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OPTIMIZING MULTIPLE SELECTIONS WITH SEQUENTIAL OBSERVATIONS

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Abstract

The optimal stopping rules with multiple selections of $m \geq 1$ objects with the objective to maximize the probability of obtaining the best object are studied for two problems with an unknown number of objects:the problem with random number of objects, and the problem where the objects arrive according to a homogeneous Poisson process with unknown intensity λ. These two problems are variation of the so-called secretary problem. This article introduces easier method based on the one-stage look-ahead function (defined herein) depending on m and its recursive relation on the number m, to find the optimal stopping rule for all m, without direct solution of equations suggested by a common dynamic programming approach.

SECRETARY PROBLEM; OPTIMAL STOPPING

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1. Introduction

A man observes the sequence of independent random variables, $X_1, X_2, \cdots, X_n$ and must decide whether to accept or not after each observations with the objective to maximize the probability of obtaining the best object, that is, $\max\{X_1, \cdots, X_n\}$ when at most m (≥ 1) selections are allowed, where m is a predetermined number. This problem has been studied by Gilbert and Mosteller [1]. When $m = 1$, this is a well-known classical secretary problem. However, they are not the originators of the classical secretary problem. Information on the foundations of the problem can be found in Ferguson [2]. They have investigated the multiple selection models for the so-called no-information case of secretary problem or optimal selection problem. In the no-information case, X_i is regarded the relative rank of the ith objects among the first i objects (rank 1 being best) under the assumption that the objects are observed sequentially in random order with all $n!$ orderings being equally likely and all that can be observed are the relative rank of the objects as they are presented. Thus X_i are independent random variables and the distribution of X_i is given by $P(X_i = j) = 1/i$ for $j = 1, 2, \cdots, i$ for $i = 1, 2, \cdots, n$.

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For the problem with two selections, Haggström [3] has studied in the general setting and Sakaguchi [4] has resolved the results of Gilbert and Mosteller [1] on the no-information problem using one-stage look-ahead rule. Tamaki [5] has solved the full-information problem with two selections by dynamic programming approach. The one-stage look-ahead rule is a special case of the monotone case of Chow et al [6] and is generally quite effective to find the optimal stopping rule. For theorem and usage of the one-stage look-ahead rule, Ferguson [7] has provided much insight and many examples.

The optimal rule for the no-information secretary problem with $m(\geq 1)$ selections can be summarized as follows: stop (=accept) the first relatively best object which appears after or on s_m^*, where s_m^* is a determined sequence of integers, non-increasing in m. It is known that for large n, $s_1^* \approx ne^{-1}, s_2^* \approx ne^{-3/2}$, $s_3^* \approx ne^{-47/24}$, and $s_4^* \approx ne^{-2761/1152}$ (see Sakaguchi [4]), and for large n, the maximum probability of obtaining the best object with m selections under the optimal rule is $s_1^*/n + s_2^*/n + \cdots + s_m^*/n$.

Our motivation has come from that even if the optimal stopping rule of the problem with m (≥ 3) selections may be estimated without difficulty but it doesn't seem to be easy to prove it. One of this difficulty may be the fact that when we employ the one-stage look-ahead approach we have been confronted by complicated calculation to find explicit solutions of corresponding differential or integral equations suggested by a common dynamic programming principle. This article introduces easier method based on the *one-stage look-ahead function* (defined later) depending on the number m, of selections and its recursive relation on m to find the optimal stopping rule for all $m \geq 1$, without direct solutions of the equations. Ano and Tamaki [8] seems to be first to use this method.

In Section 2, we apply our method to the problem where the number of objects is a random variable with known distribution $\delta_k = P(N = k), k = 0, 1, \cdots$ and $\pi_0 = 1, \pi_k = \sum_{s \geq k} \delta_s$. This problem has been studied by Presman and Sonin [11] who investigate the case with a single selection and show that under the following Presman & Sonin condition

(PS) $\{d_i\}_{-1}^{\infty}$ changes sign exactly once from negative to non-negative,

(i.e., if whenever $d_i \geq 0$ then $d_j \geq 0$ for $j = i+1, i+2, \cdots$) where $d_i \equiv \delta_i - \sum_{j \geq i+1} \delta_j / j$ for $i = 0, 1, \cdots$ and $d_{-1} \equiv -1$, the one-stage look-ahead rule is optimal. We say that the sequence changes sign once from negative to non-negative, if and only if there exists a i^* such that $d_i \geq 0$ for all $i \geq i^*$ and $d_i < 0$ for all $i < i^*$. We show under their condition (PS) the optimal rule for the problem with m selections is the same form as the one for the no-information secretary problem with m selections. As an example, we investigate in details the case in which the total number, N of objects is uniformly distributed on $[1, N_0]$. In this case, we see that as $N_0 \to \infty$, $s_1^*/N_0 \to e^{-2} \approx .135335, s_2^*/N_0 \to e^{-(1+\sqrt{21}/3)} \approx .079856$ and $s_3^*/N_0 \to e^{-(1+(\sqrt{135+42\sqrt{21}})/9)} \approx .04951742$ and for large N_0 the maximum probability of obtaining the best under the optimal stopping rule is $-((s_1^*/N_0)\log(s_1^*/N_0) + (s_2^*/N_0)\log(s_2^*/N_0) + \cdots + (s_m^*/N_0)\log(s_m^*/N_0))$.

Section 3 considers another problem with unknown number of objects where the objects arrive according to a homogeneous Poisson process with unknown intensity λ and a prior exponential distribution, $a\exp\{-a\lambda\}I(\lambda > 0)$ where a is a known nonnegative parameter. The objective is to maximize the probability of obtaining the best object from those (if any) available in the given interval $[0, T]$. The no-information version with single selection is the problem studied by Bruss [12], which has succeeded to extend the results of Cowan and Zabczyk [13] with known intensity λ. Bruss [12] has shown that the optimal rule for single selection is stationary and to accept (if possible) the first relatively best object after time $(T+a)/e-a$. Using our approach based on his developments and results with single selection to which we refer in details, we see that the optimal stopping rules with multiple selections have the following form: the optimal rule is to accept(if possible) the first relatively best object after time $s_m^* = (T+a)/e^{C^{(m)}} - a$, where $C^{(m)}$ is constant. For $a = 0$, it is interesting to see $s_1^* = T/e$, $s_2^* = T/e^{3/2}$, $s_3^* = T/e^{47/24}, \cdots$ compared with the values n/e, $n/e^{3/2}$, $n/e^{47/24}$, $n/e^{2761/1152}$ of the no-information secretary problem.

2. Random number of objects

For the problem with random number, N of objects, let $W_i^{(m)}$ be the maximum probability of obtaining the best object among all N objects when we confront a relatively best object at ith observation and we can make more m selections hereafter. Similarly when we cam make more m selections in the future, let $U_i^{(m)}(V_i^{(m)})$ be the corresponding probability when we accept(reject) the relatively best object at ith observation. Suppose that ith object is a relatively best object ($X_i = 1$). Then the conditional probability that ith object is best of all N given $N \geq i$ is

$$\sum_{j\geq i} P(X_{i+1} > 1, \cdots, X_j > 1 | N = j) P(N = j | N \geq i) = \sum_{j\geq i} \frac{i}{j}\frac{\delta_j}{\pi_i}. \tag{2.1}$$

Therefore

$$U_i^{(m)} = \sum_{j\geq i} \frac{i\delta_j}{j\pi_i} + V_i^{(m-1)}, \tag{2.2}$$

where $V_i^{(0)} = 0$ for all i. Assume we confront a relatively best object at ith observation, then since the conditional probability that jth object is a first relatively best object after ith object given $N \geq j$ is $(i\pi_j)/(j(j-1)\pi_i)$,

$$V_i^{(m)} = \sum_{j\geq i} \frac{i\pi_j}{j(j-1)\pi_i} W_j^{(m)}. \tag{2.3}$$

Throughout this article, the vacuous sum is assumed to be zero. By the principle of optimality, we get the dynamic programming equation

$$W_i^{(m)} = \max\{U_i^{(m)},\ V_i^{(m)}\}, \quad \text{for } i = 1, 2, \cdots, \text{and } m \geq 1. \tag{2.4}$$

The one-stage look-ahead rule is the rule that calls for selecting when selecting immediately is at least as good as waiting for the next relatively best to appear and then selecting. Thus for $i = 1, \cdots, n-1$ and $m \geq 1$, it requires us to select the ith object if

$$g_i^{(m)} \equiv U_i^{(m)} - \sum_{j=i+1} \frac{i\pi_j}{j(j-1)\pi_i} U_j^{(m)} \geq 0, \tag{2.5}$$

where $g_i^{(0)} = 0$ for all i and $g_{-1}^{(m)} \equiv -1$ for all $m \geq 1$. We define and call $g_i^{(m)}$ the *one-stage look-ahead function.* It is well-known that if for fixed m, $\{g_i^{(m)}\}_{-1}^{\infty}$ changes sign exactly once from negative to non-negative, then the problem is monotone in the sense of Chow et al. [5] and one-stage look-ahead rule is optimal having the following form of a threshold stopping rule with threshold $s_m \equiv \min\{i \geq 1 : g_i^{(m)} \geq 0\}$ given a fixed m:

$$\tau_{r_m}^{(m)} \equiv \min\{k \geq s_m : X_k = 1\}. \tag{2.6}$$

A stopping problem is defined to be monotone if the sets for a fixed m, $G_i^{(m)} = \{U_i^{(m)} \geq E(U_{i+1}^{(m)}|X_1, \cdots, X_i)\}$ are monotone non-decreasing, i.e., $G_0^{(m)} \subset G_1^{(m)} \subset \cdots$ a.s. When the condition (PS) holds, Presman & Sonin problem with single selection is monotone and one-stage look-ahead rule is optimal which is a threshold rule $\tau_{s_1^*}^{(1)}$ with threshold $s_1^* = \min\{i \geq 1 : g_i^{(1)} \geq 0\}$. The following theorem tells us that under the condition (PS), Presman & Sonin problem with multiple selections is also monotone.

Theorem 1. If the distribution of the number of objects satisfies the Presman & Sonin's condition (PS), then the optimal rule for the problem with random number of objects when we make m more selections is a threshold rule $\tau_{s_m^*}^{(m)}$, where s_m^* can be specified as $s_m^* = \min\{i \geq 1 : g_i^{(m)} \geq 0\}$. And s_m^* is non-increasing in m.

Proof. It is shown by induction on m based on one-stage look-ahead function. When $m = 1$, the assertion is the result of Presman & Sonin [11]. As induction hypotheses, we assume that for fixed $m \geq 1$, $\{g_i^{(m)}\}_{-1}^{\infty}$ changes sign once from negative to non-negative,and for fixed $i \geq 1$ and all $m \geq 1$, $g_i^{(m+1)} \geq g_i^{(m)}$. Consequently we assume $\tau_{s_m^*}^{(m)}$ is optimal rule and $s_m^* \geq s_{m+1}^*$. These hypotheses imply that since when $i \geq s_m^*$, $W_i^{(m)} = U_i^{(m)}$ and $V_i^{(m)} = \sum_{j \geq i}(i\pi_j/(j(j-1)\pi_i))U_j^{(m)}$ and when $i < s_m^*$, $W_i^{(m)} = V_i^{(m)}$,

$$W_i^{(m)} - V_i^{(m)} = g_i^{(m)} I(i \geq s_m^*), \quad i = 1, 2, \cdots, \tag{2.7}$$

where $I(A)$ represents the indicator function of the event A.

On the other hand, from (2.5)

$$\begin{aligned} g_i^{(m+1)} &= \sum_{j \geq i} \frac{i\delta_j}{j\pi_i} + \sum_{j \geq i} \frac{i\pi_j}{j(j-1)\pi_i} W_j^{(m)} - \sum_{j \geq i} \frac{i\pi_j}{j(j-1)\pi_i} \{\sum_{k \geq j} \frac{j\delta_k}{k\pi_j} + V_j^{(m)}\} \\ &= g_i^{(1)} + \sum_{j \geq i} \frac{i\pi_j}{j(j-1)\pi_i} \{W_j^{(m)} - V_j^{(m)}\}, \end{aligned}$$

where $g_i^{(1)} = \sum_{j \geq i}(id_j)/(j\pi_i)$.

Substituting (2.7) into the above equation,

$$g_i^{(m+1)} = g_i^{(1)} + \sum_{j \geq \max(i+1, s_m^*)} \frac{i\pi_j}{j(j-1)\pi_i} g_j^{(m)}. \tag{2.8}$$

It is convenient for the induction to condider the function $h_i^{(m)} = (\pi_i/i)g_i^{(m)}$ for $i \geq 1$ and $m \geq 1$. Then the induction hypotheses reduce to:(A1) for fixed $m \geq 1$, $\{h_i^{(m)}\}_{-1}^{\infty}$ changes sign exactle once from negative to non-negative, and (A2) for fixed $i \geq 1$ and all $m \geq 1$, $h_i^{(m+1)} \geq h_i^{(m)}$. Note that s_m^* can be written as $s_m^* = \min\{i \geq 1 : h_i^{(m)} \geq 0\}$ and $s_m^* \geq s_{m+1}^*$. Now equation (2.8) reduces to

$$h_i^{(m+1)} = h_i^{(1)} + \sum_{j \geq \max(i+1, s_m^*)} \frac{1}{j-1} h_j^{(m)}, \tag{2.9}$$

where $h_i^{(1)} = \sum_{j \geq i} d_j/j$ and $h_{-1}^{(m)} \equiv -1$ for all $m \geq 1$.

When $m = 1$, under the condition (PS), $h_i^{(1)}$ satisfies (AR1). By virtue of (2.7) we find

$$h_i^{(2)} - h_i^{(1)} = \sum_{j \geq \max(i+1, s_1^*)} \frac{1}{j} h_j^{(1)} \geq 0, \tag{2.10}$$

because for $j \geq s_1^*$, $h_j^{(1)}$ is non-negative. Hence the hypothesis (AR2) holds for $m = 1$.

We shall continue the induction. When $i + 1 \geq s_m^*$, from the induction hypotheses we have $h_i^{(m)} \geq 0$. Then the second hypothesis (AR2) implies that for $j \geq i + 1$,

$$0 \leq h_i^{(m)} \leq h_i^{(m+1)} \Rightarrow 0 \leq h_j^{(m)} \leq h_j^{(m+1)} \tag{2.11}$$

When $i + 1 \leq s_m^*$, we have $h_i^{(m)} \leq 0$, which implies from the hypothesis (AR2) that $h_i^{(1)} \leq 0$. Then since $h_i^{(1)} \leq 0$ implies $d_i \leq 0$, we have for $i + 1 \leq s_m^*$,

$$h_{i+1}^{(m+1)} - h_i^{(m+1)} = h_{i+1}^{(1)} - h_i^{(1)} = -d_i/i \geq 0. \tag{2.12}$$

Therefore the first hypothesis holds with m replaced by $m+1$. Now $h_i^{(m+2)}$ can be written as

$$h_i^{(m+2)} = h_i^{(1)} + \sum_{j \geq \max(i+1, s_{m+1}^*)} \frac{1}{j-1} h_j^{(m+1)}. \tag{2.13}$$

Taking the difference the above equation from (3.7),

$$h_i^{(m+2)} - h_i^{(m+1)} \geq \sum_{j \geq \max(i+1, s_{m+1}^*)} \frac{1}{j-1} \{h_j^{(m+1)} - h_j^{(m)}\} \geq 0, \tag{2.14}$$

The first inequality follows from $s_m^* \geq s_{m+1}^*$ and the last one follows from the hypothesis (AR2). Hence the proof is completed.

Poisson, geometric and uniform distributions satisfy the condition (PS) (see Presman and Sonin [11]). As an example we study the uniform distribution in details.

Uniform case: The total number, N of objects is assumed to be uniformly distributed on $[1, N_0]$. Thus for $k = 1, 2, \cdots$, $\delta_k = 1/N_0$ and $\pi_k = (N_0 - k + 1)/N_0$. Then the condition (PS) is easily verified, since $d_i = (1/N_0)(1 - \sum_{j=i+1}^{N_0}(1/j))$, $i = 0, 1, \cdots, N_0$, which is increasing in i. We need another modification. Let $H_i^{(m)} = N_0 h_i^{(m)}$ $(= ((N_0 - i + 1)/i) g_i^{(m)})$ for all m. Then from (2.9)

$$H_i^{(m+1)} = H_i^{(1)} + \sum_{j=\max(i+1, s_m^*)}^{N_0} \frac{1}{j} H_j^{(m)}, \tag{2.15}$$

where $H_i^{(1)} = \sum_{j=1}^{N_0}(1/j)(1 - \sum_{k=j+1}^{N_0}(1/k))$. If we let $i/N_0 \to x$ and write $H^{(m)}(x) = \lim_{N_0} H_i^{(m)}$, where $i = i(N_0)$. and $\bar{s}_m^* = \lim_{N_0} s_m^*/N_0$, a Riemann approximation to the equation (2.15) yields

$$H^{(m+1)}(x) = H^{(1)}(x) + \int_{\max(x, s_m^*)}^{1} \frac{1}{y} H^{(m)}(y) dy, \tag{2.16}$$

where

$$H^{(1)}(x) = -\frac{1}{2}\log^2 x - \log x. \tag{2.17}$$

Since $\bar{s}_m^* (\bar{s}_0^* \equiv 1)$ is unique solution x between 0 and $\bar{s}_{m-1}^*$ of the equation $H^{(m)}(x) = 0$,

$$\bar{s}_m^* = \exp\{-(1 + \sqrt{1 + 2C^{(m)}})\}, \tag{2.18}$$

where $C^{(1)} \equiv 0$ and

$$C^{(m)} = \int_{\bar{s}_{m-1}^*}^{1} \frac{1}{y} H^{(m-1)}(y) dy. \tag{2.19}$$

Therefore we have $\bar{s}_1^* = e^{-2} \approx .135335$ and

$$C^{(2)} = \int_{e^{-2}}^{1} \frac{1}{y}(-\frac{1}{2}\log^2 y - \log y) dy = \frac{2}{3}. \tag{2.20}$$

Then by (2.18),we see $\bar{s}_2^* = e^{-(1+\sqrt{21}/3)} \approx .079856$. Using (2.16) and (2.17), we have

$$H^{(2)}(x) = \begin{cases} -\frac{1}{2}\log^2 x - \log x + \frac{2}{3}, & x \le e^{-2} \\ \frac{1}{6}\log^3 x - \log x, & x \ge e^{-2}. \end{cases}$$

Substituting $H^{(2)}(x)$ into (2.19),

$$C^{(3)} = \int_{\bar{s}_2^*}^{e^{-2}} \frac{1}{y}(-\frac{1}{2}\log^2 y - \log y + \frac{2}{3})dy + \int_{e^{-2}}^{1} \frac{1}{y}(\frac{1}{6}\log^3 y - \log y)dy$$

$$= \frac{1}{3} + \frac{7}{27}\sqrt{21}, \tag{2.21}$$

where we use the relation $-\frac{1}{2}\log^2 \bar{s}_2^* - \log \bar{s}_2^* + \frac{2}{3} = 0$. By (2.18), we have $\bar{s}_3^* = \exp\{-(1 + (\sqrt{35 + 42\sqrt{21}})/9)\} \approx .04951742$.

Corollary 1. When the total number objects has a uniform distribution on $[1, N_0]$, *the limiting maximum probability of obtaining the best object under the optimal rule for the problem with m selections is given by* $-(\bar{s}_1^* \log \bar{s}_1^* + \bar{s}_2^* \log \bar{s}_2^* + \cdots + \bar{s}_m^* \log \bar{s}_m^*)$.

Proof. Let $v_i^{(m)} = ((N_0 - i + 1)/i)V_i^{(m)}$ and $u_i^{(m)} = ((N_0 - i + 1)/i)U_i^{(m)}$, then we have

$$v_i^{(m)} = \sum_{j=i+1}^{N_0} \frac{1}{j-1} \max\{u_j^{(m)}, v_j^{(m)}\} \text{ and } u_i^{(m)} = \sum_{j=i}^{N_0} \frac{1}{j} + v_i^{(m-1)}. \tag{2.22}$$

The optimal rule with a threshold s_m^* gives

$$v_i^{(m)} = \begin{cases} \sum_{j=i+1}^{s_m^*-1} \frac{1}{j-1} v_j^{(m)} + \sum_{j=s_m^*}^{N_0} \frac{1}{j} u_j^{(m)}, & i \le s_m^* - 1, \\ \sum_{j=s_m^*}^{N_0} \frac{1}{j} u_j^{(m)}, & i \ge s_m^* - 1. \end{cases}$$

Then we have the following relation

$$v_1^{(m)} = 2v_2^{(m)} = 3v_3^{(m)} = \cdots = (s_m^* - 1)v_{s_m^*-1}^{(m)}. \tag{2.23}$$

Thus the maximum probability is given by

$$V_1^{(m)} = \frac{1}{N_0} v_1^{(m)} = \frac{s_m^* - 1}{N_0} v_{s_m^*-1}^{(m)}. \tag{2.24}$$

If we let $i/N_0 \to x$ and write $v^{(m)}(x), u^{(m)}(x)$ and $\bar{s}_m^*$ as $\lim_{N_0} v_{i/N_0}^{(m)}, \lim_{N_0} u_{i/N_0}^{(m)}$ and $\lim_{N_0} s_m^*/N_0$, we have

$$v^{(m)}(x) = \begin{cases} \int_x^{\bar{s}_m^*} \frac{1}{y} v^{(m)}(y)dy + \int_{\bar{s}_m^*}^1 \frac{1}{y} u^{(m)}(y)dy, & x \le \bar{s}_m^*, \\ \int_{\bar{s}_m^*}^1 \frac{1}{y} u^{(m)}(y)dy, & x \ge \bar{s}_m^*, \end{cases}$$

where

$$u^{(m)}(x) = \int_x^1 \frac{1}{y} dy + v^{(m-1)}(x).$$

From (2.24), the limiting probability is given by $\bar{s}^*_m v^{(m)}(\bar{s}^*_m)(\equiv a^{(m)}, a^{(0)} \equiv 0)$. Thus

$$(2.25) \qquad a^{(m)} = \bar{s}^*_m \int_{\bar{s}^*_m}^1 \frac{1}{y} u^{(m)}(y)dy.$$

On the other hand, $\bar{s}^*_m$ satisfies the equation

$$(2.26) \qquad \int_{\bar{s}^*_m}^1 \frac{1}{y} dy + v^{(m-1)}(\bar{s}^*_m) - \int_{\bar{s}^*_m}^1 \frac{1}{y} u^{(m)}(y)dy = 0.$$

Now we know from (2.23) that $v^{(m-1)}(0+) = xv^{(m-1)}(x)$ for $x \in (0, \bar{s}^*_{m-1}]$. Hence

$$(2.27) \qquad a^{(m-1)} = \bar{s}^*_{m-1} v^{(m-1)}(\bar{s}^*_{m-1}) = \bar{s}^*_m v^{(m-1)}(\bar{s}^*_m).$$

Substituting (2.25) into (2.26) and using (2.27),

$$a^{(m)} = a^{(m-1)} - \bar{s}^*_m \log \bar{s}^*_m,$$

which yields the desired result.

From this corollary, as $N_0 \to \infty$ we see the maximum probabilities $W_1^{(1)} \to .270670$, $W_1^{(2)} \to .472509$, and $W_1^{(3)} \to .621329$ for the problem with one, two and three selections respectively.

3. Poisson arrival model

Let $\tau_1, \tau_2, \cdots$ denote the arrival times of a Poisson process in chronological order and let $\{N(t)\}_{t \geq 0}$ be the corresponding counting process. For the unknown intensity λ of the process, we suppose a prior exponential distribution, $a \exp\{-a\lambda\} I(\lambda > 0)$ where a is a known nonnegative parameter. Bruss [12] has succeeded to show that the optimal stopping rule which maximizes the probability of obtaining the best object in the given time interval $[0, T]$ with single selection is to accept (if possible) the relatively best object after time $(T+a)/e - a$. Here we consider the Bruss's problem with multiple selections. As is shown in Bruss, the posterior distribution of $N(T)$ generated by $\tau_1, \cdots, \tau_i$ only depends on the values of i and τ_i and equals negative binomial distribution with parameters $(i, (s+a)/(T+a))$, that is, for $0 \leq s \leq T$,

$$\begin{aligned} P(N(T) = n | \tau_1 = t_1, \cdots, \tau_{i-1} = t_{i-1}, \tau_i = s) &= P(N(T) = n | \tau_i = s) \\ (3.1) \qquad &= \binom{n}{i} \left(\frac{s+a}{T+a}\right)^{i+1} \left(1 - \frac{s+a}{T+a}\right)^{n-i}. \end{aligned}$$

Let $W_i^{(m)}(s)$ denote the maximum probability of obtaining the best object when we confront the relatively best object which is ith object arriving at time s $(0 < s \leq T)$ and we can select more m (≥ 1) objects hereafter. Similarly if m more selections are allowed, let

$U_i^{(m)}(s)$ ($V_i^{(m)}(s)$) be the corresponding probability when we accept (reject) the relatively best object which is jth object arriving at time s. Using Bruss's result we have

$$U_i^{(m)}(s) = \sum_{n\geq i}(i/n)P(N(T) = n|\tau_i = s) + V_i^{(m-1)}(s)$$

$$= \frac{s+a}{T+a} + V_i^{(m-1)}(s). \tag{3.2}$$

Denote the transition probability given prior exponential distribution that $(i+k)$th object arriving at time $s+u$ is the first relatively best object after ith object which is the relatively best arrived at time s by $p_{(i,s)}^{(k,u)}$, then we have

$$V_i^{(m)}(s) = \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} W_{i+k}^{(m)}(s+u)du \tag{3.3}$$

and for $k \geq 1,\ 0 < u < T - s$,

$$p_{(i,s)}^{(k,u)} = \int_0^\infty \frac{\lambda e^{-\lambda u}(\lambda u)^{k-1}}{(k-1)!} \frac{i}{(i+k-1)(i+k)} \frac{e^{-\lambda(s+a)}(\lambda)^i(s+a)^{i+1}}{i!} d\lambda$$

$$= \frac{s+a}{(s+a+u)^2}\binom{i+k-2}{k-1}(\frac{s+a}{s+a+u})^i(\frac{u}{s+a+u})^{k-1}, \tag{3.4}$$

where we apply the equation $\int_0^\infty \lambda^{k+i}\exp\{-\lambda(s+a+u)\}d\lambda = \Gamma(k+i+1)/(s+a+u)^{k+i+1}$ to the right hand side of the first equation above. Then we have the dynamic programming equation for $i, m \geq 1, 0 < s \leq T$,

$$W_i^{(m)}(s) = \max\{U_i^{(m)}(s), V_i^{(m)}(s)\}, \tag{3.5}$$

with boundary conditions $W_i^{(m)}(T) = 1$ for $i, m \geq 1$ and $W_i^{(0)}(s) = 0$ for all i and s. Let $g_i^{(m)}(s)$ be the *one-stage look-ahead function*, that is,

$$g_i^{(m)}(s) \equiv U_i^{(m)}(s) - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} U_{i+k}^{(m)}(s+u)du$$

$$= \frac{s+a}{T+a} - \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)}(\frac{s+a+u}{T+a})du$$

$$+ \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)}\{W_{i+k}^{(m-1)}(s+u) - V_{i+k}^{(m-1)}(s+u)\}du$$

$$= (\frac{s+a}{T+a})\{1 + \log(\frac{s+a}{T+a})\}$$

$$+ \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)}\{W_{i+k}^{(m-1)}(s+u) - V_{i+k}^{(m-1)}(s+u)\}du, \tag{3.6}$$

where we use $\sum_{k\geq 1} p_{(i,s)}^{(k,u)} = (s+a)/(s+a+u)^2$ (independently of i), since $p_{(i,s)}^{(k,u)} = (s+a)/(s+a+u)^2 \times \{$negative binomial distribution with parameters $(k,\ u/(s+a+u))\}$.

Theorem 2. The optimal rule for the problem with random arrivals on $[0,T]$ following a Poisson process at intensity $\lambda > 0$ having an exponential distribution with rate parameter $a \geq 0$ when we can select m more objects hereafter is to accept(if possible) the first relatively best object after time $s_m^* = (T+a)/e^{C^{(m)}} - a$ ($s_0^* \equiv T$), where $C^{(m)}$ is constant.

Proof. Let $h_i^{(m)}(s) = ((T+a)/(s+a))g_i^{(m)}(s)$. As induction hypotheses, we assume that $h_i^{(m)}(s)$ is independent of i and for fixed m

$$(AP1) \qquad h^{(m)}(s) \geq 0 \Rightarrow h^{(m)}(s+u) \geq 0 \quad \text{for} \quad u \in [0, T-s],$$

$h^{(m)}(s)$ for $s \in (0, s_{m-1}^*]$ has the following form,

$$(AP2) \qquad h^{(m)}(s) = C^{(m)} + \log(\frac{s+a}{T+a}),$$

where $C^{(m)}$ is constant, and for all m

$$(AP3) \qquad h^{(m+1)}(s) \geq h^{(m)}(s).$$

From these hypotheses, we have $s_m^* = \inf\{0 < s \leq s_{m-1}^* : h^{(m)}(s) \geq 0\} = (T+a)/e^{C^{(m)}} - a$ and

$$\begin{aligned} W_{i+k}^{(m)}(s+u) - V_{i+k}^{(m)}(s+u) &= g^{(m)}(s+u)I(s+u \geq s_m^*), \\ &= (\frac{s+u+a}{T+a})h^{(m)}(s+u)I(s+u \geq s_m^*), \end{aligned} \tag{3.7}$$

which follows when $s+u \geq s_m^*$,

$$W_{i+k}^{(m)}(s+u) = U_{i+k}^{(m)}(s+u),\ V_{i+k}^{(m)}(s+u) = \int_0^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} U_{i+k}^{(m)}(s+u)du.$$

Substituting (3.7) into (3.6),

$$\begin{aligned} h_i^{(m+1)}(s) &= h^{(1)}(s) + (\frac{T+a}{s+a})\int_{(s_m^*-s)^+}^{T-s} \sum_{k\geq 1} p_{(i,s)}^{(k,u)} h^{(m)}(s+u)du \\ &= h^{(1)}(s) + (\frac{T+a}{s+a})\int_{(s_m^*-s)^+}^{T-s} \frac{s+a}{(s+u+a)^2}\frac{s+u+a}{T+a}h^{(m)}(s+u)du \\ &= h^{(1)}(s) + \int_{(s_m^*-s)^+}^{T-s} \frac{1}{s+u+a} h^{(m)}(s+u)du (\equiv h^{(m+1)}(s)), \end{aligned} \tag{3.8}$$

being independently i, where

$$h^{(1)}(s) = 1 + \log(\frac{s+a}{T+a}), \tag{3.9}$$

which is increasing in s. Therefore $h^{(1)}(s)$ satisfies the hypotheses (AP1) and (AP2) with $C^{(1)} \equiv 1$. Since $h^{(1)}(s)$ is non-negative for $s \geq s_1^*$, by virtue of (3.8)

$$h^{(2)}(s) - h^{(1)}(s) = \int_{(s_1^*-s)^+}^{T-s} \frac{1}{s+u+a} h^{(1)}(s+u)du \geq 0.$$

Thus the hypothesis (AP3) holds for $m = 1$.

To complete the induction, we shall show that these hypotheses hold for m replaced by $m+1$. Recalling (3.8), for $s \leq s_m^* = (T+a)/e^{C^{(m)}} - a$

$$\begin{aligned} h^{(m+1)}(s) &= h^{(1)}(s) + \int_{(T+a)/e^{C^{(m)}}-a-s}^{T-s} \frac{1}{s+u+a} h^{(m)}(s+u)du \\ &= \log(\frac{s+a}{T+a}) + C^{(m+1)}, \end{aligned} \tag{3.10}$$

where

$$C^{(m+1)} = 1 + \int_{e^{-C^{(m)}}}^{1} \frac{1}{v} h^{(m)}((T+a)v - a)dv, \tag{3.11}$$

where we change the variable from $(s+u+a)/(T+a)$ to v in the integrand in (3.10). (3.10) states (AP2) holds with m replaced by $m+1$. Now we see $h^{(m+1)}(s)$ is increasing in $s \in (0, s_m^*]$. On the other hand, for $s \in [s_m^*, T]$, $h^{(m+1)}(s)$ is non-negative because by the hypothesis (AP3)

$$0 \leq h^{(m)}(s) \leq h^{(m+1)}(s).$$

Hence we have

$$h^{(m+1)}(s) \geq 0 \Rightarrow h^{(m+1)}(s+u) \geq 0 \quad \text{for} \quad u \in [0, T-s], \tag{3.12}$$

which states (AP1) holds with m replaced by $m+1$. Now $h^{(m+2)}(s)$ can be written as

$$h^{(m+2)}(s) = h^{(1)}(s) + \int_{(s_{m+1}^*-s)^+}^{T-s} \frac{1}{s+u+a} h^{(m+1)}(s+u)du. \tag{3.13}$$

Taking the difference the above equation from (3.8)

$$h^{(m+2)}(s) - h^{(m+1)}(s) \geq \int_{(s_{m+1}^*-s)^+}^{T-s} \frac{1}{s+u+a} \{h^{(m+1)}(s+u) - h^{(m)}(s+u)\}du \geq 0,$$

where the first inequality comes from $s^*_m \geq s^*_{m+1}$ and the second one comes from the hypothesis (AP3). Thus (AP3) holds for all m and the proof completes.

As shown in the proof, $s^*_1 = (T+a)/e - a$. From (3.11),

$$C^{(2)} = 1 + \int_{e^{-1}}^{1} \frac{1}{v} h^{(1)}((T+a)v - a)dv = 1 + \int_{e^{-1}}^{1} \frac{1}{v}\{1 + \log v\}dv = 1 + \frac{1}{2}. \tag{3.14}$$

Then $s^*_2 = (T+a)/e^{3/2} - a$. By virtue of (3.8),

$$h^{(2)}(s) = \begin{cases} \frac{3}{2} + \log(\frac{s+a}{T+a}), & 0 < s \leq s^*_2, \\ 1 - \frac{1}{2}\log^2(\frac{s+a}{T+a}), & s^*_2 \leq s \leq T. \end{cases}$$

Substituting the above into (3.11), for $s \leq s^*_2$

$$C^{(3)} = 1 + \int_{e^{-3/2}}^{e^{-1}} \frac{1}{v}(\frac{3}{2} + \log v)dv + \int_{e^{-1}}^{1} \frac{1}{v}(1 - \frac{1}{2}\log^2 v)dv = 1 + \frac{23}{24}. \tag{3.15}$$

Then $s^*_3 = (T+a)/e^{47/24} - a$.

For $a = 0$, it is of interest to compare the values $s^*_1 = T/e \approx .367879T$, $s^*_2 = T/e^{3/2} \approx .22313T$, $s^*_3 = T/e^{47/24} \approx .141093T$, with the threshold values $n/e \approx .367879n$, $n/e^{3/2} \approx .22313n$, $n/e^{47/24} \approx .141093n$, of the no-information case.

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