# Zeros and Poles of Linear Continuous-Time Periodic Systems: Definitions and Properties

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Abstract—The paper deals with definitions of zeros and poles and their features in finite-dimensional linear continuous-time periodic (FDLCP) systems under a harmonic framework. More precisely, system and transfer zeros and poles in the harmonic wave-to-wave sense are defined on what we call the regularized harmonic system operators and the harmonic transfer operators of FDLCP systems by means of regularized determinants; then their composition and properties related to system structures are examined via the Floquet theory and controllability/observability decompositions of FDLCP systems. The study shows that under mild assumptions, the harmonic transfer operators of FDLCP systems are analytic and meromorphic, on which zeros and poles are well-defined. Basic zero/pole relationships are established, which are similar to their linear time-invariant counterparts and in particular explicate some interesting harmonic wave-to-wave behaviors of FDLCP systems. The results are significant in analysis and synthesis of FDLCP systems when the harmonic approach is adopted.

*Index Terms*—Continuous-time periodic system, Floquet factorization, harmonic transfer operator, zero and pole.

#### I. INTRODUCTION

T HE paper is devoted to studying definitions of zeros and poles and their composition properties in finite-dimensional linear continuous-time periodic (FDLCP) systems under a harmonic framework that admit state-space differential equation descriptions given by

$$\begin{cases} \dot{x} = A(t)x + B(t)u\\ y = C(t)x + D(t)u \end{cases}$$
(1)

where A(t), B(t), C(t), and D(t) are, respectively,  $n \times n$ ,  $n \times m$ ,  $l \times n$ , and  $l \times m$  h-periodically time-varying. The dimension subscripts will be suppressed whenever no confusion is caused. FDLCP systems constitute a big class of practical control systems, among which stabilization of helicopter rotors and rolling ships, and reduction of electro-mechanical oscillations in synchronous generators [1], [8], [9], [15], [28] are representatives, among many others.

The Floquet theorem [9], [22], [24] says in the FDLCP system (1) that if A(t) is piecewise continuous, its transition matrix  $\Phi(t,0)$  always possesses a Floquet factorization  $\Phi(t,0) = P(t,0)e^{Qt}$ , where P(t,0) is absolutely continuous, nonsingular, and *h*-periodic with respect to time *t*, and *Q* is constant but probably complex. Floquet factorizations will be frequently used in the discussion.

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## A. Retrospect to Previous Work

Zeros and poles are essential in the control theory and play an important role in describing structures of dynamic systems and connecting frequency-domain features of system modelings with their time-domain behaviours. This is especially true in analysis and synthesis for linear time-invariant (LTI) systems [23], [30], [31]; for example, controllability/observability decomposition, robust stablization (pole assignment), LQR problem [19],  $H_2/H_{\infty}$  performance [14], [45] and linear matrix inequalities. More than a dozen of zeros and poles definitions in LTI systems can be found [10], [31], among which transmission zeros and poles of transfer functions, invariant zeros, input (respectively, output and input/output) decoupling zeros, blocking zeros are most frequently mentioned. One can get a general picture from the survey papers of [23] and [31].

Significance of zeros and poles in periodically time-varying systems is almost the same as that in LTI systems [18], [20]. In the latest two decades, numerous efforts have been made in defining zeros and poles and determining their features in linear/nonlinear time-varying systems. As a matter of fact, most of the efforts are devoted systematically to linear periodic time-varying systems such as sampled-data and discrete-time periodic [3], [13], [16], [26], [36]. In comparison, zeros and poles and their characteristics in FDLCP systems are attacked only in scattering reports [6], [21], [27], [35], [38], [42].

#### B. Harmonic Framework and Motivation

Recently, a harmonic framework is adopted to establish the so-called harmonic transfer operators in FDLCP systems [37], [38]. Relevant results are reported in [33], [34] for general linear time-periodic systems. Existence conditions and important properties of the harmonic transfer operators are thoroughly explicated in [39], which have brought in fruitful results [40], [41], [43]. Basic spectral characteristics of the so-called harmonic state operator, or implicitly aspects of the system poles in FDLCP systems, are considered in [42]. All these results allude to possibility and necessity of zeros and poles under the harmonic framework.

It is well known in multivariable systems [23], [31], [44] that subsystem zeros and poles form a group of hierarchical relationships; namely, common zeros among all subsystems are zeros of the system as a whole, while a pole of the system must be a pole in some subsystems. In the FDLCP setting, one faces a similar situation when FDLCP systems are approximated by truncating harmonics as one can see from stabilization of an FDLCP example system through pole assignment in Section V, which results in subsystems in infinite-dimensional spaces. From these

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observations, it is imperative for us to define zeros and poles in between harmonics of FDLCP systems.

#### C. Contributions and Organization

This paper examines definitions and characteristics related to zeros and poles in FDLCP systems under a harmonic framework by means of the regularized determinant technique. More precisely, zeros and poles in the harmonic wave-to-wave sense are defined in FDLCP systems, and basic properties are examined carefully. In the light of zeros and poles of LTI continuous-time systems, the study deepens our perception about the harmonic modelings of FDLCP systems and it is a harmonic and structural explanation for dynamics of FDLCP systems. The results could be helpful in exploiting the LTI analysis and synthesis techniques in the FDLCP field.

Now we outline the paper. Section II lists preliminaries for our discussions. Sections III and IV are the main context. In particular, Section III introduces zeros and poles in FDLCP systems and examines their existence, while their compositions and basic properties are attacked in Section IV. Examples are given in Section V to illustrate the main results. Notes and remarks are included in Section VI. To keep the arguments concise, all the proofs are given as Appendices.

#### **II.** PRELIMINARIES

In this section, we first list notations, and basic points on the Hilbert-Schmidt operators and the regularized determinants for our later arguments. Next we review facts for controllability/observability decompositions in FDLCP systems. Then we describe what we call the harmonic state, system and transfer operators associated with an FDLCP system and their Floquet counterpart expressions.

#### A. Notations, Terminologies and Regularized Determinants

 $\mathbb{C}$  represents the field of complex numbers and  $\mathbb{Z}$  is the ring of integers. The Euclidean norm of a vector and the norm of a matrix induced by this norm are denoted by  $|| \cdot || . l_2$  is the set of infinite-dimensional vectors  $\underline{x}$  such that  $||\underline{x}||_{l_2}^2 := \underline{x}^* \underline{x} < \infty$ , where \* denotes the complex conjugate transpose.  $L_p[0,h]$  is the linear space of vector measurable functions x defined on [0,h] such that  $||x(\cdot)||_{L_p[0,h]}^p := \int_0^h ||x(t)||^p dt < \infty$ . Also,  $||\cdot||_{X_2/X_1}$  is the induced norm from a linear space  $X_1$  to another linear space  $X_2$ .  $X_1 \oplus X_2$  denotes the direct sum of  $X_1$  and  $X_2$ .  $\rho\{\cdot\}$  and  $\zeta\{\cdot\}$  denote the sets of all singular points and all zeros of a complex function  $(\cdot)$ , respectively.

We introduce set operations:  $A + B, A - B, A \wedge B$  and  $A \vee B$  and a set relationship:  $C \prec A$ . A + B consists of all elments in A and B; A - B is a subset of A, whose elements appearing also in B are removed by multiplicities in B;  $A \wedge B$  is a collection of elements both in A and B by lower multiplicities;  $A \vee B$  denotes a set of elements in A and/or B by higher multiplicities.  $C \prec A$  means all elements of C are in A by their multiplicities in C. For instance, if  $A = \{1,1,2,2,2,3,4\}, B = \{1,1,2,2,2,3,4,5\}, A - B = \{2,3,4\}, A \wedge B = \{1,1,2,2\}, A \vee B = \{1,1,2,2,3,4,5\},$  and  $C \prec A$ .

Now we collect facts about the 2-regularized determinant of Hilbert-Schmidt operators. Let  $\lambda_i(A)$  be the *i*-th eigenvalue of a compact linear operator  $A: X \to X$  (X is a separable infinite-dimensional Hilbert space) and  $s_i(A) := (\lambda_i(A^*A))^{1/2}$  be its *i*-th singular value. For p = 1 and 2, the set of all compact operators  $A: X \to X$  satisfying  $||A||_p := (\sum_i s_i(A)^p)^{1/p} < \infty$ is denoted by  $C_1(X)$  and  $C_2(X)$ , respectively. In particular, the operators in  $C_1(X)$  are called trace class operators while those in  $C_2(X)$  are called Hilbert-Schmidt operators [4]. Clearly,  $C_1(X) \subset C_2(X)$ . For  $A \in C_1(X)$ , the operator trace and determinant are well-defined in the sense that the following infinite series and product converge; that is, we have

$$\begin{cases} \operatorname{tr}(A) \coloneqq \sum \lambda_i(A) \\ \det(I+A) \coloneqq \prod (1+\lambda_i(A)). \end{cases}$$
(2)

Note that for  $A \in C_2(X), R_2(A) := (I + A) \exp\{-A\} - I \in C_1(X)$ . The determinant of  $I + R_2(A)$  in the sense of (2), denoted by  $\det_2(I + A) := \det(I + R_2(A))$ , is called the 2-regularized determinant of I + A. For our aim, assume also that  $B \in C_2(X)$ . Then

$$\det_2(I+A)\det_2(I+B) = \det_2[(I+A)(I+B)]\exp\{\operatorname{tr}(AB)\}.$$
(3)

 $F(t) \in L_p[0, h]$  means that  $F(\cdot)$  is a matrix function, whose element are *h*-periodic and belong to  $L_p[0, h]$  when restricted to [0, h]. Similarly for subsets of  $L_p[0, h]$ . To validate the Toeplitz transformation on periodic functions (see Appendix A and [39]), subsets of  $L_1[0, h]$  such as  $L_{PCD}[0, h], L_{CPCD}[0, h]$ and  $L_{CAC}[0, h]$  are used.  $L_{PCD}[0, h]$  is the set of all piecewise continuous functions that are differentiable almost everywhere in  $[0, h]; L_{CPCD}[0, h]$  is the set of all continuous functions whose first-order derivatives are piecewise continuous in [0, h], while  $L_{CAC}[0, h]$  is that of all continuous functions fourier series are absolutely convergent. Namely, PCD stands for piecewise continuous and differentiable; CPCD stands for continuous and piecewise continuously differentiable, while CAC stands for continuous and absolute convergent.

## B. Toeplitz Expressions of FDLCP Systems

To understand what we call the harmonic state, system and transfer operators of the FDLCP system (1), let us denote the Toeplitz transformation of A(t), B(t), C(t) and D(t) by  $\underline{A} = \mathcal{T}\{A(t)\}, \underline{B} = \mathcal{T}\{B(t)\}, \underline{C} = \mathcal{T}\{C(t)\}$ , and  $\underline{D} = \mathcal{T}\{D(t)\}$ , respectively. Toeplitz transformation is described in Appendix A. And also, we write

$$\underline{E}(s) := \operatorname{diag}[\dots, \varphi_{-1}(s)I, \varphi_0(s)I, \varphi_1(s)I, \dots] : l_E \to l_2 \quad (4)$$

where  $\varphi_k(s) := s + jk\omega_h, \omega_h =: 2\pi/h, k \in \mathbb{Z}, s \in \mathbb{C}$  and  $l_E := \{\underline{x} \in l_2 : \underline{E}(j0)\underline{x} \in l_2\}$ . The linear space  $l_E$  is a proper subset of  $l_2$  and dense in  $l_2$  [39]. Clearly,  $\underline{E}(s)$  is unbounded on  $l_2$  and thus it must be restricted to  $l_E$ . This is also the case for the harmonic state operator  $\underline{E}(s) - \underline{A} : l_E \to l_2$  and the harmonic system operator  $\underline{\Sigma}(s)$  given by

$$\underline{\Sigma}(s) := \begin{bmatrix} \underline{E}(s) - \underline{A} & -\underline{B} \\ \underline{C} & \underline{D} \end{bmatrix} : l_E \oplus l_2 \to l_2 \oplus l_2.$$
(5)

In other words,  $\underline{E}(s) - \underline{A}$  and  $\underline{\Sigma}(s)$  are merely densely defined on the Hilbert spaces  $l_2$  and  $l_2 \oplus l_2$  for each specific  $s \in \mathbb{C}$ . Furthermore, we define

$$\underline{G}(s) := \underline{C}(\underline{E}(s) - \underline{A})^{-1}\underline{B} + \underline{D} : l_2 \to l_2$$
(6)

which is called the harmonic transfer operator [37], [39] of the FDLCP system (1). To guarantee that  $\underline{G}(s)$  is well-defined on  $l_2$ , the inverse of  $\underline{E}(s) - \underline{A}$  must exist. This will soon be answered by Proposition 1.

Now we state the Floquet similarity formula and other eigenvalues properties of FDLCP systems [39], based on which  $\underline{G}(s)$  can be validated in the sense that  $(\underline{E}(s)-\underline{A})^{-1}$  exists almost everywhere on the complex plane  $\mathbb{C}$  except for countably infinite number of points (which are actually eigenvalues of  $\underline{E}(j0)-\underline{A}$ ). The results in Proposition 1 lend themselves to mathematical convenience in our arguments.

Proposition 1: In the FDLCP system (1), let  $A(t) \in L_{PCD}[0,h]$  and  $\Phi(t,0) = P(t,0)e^{Qt}$  be a Floquet factorization. Then, P(t,0) and  $P^{-1}(t,0)$  belong to  $L_{CAC}[0,h]$ .  $\underline{P}$  is invertible on  $l_2$  and  $l_E$ . Also, the unbounded operator  $\underline{P}(\underline{E}(j0) - \underline{Q})\underline{P}^{-1}$  and  $\underline{E}(j0) - \underline{A}$  are densely defined on  $l_2$ (or more precisely, well-defined on the subset  $l_E \subset l_2$ ) and coincide with each other

$$\underline{P}(\underline{E}(j0) - Q)\underline{P}^{-1} = \underline{E}(j0) - \underline{A}$$

which is called the Floquet similarity formula. Moreover, the harmonic state operator  $\underline{E}(s) - \underline{A}$  and the Floquet state operator  $\underline{E}(s) - Q$  are invertible for each  $s \in \mathbb{C} \setminus \Lambda$ . Here

$$\Lambda := \{ jk\omega_h + \lambda(Q) : k \in \mathbb{Z} \}$$
<sup>(7)</sup>

with  $\lambda(Q)$  being the set of all eigenvalues of Q.

Furthermore, assume that  $A(t) \in L_{CAC}[0,h] \cap L_{PCD}[0,h]$ . Let us choose a number  $\epsilon > 0$  such that  $\Lambda \wedge \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\} = \emptyset$ . Then, for each  $s \in \mathbb{C} \setminus \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\}$ ,  $\underline{E}^{-1}(s,\epsilon) =: (\underline{E}(s) + \underline{\epsilon})^{-1}$  and  $\underline{E}^{-1}(s,\epsilon)\underline{A}$  belong to  $\mathcal{C}_2(l_2)$ , and it holds that

$$\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon)$$
  
=  $g_A(s, \epsilon) \prod_{k=1}^n \prod_{\tau=-\infty}^\infty \left(1 - \frac{\epsilon + \lambda_k(Q)}{s + \epsilon + j\tau\omega_h}\right)$   
 $\cdot \exp\{\frac{\epsilon + \lambda_k(Q)}{s + \epsilon + j\tau\omega_h}\}$ 

where  $\underline{A + \epsilon} =: \underline{A} + \underline{\epsilon}$ . More precisely,  $\det_2(\underline{I} - \underline{E}^{-1}(s, \underline{\epsilon})\underline{A + \epsilon})$  is analytic on  $\mathbb{C} \setminus \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\}$ and possesses a zero at each point  $\lambda_k(Q) + j\tau\omega_h$  with  $k = 1, 2 \cdots, n$  and  $\tau \in \mathbb{Z}; \epsilon + j\tau\omega_h$  is a *n*-multiple removeable singular points; i.e.,

$$\begin{cases} \rho\{\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon)\} \\ = \sum_{i=1}^n \{\epsilon + j\tau_i\omega_h : \tau_i \in \mathbb{Z}\} \\ \zeta\{\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon)\} = \Lambda \end{cases}$$
(8)

In the above,  $g_A(s, \epsilon)$  is analytic and vanishes nowhere on the whole complex plane  $\mathbb{C}$ .

Proposition 1 follows readily by slightly modifying the arguments of [29], [39] and [40]. To aviod repetition, the details are omitted. The number  $\epsilon > 0$  is introduced to guarantee that the regularization operator  $\underline{E}^{-1}(s, \epsilon) : l_2 \to l_2$  is well-defined at each point of  $s \in \mathbb{C} \setminus \{\epsilon + j\tau \omega_h : \tau \in \mathbb{Z}\}$ . Such a regularization approach in dealing with unbounded harmonic operators was first suggested in [42], based on some regularization theorems in [4] and [7].

## C. Controllability/Observability Canonical Forms of FDLCP Systems

Now we state a controllability/observability decomposition theorem in the FDLCP setting. An algorithm to construct such a canonical form can be found in [43]. The decomposition theorem plays a role in clarifying structural relationships of inputand/or output-decoupling zeros in FDLCP systems with noncontrollability/nonobservability modes.

Proposition 2: In the FDLCP system (1), suppose that  $A(t) \in L_{PCD}[0,h], B(t) \in L_{CAC}[0,h]$  and  $C(t) \in L_{CAC}[0,h]$ . Then there exists  $T(t) \in L_{CPCD}[0,h]$ , which is invertible uniformly over  $t \in [0,h)$  and  $T^{-1}(t) \in L_{CPCD}[0,h]$ , such that the state transformation  $x = T(t)\tilde{x}$  transforms the FDLCP system (1) into the canonical form

$$\begin{cases} \dot{x} = \begin{bmatrix} \dot{Q}_{co} & 0 & \dot{Q}_{13} & 0\\ \dot{Q}_{21} & \ddot{Q}_{c\bar{o}} & \ddot{Q}_{23} & \ddot{Q}_{24}\\ 0 & 0 & \ddot{Q}_{c\bar{o}} & 0\\ 0 & 0 & \ddot{Q}_{43} & \ddot{Q}_{\bar{c}\bar{o}} \end{bmatrix} \tilde{x} + \begin{bmatrix} B_{co}(t)\\ B_{c\bar{o}}(t)\\ 0\\ 0 \end{bmatrix} u \quad (9) \\ y = \begin{bmatrix} C_{co}(t) & 0 & C_{\bar{c}o}(t) & 0 \end{bmatrix} \tilde{x} + D(t)u \end{cases}$$

where  $\tilde{Q}_{co}, \tilde{Q}_{\bar{c}o}, \tilde{Q}_{13}$  and etc., are constant matrices of compatible dimensions;  $B_{co}(t), B_{c\bar{o}}(t), C_{co}(t)$  and  $C_{\bar{c}o}(t)$  belong to  $L_{CAC}[0, h]$ , and the pairs

$$\left(\begin{bmatrix} \tilde{Q}_{co} & 0\\ \tilde{Q}_{21} & \tilde{Q}_{c\bar{o}} \end{bmatrix}, \begin{bmatrix} B_{co}(t)\\ B_{c\bar{o}}(t) \end{bmatrix}\right)$$

$$\left(\begin{bmatrix} \ddot{Q}_{co} & \ddot{Q}_{13} \\ 0 & \tilde{Q}_{\bar{c}o} \end{bmatrix}, \begin{bmatrix} C_{co}(t) & C_{\bar{c}o}(t) \end{bmatrix}\right)$$

are completely controllable and observable, respectively.

Actually, one can also find other ways of controllability/observability decomposition for FDLCP systems; for example, the algorithms suggested in [2] and [17] are typical. However, since the algorithm of [17] is constructed in a pointwise fashion, the resulted decomposition canonical form lacks analytical properties needed in the harmonic analysis; the decomposition canonical form of [2] usually has a periodically time-varying state matrix that does not fit our purposes. This is the reason why Proposition 2 is introduced.

#### **III. ZEROS/POLES DEFINITIONS IN FDLCP SYSTEMS**

Comparing the harmonic system operator  $\underline{\Sigma}(s)$  and the harmonic transfer operator  $\underline{G}(s)$  in (5) and (6), respectively, to the so-called system matrix and the transfer function of an LTI continuous-time system [30], [31], one would come to some simi-

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larities between them. These similarities might invite someone to draw a conclusion that it is trivial to extend zeros and poles definitions of LTI systems to FDLCP systems via  $\underline{\Sigma}(s)$  and  $\underline{G}(s)$ . However, careful observations about  $\underline{\Sigma}(s)$  and  $\underline{G}(s)$  soon reveal that this is not the case.

In fact, there are at least two pending difficulties in any direct extension of zeros and poles definitions. One stems from the fact that  $\underline{\Sigma}(s)$  and  $\underline{G}(s)$  are infinite-dimensional. Therefore, the conventional unimodular transformation theory and matrix determinants do not work. Another difficulty is that  $\underline{\Sigma}(s)$  is unbounded. It means that if we compulsively extended LTI zeros and poles definitions to FDLCP cases via certain truncations, we would inevitably encounter some convergence issues that are hard to verify.

In this section, we first introduce what we call the regularized harmonic system operator via the regularization technique of [4], [7], [42]. This will equip us with important results, that is, Propositions 3 and 4 in Sections III-A and III-B, which will eventually surmount the difficulties mentioned in the above. This, together with the 2-regularized determinant theory, helps us in defining zeros and poles in a big class of FDLCP systems in Section III-C. Compositions and properties related to zeros and poles will be examined in the next section.

#### A. Regularized Harmonic System Operators

Let  $s \in \mathbb{C} \setminus \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\}$  and define the regularized harmonic system operator  $\underline{\Sigma}_R(s,\epsilon) : l_2 \oplus l_2 \to l_2 \oplus l_2$  by

$$\underline{\Sigma}_{R}(s,\epsilon) := \begin{bmatrix} \underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & -\underline{E}^{-1}(s,\epsilon)\underline{B} \\ \underline{C} & \underline{D} \end{bmatrix}$$
(10)

where the regularization operator  $\underline{E}^{-1}(s,\epsilon)$  is well-defined. Clearly,  $\underline{\Sigma}_R(s,\epsilon)$  is bounded on  $l_2 \oplus l_2$  for each  $s \in \mathbb{C} \setminus \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\}.$ 

In the sequel,  $(\cdot)((i/\cdot)), (\cdot)(\cdot/k)$  and  $(\cdot)(i/k)$  denote the *i*-th blockwise column, the *k*-th blockwise row and the (i, k)-th block in an (infinite-dimensional) blockwise matrix  $(\cdot)$ , where  $i, k \in \mathbb{Z}$ . Furthermore,  $(\cdot)(i: r/\cdot)$  is a submatrix of *r* column vectors in  $(\cdot)(i/\cdot)$ , say its  $i_1$ -th,  $\cdots, i_r$ -th column vectors;  $(\cdot)(\cdot/k: r)$  is a submatrix of *r* row vectors in  $(\cdot)(i/k)$ , say its  $k_1$ -th,  $\ldots, k_r$ -th row vectors; and  $(\cdot)((i: r)/(k: r))$  denotes a  $r \times r$  submatrix in  $(\ldots)(i/k)$ , say the submatrix  $(i_1, \ldots, i_r) \times (k_1, \ldots, k_r)$ . Here,  $i_1, \ldots, i_r \in \{1, \ldots, m\}$  and  $k_1, \ldots, k_r \in \{1, \ldots, l\}$ .

Based on these notations, we define

$$\begin{cases} \underline{\Sigma}_{R}(s,\epsilon)(\frac{i}{k}) = \\ \begin{bmatrix} \underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & -\underline{E}^{-1}(s,\epsilon)\underline{B}(\frac{i}{\cdot}) \\ \underline{C}(\frac{i}{k}) & \underline{D}(\frac{i}{k}) \end{bmatrix} \\ : l_{2} \oplus \mathbb{C}^{m} \to l_{2} \oplus \mathbb{C}^{l} \\ \underline{G}(s)(\frac{i}{k}) = \underline{C}(\frac{i}{k})(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i}{\cdot}) + \underline{D}(\frac{i}{k}) \\ : \mathbb{C}^{m} \to \mathbb{C}^{l}. \end{cases}$$
(11)

It is obvious that  $\underline{D}(i/k) = D_{k-i}$  with  $\{D_{\tau}\}_{-\infty}^{\infty}$  being the Fourier coefficients of D(t). It is straightforward to see by Proposition 1 that  $\underline{G}(s)$  is well-defined on  $l_2$  over  $s \in \mathbb{C} \setminus \Lambda$ . This in turn means that  $\underline{\Sigma}_R(s,\epsilon)(i/k)$  and  $\underline{G}(s)(i/k)$  are well-defined over  $s \in \mathbb{C} \setminus (\{\epsilon + j\tau\omega_h : \tau \in \mathbb{C}\} \cup \Lambda)$ .

For  $r = \min\{m, l\}$ , we further define

$$\begin{cases} \underline{\Sigma}_{R}(s,\epsilon)\left(\frac{i:r}{k:r}\right) = \\ \begin{bmatrix} \underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & -\underline{E}^{-1}(s,\epsilon)\underline{B}\left(\frac{i:r}{\cdot}\right) \\ \underline{C}\left(\frac{i}{k:r}\right) & \underline{D}\left(\frac{i:r}{k:r}\right) \end{bmatrix} \\ : l_{2} \oplus \mathbb{C}^{r} \to l_{2} \oplus \mathbb{C}^{r} \\ \underline{G}(s)\left(\frac{i:r}{k:r}\right) = \underline{C}\left(\frac{i}{k:r}\right)(\underline{E}(s) - \underline{A})^{-1}\underline{B}\left(\frac{i:r}{\cdot}\right) + \underline{D}\left(\frac{i:r}{k:r}\right) \\ : \mathbb{C}^{r} \to \mathbb{C}^{r}. \end{cases}$$
(12)

Since we will define zeros and poles in FDLCP systems through the 2-regularized determinants of submatrices of  $\underline{\Sigma}_R(s)(i/k)$  and  $\underline{G}(s)(i/k)$ , we show in Proposition 3 that such 2-regularized determinants make sense and are analytic and meromorphic functions. This is nontrivial since  $\underline{\Sigma}_R(s,\epsilon)(i/k)$ is also infinite-dimensional and entries in  $\underline{G}(s)(i/k)$  contains infinite summations.

Proposition 3: In the FDLCP system (1), assume that  $A(t) \in L_{PCD}[0, h]$  and B(t) and C(t) belong to  $L_{CAC}[0, h], D(t) \in L_2[0, h]$ . Then for each pair (i, k) with  $i, k \in \mathbb{Z}$ ,

- i) For any fixed  $s \in \mathbb{C} \setminus \{\epsilon + j\tau \omega_h : \tau \in \mathbb{Z}\}, \underline{\Sigma}_R(s, \epsilon)(i/k) \text{diag}[\underline{I}, 0_{m \times l}]$  is a Hilbert-Schmidt operator from  $l_2 \oplus \mathbb{C}^m$  to  $l_2 \oplus \mathbb{C}^l$ , while  $\underline{\Sigma}_R(s, \epsilon)((i : r)/(k : r)) \text{diag}[\underline{I}, 0_{r \times r}]$  is a Hilbert-Schmidt operator on  $l_2 \oplus \mathbb{C}^r$ ;
- ii) Each (scalar) entry of  $\underline{G}(s)(i/k)$  is analytic on  $\mathbb{C} \setminus \Lambda$  and meromorphic.

Here  $0_{m \times l}$  and  $0_{r \times r}$  denote  $m \times l$  and  $r \times r$  zero matrices, respectively, while  $r = \min\{m, l\}$ .

## B. Properties Related to $\underline{G}(s)(i/k)$ and $\Delta(s,\epsilon)((i:r)/(k:r))$

Note from the assertion i) of Proposition 3 that  $\underline{\Sigma}_R(s,\epsilon)((i:r)/(k:r)) - \text{diag}[\underline{I},I_r] =$  $\underline{\Sigma}_R(s,\epsilon)((i:r)/(k:r)) - \text{diag}[\underline{I},0_{r\times r}] - \text{diag}[0,I_r] \in$  $C_2(l_2 \oplus \mathbb{C}^r)$ . Hence, it makes sense to talk about the 2-regularized determinant on  $\underline{\Sigma}_R(s,\epsilon)((i:r)/(k:r)) - \text{diag}[\underline{I},I_r]$  for each  $s \in \mathbb{C} \setminus \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\}$ . Now we see how to compute such 2-regularized determinants and what features they have.

After algebras described in Appendix B, we can show that

$$\det_2\left(\underline{\Sigma}_R(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right) = \Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\exp\left\{r - \operatorname{tr}\left(\underline{D}\left(\frac{i:r}{k:r}\right)\right)\right\} \quad (13)$$

where

$$\Delta(s,\epsilon) \left(\frac{i:r}{k:r}\right) = \det_2(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A+\epsilon}) \\ \times \det\left(\underline{G}(s) \left(\frac{i:r}{k:r}\right)\right).$$

Evidently,  $\Delta(s,\epsilon)((i:r)/(k:r))$  in  $\det_2(\underline{\Sigma}_R(s,\epsilon)((i:r)/(k:r)))$  is the kernel part since the last term of (13) is independent of *s* and nonzero no matter which *r* rows and *r* columns are taken in D(i/k).

Proposition 4 claims properties on the operators  $\underline{G}(s)(i/k)$ and  $\Delta(s,\epsilon)((i:r)/(k:r))$  that guarantee existence of zeros and poles we will introduce for the FDLCP system (1). Proposition 4: In the FDLCP system (1), assume that  $A(t) \in L_{PCD}[0, h]$  and B(t) and C(t) belong to  $L_{CAC}[0, h], D(t) \in L_2[0, h]$ . Then for each pair (i, k) with  $i, k \in \mathbb{Z}$ , we have the following.

i). For any  $(i_1, \ldots, i_r) \times (k_1, \ldots, k_r)$  submatrix in  $\underline{G}(s)(i/k)$  with  $i_1, \ldots, i_r \in \{1, 2, \ldots, m\}$ and  $k_1, \ldots, k_r \in \{1, 2, \ldots, l\}$ , namely  $\underline{G}(s)((i : r)/(k : r)), \quad \det(\underline{G}(s)((i : r)/(k : r)))$ is analytic on  $\mathbb{C} \setminus \Lambda$ , meromorphic, and satisfies

$$\rho\left\{\det\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)\right\} \prec \Lambda.$$
 (14)

ii). Equation (13) holds for each  $s \in \mathbb{C} \setminus (\{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\} \cup \Lambda)$  with  $\Delta(s,\epsilon)((i:r)/(k:r))$  being analytic on  $\mathbb{C} \setminus (\{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\} \cup \Lambda)$ , meromorphic, and satisfying

$$\rho\left\{\Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right\} \prec \sum_{q=1}^{n} \{\epsilon + j\tau_q \omega_h : \tau_q \in \mathbb{Z}\}.$$
 (15)

In the above, the set  $\Lambda$  is given in Proposition 1.

## C. Zeros/Poles Definitions and Remarks

With the above preparations, we further define

$$\begin{cases} \mathcal{Z}_{S}\left(\frac{i}{k}\right) = \wedge \left[\zeta\left\{\Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right\} \\ +\sum_{q=1}^{n}\left\{\epsilon + j\tau_{q}\omega_{h}:\tau_{q}\in\mathbb{Z}\right\} \\ -\rho\left\{\Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right\}\right] & (16) \\ \mathcal{P}_{S}\left(\frac{i}{k}\right) = \zeta\left\{\det_{2}\left(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon\right)\right\} \\ \begin{cases} \mathcal{Z}_{T}\left(\frac{i}{k}\right) = \wedge \left[\zeta\left\{\det\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right\}\right)\right] \\ \mathcal{P}_{T}\left(\frac{i}{k}\right) = \vee \left[\rho\left\{\det\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)\right\}\right] \end{cases} \end{cases}$$

where  $\wedge [\cdot]$  and  $\vee [\cdot]$  are defined over  $i_1, \ldots, i_r \in \{1, 2, \ldots, m\}$  and  $k_1, \ldots, k_r \in \{1, 2, \ldots, l\}$ .

Definition 1: The elements in  $\mathcal{Z}_S(i/k)$  and  $\mathcal{P}_S(i/k)$  are called the (i, k)-th class system zeros and poles, respectively, of the regularized harmonic system operator  $\underline{\Sigma}_R(s, \epsilon)$  of the FDCLP system (1). The elements in  $\mathcal{Z}_T(i/k)$  and  $\mathcal{P}_T(i/k)$  are termed the (i, k)-th class transfer zeros and poles, respectively, of the harmonic transfer operator  $\underline{G}(s)$  of the FDCLP system (1).

In the above,  $Z_S(i/k)$ ,  $\mathcal{P}_S(i/k)$ ,  $Z_T(i/k)$  and  $\mathcal{P}_T(i/k)$  are defined in (16) and (17), and the subscripts S and T mean "system" and "transfer", respectively.

Remark 1: In  $\mathcal{Z}_{S}(i/k)$ ,  $\sum_{q=1}^{n} \{\epsilon + j\tau_{q}\omega_{h} : \tau_{q} \in \mathbb{Z}\}$  $- \rho\{\Delta(s,\epsilon)((i:r)/(k:r))\}$  collects zeros of  $\det(\underline{G}(s)((i:r)/(k:r)))$  that are situating in  $\sum_{q=1}^{n} \{\epsilon + j\tau_{q}\omega_{h} : \tau_{q} \in \mathbb{Z}\}$  but cancelled by singular points of  $\Delta(s,\epsilon)((i:r)/(k:r))$  situating at the same points. Such cancellations may occur due to the fact that

$$\sum_{q=1}^{n} \{ \epsilon + j\tau_{q}\omega_{h} : \tau_{q} \in \mathbb{Z} \} - \rho \left\{ \Delta(s,\epsilon) \left( \frac{i:r}{k:r} \right) \right\}$$
$$= \zeta \left\{ \det \left( \underline{G}(s) \left( \frac{i:r}{k:r} \right) \right) \right\}$$
$$\wedge \sum_{i=1}^{n} \{ \epsilon + j\tau_{i}\omega_{h} : \tau_{i} \in \mathbb{Z} \}.$$
(18)

Such a modification is needed because the singular points of  $\Delta(s,\epsilon)((i:r)/(k:r))$  in  $\sum_{q=1}^{n} \{\epsilon + j\tau_q \omega_h : \tau_q \in \mathbb{Z}\}$  are introduced when  $\underline{E}^{-1}(s,\epsilon)$  regularizes  $\underline{\Sigma}(s)$  to  $\underline{\Sigma}_R(s,\epsilon)$ . These

singular points associated with the regularization must be removed from zero and pole definitions.

Remark 2: By (16) and Proposition 1,  $\mathcal{P}_S(i/k)$  is independent of  $\epsilon$ . It should be stressed that  $\mathcal{Z}_S(i/k)$  has nothing to do with  $\epsilon$ , either, which follows from (16) and (18). In other words,  $\mathcal{P}_S(i/k)$  and  $\mathcal{Z}_S(i/k)$  will not be affected by the regularization operator  $\underline{E}^{-1}(s,\epsilon)$  as long as  $\epsilon > 0$  is chosen such that the eigenvalues, i.e., the elements in  $\Lambda$ , of the harmonic state operator  $\underline{E}(j0) - \underline{A}$  concide with none of the elements in  $\{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\}$ .

*Remark 3:* Under the assumptions of Proposition 4, det( $\underline{G}(s)((i:r)/(k:r))$ ) and  $\Delta(s,\epsilon)((i:r)/(k:r))$  are analytic and meromorphic. Therefore, they possess only isolated zeros and removable singular points by complex theory. Namely,  $\mathcal{Z}_S(i/k)$ ,  $\mathcal{P}_S(i/k)$ ,  $\mathcal{Z}_T(i/k)$  and  $\mathcal{P}_T(i/k)$  are sets of isolated numbers that are at most countably infinite. In other words, these sets have algebraic characteristics similar to the zeros and poles sets in LTI systems.

By the matrix expressions,  $\underline{\Sigma}_R(s,\epsilon)(i/k)$  and  $\underline{G}(s)(i/k)$ seemingly can be viewed as relationships from the *i*-th input to the *k*-th output if  $\underline{\Sigma}_R(s,\epsilon)$  and  $\underline{G}(s)$  as we do in multivariable systems. We point out that  $\underline{\Sigma}_R(s)$  and  $\underline{G}(s)$  are defined by lifting the harmonics in the Hilbert space  $l_2$ . Hence,  $\underline{\Sigma}_R(s,\epsilon)(i/k)$  and  $\underline{G}(s)(i/k)$  reflects the relationship between the *i*-th harmonic wave of the input and the *k*-th harmonic wave of the output. In this sense, the elements in  $\mathcal{Z}_S(i/k)$  possess a harmonic wave-to-wave meaning. This is why we term an element of  $\mathcal{Z}_S(i/k)$  an (i, k)-th "class" system zero instead of a "subsystem" system zero. Similar words can be said for other zeros and poles.

*Remark 4:* In [38], poles and transmission zeros are defined through integral operators of FDLCP systems, which are formally frequency/time-domain mixed. Based on the definitions [38] the so-called eigenstructures and associated directions of FDLCP systems are studied. It is shown that the poles are the characteristic multipliers (see [22] for definition) of the monodromy  $\Phi(h, 0)$ . Such zeros [38] can be connected with identically zero outputs under geometrical periodic signals.

In [6] and [27], a derivative operator framework (thus timedomain essentially) is suggested to deal with zeros and blocking properties in square FDLCP systems. The definitions there are stated under strong conditions on A(t) and B(t), and a uniform relative degree assumption of the system concerned is required. It is shown that zeros are unobservable exponents of some associated periodic pair and can be connected to zero outputs, together with appropriate initial states.

The zeros and poles of Definition 1 are given under a harmonic framework, which is frequency-domain essentially. In particular, the definitions here can be viewed as direct extensions of those in LTI systems with help of the regularization technique. It is expected that relationships between dynamic behaviors and zeros/poles can also be established under the harmonic framework in FDLCP systems, as we have seen in [27] and [38]. To clarify such relationships in the harmonic framework needs many more notations and further preparations, and thus is left for another paper.

*Remark 5:* Also in connection with Remark 3, we must say that it is a nontrivial task to extend Definition 1 in order to in-

clude other zero facets in the FDLCP setting, say blocking zeros, transmission zeros and invariant zeros, as we do in multivariable systems [23], [31]. There are big mathematical gaps in such extensions, although these extensions are natural and intuitive in form.

An essential difficulty in such definition extensions is that  $\underline{\Sigma}_R(s,\epsilon)$  is not a Hilbert-Schmidt operator, and thus it would be meaningless to even talk about any regularized determinants on  $\underline{\Sigma}_R(s,\epsilon)$ . Actually, we know at most that  $\underline{\Sigma}_R(s,\epsilon)$  is bounded and Fredholm [7]. To make things worse, compactness of all the submatrices  $\underline{\Sigma}_R(s,\epsilon)(i/k)$  does not lead compactness of  $\underline{\Sigma}_R(s,\epsilon)$  due to convergence caused by  $\underline{D}$  (see a possible explanation by [25, Theor. 5.24.8 ]).

## IV. PROPERTIES OF ZEROS/POLES IN FDLCP SYSTEMS

In this section, properties related to zeros and poles in FDLCP systems are examined rigorously, which clarify some structural features about FDLCP systems. In particular, zeros/poles relationships similar to those in LTI systems are derived, which greatly enrich our understanding to analysis and synthesis problems in FDLCP systems where zeros and poles are involved. Based on the preparations in the previous subsection, we first examine composition and distribution features of the zeros and poles introduced in Definition 1.

Theorem 1: In the FDCLP system (1), assume that  $A(t) \in L_{PCD}[0, h]$  and that B(t) and C(t) belong to  $L_{CAC}[0, h]$ . Assume also that  $D(t) \in L_2[0, h]$ . Then for each pair (i, k) with  $i, k \in \mathbb{Z}$ , it holds:

i) 
$$\mathcal{P}_{S}(i/k) = \mathcal{P}_{T}(i/k) + \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \mathcal{Z}_{c.d}(i/k) = \Lambda$$

ii)  $\mathcal{Z}_{S}(i/k) = \mathcal{Z}_{T}(i/k) + \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \mathcal{Z}_{c.d}(i/k)$ where

$$\begin{cases} \mathcal{Z}_{i.d} := \left\{ j\tau\omega_h + \lambda : \tau \in \mathbb{Z} \text{ and} \\ \lambda \text{ is an eigenvalue of } \begin{bmatrix} \tilde{Q}_{\bar{c}0} & 0 \\ \tilde{Q}_{43} & \tilde{Q}_{\bar{c}5} \end{bmatrix} \right\} \\ \mathcal{Z}_{o.d} := \left\{ j\tau\omega_h + \lambda : \tau \in \mathbb{Z} \text{ and} \\ \lambda \text{ is an eigenvalue of } \begin{bmatrix} \tilde{Q}_{c\bar{o}} & \tilde{Q}_{24} \\ 0 & \tilde{Q}_{\bar{c}5} \end{bmatrix} \right\} \\ \mathcal{Z}_{i.o.d} := \left\{ j\tau\omega_h + \lambda : \tau \in \mathbb{Z} \text{ and} \\ \lambda \text{ is an eigenvalue of } \tilde{Q}_{\bar{c}5} \right\} \end{cases}$$

The set  $\mathcal{Z}_{c.d}(i/k)$  denotes all the system poles which disappear from  $\mathcal{P}_T(i/k)$  but do not belong to the set  $\mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d}$ . Here,  $\tilde{Q}_{\bar{c}o}, \tilde{Q}_{24}$  and so on are defined in Proposition 2.

*Remark 6:* Theorem 1 shows that zeros and poles of FDLCP systems has set relationships similar to those of LTI systems. For example, the harmonic transfer operator  $\underline{G}(s)$  only represents the controllable and observable structures of an FDLCP system. That is, uncontrollable and/or unobservable modes of the FDLCP system can only be treated in the regularized harmonic system operator. Following suit to some terminologies in LTI systems, the elements belonging to  $\mathcal{Z}_{i.d}$ ,  $\mathcal{Z}_{o.d}$  and  $\mathcal{Z}_{i.o.d}$  are termed, respectively, the input-, output- and input/output-decoupling zeros. The meaning of the elements in  $\mathcal{Z}_{c.d}$  will be explained in Remark 7.

Theorem 1 also says that the system poles (including decoupling zeros) and the transfer poles distribute themselves in a strip region parallel to the imaginary axis of the complex plane. However, as we have explained in Remark 3, the zeros and poles of FDLCP systems must be interpreted as set relationships in between the harmonic waves of input and output. This is substantially different from the well-known results in LTI systems.

In the following, we do some deeper observations about properties of the zeros and poles in the FDLCP setting. The observations are summarized in Theorem 2

Theorem 2: In the FDLCP system (1), assume that  $A(t) \in L_{PCD}[0,h]$ . Assume also that B(t) and C(t) belong to  $L_{CAC}[0,h]$ , and  $D(t) \in L_2[0,h]$ . Then it holds

- i)  $\mathcal{Z}_{c,d}(i/k) + \mathcal{P}_T(i/k) = \{\lambda + j\tau\omega_h : \lambda \text{ is an eigenvalue of } \hat{Q}_{co}, \tau \in \mathbb{Z}\};$
- ii) for any  $\gamma \in \mathbb{Z}$ , it holds that  $\mathcal{Z}_S((i+\gamma)/(k+\gamma)) = \mathcal{Z}_S(i/k) + j\gamma\omega_h, \mathcal{Z}_T((i+\gamma)/(k+\gamma)) = \mathcal{Z}_T(i/k) + j\gamma\omega_h$  and  $\mathcal{Z}_{c.d}((i+\gamma)/(k+\gamma)) = \mathcal{Z}_{c.d}(i/k) + j\gamma\omega_h;$
- iii) for any  $k \in \mathbb{Z}, \wedge_{\forall i \in \mathbb{Z}}[\mathcal{Z}_{c.d}(i/k)] = \emptyset$ ; for any  $i \in \mathbb{Z}$ , and  $\wedge_{\forall k \in \mathbb{Z}}[\mathcal{Z}_{c.d}(i/k)] = \emptyset$ ; thus it holds that  $\wedge_{\forall i,k \in \mathbb{Z}}[\mathcal{Z}_{c.d}(i/k)] = \emptyset$ ;
- iv) if there exists at least one point  $s_0 \in \mathbb{C} \setminus \Lambda$ , rank  $(\underline{G}(s_0)(i/k)) = \min\{m, l\} =: r \text{ and } \mathcal{Z}_T(i/k)$ is nonempty, then  $\mathcal{Z}_T(i/k)$  contains no limit points; i.e., a nonempty transfer zeros set  $\mathcal{Z}_T(i/k)$  does not contain any convergent transfer zeros sequences.

Remark 7: The first three assertions of Theorem 2 tell that the elements in  $\mathcal{Z}_{c.d}(i/k)$  represent transfer poles that are cancelled by harmonic waves cross combinations in A(t), B(t), C(t) and D(t). Here, the phrase "cross combination" should be understood as mutual position correspondence of the Fourier coefficient sequences  $\{A_{\tau}\}_{\tau=-\infty}^{+\infty}, \{B_{\tau}\}_{\tau=-\infty}^{+\infty}, \{C_{\tau}\}_{\tau=-\infty}^{+\infty}, and \{D_{\tau}\}_{\tau=-\infty}^{+\infty}$  in  $\underline{G}(s)(i/k)$ . Hence, we call the elements in  $\mathcal{Z}_{c.d}(i/k)$  the harmonic combination decoupling zeros. In other words, if  $\mathcal{Z}_{c.d}(i/k)$  is not empty, its elements reflect noncontrollable/nonobservable characteristics cancellations of a concerned FDLCP system that are caused by cross combinations of the harmonic waves in A(t), B(t), C(t) and D(t). This cross wave noncontrollability/nonobservability phenomenon may also be observed in LTI continuous-time systems but in a subsystem sense [32], [44].

The assertion ii) of Theorem 2 implies that the system and transfer zeros of  $\underline{\Sigma}_R(s,\epsilon)(i/k)$  and  $\underline{G}(s)(i/k)$  that are situating along a same skew line in  $\underline{\Sigma}_R(s,\epsilon)$  and  $\underline{G}(s)$  distribute themselves in a same vertical strip region parallel to the imaginary axis.

As for the assertion iv) of Theorem 2, it is usually true that  $\operatorname{rank}(\underline{G}(s)(i/k)) = r$  for some  $s \in \mathbb{C} \setminus \Lambda$ . With less rigorous words, it means that the transfer zeros of general FDLCP systems do not aggregate locally on the complex plane. This, together with the distribution patterns of the poles, implies in figurative words that FDLCP systems are not "compact."

Now we claim a corollary of Theorem 2, which states some interesting observations about "blocking zeros" in FDLCP systems. Here we use "blocking zeros" simply for lack of better words. Blocking zeros in LTI multivariable systems can be found in [10], [44].

Corollary 1: In the FDLCP system (1), suppose that all the assumptions of Theorem 1 are satisfied. If it holds for a pair (i,k) that  $\mathcal{Z}_S(i/k)$  (respectively,  $\mathcal{Z}_T(i/k)$ ) contains at most finitely many elements, then  $\wedge \mathcal{Z}_S(i/k) = \emptyset$  (respectively,  $\wedge \mathcal{Z}_T(i/k) = \emptyset$ ). Here,  $\wedge$  is over  $i \in \mathbb{Z}$  and/or over  $k \in \mathbb{Z}$ .

*Proof:* Since there exists a pair (i, k) at which  $\mathcal{Z}_S(i/k)$  contains at most finitely many elements, it follows from the assertion ii) of Theorem 2 that  $\wedge_{\gamma} \mathcal{Z}_S((i+\gamma)/(k+\gamma)) = \emptyset$  with  $\gamma \in \mathbb{Z}$ . Examining  $\wedge \mathcal{Z}_S(i/k)$  all over  $i, k \in \mathbb{Z}$ , the assertion follows. Similarly, we can show the assertion about  $\mathcal{Z}_T(i/k)$ .

*Remark 8:* The results of Corollary 1 have a harmonic interpretation. If  $\wedge \mathbb{Z}_T(i/k) = \emptyset$ , then for any nonzero harmonic waves in the input to the FDLCP system (1), their effects cannot be blocked out completely from all the harmonic waves in the output. This coincides with what we have known in LTI systems, which are FDLCP systems with arbitrary periods. Indeed, the harmonic transfer operator  $\underline{G}(s)$  for an LTI system is blockwise diagonal and can be seen as a group of transfer functions defined from the same LTI system G(s) but restricted to different frequency bands. If one imposes a same sinsuidal signal to each of these transfer functions, at least one response is nonzero; otherwise, we can assert from the complex analysis theory that G(s) = 0.

#### V. ILLUSTRATIVE EXAMPLES

The first example indicates that the zero/pole concepts defined in the paper are closely related to contorl problems. More precisely, we show that FDLCP control systems can be stabilized through pole assignment as we have seen in the LTI control theory. We consider stabilization of the periodic differential equation  $\ddot{y} + 2\zeta \dot{y} = [1 - 2\cos 2t]u$ , where  $\zeta > 0$  is a damping factor and u is an input. Simple algebras produce us the state-space differential equation

$$\dot{x} = Ax + B(t)u, \quad y = [0, 1]x$$
 (19)

which is FDLCP with period  $h = \pi$  and

$$A = \begin{bmatrix} 0 & 1\\ 0 & -2\zeta \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0\\ 1 - 2\cos 2t \end{bmatrix}$$

The system is not asymptotically stable. Let us stabilize it via a feedback  $u = -Kx + v = -[k_1, k_2]x + v$ , where  $k_1$  and  $k_2$  are constants. Implementing u = -Kx + v in (19) gives a closed-loop FDLCP system  $\dot{x} = A_c(t)x + B(t)v$  with  $A_c(t) =$ A - B(t)K. We will fix K such that  $A_c(t)$  is stable. This is nontrivial since  $A_c(t)$  is periodically time-varying.

As a solution, we implement u = -Kx + v in an approximate system  $\dot{x} = Ax + B_a u$ , where  $B_a = [0, 1]^T$ . This yields an LTI closed-loop system  $\dot{x} = A'_c x + B_a v$  with  $A'_c = A - B_a K$ . Clearly,  $B_a$  is obtained by dropping all high-order harmonics in B(t). Hence, the poles of the approximate LTI system are ones defined on the 0-th harmonic structure of the FDLCP system. Since the pair  $(A, B_a)$  is controllable, there are K's such that the LTI closed-loop system  $\dot{x} = A'_c x + B_a v$  can be stabilized via u = -Kx + v; for example, when  $\zeta = 0.1$  and we take specifically K = [0.1, 0.5], then the egienvalues of  $A'_c$  are -0.2000 and -0.5000.

Now our question is: does stability of the approximate LTI system  $\dot{x} = A'_c x + B_a v$  guarantee that of the closed-loop

FDLCP system  $\dot{x} = A_c(t)x + B(t)v$  under the same state feedback gain K? There are two ways to answer this question.

Firstly, we note that stability of the closed-loop FDLCP system is reflected by the eigenvalues of the so-called monodromy matrix by the Floquet theory. In other words, we compute the monodromy matrix  $\Phi(h, 0)$  of  $A_c(t)$  and examine its eigenvalues distribution. Stability of the closed-loop FDLCP system follows if all eigenvalues of  $\Phi(h, 0)$  fall inside the open unit disc. For the specific case, the monodromy matrix  $\Phi(h, 0)$  possesses two eigenvalues: 0.2079 and 0.5335, where  $\Phi(h, 0)$  is calculated by piecewise constant approximation on  $A_c(t)$  as in [9], [41].

Secondly, by means of the system poles in FDLCP systems and observing their distribution. In other words, stability of the closed-loop FDLCP system are revealed by the closed-loop system poles, which can be obtained by truncating the harmonic state operator  $\underline{A}_c - \underline{E}(j0)$  (where  $\underline{A}_c =: \mathcal{T}(A_c(t))$ ) and calculating its eigenvalues in the fundamental strip  $\mathbb{C}_f =: \{z \in \mathbb{C} : -\omega_h/2 < \text{Im}(z) \le \omega_h/2\}$ , as explained in [42]. For the aforementioned specific case, the system poles of  $\underline{A}_c - \underline{E}(j0)$  in  $\mathbb{C}_f$  are -0.2000 + 0.0000j and -0.5000 + 0.0000j.

Also related to the second method, we note that  $\underline{A}_c - \underline{E}(j0) = \underline{A} - \underline{E}(j0) - \underline{K} \underline{B}$ , which in form is the same as we see in LTI cases. This means that the second method can be interpreted as pole assignment in the harmonic sense (details are omitted for brevity). In contrast, if stability is tested via the Floquet theory as discussed in the first method, it is not a simple task to do any pole assignment due to the involement of the monodromy matrix.

In the next example, we consider how to numerically treat zeros and poles of the following  $\pi$ -periodic FDLCP system. Here,  $\beta$  is an input weighting parameter

$$\begin{cases} \dot{x} = \begin{bmatrix} -1 - \sin^2(2t) & 2 - \frac{1}{2}\sin(4t) \\ -2 - \frac{1}{2}\sin(4t) & -1 - \cos^2(2t) \end{bmatrix} x \\ + \begin{bmatrix} 0 \\ \beta\cos(2t) \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{cases}$$

The transition matrix of the example system has a Floquet factorization of the form

$$P(t,0) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -\sin(2t) & \cos(2t) \end{bmatrix}, \quad Q = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Since Floquet factors of the transition matrix are available, we are able to compute each scalar entry  $\underline{G}(s)(i/k)$  in  $\underline{G}(s)$  whenever i and k are specified. By trivial computations, we see that  $\hat{B}(t) = P^{-1}(t,0)B(t)$  and  $\hat{C}(t) = C(t)P(t,0)$  possess only finitely many nonzero harmonic waves. Therefore, all nonzero entries  $\underline{G}(s)(i/k)$  in  $\underline{G}(s)$  can be deduced through the following four formulas and arranged according to the pair index (i, k):

$$\begin{aligned} G_{-2}(s,q) &= -\frac{\beta}{8} \frac{1}{(s_q+1)(s_q+2)} \\ G_{-1}(s,q) &= \frac{\beta}{8} \frac{4s_q^2 + (13+8j)s_q + 10 + 14j}{(s_q-2j+2)(s_q-2j+1)(s_q+2j+2)} \\ G_1(s,q) &= \frac{\beta}{8} \frac{4s_q^2 + (13-24j)s_q - 22 - 40j}{(s_q-4j+2)(s_q-4j+1)(s_q+2)} \\ G_2(s,q) &= -\frac{\beta}{8} \frac{1}{(s_q-2j+1)(s_q-2j+2)} \end{aligned}$$

where  $s_q =: s + j2q$  is used for expression brevity. More precisely, the harmonic transfer operator can be expressed by (20), as shown at the bottom of the page, in which  $\bigcirc$  denotes the center of  $\underline{G}(s)$ , i.e.,  $\underline{G}(s)(0/0) = 0$ . By the definitions and simple computations, we can claim that for each specific  $q \in \mathbb{Z}$ 

$$\begin{cases} \mathcal{P}_{S}\left(\frac{-2+q}{1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{P}_{T}\left(\frac{-2+q}{1+q}\right) = \{-j2q - 1, -j2q - 2\} \\ \mathcal{Z}_{T}\left(\frac{-2+q}{1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ -\mathcal{P}_{T}\left(\frac{-2+q}{1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{P}_{T}\left(\frac{-1+q}{0+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{P}_{T}\left(\frac{-1+q}{0+q}\right) = \{j2 - j2q - 2, -j2 - j2q - 1, -j2 - j2q - 2\} \\ \mathcal{Z}_{T}\left(\frac{-1+q}{0+q}\right) = \{-j3.9366 - j2q - 1.4915, -j2.0634 - j2q - 1.7585\} \\ \mathcal{Z}_{c.d}\left(\frac{-1+q}{0+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ -\mathcal{P}_{T}\left(\frac{-1+q}{0+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{P}_{T}\left(\frac{0+q}{-1+q}\right) = \{-j2q - 2, -j4 - j2q - 2, -j4 - j2q - 2, -j4 - j2q - 1\} \\ \mathcal{Z}_{T}\left(\frac{0+q}{-1+q}\right) = \{-j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{P}_{T}\left(\frac{0+q}{-1+q}\right) = \{-j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{c.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{c.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{c.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{Z}_{r.d}\left(\frac{0+q}{-1+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\}$$

$$\begin{cases} \mathcal{P}_{S}\left(\frac{1+q}{-2+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ \mathcal{P}_{T}\left(\frac{1+q}{-2+q}\right) = \{j2 - j2q - 1, j2 - j2q - 2\} \\ \mathcal{Z}_{T}\left(\frac{1+q}{-2+q}\right) = \emptyset \\ \mathcal{Z}_{c.d}\left(\frac{1+q}{-2+q}\right) = \{j2(k-q) - 1, j2(k-q) - 2 : k \in \mathbb{Z}\} \\ -\mathcal{P}_{T}\left(\frac{1+q}{-2+q}\right) \end{cases}$$

The numerical results show that  $Z_{i.d} + Z_{o.d} - Z_{i.o.d}$  is empty. This reflects the fact that the FDLCP example system is completely controllable and observable. For any pairs in (-2+q, 1+q), (-1+q, 0+q), (0+q, -1+q) and (1+q, -2+q) over  $q \in \mathbb{Z}$ , the system poles are the same and distribute themselves equitably in a vertical strip parallel to the imaginary axis. The transfer poles in each pair (i, k) form just a fragment of the strip.

#### VI. CONCLUSION

Zeros and poles in the harmonic wave-to-wave sense are introduced in Section III-A for FDLCP systems through the regularized harmonic system operators and the harmonic transfer operators, which are well-defined in the sense of Propositions 3 and 4. Interesting zeros and poles relationships are established in Theorems 1 and 2 and the relevant remarks, which in form are similar to the counterparts in the LTI systems. We see that harmonic characteristics of FDLCP systems are essentially LTI. Hence, we can exploit analysis and synthesis techniques developed in the LTI setting for FDLCP systems whenever zeros and poles are necessary.

As pointed out by [27], it is difficult to define zeros and poles in simple and numerically tractable fashion on infinite-dimensional operators. This is also the case under the harmonic framework of FDLCP systems. This paper surmounts most difficulties related to zeros and poles of FDLCP systems by working on a harmonic wave-to-wave approach and exploiting the regularization technique. To clarify relationships between dynamical behaviours of FDLCP systems with zeros and poles defined in the paper is left for our subsequent study.

## APPENDIX A TOEPLITZ TRANSFORMATION

Expand the *h*-periodic function X(t) to its Fourier series  $\sum_{m=-\infty}^{+\infty} X_m e^{jm\omega_h t}$  with  $\omega_h = 2\pi/h$ . The Toeplitz transformation on X(t), i.e.,  $\mathcal{T}\{X(t)\}$ , maps X(t) into a Toeplitz operator [38] given by

$$\mathcal{T}\{X(t)\} := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \cdots & X_0 & X_{-1} & X_{-2} & \cdots \\ \cdots & X_1 & X_0 & X_{-1} & \cdots \\ \cdots & X_2 & X_1 & X_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} =: \underline{X}$$

By the terminology of [11, p. 564],  $\underline{X}$  is also termed a block Laurent operator and X(t) is called the defining function of the block Laurent operator  $\underline{X}$ .

$$\underline{G}(s) = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & G_{-1}(s, -1) & 0 & G_{-2}(s, 0) & 0 \\ \ddots & G_{1}(s, -1) & 0 & G_{-1}(s, 0) & 0 & G_{-2}(s, 1) & \ddots \\ \ddots & 0 & G_{1}(s, 0) & 0 & G_{-1}(s, 1) & 0 & \ddots \\ \ddots & G_{2}(s, 0) & 0 & G_{1}(s, 1) & 0 & G_{-1}(s, 2) & \ddots \\ & 0 & G_{2}(s, 1) & 0 & G_{1}(s, 2) & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$(20)$$

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## APPENDIX B DERIVATION OF (13)

By the 2-regularized determinant definition, we have

$$\det_{2} \left[ \begin{bmatrix} I & 0 \\ 0 & I_{r} \end{bmatrix} + \left( \underline{\Sigma}_{R}(s,\epsilon) \begin{pmatrix} i:r \\ k:r \end{pmatrix} - \begin{bmatrix} I & 0 \\ 0 & I_{r} \end{bmatrix} \right) \right]$$
$$= \det_{2} \left( \underline{\Sigma}_{R}(s,\epsilon) \begin{pmatrix} i:r \\ k:r \end{pmatrix} \right)$$
$$= \det_{2} \left( \begin{bmatrix} I - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & 0 \\ 0 & I_{r} \end{bmatrix} \right)$$
$$\cdot \begin{bmatrix} I & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot}) \\ \underline{C}(\frac{i:r}{k:r}) & \underline{D}(\frac{i:r}{k:r}) \end{bmatrix} \right)$$
(21)

where we note that  $\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon$  is invertible at each  $s \in \mathbb{C} \setminus (\{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\} \cup \Lambda)$ . This can be shown through the Floquet similarity formula of Proposition 1. By some lengthy but trivial arguments, one can say

and

$$\begin{bmatrix} -\underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & 0\\ 0 & 0_{r \times r} \end{bmatrix}$$
$$\begin{bmatrix} 0 & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot})\\ \underline{C}(\frac{\cdot}{k:r}) & \underline{D}(\frac{i:r}{k:r}) - I_r \end{bmatrix}$$

are Hilbert-Schmidt operators on  $l_2 \oplus \mathbb{C}^r$  for each  $s \in \mathbb{C} \setminus \{\epsilon + j\tau\omega_h : \tau \in \mathbb{Z}\} \cup \Lambda$ . Therefore, we can expand the last 2-regularized determinant in (21) by means of (3). This gives

$$\det_{2} \left( \underline{\Sigma}_{R}(s,\epsilon) \begin{pmatrix} i:r\\k:r \end{pmatrix} \right)$$

$$= \det_{2} \left( \begin{bmatrix} \underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & 0\\ 0 & I_{r} \end{bmatrix} \right)$$

$$\cdot \det_{2} \left( \begin{bmatrix} \underline{I} & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\underline{i:r})\\ \underline{C}(\underline{i:r}) & \underline{D}(\underline{i:r})\\ \underline{C}(\underline{i:r}) & \underline{D}(\underline{i:r}) \end{bmatrix} \right)$$

$$\cdot \exp \left\{ -\operatorname{tr} \left( \begin{bmatrix} -\underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & 0\\ 0 & 0_{r \times r} \end{bmatrix} \right)$$

$$\cdot \begin{bmatrix} 0 & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\underline{i:r})\\ \underline{C}(\underline{i:r}) & \underline{D}(\underline{i:r}) - I_{r} \end{bmatrix} \right) \right\}$$

$$= \det_{2} (\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon)$$

$$\cdot \det_{2} \left( \begin{bmatrix} \underline{I} & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\underline{i:r})\\ \underline{C}(\underline{i:r}) & \underline{D}(\underline{i:r}) \end{bmatrix} \right) .$$
(22)

In the derivation of (22), we used the following facts:

$$\det_2 \left( \begin{bmatrix} \underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon & 0\\ 0 & I_r \end{bmatrix} \right)$$
$$= \det_2(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon)$$

and

$$\exp\left\{-\operatorname{tr}\left(\begin{bmatrix}-\underline{E}^{-1}(s,\epsilon)\underline{A}+\epsilon & 0\\ 0 & 0_{r\times r}\end{bmatrix}\right) \cdot \begin{bmatrix}0 & -(\underline{E}(s)-\underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot})\\\underline{C}(\frac{\cdot}{k:r}) & \underline{D}(\frac{i:r}{k:r})-I_r\end{bmatrix}\right)\right\} = 1$$

which can be proved according to the basic points of Section II-A about determinant and trace.

Carefully examining the last equation in (22), we can concentrate our attention only on the second 2-regularized determinant since properties about the first 2-regularized determinant, i.e.,  $\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon)$ , have been clarified in Proposition 1. We have

$$\det_{2} \left( \begin{bmatrix} I & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot}) \\ \underline{C}(\frac{i:r}{k:r}) & \underline{D}(\frac{i:r}{k:r}) \end{bmatrix} \right)$$

$$= \det_{2} \left( \begin{bmatrix} I & 0 \\ \underline{C}(\frac{i:r}{k:r}) & I_{r} \end{bmatrix} \right)$$

$$= \det_{2} \left( \begin{bmatrix} I & 0 \\ \underline{C}(\frac{s}{k:r}) & I_{r} \end{bmatrix} \right)$$

$$= \det_{2} \left( \begin{bmatrix} I & 0 \\ \underline{C}(\frac{i:r}{k:r}) & I_{r} \end{bmatrix} \right)$$

$$\cdot \det_{2} \left( \begin{bmatrix} I & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot}) \\ 0 & \underline{C}(s)(\frac{i:r}{k:r}) \end{bmatrix} \right)$$

$$\cdot \exp \left\{ -\operatorname{tr} \left( \begin{bmatrix} 0 & 0 \\ \underline{C}(\frac{i:r}{k:r}) & 0_{r \times r} \end{bmatrix} \right)$$

$$\cdot \begin{bmatrix} 0 & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot}) \\ 0 & \underline{C}(s)(\frac{i:r}{k:r}) \end{bmatrix} \right) \right\}$$

$$= \det_{2} \left( \begin{bmatrix} I & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot}) \\ 0 & \underline{C}(s)(\frac{i:r}{k:r}) - I_{r} \end{bmatrix} \right) \right\}$$

$$= \det_{2} \left( \begin{bmatrix} I & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}(\frac{i:r}{\cdot}) \\ 0 & \underline{C}(s)(\frac{i:r}{k:r}) \end{bmatrix} \right)$$

$$\cdot \exp \left\{ -\operatorname{tr} \left( \begin{bmatrix} 0 & 0 \\ 0 & -\underline{C}(\frac{i:r}{k:r}) \end{bmatrix} \right) \right\}$$

$$= \det_{2} \left( \underbrace{G}(s)\left(\frac{i:r}{k:r}\right) \right)$$

$$\cdot \exp \left\{ \operatorname{tr} \left( \underbrace{C}(s)\left(\frac{i:r}{k:r}\right) \right) \right\}$$

$$(23)$$

In (23), the following result is used:

$$\det_2\left(\begin{bmatrix}\underline{I} & 0\\ \underline{C}(\frac{\cdot}{k:r}) & I_r\end{bmatrix}\right) = 1.$$

Besides, we assert that the operator matrix

$$\begin{bmatrix} 0 & -(\underline{E}(s) - \underline{A})^{-1}\underline{B}\left(\frac{i:r}{\cdot}\right) \\ 0 & \underline{G}(s)\left(\frac{i:r}{k:r}\right) - I_r \end{bmatrix}$$

is a Hilbert-Schmidt operator on  $l_2 \oplus \mathbb{C}^r$ , since  $\underline{G}(s)((i:r)/(k:r)) : \mathbb{C}^r \to \mathbb{C}^r$  is finite-dimensional and  $(\underline{E}(s) - \underline{A})^{-1}\underline{B}((i:r)/(\cdot)) : \mathbb{C}^r \to l_2$  is a Hilbert-Schmidt operator.

Again, note that  $\underline{G}(s)((i:r)/(k:r)): \mathbb{C}^r \to \mathbb{C}^r$  is finitedimensional. Thus  $\underline{G}(s)((i:r)/(k:r))$  is also a trace class operator. Then, Property 1.8(b) of [4, p. 17] yields that

$$\det_{2}\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)$$

$$= \det_{2}\left(I_{r} - I_{r} + \underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)$$

$$= \det\left(I_{r} - I_{r} + \underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)$$

$$\cdot \exp\left\{-\operatorname{tr}\left(-I_{r} + \underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)\right\}$$

$$= \det\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)$$

$$\cdot \exp\left\{r - \operatorname{tr}\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)\right\}$$
(24)

where  $det(\cdot)$  denotes the standard determinant of a matrix  $(\cdot)$  (see (2) for the details).

Now let us substitute (23) and (24) back into (22). Then simple algebra brings us that

$$\det_2\left(\underline{\Sigma}_R(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right)$$
  
=  $\det_2\left(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon\right) \det(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right)$   
 $\cdot \exp\left\{r - \operatorname{tr}\left(\underline{D}\left(\frac{i:r}{k:r}\right)\right)\right\}.$ 

Finally, if we write  $\Delta(s,\epsilon)((i:r)/(k:r)) = \det_2(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A}+\epsilon) \det(\underline{G}(s)((i:r)/(k:r)))$ , then (13) follows.

## APPENDIX C PROOFS FOR PROPOSITIONS AND THEOREMS

Proof of Proposition 3: Since the latter part of the assertion i) is merely a special case of the former part of the assertion i), it is sufficient only to show that  $\sum_{R}(s)(i/k) - \text{diag}[\underline{I}, 0_{m \times l}]$  is a Hilbert-Schmidt operator from  $l_2 \oplus \mathbb{C}^m$  to  $l_2 \oplus \mathbb{C}^l$ . To see this, we observe that

$$\underline{\Sigma}_{R}(s,\epsilon) \left(\frac{i}{k}\right) - \operatorname{diag}[\underline{I}, 0_{m \times l}] \\
= \begin{bmatrix} \underline{E}^{-1}(s,\epsilon) & 0\\ 0 & I_{l} \end{bmatrix} \begin{bmatrix} -\underline{A} + \epsilon & -\underline{B}(\frac{i}{\cdot})\\ \underline{C}(\frac{i}{k}) & \underline{D}(\frac{i}{k}) \end{bmatrix} \quad (25)$$

where  $I_l$  denotes an  $l \times l$  identity matrix. We have

$$\begin{split} \left\| \begin{bmatrix} -\underline{A} + \epsilon & -\underline{B}(\frac{i}{k}) \\ \underline{C}(\frac{i}{k}) & \underline{D}(\frac{i}{k}) \end{bmatrix} \right\|_{(l_{2} \oplus \mathbb{C}^{l})/(l_{2} \oplus \mathbb{C}^{m})} \\ &\leq \left\| \underline{A} + \epsilon \right\|_{l_{2}/l_{2}} + \left\| \underline{B}\left(\frac{i}{\cdot}\right) \right\|_{l_{2}/\mathbb{C}^{m}} \\ &+ \left\| \underline{C}\left(\frac{i}{k}\right) \right\|_{\mathbb{C}^{l}/l_{2}} + \left\| \underline{D}\left(\frac{i}{k}\right) \right\|_{\mathbb{C}^{l}/\mathbb{C}^{m}} \\ &\leq \sup_{t \in [0,h)} \left\| A(t) + \epsilon \right\| + \left( \sum_{\tau = -\infty}^{\infty} \|B_{\tau}\|^{2} \right)^{1/2} \\ &+ \left( \sum_{\tau = -\infty}^{\infty} \|C_{\tau}\| \right)^{1/2} + \|D_{i-k}\| < \infty \end{split}$$

where  $||\underline{A} + \epsilon||_{l_2/l_2} = \sup_{t \in [0,h)} ||A(t) + \epsilon||$  follows from Corollary XXIII.2.2 of [12, p. 567] and  $\{B_{\tau}\}_{\tau=-\infty}^{\infty}, \{C_{\tau}\}_{\tau=-\infty}^{\infty}$  and  $\{D_{\tau}\}_{\tau=-\infty}^{\infty}$  are the Fourier coefficients sequences of B(t), C(t) and D(t). The last inequality follows from the assumptions on A(t), B(t), C(t) and D(t). The above arguments say that the second operator matrix in the right-hand side of (25) is bounded.

Note by Proposition 1 that  $\underline{E}^{-1}(s) \in C_2(l_2)$ . This, together with the fact that  $I_l$  has finite-dimensional range, tells us that the first operator matrix in the right-hand side of (25) represents a Hilbert-Schmidt operator on  $l_2 \oplus \mathbb{C}^l$ . Then, by Property 1.3(b) of [4 p. 14], we have that  $\underline{\Sigma}_R(s)(i/k) - \text{diag}[\underline{I}, 0_{m \times l}]$  is compact and  $||\underline{\Sigma}_R(s)(i/k) - \text{diag}[\underline{I}, 0_{m \times l}]||_2 < \infty$ .

To show the assertion ii), let us notice by the Floquet similarity formula of Proposition 1 that

$$\underline{G}(s)\left(\frac{i}{k}\right) = \underline{\hat{C}}\left(\frac{\cdot}{k}\right)(\underline{E}(s) - \underline{Q})^{-1}\underline{\hat{B}}\left(\frac{i}{\cdot}\right) + \underline{D}\left(\frac{i}{k}\right) \quad (26)$$

where  $\underline{Q} = \mathcal{T}\{Q\}, \underline{\hat{B}} = \underline{P}^{-1}\underline{B}$ , and  $\underline{\hat{C}} = \underline{C}\underline{P}$ . If we denote the defining functions of  $\underline{\hat{B}}$  and  $\underline{\hat{C}}$  by  $\hat{B}(t)$  and  $\hat{C}(t)$ , respectively, from the assumptions on B(t), C(t) and Proposition 1, it follows that  $\hat{B}(t)$  and  $\hat{C}(t)$  also belong to  $L_{\text{CAC}}[0,h]$ . Hence, it can be validated [39] that  $\hat{B}(t) = P^{-1}(t,0)B(t)$  and  $\hat{C}(t) = C(t)P(t,0)$ .

Now let us denote the Fourier coefficients sequences of  $\hat{B}(t)$ and  $\hat{C}(t)$  by  $\{\hat{B}_{\tau}\}_{\tau=-\infty}^{\infty}$  and  $\{\hat{C}_{\tau}\}_{\tau=-\infty}^{\infty}$ . Then, (26) can be equivalently re-written as follows:

$$\underline{G}(s)\left(\frac{i}{k}\right) = \sum_{\tau=-\infty}^{\infty} \hat{C}_{k-\tau} \times (sI + j\tau\omega_h I - Q)^{-1} \hat{B}_{\tau-i} + D_{k-i}$$
(27)

whose partial summation is given by

$$\underline{G}_N(s)\left(\frac{i}{k}\right) =: \sum_{|\tau| \le N} \hat{C}_{k-\tau} \times (sI + j\tau\omega_h I - Q)^{-1} \hat{B}_{\tau-i} + D_{k-i}.$$

By the definition of  $\underline{G}_N(s)(i/k)$ , it is straightforward to see that for each specific  $N, \underline{G}_N(s)(i/k)$  is analytic over any bounded domain  $\Omega \subset \mathbb{C} \setminus \Lambda$ . Now we observe that

$$\begin{split} \left\| \frac{\underline{G}(s)\left(\frac{i}{k}\right) - \underline{G}_{N}(s)\left(\frac{i}{k}\right)}{\leq} \sum_{|\tau| > N} \left\| \hat{C}_{k-\tau} \right\| \cdot \left\| (sI + j\tau\omega_{h}I - Q)^{-1} \right\| \cdot \left\| \hat{B}_{\tau-i} \right\| \\ \leq \sup_{\tau \in \mathbb{Z}} \left\| (sI + j\tau\omega_{h}I - Q)^{-1} \right\| \left( \sum_{|\tau| > N} \left\| \hat{C}_{k-\tau} \right\|^{2} \right)^{1/2} \\ \cdot \left( \sum_{|\tau| > N} \left\| \hat{B}_{\tau-i} \right\|^{2} \right)^{1/2}. \end{split}$$

In the derivation the Cauchy-Schwarz inequality is used. It is not hard to see that there exists K > 0 such that for any  $s \in \Omega \subset \mathbb{C} \setminus \Lambda$ ,  $\sup_{\tau \in \mathbb{Z}} ||(sI+j\tau\omega_h I-Q)^{-1}|| < K$ . This, together with the facts that  $\sum_{|\tau|>N} ||\hat{B}_{\tau-i}||^2 \to 0$  and  $\sum_{|\tau|>N} ||\hat{C}_{k-\tau}||^2 \to 0$  as  $N \to \infty$ , implies that

$$\left\| \underline{G}(s) \left( \frac{i}{k} \right) - \underline{G}_N(s) \left( \frac{i}{k} \right) \right\| \to 0$$

as  $N \to \infty$ . This, in particular, means that  $\underline{G}_N(s)(i/k)$  converges to  $\underline{G}(s)(i/k)$  uniformly on  $\Omega \subset \mathbb{C} \setminus \Lambda$ . Hence, one can conclude by the complex analysis theory that  $\underline{G}(s)(i/k)$  is also analytic on  $\Omega \subset \mathbb{C} \setminus \Lambda$ . Note that  $\Omega$  can be any bounded subset of  $\mathbb{C} \setminus \Lambda$ . Then analytic continuation arguments lead us immediately that  $\underline{G}(s)(i/k)$  is analytic on  $\mathbb{C} \setminus \Lambda$ .

The uniform convergence argued in the above can validate the order interchange between the limit and the infinite summation in the following deductions:

$$\lim_{s \to s_0} (s - s_0)^{\mu} \underline{G}(s) \left(\frac{i}{k}\right)$$
$$= D_{k-i} + \sum_{\tau = -\infty}^{\infty} \hat{C}_{k-\tau}$$
$$\cdot \lim_{s \to s_0} (s - s_0)^{\mu} (sI + j\tau\omega_h I - Q)^{-1} \hat{B}_{\tau-i}$$

Clearly, for any  $s_0 \in \mathbb{C} \setminus \Lambda$ , one can take  $\mu = 0$  so that  $||\lim_{s \to s_0} (s - s_0)^{\mu} \underline{G}(s)(i/k)|| < \infty$ . On the other hand, for each  $s_0 \in \Lambda$  one can always take an appropriate integer  $0 \leq \mu \leq n$  such that  $||\lim_{s \to s_0} (s - s_0)^{\mu} \underline{G}(s)(i/k)|| < \infty$ . This implies that  $\underline{G}(s)(i/k)$  possesses only removeable singular points on  $\mathbb{C}$ ; or  $\underline{G}(s)(i/k)$  is meromorphic by definition.

*Proof of Proposition 4:* By definition,  $\det(\underline{G}(s)((i:r)/(k:r)))$  is the summation of finitely many muplications of r scalar entries of  $\underline{G}(s)((i:r)/(k:r))$ . Then from the assertion ii) of Proposition 3, analyticity and meromorphism of  $\det(\underline{G}(s)((i:r)/(k:r)))$  follows.

To see (14), assume that  $s_0$  is a singular point of multiplicity  $n_0$  in  $\rho$ {det( $\underline{G}(s)((i : r)/(k : r))$ )}. Note that  $\underline{G}(s)(i/k)$  has the expression of (27). By simple contradiction arguments, one can conclude that  $s_0$  must be a  $n_0$ -multiple singular point of at least one term in (27), say

$$\hat{C}_{k-\tau_0}(sI+j\tau_0\omega_hI-Q)^{-1}\hat{B}_{\tau_0-i}=:\hat{G}(s,\tau_0).$$

To show (14), suppose that (14) is the contrary. That is,  $s_0$  is only a  $n'_0$ -multiple eigenvalue of  $Q - j\tau_0\omega_h I$  with  $n'_0 < n_0$ . Then basic knowledge about poles of transfer functions of multivariable systems [30], [31] tells us that any singular point at  $s_0$ , if any, in each scalar entry of  $\hat{G}(s, \tau_0)$  is at most of multiplicity  $n'_0$ . From this, it follows readily that

$$|\lim_{s \to s_0} (s - s_0)^{\mu} \hat{G}(s, \tau_0)| = 0, \quad \forall \mu > n'_0, \ \tau_0 \in \mathbb{Z}$$

where  $\mu$  is an nonnegative integer. Using this back to (27), it follows after trivial arguments that  $s_0$  is a singular point of  $\det(\underline{G}(s)((i:r)/(k:r)))$  at most with multiplicity  $n'_0$ . This is contradictory to the multiplicity assumption about  $s_0$ . From this contradiction, (14) follows.

Analyticity and meromorphism of  $\Delta(s,\epsilon)((i:r)/(k:r))$ follows from those of  $\det_2(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{A} + \epsilon)$  and  $\det(\underline{G}(s)((i:r)/(k:r)))$ , which have already been claimed in Proposition 1 and the assertion i) of Proposition 4, respectively.

To complete the proof of the assertion ii), it remains only to show (15). By Proposition 1, we have  $\zeta \{ \det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon) \} = \Lambda$ . Taking (14) into account, we can assert that any elements in  $\rho \{ \det(\underline{G}(s)((i:r)/(k:r))) \}$  cannot be in  $\rho \{ \Delta(s, \epsilon)((i:r)/(k:r)) \}$ . Then (15) follows from the definition of  $\Delta(s, \epsilon)((i:r)/(k:r))$  and (8) claimed in Proposition 1.

*Proof of Theorem 1:* By the assumptions on A(t), B(t), C(t) and D(t), Propositions 1 and 2 say that we can define the following regularized harmonic system operator

based on the controllability/observability canonical form of the system (1)

$$\tilde{\underline{\Sigma}}_{R}(s,\epsilon) \left( \frac{i}{k} \right) = \begin{bmatrix} I - \underline{E}^{-1}(s,\epsilon)\underline{\hat{Q}} + \epsilon & -\underline{E}^{-1}(s,\epsilon)\underline{\hat{B}}\left(\frac{i}{k}\right) \\ \underline{\tilde{C}}\left(\frac{i}{k}\right) & \underline{D}\left(\frac{i}{k}\right) \end{bmatrix} \quad (28)$$
where  $\underline{\tilde{Q}} + \epsilon =: \underline{\tilde{Q}} + \epsilon$  and
$$\tilde{\underline{Q}} = \mathcal{T} \left\{ \begin{bmatrix} \tilde{Q}_{co} & 0 & \tilde{Q}_{13} & 0 \\ \tilde{Q}_{21} & \tilde{Q}_{c\bar{o}} & \tilde{Q}_{23} & \tilde{Q}_{24} \\ 0 & 0 & \tilde{Q}_{\bar{c}o} & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix}
 0 & 0 & Q_{43} & Q_{\overline{co}} \\
 \underline{\tilde{B}} = \mathcal{T} \left\{ \begin{bmatrix} B_{co}^T(t) & B_{c\overline{o}}^T(t) & 0 & 0 \end{bmatrix}^T \right\}$$

$$\underline{\tilde{C}} = \mathcal{T} \left\{ \begin{bmatrix} C_{co}(t) & 0 & C_{\overline{co}}(t) & 0 \end{bmatrix} \right\}$$

with  $\tilde{B}_{co}(t), \tilde{B}_{c\bar{o}}(t), \tilde{C}_{co}(t)$  and  $\tilde{C}_{\bar{c}o}(t)$  belonging to  $L_{CAC}[0, h]$ . We also define

$$\underline{\tilde{G}}(s)\left(\frac{i}{k}\right) = \underline{\tilde{C}}\left(\frac{\cdot}{k}\right)(\underline{E}(s) - \underline{\tilde{Q}})^{-1}\underline{\tilde{C}}\left(\frac{i}{\cdot}\right) + \underline{D}\left(\frac{i}{k}\right).$$
 (29)

By (28) and (29) and Proposition 1, it follows that

$$\begin{cases} \det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{A} + \epsilon) \\ = q(s, \epsilon)\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\tilde{Q} + \epsilon) \\ \det(\tilde{\underline{G}}(s)(\frac{i:r}{k:r})) = \det(\underline{G}(s)(\frac{i:r}{k:r})) \end{cases}$$
(30)

where  $q(s, \epsilon)$  is an analytic function on the whole complex plane and vanishes nowhere. These equations indicate that one can complete the proof by working on  $\tilde{\Sigma}_R(s, \epsilon)$  and  $\tilde{G}(s)$ .

To see the assertion i), we first observe by Definition 1 and Proposition 1 that

$$\mathcal{P}_S\left(\frac{i}{k}\right) = \Lambda, \quad \forall i, k \in \mathbb{Z}.$$
 (31)

Next, by the specific expressions of  $\underline{\tilde{Q}}$ ,  $\underline{\tilde{B}}$ , and  $\underline{\tilde{C}}$ , we have

$$\tilde{\underline{G}}(s) = D_{k-i} + \sum_{\tau = -\infty}^{\infty} \tilde{C}_{\mathrm{co},k-\tau} \\
\cdot (sI_1 + j\tau\omega_h I_1 - \tilde{Q}_{\mathrm{co}})^{-1} \tilde{B}_{\mathrm{co},\tau-i} \quad (32)$$

where  $\{\tilde{B}_{\mathrm{co},\tau}\}_{\tau=-\infty}^{\infty}$  and  $\{\tilde{C}_{\mathrm{co},\tau}\}_{\tau=-\infty}^{\infty}$  denote the Fourier coefficients sequences of  $\tilde{B}_{\mathrm{co}}(t)$  and  $\tilde{C}_{\mathrm{co}}(t)$ , respectively. Equation (32) clearly says that any eigenvalues of  $\underline{Q} - \underline{E}(j0)$  corresponding to  $\tilde{Q}_{c\bar{o}} - j\tau\omega_h I_2$ ,  $\tilde{Q}_{\bar{c}o} - j\tau\omega_h I_3$  and  $\tilde{Q}_{\bar{c}\bar{o}} - j\tau\omega_h I_4$  for each  $\tau \in \mathbb{Z}$  do not emerge in any scalar entries of  $\underline{\tilde{G}}(s)$ . Thus, these eigenvalues of  $\underline{\tilde{Q}} - \underline{E}(j0)$  cannot emerge in  $\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))$  as singular points. Bearing this in mind, the definitions of  $\mathcal{Z}_{i.d}$ ,  $\mathcal{Z}_{o.d}$  and  $\mathcal{Z}_{i.o.d}$  imply immediately that

$$\mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} \prec \zeta \{ \det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{\tilde{Q}} + \epsilon) \} = \mathcal{P}_S\left(\frac{i}{k}\right)$$
(33)

Further, by the assertion i) of Proposition 4. one can see

$$o\{\det(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right))\} \prec \mathcal{P}_S\left(\frac{i}{k}\right).$$
 (34)

Note by (32) that the singular points in  $\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))$  reflects modes of the

system (1) that are different from those in  $Z_{i.d} + Z_{o.d} - Z_{i.o.d}$ in the sense of (28) and (32). This, together with (33) and (34), implies that there exists a set  $Z_{c.d}((i:r)/(k:r)) \prec \mathcal{P}_S(i/k)$ satisfying

$$\mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \rho \left\{ \det \left( \frac{\tilde{G}(s) \left( \frac{i:r}{k:r} \right)}{k:r} \right) \right\} + \mathcal{Z}_{c.d} \left( \frac{i:r}{k:r} \right) = \mathcal{P}_S \left( \frac{i}{k} \right) \quad (35)$$

where  $\mathcal{Z}_{c.d}((i:r)/(k:r))$  may be empty but satisfies

$$\wedge \left[ \mathcal{Z}_{\mathrm{c.d}} \left( \frac{i:r}{k:r} \right) \right] =: \mathcal{Z}_{\mathrm{c.d}} \left( \frac{i}{k} \right).$$

In the above,  $\wedge [\cdot]$  runs over all  $i_1, \ldots, i_r \in \{1, 2, \ldots, m\}$  and  $k_1, \ldots, k_r \in \{1, 2, \ldots, l\}$ .

Now we examine the operation  $\forall [\cdot]$  on (35) over all  $i_1, \ldots, i_r \in \{1, 2, \ldots, m\}$  and  $k_1, \ldots, k_r \in \{1, 2, \ldots, l\}$ . By (31), we obtain  $\forall [\mathcal{P}_S(i/k)] = \mathcal{P}_S(i/k)$  and thus

$$\mathcal{P}_{S}\left(\frac{i}{k}\right) = \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \vee \left[\rho \left\{ \det\left(\frac{\tilde{G}(s)\left(\frac{i:r}{k:r}\right)\right)\right\} + \mathcal{Z}_{c.d}\left(\frac{i:r}{k:r}\right) \right] = \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \vee \left[\rho \left\{ \det\left(\frac{\tilde{G}(s)\left(\frac{i:r}{k:r}\right)\right)\right\} \right] + \wedge \left[\mathcal{Z}_{c.d}\left(\frac{i:r}{k:r}\right)\right] = \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \mathcal{P}_{T}\left(\frac{i}{k}\right) + \mathcal{Z}_{c.d}\left(\frac{i}{k}\right).$$

In the deduction, used the fact that we  $\rho\{\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))\}$ + $\mathcal{Z}_{c.d}((i:r)/(k:r))$ a fixed set that is actually independent is of i,ke  $\mathbb{Z}, i_1, \ldots, i_r$  $\in$  $\{1, 2, \ldots, m\}$ and  $k_1,\ldots,k_r$  $\in$  $\{1, 2, \ldots, l\}$ . With this, the proof for the assertion i) is accomplished.

To show the assertion ii), let us take into account possible zero/singular-points cancellations between  $\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{\tilde{Q}} + \epsilon)$  and  $\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))$  and observe that

$$\begin{split} & \zeta \left\{ \det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{\tilde{Q}} + \epsilon) \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ &= \zeta \{\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{\tilde{Q}} + \epsilon)\} \\ &- \zeta \{\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{\tilde{Q}} + \epsilon)\} \wedge \rho \left\{ \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ &+ \zeta \left\{ \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ &- \rho \{\det_2(\underline{I} - \underline{E}^{-1}(s, \epsilon)\underline{\tilde{Q}} + \epsilon)\} \wedge \zeta \left\{ \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \end{split}$$

$$= \mathcal{P}_{S}\left(\frac{i}{k}\right) - \mathcal{P}_{S}\left(\frac{i}{k}\right) \wedge \rho \left\{ \det\left(\frac{\tilde{G}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ + \zeta \left\{ \det\left(\frac{\tilde{G}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ - \sum_{q=1}^{n} \left\{ \epsilon + j\tau_{q}\omega_{h} : \tau_{q} \in \mathbb{Z} \right\} \\ + \rho \left\{ \det\left(\Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ = \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \mathcal{Z}_{c.d}\left(\frac{i:r}{k:r}\right) \\ + \zeta \left\{ \det\left(\frac{\tilde{G}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ - \sum_{i=1}^{n} \left\{ \epsilon + j\tau_{i}\omega_{h} : \tau_{i} \in \mathbb{Z} \right\} \\ + \rho \left\{ \det\left(\Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right) \right\}.$$
(36)

In the above, some arguments in proving the assertion i) are used. In addition, by the assertion ii) of Proposition 4, we have

$$\rho\{\det_2(\underline{I} - \underline{E}^{-1}(s,\epsilon)\tilde{Q} + \epsilon)\}$$
  
 
$$\wedge \zeta \left\{ \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\}$$
  
 
$$= \sum_{q=1}^n \{\epsilon + j\tau_q \omega_h : \tau_q \in \mathbb{Z}\}$$
  
 
$$- \rho \left\{ \det\left(\Delta(s,\epsilon)\left(\frac{i:r}{k:r}\right)\right) \right\}$$

Combining (36) with (16) and (30), it follows that

$$\begin{aligned} \mathcal{Z}_{S}\left(\frac{i}{k}\right) &= \wedge \left[\zeta \left\{ \det_{2}(\underline{I} - \underline{E}^{-1}(s,\epsilon)\underline{\tilde{Q}} + \epsilon) \right. \\ &\quad \cdot \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \\ &\quad + \sum_{q=1}^{n} \{\epsilon + j\tau_{q}\omega_{h} : \tau_{q} \in \mathbb{Z} \} \\ &\quad -\rho \left\{ \Delta \left(s,\epsilon\right)\left(\frac{i:r}{k:r}\right)\right) \right\} \right] \\ &= \wedge \left[ \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \mathcal{Z}_{c.d}\left(\frac{i:r}{k:r}\right) \\ &\quad + \zeta \left\{ \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \right] \\ &= \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \wedge \left[ \mathcal{Z}_{c.d}\left(\frac{i:r}{k:r}\right) \right] \\ &\quad + \wedge \left[ \zeta \left\{ \det\left(\underline{\tilde{G}}(s)\left(\frac{i:r}{k:r}\right)\right) \right\} \right] \\ &= \mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d} + \mathcal{Z}_{c.d}\left(\frac{i}{k}\right) \\ &\quad + \mathcal{Z}_{T}\left(\frac{i}{k}\right). \end{aligned}$$

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In the above deduction, we used the following assertion:

$$\wedge \left[ \mathcal{Z}_{c.d} \left( \frac{i:r}{k:r} \right) + \zeta \left\{ \det \left( \frac{\tilde{G}(s) \left( \frac{i:r}{k:r} \right)}{k:r} \right) \right\} \right] \\ = \wedge \left[ \mathcal{Z}_{c.d} \left( \frac{i:r}{k:r} \right) \right] + \wedge \left[ \zeta \left\{ \det \left( \frac{\tilde{G}(s) \left( \frac{i:r}{k:r} \right)}{k:r} \right) \right\} \right].$$

To see this, we note that  $(\cdot) \succ (\cdot)$  is obvious. It remains to show  $(\cdot) \prec (\cdot)$ . Assume that  $s_0$  belongs to  $\wedge [\mathbb{Z}_{c.d}((i:r)/(k:r)) + \zeta \{\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))\}]$  but  $s_0$  belongs to neither  $\wedge [\mathbb{Z}_{c.d}((i:r)/(k:r))]$  nor  $\wedge [\zeta \{\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))\}]$ . It is easy to see that such  $s_0$  must appear in  $\zeta \{\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))\}$  whenever  $s_0$  is not in  $\mathbb{Z}_{c.d}((i:r)/(k:r))$  and vice versa. It is also clear that  $s_0$  must belong at least to the set  $\mathbb{Z}_{c.d}(\cdot)$  of another  $r \times r$  substructure in  $\underline{\tilde{G}}(s)(i/k)$  that is different from  $\underline{\tilde{G}}(s)((i:r)/(k:r))$ . In other words,  $s_0$  must belong to  $\mathcal{P}_S(i/k)$  since  $\mathbb{Z}_{c.d}(\cdot) \prec \mathcal{P}_S(i/k)$  by definition. Recalling that  $s_0$  is not in  $\mathbb{Z}_{c.d}((i:r)/(k:r))$ , it follows that  $s_0$  must be in  $\rho \{\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))\}$ . However, this is impossible since  $s_0$  is already in  $\zeta \{\det(\underline{\tilde{G}}(s)((i:r)/(k:r)))\}$ . This completes the proof for the assertion ii).

Proof of Theorem 2: The assertion i) is obvious. To see the assertion ii), it is enough to notice that for any  $\gamma \in \mathbb{Z}$ ,  $\underline{\Sigma}_R(s,\epsilon)((i+\gamma)/(k+\gamma))$  and  $\underline{G}(s)((i+\gamma)/(k+\gamma))$  can be expressed by  $\underline{\Sigma}_R(s+j\gamma\omega_h,\epsilon)(i/k)$  and  $\underline{G}(s+j\gamma\omega_h)(i/k)$ , respectively, if s in  $\underline{\Sigma}_R(s,\epsilon)(i/k)$  and  $\underline{G}(s)(i/k)$  is replaced by  $s+j\gamma\omega_h$ .

To see the assertion iii), it suffices to show that for any  $k \in \mathbb{Z}$ ,  $\wedge_{\forall i \in \mathbb{Z}} [\mathcal{Z}_{c.d}(i/k)] = \emptyset$ . To this end, let us suppose the contrary. That is, there exists  $z_0 \in \mathcal{Z}_{c.d}(i/k)$  but  $z_0 \notin \mathcal{P}_T(i/k)$  for all  $i \in \mathbb{Z}$ . By the definition of  $\mathcal{Z}_{c.d}(i/k), z_0$  is not a mode belonging to  $\mathcal{Z}_{i.d} + \mathcal{Z}_{o.d} - \mathcal{Z}_{i.o.d}$ . Interpreting this along terms in (32), it follows that at least for one triple  $(i, k_0, \tau_0), z_0$  is a singular point of

$$\tilde{C}_{\mathrm{co},k_0-\tau_0} (sI_1 + j\tau_0 \omega_h I_1 - \tilde{Q}_{\mathrm{co}})^{-1} + \tilde{B}_{\mathrm{co},\tau_0-i} + D_{k_0-i} =: \{a_{\mu\nu}(s)\}$$

Then basic knowledge about zeros and poles in multivariable systems [23], [30], [31] leads that  $z_0$  must be a transfer function pole of  $\{a_{\mu\nu}(s)\}$ ; or equivalently, there is at least one  $r \times r$  submatrix in  $\{a_{\mu\nu}(s)\}$ , say  $\{a_{\mu\nu}(s)\}((\mu : r)/(\nu : r))$ , such that det $(\{a_{\mu\nu}(s)\}((\mu : r)/(\nu : r)))$  possesses a singular point at  $z_0$ .

On the other hand, (32) tells that it is always possible to re-write  $\underline{G}(s)(i/k) = \{a_{\mu\nu}(s) + b_{\mu\nu}(s)\}$  with an appropriate matrix  $\{b_{\mu\nu}(s)\}$ . After trivial computations, we see that

$$\det\left(\underline{G}(s)\left(\frac{i:r}{k:r}\right)\right) = \det\left(\{a_{\mu\nu}(s)\}\left(\frac{\mu:r}{\nu:r}\right)\right) + f(s)$$

where f(s) is a complex function whose exact expression has no significance for our arguments. It follows that  $z_0$  is a singular point of det $(\underline{G}(s)((i : r)/(k : r)))$  and thus  $z_0 \in \mathcal{P}_T(i/k)$  by Definition 1. Thus, the assumption on  $z_0$  cannot be true.

Finally, to show the assertion iv) let us suppose the contrary. It follows that  $Z_T(i/k)$  contains a convergent points sequence, on each point of which  $\det(\underline{G}(s)((i:r)/(k:r))) = 0$  for any  $i_1, \ldots, i_r \in \{1, 2, \ldots, m\}$  and  $k_1, \ldots, k_r \in \{1, 2, \ldots, l\}$ . Note that  $\det(\underline{G}(s)((i:r)/(k:r)))$  is analytical on  $\mathbb{C} \setminus \Lambda$ . This implies by the theorem of identity that  $\det(\underline{G}(s)((i:r)/(k:r))) \equiv 0$  for all  $s \in \mathbb{C} \setminus \Lambda$  for any  $i_1, \ldots, i_r \in \{1, 2, \ldots, m\}$  and  $k_1, \ldots, k_r \in \{1, 2, \ldots, l\}$ . However, this is contradictory to  $\operatorname{rank}(\underline{G}(s_0)(i/k)) = r \neq 0$  for some  $s_0 \in \mathbb{C} \setminus \Lambda$ .

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