : \lambda_q \leq (1 - \epsilon_0)/t\}$ and $r = \min_q \{q : \lambda_q \geq (1 + \epsilon_0)/t\}$. If $t_0$ is sufficiently small, $p$ and $r$ are sufficiently large natural number and $p + 1 = r$. From the assumption A-2) and (1.1) we get

$C p^\alpha + o(p^\alpha) \leq (1 - \epsilon_0)/t$ and $C(p + 1)^\alpha + o((p + 1)^\alpha) \geq (1 + \epsilon_0)/t$.

Then it follows

$(1 + \epsilon_0)(C(p + 1)^\alpha + o((p + 1)^\alpha))^{-1} \leq t \leq (1 - \epsilon_0)(C p^\alpha + o(p^\alpha))^{-1}$.

Since $p$ is sufficiently large natural number we obtain that the above inequalities are contradiction. Thus the proof is complete.

Let $\theta$ and $N$ be $1/3 - 4/(3N)$ and $10^3$ respectively.

We choose a sequence $\{t_n\}$ such that $t_1 = t_0/2$ and $0 < t_{n+1} < t_n \theta^n/2$ for any $n = 1, 2, 3, 4, \ldots$.

where $t_0$ is of lemma 1

**Lemma 2.** Let $j$ and $n$ be natural number such that $0 < j \leq n$. Thus there exists a natural number $\ell(n, j)$ such that

$1 - \epsilon_0 < (\theta^j t_n) \lambda_{\ell(n, j)} < 1 + \epsilon_0.$
and if \((n_1, j_1) \neq (n_2, j_2)\) then \(\lambda_{t(n_1, j_1)} \neq \lambda_{t(n_2, j_2)}\).

where \(\epsilon_0 = 10^{-1}\).

Proof. Since \(t_0\) is sufficiently small positive number, from Lemma 1, we see that there exists \(\lambda_t\). Next we shall show the eigenvalue is unique. Suppose \((n_1, j_1) \neq (n_2, j_2)\) and \(n_1 \geq n_2\). Then if \(n_1 > n_2\) it follows \(t_{n_2} \theta^{j_2} > 2t_{n_1} \theta^{j_1}\).

If \(n_1 = n_2\) and \(j_1 > j_2\) it also follows \(t_{n_2} \theta^{j_2} > 2t_{n_1} \theta^{j_1}\). From (1.1) and the above inequalities we have

\[
\lambda_{t(n_2, j_2)} < (1+\epsilon_0)(t_{n_2} \theta^{j_2})^{-1} < (1+\epsilon_0)2^{-1}(t_{n_1} \theta^{j_1})^{-1} < (1+\epsilon_0)(1-\epsilon_0)^{-1}2^{-1}\lambda_{t(n_1, j_1)}.
\]

Thus it follows \(\lambda_{t(n_2, j_2)} < \lambda_{t(n_1, j_1)}\).

2. Constitution of functions.

We shall constitute our aim’s function which satisfies the following conditions:

\[
f \in F_{m-1} \cap L_{loc}^2((0, h]; H) \quad \text{but} \quad f \notin F_m.
\]

For the sake of simplicity we suppose \(h = 1\).

We first take a sequence \(\{x_{n,j}\}\) such that

\[
x_{n,0} = 2^{-1}t_n \quad \text{and} \quad x_{n,j} = x_{n,j-1} + (1+2/N)\theta^{j-1}t_n/3
\]

where \(n = 1, 2, \ldots\) and \(j = 1, 2, \ldots \leq n\).

REMARK 1. Since \(\sum_{j=1}^{n}(1+2/N)\theta^{j-1}/3 \leq 1/2\) it follows \(t_n/2 \leq x_{n,j} < t_n\) where \(j = 0, 1, 2, \ldots, n\).

For the sake of the simplicity we put \(\gamma_{n,j} = \theta^j t_n/(3N)\), and \(\Gamma_{n,j} = (1 + 1/N)\theta^j t_n/3\).

Let \(\chi_1\) and \(\chi_2\) be functions such that
1) $\chi_1, \chi_2 \in C^\infty([0,1])$,
2) $\text{Supp } \chi_1 \subset [2^{-1}, 1]$ and $\text{Supp } \chi_2 \subset [0, 2^{-1}]$,
3) $\chi_1(\cdot) = 1$ on $[2/3, 1]$ and $\chi_2(\cdot) = 1$ on $[0, 1/3]$.

We denote $\chi_1((t-x_{n,j})/\gamma_{n,j})$ and $\chi_2((t-x_{n,j}-\Gamma_{n,j})/\gamma_{n,j})$ by $\chi_{1,n,j}(t)$ and $\chi_{2,n,j}(t)$ respectively.

Let $p$ be an arbitrary natural number. We define a function $f_{n,j}^p(t) \in C([0,1]; H)$ by

$$
0 \quad \text{if } t \in [0, x_{n,j}] \cup [x_{n,j}+1, 1],
$$

$$
\sum_{\alpha=0}^{p}(t-x_{n,j}-\gamma_{n,j})^\alpha A^{-p}a_\alpha \chi_{1,n,j}(t) \quad \text{if } t \in [x_{n,j}, x_{n,j}+\gamma_{n,j}],
$$

$$
A^{-p}S(t-x_{n,j}-\gamma_{n,j}+\epsilon_0 \theta^j t_{n}/3)\varphi_{t(n,j)} \quad \text{if } t \in [x_{n,j}+\gamma_{n,j}, x_{n,j}+\Gamma_{n,j}],
$$

$$
\sum_{\alpha=0}^{p}(t-x_{n,j}-\Gamma_{n,j})^\alpha A^{-p}b_\alpha \chi_{2,n,j}(t) \quad \text{if } t \in [x_{n,j}+\Gamma_{n,j}, x_{n,j+1}]
$$

where

$$a_\alpha = (\alpha!)^{-1}(-A)^\alpha S(\epsilon_0 3^{-1} \theta^j t_n)\varphi_{t(n,j)} \quad \text{and} \quad b_\alpha = (\alpha!)^{-1}(-A)^\alpha S((1+\epsilon_0) 3^{-1} \theta^j t_n)\varphi_{n,j}.$$ 

**Remark 2.** 1) $a_\alpha$ and $b_\alpha$ are $\alpha$ order's coefficients of Taylor expansion of the functions $S(s)\varphi_{n,j}$ at $s = \epsilon_0 \theta^j t_n/3$ and $s = (1+\epsilon_0) \theta^j t_n/3$ respectively.

2) From the constructive method of the function $f_{n,j}^p$ we see

$$(\text{Supp } f_{n_1,j_1}^p) \cap (\text{Supp } f_{n_2,j_2}^p) = \emptyset \quad \text{if} \quad (n_1, j_1) \neq (n_2, j_2).$$

3) $f_{n,j}^p \in C^p([0,1]; D(A^\infty))$ and it is piecewise sufficiently smooth at $t \in [0,1]$.

**Lemma 3.** Let $q$ and $k$ be nonnegative integers such that $q \leq p$. Then we have

$$
| (d/dt)^q A^k f_{n,j}^p(t) |_H \leq \text{Const} \lambda_{n,j}^{q+k-p}.
$$

$$(d/dt)(d/dt)^q A^k f_{n,j}^p(t) \in L^2(0,1; H).$$

**Proof.** We first shall show the former.

Let $t \in [x_{n,j}, x_{n,j}+\gamma_{n,j}]$. From the definition of $\chi_{1,n,j}$ and Lemma 1 it follows

$$
| (d/ds)^p \chi_{1,n,j} | \leq \text{Const}/\gamma_{n,j}^p \leq C\lambda_{t(n,j)}^p.
$$
If $\beta \leq \alpha$ we have

\begin{equation}
| (d/dt)^{\beta} (t - x_{n,j} - \gamma_{n,j})^{\alpha} | \leq \text{Const} \nu_{n,j}^{\alpha - \beta} \leq C \lambda_{\ell(n,j)}^{\beta - \alpha}.
\end{equation}

From the semigroup properties we see

\begin{equation}
| A^{k} S(s) \varphi_{n,j} |_{H} \leq \text{Const} \lambda_{t(n,j)}^{k} \exp(-s \lambda_{t(n,j)})
\end{equation}

Combining (2.1), (2.2) and (2.3) we get

\begin{equation}
| (d/dt)^{q} A^{k} f_{n,j}^{p} |_{H} \leq \text{Const} \lambda_{t(n,j)}^{k - p} \exp(-\gamma_{n,j} \lambda_{\ell(n,j)}) \sum_{\alpha=0}^{p} \sum_{\beta=0}^{q \wedge \alpha} \lambda_{t(n,j)}^{\beta - \alpha} \lambda_{t(n,j)}^{q - \beta} \leq \text{Const} \lambda_{t(n,j)}^{-p + q + k}.
\end{equation}

Using the similar method to the above, for $t \in [x_{n,j} + \Gamma_{n,j}, x_{n,j+1}]$, we also get the same estimate as the above.

For $t \in [x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j}]$, from (2.3), we also get the same estimate as (2.4).

Then the former is proved.

Next we shall show the latter.

If $q + 1$ is smaller than $p$, from the above, it is trivial. We suppose $q = p$. If $t \in (x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j})$ it follows

\begin{equation}
| (d/dt)(d/dt)^{p} A^{k} f_{n,j}^{p} (t) |_{H} \leq C \text{Const} \lambda_{t(n,j)}^{k + 1}.
\end{equation}

If $t \in (x_{n,j}, x_{n,j} + \gamma_{n,j}) \cup (x_{n,j} + \Gamma_{n,j}, x_{n,j+1})$ it follows

\begin{equation}
(d/dt)(d/dt)^{q} A^{k} f_{n,j}^{p} (t) = 0.
\end{equation}

Then the latter is proved.

Let $b_{n}$ be a decreasing sequence such that

\begin{equation}
\lim_{n \to \infty} b_{n} = 0, \quad \inf_{n} n^{1/2} b_{n} \geq \delta_{0} > 0.
\end{equation}

From 2) of Remark 2 we know that there exists $\sum_{n=1}^{\infty} \sum_{j=1}^{n} f_{n,j}^{p}(t) b_{n}$. Thus we denote the above function by $f_{p}(t)$. 


LEMMA 4. The function $f^p(\cdot)$ holds the following properties:

1) $f^p \in C^q([0, 1]; D(A^k)) \cap C^p((0, 1]; D(A^\infty))$ where $q + k \leq p$.

2) Let $\delta$ be any positive small number. This function is piecewise sufficiently smooth on $[\delta, 1]$.

3) $(d/dt + A)^k f^p \in C([0, 1]; H)$ and $\lim_{t \to 0} (d/dt + A)^k f^p(t) = 0$

where $k = 0, 1, \cdots, p$.

4) $(d/dt)(d/dt + A)^p f^p \in L^2_{loc}((0, 1]; H)$.

Proof. Combining 2), 3) of Remark 2 and lemma 3 and noting (2.5) we get the proof of 1). Since the sum of $f^p$ is finite on $[\delta, 1]$, from 3) of Remark 2, the proof of 2) is complete. From Lemma 3 and (2.5) the proof of 3) is complete. Noting the sum of $f^p$ is finite on $[\delta, 1]$ and Lemma 3 we can prove 4).

LEMMA 5. Let $t$ be any positive number such that $0 < t \leq 1$. Then there exists

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{t} (d/ds)(d/ds + A)^k f^p(s) ds = 0$$

where $k = 0, 1, \cdots, p$.

Proof. From 2) and 3) of Lemma 4 it is easy to prove this lemma.

LEMMA 6.

$$| A \int_{t_{n}/2}^{t_{n}} S(t_{n} - s) A^p f^p(s) ds | \geq \delta n^{1/2} b_n$$

where $\delta$ is a positive constant independent of $n$.

Proof. From the definition of $f^p$ we have $f^p = \sum_{j=1}^{n} f^p_{n,j} b_n$ on $[t_{n}/2, t_{n}]$. We put

$$\int_{x_{n,j}}^{x_{n,j+1}} A S(t_{n} - s) A^p f^p_{n,j} ds =$$

$$( \int_{x_{n,j}}^{x_{n,j} + \gamma_{n,j}} + \int_{x_{n,j} + \gamma_{n,j}}^{x_{n,j} + \Gamma_{n,j}} + \int_{x_{n,j} + \Gamma_{n,j}}^{x_{n,j+1}} ) \{ A S(t_{n} - s) A^p f^p_{n,j}(s) \} ds$$
\[= I_1 + I_2 + I_3.\]

We first shall estimate \(I_1\). From the definition of \(f_{n,j}^p\) on \([x_{n,j}, x_{n,j} + \gamma_{n,j}]\) and semigroup properties we have

\[
| AS(t_n - s)A^p f_{n,j}^p |_H 
\leq \sum_{\alpha=0}^{p} 1/(\alpha!) |s - x_{n,j} - \gamma_{n,j}|^\alpha \lambda_{n,j}^{\alpha+1} \exp(-(t_n - s + \epsilon_0 \theta^j t_n/3) \lambda_{n,j}).
\]

Since \(s - x_{n,j} \geq \lambda_{n,j}\) and \(\gamma_{n,j} \lambda_{t(n,j)} \leq 1/N\) we see

(2.6) \[| I_1 |_H \leq \sum_{\alpha=0}^{p} \text{Const}(\gamma_{n,j})^{\alpha+1} \lambda_{t(n,j)}^{\alpha+1} \leq \text{Const}/N.\]

where \text{Const} is a constant independent of \(n, j\) and \(N\). Using the similar method to the above we get

(2.7) \[| I_3 |_H \leq \text{Const}/N.\]

Let us estimate \(I_2\). Using the semigroup properties we get

\[AS(t_n - s)A^p f_{n,j}^p = \exp(-(t_n - x_{n,j} + (\epsilon_0 - 1/N)\theta^j t_n/3) \lambda_{n,j})) \lambda_{n,j} \varphi_{n,j}.\]

Since \(t_n - x_{n,j} = (1 + 2/N)(1 - \theta)^{-1} \theta^j t_n/3\), from lemma 2 and the above equality we have

\[| I_2 |_H \geq (1 - \epsilon_0) \exp(-\delta_1)/3\]

where \(\delta_1 = (1 - \epsilon_0)\{1/3(1+2/N)(1-\theta)^{-1} + (\epsilon_0 - 1/N)\}\). Then combining (2.6),(2.7) and the above inequality and noting \(N\) is a sufficiently large number there exists a constant \(\delta_0\) such that

\[| I_1 + I_2 + I_3 |_H^2 \geq (| I_2 |_H - | I_1 |_H - | I_3 |_H)^2 \geq ((1 - \epsilon_0) \exp(-\delta_1) - 2\text{Const}/N)^2 = \delta_0^2.\]
Thus we complete the proof of this lemma.

**Lemma 7.** Let $k$ be a nonnegative integer such that $k \leq p$. Then we get the following equality:

\[
\int_{t/2}^{t} (t - s)^{k} A^{k+1} S(t - s) (d/dt + A)^{p} f^{p}(s) ds \\
= - \sum_{q=0}^{k-1} (t/2)^{k-q} A^{k-q} S(t/2) (d/ds + A)^{p-q-1} A^{k+1} f^{p}(t/2) C_{q} \\
+ C_{k} \int_{t/2}^{t} S(t - s) (d/ds + A)^{p-k} A^{k+1} f^{p}(s) ds
\]

where $C_{q} = k!/(k-q)!$.

Proof. Using the integration by parts we get the following recurrence formula for $q$.

\[
\int_{t/2}^{t} (t - s)^{k-q} A^{k+1} S(t - s) (d/ds + A)^{p-q} f^{p}(s) ds \\
= -(t/2)^{k-q} A^{k+1} S(t/2) (d/ds + A)^{k-q-1} f^{p}(t/2) \\
+(k-q) \int_{t/2}^{t} (t - s)^{k-q-1} A^{k+1} S(t - s) (d/ds + A)^{p-q-1} f^{p}(s) ds.
\]

Solving the above recurrence formula we get the proof of this lemma.

**Lemma 8.** We get the following inequality:

\[
\lim_{t \to 0} \sup_{t} | \int_{t/2}^{t} (t - s)^{p} A^{p} S(t - s) d/ds (d/ds + A)^{p} f^{p}(s) ds |_{H} > 0.
\]

Proof. From the definition of $f^{p}$ it follows, for any nonnegative integer $\alpha$,

\[
(2.8) \quad ((d/dt)^{\alpha} f^{p})(t_{n}/2) = 0 \quad \text{and} \quad ((d/dt)^{\alpha} f^{p})(t_{n}) = 0.
\]

Let $p$ be 0. Using the integration by parts and (2.8) we see

\[
| \int_{t_{n}/2}^{t_{n}} S(t_{n} - s) d/ds f^{0}(s) ds |_{H} = | -A \int_{t_{n}/2}^{t_{n}} S(t_{n} - s) f^{0}(s) ds |_{H}.
\]
From Lemma 6 it follows the right term of the above equation is uniformly positive about \( n \).

Let \( p \) be larger than 1. Then from the integration by parts and (2.8) we have

\[
\int_{t/2}^{t} (t-s)^p A^p S(t-s) d/ds (d/ds + A)^p f^p(s) ds
\]

\[= p \int_{t/2}^{t} (t-s)^{p-1} A^p S(t-s) (d/ds + A)^p f^p(s) ds
\]

\[- \int_{t/2}^{t} (t-s)^p A^{p+1} S(t-s) (d/ds + A)^p f^p(s) ds = I_1 + I_2.\]

From Lemma 7 and (2.8) we get

\[I_1 = \text{Const} \int_{t/2}^{t} S(t-s) (d/ds + A) A^p f^p(s) ds.\]

On the other hand from the integration by parts it follows

\[\int_{t/2}^{t} S(t-s) (d/ds + A) A^p f^p(s) ds = 0.\]

Then \( I_1 = 0 \).

Combining Lemma 6 we obtain \( |I_2| \geq \delta_0 \). The proof is complete.

**Lemma 9.** Let \( k \) be a nonnegative integer smaller than \( p - 1 \). Then it follows

\[\lim_{t \searrow 0} \left| \int_{t/2}^{t} (t-s)^k A^k S(t-s) d/ds (d/ds + A)^p f^p(s) ds \right|_H = 0.\]

**Proof.** From the integration by parts we get

\[
\int_{t/2}^{t} (t-s)^k A^k S(t-s) d/ds (d/ds + A)^p f^p(s) ds = -(t/2)^k A^k S(t/2) (d/ds + A)^p f^p(t/2)
\]

\[+ k \int_{t/2}^{t} (t-s)^{k-1} A^k S(t-s) (d/ds + A)^p f^p(s) ds = I_1 + I_2.\]

On the other hand we have the operator norm: \( |s^k A^k S(s)|_{H \rightarrow H} \geq \text{Const} \). Combining 3) of Lemma 4 and the above result we obtain \( \lim_{t \searrow 0} I_1 = 0 \). From Lemma 7 and 3) of Lemma 4 we get \( \lim_{t \searrow 0} I_2 = 0 \). Thus the proof is complete.
3. Proof of Theorem.

We take a function $f$ defined on $[0,1]$ such that

$$f(t) = (d/dt)(d/dt + A)^p f^p(t).$$

From then 4) of Lemma 4, Lemma 5, Lemma 8 and Lemma 9 we get

$$f \in F_{p-1} \quad \text{and} \quad f \notin F_p.$$

Combining Proposition 6.9 in [3] and the above result we obtain the proof of Theorem is complete.

References


