Title

DISCONTINUITY OF SOLUTIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE (Nonlinear Evolution Equations and Their Applications)

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DISCONTINUITY OF SOLUTIONS OF PARABOLIC INTEGRO-DIFFERENTIAL EQUATIONS WITH TIME DELAY IN HILBERT SPACE

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0. Introduction and Theorem.

In this paper we consider the following integro-differential equation with time delay in a real Hilbert space $H$:

\( (0.1) \quad \frac{d}{dt}u(t) + Au(t) + A_1 u(t-h) + \int_{-h}^{0} a(-s)A_2 u(t+s)ds = f(t) \)

\[ u(0) = x, \quad u(s) = y(s) \quad -h \leq s < 0. \]

Here, $A$ is a positive definite self-adjoint operator and $A_1, A_2$ are closed linear operators with domains containing that of $A$. The notations $h$ and $N$ denote a fixed positive number and a large natural number respectively. Let $a(\cdot)$ is a real valued function belonging to $C^3([0, h])$.

The equations of the type (0.1) were investigated by G.Di Blasio, K.Kunisch and E.Sinestrari [2], S.Nakagiri [4], H.Tanabe [6] and D.G.Park and S.Y.Kim [5]. Particulary, G.Di Blasio, K.Kunisch and E.Sinestrari [2] showed the existence and uniqueness of a solution for $f \in L^2(0,T;H)$, $Ay \in L^2(-h,0;H)$ and $x \in (D(A), H)_{1/2,2}$ where $(D(A), H)_{1/2,2}$ is a interpolation space.

Since the equation (0.1) is of parabolic type, we want $x$ to be an arbitrary element of $H$. Then the integral in (0.1) exists only in the improper sense no
matter what nice functions $f$ and $Ay$ may be. Hence, it would be considered natural to investigate our problem under the following hypothesis:

\[ f \in \cap_{\delta>0}L^2(\delta, T; H) \quad \text{and} \quad Ay \in \cap_{\delta>0}L^2(-h + \delta, 0; H), \]

\[ f(t) \text{ and } Ay(t - h) \text{ are improperly integrable at } t = 0. \]

For the sake of simplicity we put

\[ L^2_{loc}((0, T]; H) = \cap_{\delta>0}L^2(\delta, T; H). \]

We first shall state the definition of a weak solution of (0.1).

DEFINITION. We say that a function $u$ defined on $[-h, T]$ is a weak solution of the equation (0.1) if the following four conditions satisfied:(see Definition 1.1 in [3])

1) $u \in L^2_{loc}((nh, (n+1)h]; D(A)) \cap W^{1,2}_{loc}((nh, (n+1)h]H) \cap C([0, Nh]; D(A^{-\alpha}))$
   for $n = 0, 1, 2, \ldots, N - 1$ and any $\alpha > 0$.

2) $\lim_{t \to 0}A^{-\alpha}u(t) = A^{-\alpha}x$
   for any $\alpha > 0$ and $u(s) = y(s)$ for $-h \leq s < 0$.

3) $Au(\cdot + nh) \in L^2_{loc}((0, h]; H)$ and $A^{1-\alpha}u(\cdot + nh)$ is improper integrable
   at $t = 0$.

4) The function $u$ satisfies the equation (0.1) for a.e $t$.

In Theorem 1 in [3] we showed the existence and uniqueness of a weak solution for which $A^{-\alpha}u$ is continuous in $[0, T]$ for an arbitrary positive number $\alpha$ but this solution is not always in $C([0, T]; H)$.

As the notations we put

\[ F_{-1} = \{ g \in L^2_{loc}((0, h]: H); \text{ there exists } \lim_{\epsilon \downarrow 0} \int_{\epsilon}^{h} g(s)ds \}. \]
\[ F_m = \{ g \in F_{m-1}; \lim_{t \searrow 0} \int_{t/2}^{1} (t - s)^m A_1^m S(t - s)g(s)ds = 0 \} \]

where \( S(\cdot) \) is an analytic semigroup of the positive defined self-adjoint operator \( A \) and \( m = 1, 2, \ldots, N - 1 \).

In Proposition 6.9 of [3] we also showed the following resultant.

Let \( f \) belong to \( F_{-1} \cap L^2_{loc}((0, Nh] : H) \) and \( m \) is a nonnegative integer such that \( 0 \leq m \leq N - 1 \). Then following two conditions are equivalent.

1) A weak solution of (0.1) is continuous on \([0, mh]\), but at \( t = mh \) this solution is discontinuous.

2) \( f - A_1 y(\cdot - h) \in F_{m-1} \), but \( f - A_1 y(\cdot - h) \notin F_m \).

In [3] we could not show that \( F_m \) is a proper subset in \( F_{m-1} \). The object in this paper is to show that \( F_m \) is a proper subset in \( F_{m-1} \) (i.e there exists a inhomogeneous function \( f \) and a initial data function \( y \) such that the solution of (0.1) is continuous on \([0,mh]\), but at \( t = mh \) this solution is discontinuous on \( H \)).

Throughout this paper we assume

\[ A - 1) \quad A = A_1 = A_2, \]

\[ A - 2) \quad \text{the operator} \ A \ \text{holds eigenvalues} \ \{ \lambda_q \}_{q=1}^\infty \ \text{such that} \]

\[ (0.2) \quad \lambda_q = C q^\alpha + o(q^\alpha), \quad \lambda_q \leq \lambda_{q+1} \]

where \( \alpha \) and \( C \) are some positive numners. We denote normal eigenfuctions of eigenvalues \( \lambda_q \) by \( \varphi_q \).

\textbf{Theorem} Under the assumptions \textit{A-1)} and \textit{A-2)} there exist a inhomogeneous function \( f \) and the initial valued function \( y \) such that the weak solution of \((0.1)\) is continuous on \([0, mh]\), but at \( t = mh \) it is discontinuous.

\textbf{1. Properties of eigenvalues.}

We denote \( 10^{-1} \) by \( \epsilon_0 \).
**Lemma 1.** Let $\epsilon_0$ be a small positive number and $t_0$ be sufficiently small positive number. Then there exists a eigenvalue $\lambda_q$ such that

(1.1). $1 - \epsilon_0 < t\lambda_q < 1 + \epsilon_0$ for any $t : 0 < t < t_0$.

Proof. We suppose that there exists a small positive number $t_0$ such that

$t\lambda_q \leq 1 - \epsilon_0$ or $t\lambda_q \geq 1 + \epsilon_0$ for any natural number $q$.

We put $p = \max_q \{q : \lambda_q \leq (1 - \epsilon_0)/t\}$ and $r = \min_q \{q : \lambda_q \geq (1 + \epsilon_0)/t\}$. If $t_0$ is sufficiently small, $p$ and $r$ are sufficiently large natural number and $p + 1 = r$. From the assumption A-2) and (1.1) we get

$$Cp^\alpha + o(p^\alpha) \leq (1 - \epsilon_0)/t \quad \text{and} \quad C(p + 1)^\alpha + o((p + 1)^\alpha) \geq (1 + \epsilon_0)/t.$$ 

Then it follows

$$(1 + \epsilon_0)(C(p + 1)^\alpha + o((p + 1)^\alpha))^{-1} \leq t \leq (1 - \epsilon_0)(Cp^\alpha + o(p^\alpha))^{-1}.$$ 

Since $p$ is sufficiently large natural number we obtain that the above inequalities are contradiction. Thus the proof is complete.

Let $\theta$ and $N$ be $1/3 - 4/(3N)$ and $10^3$ respectively.

We choose a sequence $\{t_n\}$ such that $t_1 = t_0/2$ and $0 < t_{n+1} < t_n\theta^n/2$ for any $n = 1, 2, 3, 4, \cdots$.

where $t_0$ is of lemma 1

**Lemma 2.** Let $j$ and $n$ be natural number such that $0 < j \leq n$. Thus there exists a natural number $\ell(n, j)$ such that

$$1 - \epsilon_0 < (\theta^j t_n)\lambda_{\ell(n, j)} < 1 + \epsilon_0,$$
and if \((n_1, j_1) \neq (n_2, j_2)\) then \(\lambda_{t(n_1,j_1)} \neq \lambda_{t(n_2,j_2)}\).

where \(\epsilon_0 = 10^{-1}\).

Proof. Since \(t_0\) is sufficiently small positive number, from Lemma 1, we see that there exists \(\lambda_t\). Next we shall show the eigenvalue is unique. Suppose \((n_1, j_1) \neq (n_2, j_2)\) and \(n_1 \geq n_2\). Then if \(n_1 > n_2\) it follows \(t_{n_2}\theta^{j_2} > 2t_{n_1}\theta^{j_1}\). If \(n_1 = n_2\) and \(j_1 > j_2\) it also follows \(t_{n_2}\theta^{j_2} > 2t_{n_1}\theta^{j_1}\). From (1.1) and the above inequalities we have

\[
\lambda_{t(n_2,j_2)} < (1+\epsilon_0)(t_{n_2}\theta^{j_2})^{-1} < (1+\epsilon_0)2^{-1}(t_{n_1}\theta^{j_1})^{-1} < (1+\epsilon_0)(1-\epsilon_0)^{-1}2^{-1}\lambda_{t(n_1,j_1)}.
\]

Thus it follows \(\lambda_{t(n_2,j_2)} < \lambda_{t(n_1,j_1)}\).

2. Constitution of functions.

We shall constitute our aim’s function which satisfies the following conditions:

\[
f \in F_{m-1} \cap L_{loc}^2((0, h]; H) \quad \text{but} \quad \not\in F_m.
\]

For the sake of simplicity we suppose \(h = 1\).

We first take a sequence \(\{x_{n,j}\}\) such that

\[
x_{n,0} = 2^{-1}t_n \quad \text{and} \quad x_{n,j} = x_{n,j-1} + (1+2/N)\theta^{j-1}t_n/3
\]

where \(n = 1, 2, \cdots\) and \(j = 1, 2, \cdots \leq n\).

Remark 1. Since \(\sum_{j=1}^{n}(1+2/N)\theta^{j-1}/3 \leq 1/2\) it follows \(t_n/2 \leq x_{n,j} < t_n\) where \(j = 0, 1, 2, \cdots, n\).

For the sake of the simplicity we put \(\gamma_{n,j} = \theta^j t_n/(3N)\), and \(\Gamma_{n,j} = (1 + 1/N)\theta^j t_n/3\).

Let \(\chi_1\) and \(\chi_2\) be functions such that
1) \( \chi_{1}, \chi_{2} \in C^{\infty}([0, 1]), \)
2) \( \text{Supp } \chi_{1} \subset [2^{-1}, 1] \) and \( \text{Supp } \chi_{2} \subset [0, 2^{-1}] \),
3) \( \chi_{1}(\cdot) = 1 \) on \([2/3, 1]\) and \( \chi_{2}(\cdot) = 1 \) on \([0, 1/3]\).

We denote \( \chi_{1}((t-x_{n,j})/\gamma_{n,j}) \) and \( \chi_{2}((t-x_{n,j}-\Gamma_{n,j})/\gamma_{n,j}) \) by \( \chi_{1,n,j}(t) \) and \( \chi_{2,n,j}(t) \) respectively.

Let \( p \) be an arbitrary natural number. We define a function \( f_{n_{2}j}^{p}(t) \in C([0, 1]; H) \) by

\[
\begin{align*}
0 & \quad \text{if } t \in [0, x_{n,j}] \cup [x_{n,j}+1, 1], \\
\sum_{\alpha=0}^{p}(t-x_{n,j}-\gamma_{n,j})^{\alpha}A^{-p}a_{\alpha}\chi_{1,n,j}(t) & \quad \text{if } t \in [x_{n,j}, x_{n,j}+\gamma_{n,j}], \\
A^{-p}S(t-x_{n,j}-\gamma_{n,j}+\epsilon_{0}\theta^{j}t_{n}/3)\varphi_{t(n,j)} & \quad \text{if } t \in [x_{n,j}+\gamma_{n,j}, x_{n,j}+\Gamma_{n,j}], \\
\sum_{\alpha=0}^{p}(t-x_{n,j}-\Gamma_{n,j})^{\alpha}A^{-p}b_{\alpha}\chi_{2,n,j}(t) & \quad \text{if } t \in [x_{n,j}+\Gamma_{n,j}, x_{n,j+1}]
\end{align*}
\]

where

\[
a_{\alpha} = (\alpha!)^{-1}(-A)^{\alpha}S(\epsilon_{0}3^{-1}\theta^{j}t_{n})\varphi_{t(n,j)} \quad \text{and} \quad b_{\alpha} = (\alpha!)^{-1}(-A)^{\alpha}S((1+\epsilon_{0})3^{-1}\theta^{j}t_{n})\varphi_{n,j}.
\]

**Remark 2.**
1) \( a_{\alpha} \) and \( b_{\alpha} \) are \( \alpha \) order's coefficients of Taylor expansion of the functions \( S(s)\varphi_{n,j} \) at \( s = \epsilon_{0}\theta^{j}t_{n}/3 \) and \( s = (1+\epsilon_{0})\theta^{j}t_{n}/3 \) respectively.
2) From the constructive method of the function \( f_{n,j}^{p} \) we see

\( (\text{Supp } f_{n_{1},j_{1}}^{p}) \cap (\text{Supp } f_{n_{2},j_{2}}^{p}) = \emptyset \) if \( (n_{1}, j_{1}) \neq (n_{2}, j_{2}) \).
3) \( f_{n,j}^{p} \in C^{p}([0, 1]; D(A^{\infty}) \) and it is piecewise sufficiently smooth at \( t \in [0, 1] \).

**Lemma 3.** Let \( q \) and \( k \) be nonnegative integers such that \( q \leq p \). Then we have

\[
| (d/dt)^{q}A^{k}f_{n,j}^{p}(t) |_{H} \leq Const\lambda_{n,j}^{q+k-p}.
\]

\[
(d/dt)(d/dt)^{q}A^{k}f_{n,j}^{p}(t) \in L^{2}(0, 1; H).
\]

**Proof.** We first shall show the former.

Let \( t \in [x_{n,j}, x_{n,j}+\gamma_{n,j}] \). From the definition of \( \chi_{1,n,j} \) and Lemma 1 it follows

\[
| (d/ds)^{q}\chi_{1,n,j} | \leq Const/\gamma_{n,j} \leq C\lambda_{t(n,j)}^{q}.
\]
If $\beta \leq \alpha$ we have

\begin{equation}
| (d/dt)^\beta (t - x_{n,j} - \gamma_{n,j})^\alpha | \leq Const \gamma_{n_{t,j}}^{\alpha - \beta} \leq C \lambda_{t(n,j)}^{\beta - \alpha}.
\end{equation}

From the semigroup properties we see

\begin{equation}
| A^k S(s) \varphi_{n,j} |_H \leq Const \lambda_{t(n,j)}^k \exp(-s \lambda_{t(n,j)})
\end{equation}

Combining (2.1), (2.2) and (2.3) we get

\begin{equation}
| (d/dt)^q A^k f_{n,j}^p |_H \leq Const \lambda_{t(n,j)}^{k-p} \exp(-\gamma_{n,j} \lambda_{t(n,j)}) \sum_{\alpha=0}^{p} \sum_{\beta=0}^{q - \alpha} \lambda_{t(n,j)}^{\beta - \alpha} \lambda_{t(n,j)}^{q - \beta} \leq Const \lambda_{t(n,j)}^{-p + q + k}.
\end{equation}

Using the similar method to the above, for $t \in [x_{n,j} + \Gamma_{n,j}, x_{n,j+1}]$, we also get the same estimate as the above.

For $t \in [x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j}]$, from (2.3), we also get the same estimate as (2.4). Then the former is proved.

Next we shall show the latter.

If $q + 1$ is smaller than $p$, from the above, it is trivial. We suppose $q = p$. If $t \in (x_{n,j} + \gamma_{n,j}, x_{n,j} + \Gamma_{n,j})$ it follows

\begin{equation}
| (d/dt)(d/dt)^p A^k f_{n,j}^p (t) |_H \leq Const \lambda_{t(n,j)}^{k+1}.
\end{equation}

If $t \in (x_{n,j}, x_{n,j} + \gamma_{n,j}) \cup (x_{n,j} + \Gamma_{n,j}, x_{n,j+1})$ it follows

\begin{equation}
(d/dt)(d/dt)^q A^k f_{n,j}^p (t) = 0.
\end{equation}

Then the latter is proved.

Let $b_n$ be a decreasing sequence such that

\begin{equation}
\lim_{n \to \infty} b_n = 0, \quad \inf_n n^{1/2} b_n \geq \delta_0 > 0.
\end{equation}

From 2) of Remark 2 we know that there exists $\sum_{n=1}^{\infty} \sum_{j=1}^{n} f_{n,j}^p (t) b_n$. Thus we denote the above function by $f^p(t)$.
Lemma 4. The function $f^p(\cdot)$ holds the following properties:

1) $f^p \in C^q([0,1]; D(A^k)) \cap C^p((0,1]; D(A^{\infty}))$ where $q + k \leq p$.

2) Let $\delta$ be any positive small number. This function is piecewise sufficiently smooth on $[\delta, 1]$.

3) $(d/dt + A)^k f^p \in C([0,1]; H)$ and $\lim_{t \to 0} (d/dt + A)^k f^p(t) = 0$

where $k = 0, 1, \ldots, p$.

4) $(d/dt)(d/dt + A)^p f^p \in L_{loc}^2((0,1]; H)$.

Proof. Combining 2), 3) of Remark 2 and lemma 3 and noting (2.5) we get the proof of 1). Since the sum of $f^p$ is finite on $[\delta, 1]$, from 3) of Remark 2, the proof of 2) is complete. From Lemma 3 and (2.5) the proof of 3) is complete. Noting the sum of $f^p$ is finite on $[\delta, 1]$ and Lemma 3 we can prove 4).

Lemma 5. Let $t$ be any positive number such that $0 < t \leq 1$. Then there exists

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{t} (d/ds)(d/ds + A)^k f^p(s) ds = 0$$

where $k = 0, 1, \ldots, p$.

Proof. From 2) and 3) of Lemma 4 it is easy to prove this lemma.

Lemma 6.

$$| A \int_{t_{n}/2}^{t_{n}} S(t_{n} - s) A^p f^p(s) ds |_{H} \geq \delta n^{1/2} b_n$$

where $\delta$ is a positive constant independent of $n$.

Proof. From the definition of $f^p$ we have $f^p = \sum_{j=1}^{n} f_{n,j}^p b_n$ on $[t_{n}/2, t_{n}]$. We put

$$\int_{x_{n,j}}^{x_{n,j} + 1} AS(t_{n} - s) A^p f_{n,j}^p ds =$$

$$(\int_{x_{n,j}}^{x_{n,j} + 1} + \int_{x_{n,j} + \gamma_{n,j}}^{x_{n,j} + 1} + \int_{x_{n,j} + \Gamma_{n,j}}^{x_{n,j} + 1}) \{AS(t_{n} - s) A^p f_{n,j}^p(s)\} ds$$
\[ = I_1 + I_2 + I_3. \]

We first shall estimate \( I_1 \). From the definition of \( f_{n,j}^p \) on \([x_{n,j}, x_{n,j} + \gamma_{n,j}]\) and semigroup properties we have

\[
|AS(t_n - s)A^p f_{n,j}^p|_H \leq \sum_{\alpha=0}^{p} \frac{1}{(\alpha!)} |s - x_{n,j} - \gamma_{n,j}|^\alpha \lambda_{n,j}^{\alpha+1} \exp(- (t_n - s + \epsilon_0 \theta^j t_n/3) \lambda_{n,j}).
\]

Since

\[
s - x_{n,j} \geq \lambda_{n,j} \quad \text{and} \quad \gamma_{n,j} \lambda_{t(n,j)} \leq 1/N
\]

we see

\[ |I_1|_H \leq \sum_{\alpha=0}^{p} \text{Const}(\gamma_{n,j})^{\alpha+1} \lambda_{t(n,j)}^{\alpha+1} \leq \text{Const}/N. \]

where \( \text{Const} \) is a constant independent of \( n, j \) and \( N \). Using the similar method to the above we get

\[ |I_3|_H \leq \text{Const}/N. \]

Let us estimate \( I_2 \). Using the semigroup properties we get

\[ AS(t_n - s)A^p f_{n,j}^p = \exp(- (t_n - x_{n,j} + (\epsilon_0 - 1/N) \theta^j t_n/3)) \lambda_{n,j}) \lambda_{n,j} \varphi_{n,j}. \]

Since \( t_n - x_{n,j} = (1 + 2/N)(1 - \theta)^{-1} \theta^j t_n/3 \), from lemma 2 and the above equality we have

\[ |I_2|_H \geq (1 - \epsilon_0) \exp(-\delta_1)/3 \]

where \( \delta_1 = (1 - \epsilon_0)\{1/3(1+2/N)(1-\theta)^{-1}+(\epsilon_0-1/N)\} \). Then combining (2.6),(2.7) and the above inequality and noting \( N \) is a sufficiently large number there exists a constant \( \delta_0 \) such that

\[ |I_1+I_2+I_3|_H^2 \geq (|I_2|_H - |I_1|_H - |I_3|_H)^2 \geq ((1-\epsilon_0)\exp(-\delta_1) - 2\text{Const}/N)^2 = \delta_0^2. \]
Thus we complete the proof of this lemma.

**Lemma 7.** Let $k$ be a nonnegative integer such that $k \leq p$. Then we get the following equality:

\[
\int_{t/2}^{t} (t-s)^k A^{k+1} S(t-s)(d/dt + A)^p f^p(s)ds = -\sum_{q=0}^{k-1} (t/2)^{k-q} A^{k-q} S(t/2)(d/ds + A)^{p-q-1} A^{k+1} f^p(t/2) C_q + C_k \int_{t/2}^{t} S(t-s)(d/ds + A)^{p-k} A^{k+1} f^p(s)ds
\]

where $C_q = k!/(k-q)!$.

Proof. Using the integration by parts we get the following recurrence formula for $q$.

\[
\int_{t/2}^{t} (t-s)^{k-q} A^{k+1} S(t-s)(d/ds + A)^{p-q} f^p(s)ds = -(t/2)^{k-q} A^{k+1} S(t/2)(d/ds + A)^{k-q-1} f^p(t/2) + (k-q) \int_{t/2}^{t} (t-s)^{k-q-1} A^{k+1} S(t-s)(d/ds + A)^{p-q-1} f^p(s)ds.
\]

Solving the above recurrence formula we get the proof of this lemma.

**Lemma 8.** We get the following inequality:

\[
\limsup_{t \searrow 0} \left| \int_{t/2}^{t} (t-s)^p A^p S(t-s) d/ds(d/ds + A)^p f^p(s)ds \right|_H > 0.
\]

Proof. From the definition of $f^p$ it follows, for any nonnegative integer $\alpha$,

(2.8) \quad \left( (d/dt)^\alpha f^p \right)(t_{n}/2) = 0 \quad \text{and} \quad \left( (d/dt)^\alpha f^p \right)(t_n) = 0.

Let $p$ be 0. Using the integration by parts and (2.8) we see

\[
\left| \int_{t_n/2}^{t_n} S(t_n-s) d/ds f^0(s)ds \right|_H = \left| -A \int_{t_n/2}^{t_n} S(t_n-s) f^0(s)ds \right|_H.
\]
From Lemma 6 it follows the right term of the above equation is uniformly positive about $n$.

Let $p$ be larger than 1. Then from the integration by parts and (2.8) we have

\[
\int_{t_n/2}^{t_n} (t_n - s)^p A^p S(t_n - s) d/ds (d/ds + A)^p f^p(s) ds = p \int_{t_n/2}^{t_n} (t_n - s)^{p-1} A^p S(t_n - s) (d/ds + A)^p f^p(s) ds
\]

\[
- \int_{t_n/2}^{t_n} (t_n - s)^p A^{p+1} S(t_n - s) (d/ds + A)^p f^p(s) ds = I_1 + I_2.
\]

From Lemma 7 and (2.8) we get

\[
I_1 = \text{Const} \int_{t_n/2}^{t_n} S(t_n - s) (d/ds + A) A^p f^p(s) ds.
\]

On the other hand from the integration by parts it follows

\[
\int_{t_n/2}^{t_n} S(t_n - s) (d/ds + A) A^p f^p(s) ds = 0.
\]

Then $I_1 = 0$.

Combining Lemma 6 we obtain $|I_2| \geq \delta_0$. The proof is complete.

**LEMMA 9.** Let $k$ be a nonnegative integer smaller than $p - 1$. Then it follows

\[
\lim_{t \searrow 0} | \int_{t/2}^{t} (t-s)^k A^k S(t-s) d/ds (d/ds + A)^p f^p(s) ds |_H = 0.
\]

Proof. From the integration by parts we get

\[
\int_{t/2}^{t} (t-s)^k A^k S(t-s) d/ds (d/ds + A)^p f^p(s) ds = -(t/2)^k A^k S(t/2) (d/ds + A)^p f^p(t/2)
\]

\[
+ k \int_{t/2}^{t} (t-s)^{k-1} A^k S(t-s) (d/ds + A)^p f^p(s) ds = I_1 + I_2.
\]

On the other hand we have the operator norm: $|s^k A^k S(s)|_{H \to H} \geq \text{Const}$. Combining 3) of Lemma 4 and the above result we obtain $\lim_{t \searrow 0} I_1 = 0$. From Lemma 7 and 3) of Lemma 4 we get $\lim_{t \searrow 0} I_2 = 0$. Thus the proof is complete.
3. Proof of Theorem.

We take a function $f$ defined on $[0,1]$ such that

$$f(t) = (d/dt)(d/dt + A)^p f^p(t).$$

From then 4) of Lemma 4, Lemma 5, Lemma 8 and Lemma 9 we get

$$f \in F_{p-1} \quad \text{and} \quad f \notin F_p.$$

Combining Proposition 6.9 in [3] and the above result we obtain the proof of Theorem is complete.

References