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Skeletons of some relatives of the $n$-cube

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Abstract

We study the skeleton of several polytopes related to the $n$-cube, the halved $n$-cube, and the folded $n$-cube. In particular, the Gale polytope of the $n$-cube, its dual and the duals of the halved $n$-cube and the complete bipartite subgraphs polytope.

1 Introduction

The general references are [2, 6, 12] for polytopes, [4] for graphs and [5] for lattices. We first recall some basic properties of the cube and the halved cube.

The vertices of the $n$-cube $\gamma_n = [0,1]^n$ are all the $2^n$ characteristic vectors $\chi^S$ for $S \subset N = \{1,2,\ldots,n\}$, that is, $\chi_i^S = 1$ for $i \in S$ and 0 otherwise. With $|S\Delta S'|$ denoting the size of the symmetric difference of the subsets $S$ and $S'$, two vertices $\chi^S$ and $\chi^{S'}$ are adjacent if and only if $|S\Delta S'| = 1$. The skeleton of $\gamma_n$ is denoted by $H(n,2)$ and the skeleton of its dual, the cross-polytope $\beta_n = \gamma_n^*$, is $K_{2\times n}$, which is also called the Cocktail-Party graph. The diameter of the $n$-cube and its dual are, respectively, $n$ and 2.

The halved $n$-cube $h\gamma_n$ (see Section 8.6 of [6]) is obtained from the $n$-cube $\gamma_n$ by selecting the vertex of even cardinality on each edge, that is, $h\gamma_n$ is the convex hull of all the $2^{n-1}$ characteristic vectors $\chi^S$ for $S \subset N = \{1,2,\ldots,n\}$ and $|S|$ even. Two vertices $\chi^S$ and $\chi^{S'}$ are adjacent if and only if $|S\Delta S'| = 2$. The skeleton of the halved $n$-cube is denoted by $\frac{1}{2}H(n,2)$; its diameter is $\lceil \frac{n}{2}\rceil$. 
2 Skeleton of the dual halved \(n\)-cube

The halved 3-cube is a regular tetrahedron \(\alpha_3\). The halved 4-cube is the simplicial polytope \(h\gamma_4 = \beta_4\). For \(n > 4\), the facets of \(h\gamma_n\)-cube are partitioned into the following two orbits of its symmetry group \(2^{n-1}\text{Sym}(n)\). The orbit \(O_1^n\) consists of the 2\(n\) facets belonging to the facets of the \(n\)-cube and defined by the inequalities:

\[
x_i \leq 1 \quad \text{for } i \in N, \\
x_i \geq 0 \quad \text{for } i \in N.
\]

The orbit \(O_2^n\) consists of the 2\(n-1\) facets cutting off the vertices of odd cardinality from the \(n\)-cube and defined by the inequalities:

\[
\sum_{i=1}^{n} x_i(1 - 2\chi_{i}^{A}) \leq |A| - 1 \quad \text{for } A \subset N \text{ and } |A| \text{ odd.}
\]

The facets defined by the inequalities (1), (2) and (3) are respectively denoted by \(F_{1}^{i}\), \(F_{0}^{i}\) and \(F^{A}\). Since the symmetries of a polytope preserve adjacency and linear independence, we can describe the properties of its facets by simply considering a representative facet of each orbit. The facets \(F_{1}^{i} \simeq F_{0}^{i} \simeq h\gamma_{n-1}\) (here and in the following " \(\simeq\) " denotes the affine equivalency) and each facet \(F^{A}\) is the simplex containing the \(n\) vertices: \(\chi_{i}^{A \cup \{i\}}\) for \(i \in \bar{A}\) and \(\chi_{i}^{A \setminus \{i\}}\) for \(i \in A\).

The skeleton of the dual halved \(n\)-cube, denoted by \(h\gamma_n^*\), is the graph whose nodes are the facets of \(h\gamma_n\), two facets being adjacent if and only if their intersection is a face of codimension 2. This skeleton is given below.

**Lemma 2.1** The facets of \(O_1^n\) and \(O_2^n\) form, respectively, the coclique \(K_{2n}\), and the coclique \(K_{{2n-1}}\); each facet \(F^{A}\) is adjacent, either to \(F_{1}^{i}\) if \(i \in A\), or to \(F_{0}^{i}\) if \(i \in \bar{A}\) for each \(i \in N\).

**Corollary 2.2** For \(n \geq 4\), the skeleton of the dual halved \(n\)-cube is a bipartite graph of diameter 4.

**Proof.** Since the valency of a facet belonging to \(O_1^n\), respectively to \(O_2^n\), is half the size of \(O_2^n\), respectively of \(O_1^n\), we have \(\delta(h\gamma_n^*) \leq 4\). On the other hand, the facets \(F_{1}^{i}\) and \(F_{0}^{i}\), having no common neighbour, we get \(\delta(h\gamma_n^*) > 3\).

**Corollary 2.3** The halved \(n\)-cube has \(n 2^{n-2}\) faces of codimension 2 which are all simplices, that is \(h\gamma_n\) is quasi-simplicial. For \(n \to \infty\), \(h\gamma_n\) is asymptotically simplicial.
3 Gale transform of the $n$-cube

Let $A$ be a $(2^n - n - 1) \times 2^n$ matrix which rows form a basis for the space of all the affine dependencies on the vertices of the $n$-cube. A Gale transform of $\gamma_n$ is the collection of the $2^n$ points in $\mathbb{R}^{2^n-n-1}$ which are the columns of $A$.

We consider the matrix $A$ induced by the following $2^n - n - 1$ affine dependencies on the vertices of $\gamma_n$:

\[(1 - |T|)\chi^\emptyset + \sum_{i \in T} \chi^{\{i\}} - \chi^T = 0 \quad \text{for } T \subset N \text{ and } |T| \geq 2. \tag{4}\]

Since each column of $A$ corresponds to a vertex $\chi^S$ of $\gamma_n$ for $S \subset N$, we simply denote by $v^S$ the vector formed by this column of $A$. For example, the first column of $A$ corresponds to $\chi^\emptyset$ and forms the vector $v^\emptyset = (1 - |T|)$, where $\mathbb{R}^{2^n-n-1}$ is naturally indexed by $T \subset N$, $|T| \geq 2$.

A *Gale polytope, Gale($P$)*, of a polytope $P$ is the convex hull of a Gale transform of $P$. In the following we consider Gale($\gamma_n$) associated to the affine dependencies (4). The polytope Gale($\gamma_n$) is a prism over a tetrahedron; see also Example 5.6 in [3] for relation with Lawrence polytopes. For $n \geq 4$, we introduce some edges and facets of Gale($\gamma_n$) in order to compute its diameter and the one of its dual.

Consider the following inequalities, where $x_T$ for $T \subset N$ and $|T| \geq 2$ are the coordinates of a point $x$ in $\mathbb{R}^{2^n-n-1}$ indexed by $T \subset N$, $|T| \geq 2$.

\[-x_A \leq 1 \quad \text{for } |A| = 2, \tag{e_1}\]
\[x_{A \setminus \{i\}} - x_A \leq 1 \quad \text{for } |A| \geq 3 \text{ and } i \in A, \tag{e_2}\]
\[x_A \leq 1 \quad \text{for } |A| = 2, \tag{e_3}\]
\[x_{A \cup \{i\}} - x_A \leq 1 \quad \text{for } |A| \geq 2 \text{ and } i \notin A, \tag{e_4}\]
\[2 \sum_{j \in N} x_{\{j\}} - 2x_{\{i\}} + (n-1)(x_N-1) \leq 0 \quad \text{for } i \in N, \tag{e_5}\]
\[\sum_{|T| \geq 2} x_T - 2^n(x_A + x_B) \leq 2^n - 1 \quad \text{for } |A|, |B| \geq 2 \text{ and } 2(|A| + |B|) \leq n + 3. \tag{e_6}\]
One can easily check that each of those inequalities induces an edge of $Gale(\gamma_n)$. More precisely, $(e_1)$ and $(e_2)$ induce the edges $[v^\emptyset, v^A]$ for $|A| \geq 2$, $(e_3), (e_4)$ and $(e_5)$ induce the edges $[v^i, v^A]$ for $|A| \geq 1$ and $i \not\in A$ or $A = N$ and $(e_6)$ induce the edges $[v^A, v^B]$ for $|A|, |B| \geq 2$ and $2(|A| + |B|) \leq n + 3$.

**Property 3.1** The diameter of $Gale(\gamma_n)$ is at most 2. Moreover, $\delta(Gale(\gamma_3)) = 2$ and $\delta(Gale(\gamma_4)) = 1$.

**Proof.** The vertices $v^\emptyset$ and $v^A$ are respectively linked by the edges $[v^\emptyset, v^N]$ and $[v^N, v^A]$ for $|A| = 1$ and by the edge $[v^\emptyset, v^A]$ for $|A| \geq 2$. The vertices $v^i$ and $v^j$ always form an edge, $v^i$ and $v^A$ are linked by $[v^i, v^j]$ and $[v^j, v^A]$ with $j \not\in A$, for $2 \leq |A| \leq n - 1$, and $[v^i, v^N]$ form an edge. Finally, the vertices $v^A$ and $v^B$ are linked by the edges $[v^A, v^\emptyset]$ and $[v^\emptyset, v^B]$ for $|A|, |B| \geq 2$.

We then consider the following $2^{n-1}$ inequalities.

$$2^{n-1}x_A - \sum_{|T| \geq 2} x_T \leq 1$$

for $A \subset N$ and $|A| \leq 1$,

$$2^{n-1}(x_A + x_{\overline{A}}) - \sum_{|T| \geq 2} x_T \leq 1$$

for $A \subset N$ and $2 \leq |A| \leq n - 1$.

One can easily check that each of those inequalities induces a facet $G^A$ of $Gale(\gamma_n)$ for $A \subset N$ and $|A| \leq n - 1$. Since each facet $G^A$ contains all vertices except the pair $\{v^S, v^S\}$, we call them the huge facets.

**Lemma 3.2** The huge facets form the clique $K_{2^{n-1}}$ in the skeleton of $Gale^*(\gamma_n)$.

**Proof.** Let us first consider $g = G^A \cap G^B$ with $A, B \subset N$ and $2 \leq |A|, |B| \leq n - 1$. The face $g$ contains all the vertices of $Gale(\gamma_n)$ except $\{v^A, v^A, v^B, v^B\}$. We show that $g$ is of codimension 2 by exhibiting a family $V$ of $2^n - n - 2$ affinely independent vertices belonging to $g$, this will imply that $G^A$ and $G^B$ are adjacent. Namely, $V$ is formed by the vertices $v^S$ with $S \not\subset \{A, \overline{A}, B, \overline{B}\}$ and $|S| \geq 2$ and the vertices $\{v^i, v^j\}$ with $1 \leq i < j \leq n$ such that $v^i_A = v^i_B = 1$ and $v^j_B = v^j_A = 0$. In the case $0 \leq |A|, |B| \leq 1$, $V$ is formed by the vertices $v^S$ with $S \not\subset \{A, \overline{B}\}$ and $|S| \geq 2$. Finally, in the case $0 \leq |A| \leq 1$ and $2 \leq |B| \leq n - 1$, $V$ is formed by the vertices $v^S$ with $S \not\subset \{\overline{A}, B, \overline{B}\}$ and $|S| \geq 2$ and the vertex $v^\emptyset$.

**Property 3.3** The huge facets form a dominating clique in the skeleton of $Gale^*(\gamma_n)$.

**Proof.** Since the pairs $\{v^S, v^S\}$ form a partition of all the vertices of $Gale(\gamma_n)$, for any facet $F$, at least one huge facet $G^A$ satisfies $|G^A \cap F| = |F| - 1$. This implies that $G^A$ is adjacent to $F$; in other words, the huge facets form a dominating clique.

\(\square\)
Corollary 3.4 The diameter of $Gale^*(\gamma_n)$ is at most 3. Moreover, it is 2 for $n = 3, 4$.

Conjecture 3.5 For $n \geq 4$, the diameters of the Gale polytope of the $n$-cube and of its dual are 1 and 2, respectively.

4 Complete bipartite subgraphs polytope

We recall that the folded $n$-cube $\square_n$ is the graph whose vertices are the $2^{n-1}$ partitions of $N = \{1, \ldots, n\}$ into two subsets, $S$ and $\bar{S}$; two partitions being adjacent when their common refinement contains a singleton. In particular, $\square_4 = K_{4,4}$ and $\square_5 = \frac{1}{2}H(5,2)$, also called the Clebsch graph.

The complete bipartite subgraphs polytope $c_n$, which is also called the cut polytope of the complete graph, is a relative of the folded $n$-cube. More precisely, the vertices of $c_n$ are the $2^{n-1}$ incidence vectors $\delta(S)$ in $IR^\left(\binom{n}{2}\right)$ of the partitions of $N$, that is, $\delta(S)_{ij} = 1$ if exactly one of $i, j$ is in $S$ and 0 otherwise for $1 \leq i < j \leq n$. It is easy to check that the squared Euclidian distance between two partitions, seen as vertices of $c_n$, is $d(n - d)$, where $d$ is their path distance, in the graph $\square_n$. Now, $c_3 = h\gamma_3 = \alpha_3$ and $c_4$ is combinatorially equivalent to the simplicial 6-dimensional cyclic polytope with 8 vertices. The symmetry group of $c_n$ is isomorphic to the automorphism group of $\square_n$, see [10]. See [11] for a detailed treatment of $c_n$.

The skeleton of $c_n$ is the clique $K_{2^n-1}$, see [1]. The determination of all the facets of $c_n$ for large $n$ seems to be hopeless, but a wide range of facets has been already found (including all for $n \leq 7$). It seems that the huge majority of them are simplices for large $n$, that is, $c_n$ is asymptotically simplicial, as well as $h\gamma_n$. In [7] it was conjectured (and proved for $n \leq 7$) that $\delta(c_4^*) \leq 4$; moreover, $\delta(c_4^*) = \delta(c_5^*) = 2$ and $\delta(c_5^*) = 3$. Actually, the skeleton of $c_4^*$ is the line graph of the folded 4-cube.

Remark 4.1 Using the basis of the space of affine dependencies on $c_5$ given in [8], we found by computer that $Gale(c_5) \simeq h\gamma_5$; recall that $\overline{\gamma_5} = \frac{1}{2}H(5,2)$. Clearly, $Gale(h\gamma_4) \simeq \alpha_3$ and $Gale(h\gamma_5) \simeq c_5$; more generally, for $n$ odd, $Gale(h\gamma_n)$ can be obtained from the following basis of $2^{n-1} - n - 1$ affine dependencies:

$$(n - 1) \sum_{i \in X} x_{N\setminus \{i\}} - |A| \sum_{i \in N} x_{N\setminus \{i\}} + (n - 1)x_A = 0 \text{ for } |A| \text{ even, } 2 \leq |A| \leq n - 2.$$
Finally, we mention $\text{cont}_m$, the contact polytope of the lattice $\mathbb{Z}(V_m)$ in $\mathbb{R}^n$ studied in [9], where $V_m$ denotes the set of vertices of $c_m$, that is, $\text{cont}_m$ is the convex hull of all vectors of this lattice having the minimal length $\mu = \min(4, m-1)$. Clearly, it comes from the construction $A$ given in Chapters 5, 7 of [5] with $V_m$ seen as a linear binary code with $n = \binom{m}{2}$, $M = 2^{m-1}$ and $d = m - 1$. We have,

- $\text{cont}_2 = \text{conv}\{\pm e_1\} = \beta_1$ and $\mathbb{Z}(V_2) = \mathbb{Z} = A_1$,
- $\text{cont}_3 = \text{conv}\{\pm e_i \pm e_j : 1 \leq i \neq j \leq 3\}$ is the cubo-octahedron (the vertices of this Archimedean solid are the midpoints of the edges of $\gamma_3$) and $\mathbb{Z}(V_3)$ is the face-centered cube lattice $A_3 \cong D_3$,
- $\text{cont}_4 = \text{conv}\{\pm \delta(i), \pm \delta(i) - 2e_{ij} : 1 \leq i \neq j \leq 4\} \simeq h\gamma_6$,
- $\text{cont}_5$ is a 10-polytope with the following 100 vertices: $\{\pm 2e_{ij} : 1 \leq i \leq j \leq 5\} \cup \{\delta(i) - 2\sum_{\{jk\} \in X} e_{jk} : 1 \leq i \leq 5, X \subset E(K_{\{1,2,3,4,5\}-i})\}$. So, $\text{cont}_5$ is the union of $2\beta_{10}$ and five 4-cubes $\gamma_4$, this polytope has 4 624 facets divided into 4 orbits of its symmetry group $2^5\text{Sym}(5)$, moreover, the orbit formed by the 384 facets equivalent to the one induced by the inequality $\sum_{\{ij\} \in C_{1,2,3,4,5}} x_{ij} \leq 2$ forms a dominating set in the skeleton of $\text{cont}_5^*$,
- for $m \geq 6$, $\text{cont}_m = \text{conv}\{\pm 2e_{ij} : 1 \leq i \leq j \leq m\} \simeq \beta_{\binom{m}{2}}$.

So, the kissing number of the lattice, that is the number of vertices of $\text{cont}_m$, is $\tau = 2, 12, 32, 100, m(m - 1)$ for $m = 2, 3, 4, 5, \geq 6$.

Figure 4.1: The contact polytope of $\mathbb{Z}(V_3)$ is a cubo-octahedron
References


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