Geometric Spanner of Objects Under L_1 Distance

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Abstract. Geometric spanner is a fundamental structure in computational geometry and plays an important role in many geometric networks design applications. In this paper, we consider the following generalized geometric spanner problem under L_1 distance: Given a set of disjoint objects S, find a spanning network G with minimum size so that for any pair of points in different objects of S, there exists a path in G with length no more than t times their L_1 distance, where t is the stretch factor. We specifically focus on three types of objects: rectilinear segments, axis aligned rectangles, and rectilinear polygons. By combining the ideas of *t-weekly dominating set* and *imaginary Steiner points*, we develop a 2-approximation algorithm for each type of objects. Our algorithms run in near quadratic time, and can be easily implemented for practical applications.

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1 Introduction

In this paper, we consider the following generalization of the classical geometric spanner problem: Given a set S of n disjoint objects in L_1^2 space (i.e., 2-dimensional space with L_1 norm) and a constant $t > 1$, construct a graph G for S of minimum size (i.e. the number of vertices and edges is minimized) so that for any pair of points $p_i \in o_i$ and $p_j \in o_j$, there exists a path $P(p_i, p_j)$ in G whose total length is at most $t \times d(p_i, p_j)$, where o_i and o_j are objects in S and $d(p_i, p_j)$ is the L_1 (or Manhanttan) distance between p_i and p_j. The path $P(p_i, p_j)$ consists of three parts, P_1, P_2 and P_3, where P_1 and P_3 are the portions of $P(p_i, p_j)$ inside o_i and o_j respectively. We assume that there implicitly exists an edge (or path) between any pair of points inside each object $o \in S$. Thus, the objective of minimizing the size of G is equivalent to minimizing the total number of vertices, and edges between vertices in different objects. In this paper, we consider the cases where objects are disjoint rectilinear segments, axis aligned rectangles, and rectilinear polygons in L_1^2 space.

Spanner is a fundamental structure in computational geometry and finds applications in many different areas. Extensive researches have been done on this structure and a number of interesting results have been obtained [1–11]. Almost all previous results consider the case in which the objects are points and seek to minimize the spanner's construction time, size, weight, maximum degree of vertex, diameter, or any combination of them.

A common approach for constructing geometric spanner is the use of Θ-graph [1–4]. In [5], Arya *et al.* showed that a t-spanner with constant degree can be constructed in $O(n \log n)$ time. In [6, 7], they gave a randomized construction of a sparse t-spanner with expected spanner diameter $O(\log n)$. In [9, 10], Das *et al.* proposed an $O(n \log^2 n)$-time greedy algorithm for a t-spanner with $O(n)$ edges and $O(1)wt(MST)$ weight in 3-D space. Gudmundsson *et al.* showed in [11] that an $O(n)$ edges, and $O(1)wt(MST)$ weight t-spanner is possible to be constructed in $O(n \log n)$ time.

In graph settings, Chandar *et al.* [8] showed that for an arbitrary positive edge-weighted graph G and any $t > 1, \epsilon > 0$, a t-spanner of G with weight $O(n^{\frac{2+\epsilon}{t-1}})wt(MST)$ can be constructed in polynomial time. They also showed that $(\log^2 n)$-spanners of weight $O(1)wt(MST)$ can be constructed.

For geometric spanners of objects other than points, Asano *et al.* considered the problem of constructing a spanner graph for a set of axis-aligned rectangles using rectilinear bridges and under L_1 distance [12]. They showed that in general it is NP-hard to minimize the dilation, and when the spanner graph is restricted to be trees with rectilinear edges, the problem can be solved using a linear program. They also considered other simple graphs such as paths and sorted paths, and presented polynomial time solution for each of them.

In [13], Yang *et al.* generalized the geometric spanner structure from points to segments and considered the problem of constructing a minimum-sized t-spanner for a set of disjoint segments in Euclidean space. They showed that a constant approximation can be obtained in $O(|Q| + n^2 \log n)$ time if the segments are relatively well separated, where Q is the set of vertices (called Steiner points) of G.

The problem considered in this paper is motivated by several applications. First, since the segment spanner in [13] can be viewed as a special case of rectangle (or polygon) spanner, its applications in architecture and wireless mesh networks imply applications for the spanners constructed in this paper. Second, the spanner of rectilinear polygons under L_1 distance also finds its own application in VLSI layout. In such applications, a set of pre-layouted modules (represented as rectangles or polygons) are to be connected by a set of (mainly rectilinear) wires (or network). To minimize the latency, for each pair of locations in different modules, it is expected that their shortest path in the network has length close to their L_1 distance, making the network design problem be a polygon spanner problem.

To solve the aforementioned problem, we further extend in this paper the concept of geometric spanner to polygons. Particularly, we consider three types of objects, rectilinear segments, axis-aligned rectangles, and rectilinear polygons. We show that our framework for constructing geometric spanner of segments in [13] can be generalized to polygons and achieves much better performance ratios. Our approach builds the spanner in two steps. First, we identify a set of points, called *Steiner points*, from each object; Then a t-spanner is constructed for the Steiner points by applying some existing algorithms for point spanners such as the ones in [14]. Thus, our focus will be only on the first step. Furthermore, since most existing spanners are sparse graphs (i.e. consist of $O(n)$ edges), minimizing the size of the spanner for rectilinear polygons is equivalent to minimizing the total number of Steiner points. Our objective is hence to obtain a spanner with a minimum number of Steiner points.

Minimizing the number of Steiner points is in general quite challenging. Part of the reason is that the position of a Steiner point on one object affects not only the positions of the Steiner points on the same object but also on other objects. To overcome this difficulty, we first generalize the concept of *weakly dominating set* in [13] to lower bound the number of Steiner points on one object. By using some imaginary Steiner points and a few other interesting techniques, we are able to find a set of strongly dominating set for each object. We show that the size of the strongly dominating set is a 2-approximation of the optimal solution. Our algorithm can be easily implemented and runs in near quadratic time. Our technique can be easily extended to higher dimensional space.

Due to space limit, we omit a lot of details in some proofs from this extended abstract.

2 Main Ideas

Let $S = \{O_1, O_2, ..., O_n\}$ be a set of n disjoint connected objects in L_1^2 space. A t-spanner G_S of S is a network which connects the objects in S and satisfies the following condition. For any two points p_i and p_j in objects $O_i \in S$ and $O_j \in S, i \neq j$, respectively, there exists a path (called spanner path) in G_S between p_i and p_j with length no more than $t|p_ip_j|$, where t is the *stretch factor* of the spanner and $|p_ip_j|$ is the L_1 distance between p_i and p_j. The spanner G_S consists of the objects, some sample points (called Steiner points) of the objects, and line segments (called *bridges*) connecting the Steiner points. We assume that there is an implicit path between p_i (or p_j) to any Steiner point in O_i (or O_j). Thus the spanner path between p_i and p_j includes an implicit path from p_i to some Steiner point $q_i \in O_i$ and and an implicit path from p_j to some Steiner point $q_j \in O_j$ (see Figure 1).

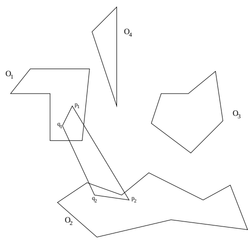

Fig. 1. Spanner Path between p_1 and p_2: $p_1 \to q_1 \to q_2 \to p_2$, where q_1 and q_2 are the Steiner points.

As mentioned in previous section, our main objective for the spanner G_S is to minimize its size. The size of G_S is the sum of the complexities of objects in S and the numbers of Steiner points and bridges. Since the total complexity of the objects is fixed, minimizing the size of G_S is equivalent to minimize the total number of Steiner points and bridges.

To simplify the optimization task, our main idea is to separate the procedure of minimizing the number of Steiner points from that of minimizing the number of bridges. In following sections, for each type of objects (i.e., rectilinear segments, axis aligned rectangles and rectilinear polygons), we first compute a set Q of Steiner points with small size, and then construct a spanner G_Q for Q to minimize the number of bridges. The spanner G_Q together with the objects forms the spanner of S (i.e. G_S). Since most existing spanner algorithms for points yield spanners with linear number of edges, the difficulty of minimizing the size of G_S lies on minimizing the number of Steiner points.

To illustrate our main ideas on minimizing Steiner points, we first briefly discuss the framework for all three types of objects inherited from our algorithm for constructing segment spanners in [13]. We start with selecting Steiner points for a pair of objects.

Let O_1 and O_2 be two different objects in S and p_1 and p_2 be a pair of arbitrary points in O_1 and O_2 respectively. Let $q_1 \in O_1$ and $q_2 \in O_2$ be two Steiner points close enough to p_1 and p_2.

Definition 1 (t-Domination). *Steiner points q_1 and q_2 t-dominate p_1 and p_2 if the path $p_1 \to q_1 \to q_2 \to p_2$ is a t-spanner path for p_1 and p_2 (i.e., the length of the path is no more than $t \times |p_1p_2|$, where $|p_1p_2|$ is the length of the segment $\overline{p_1p_2}$). q_1 and q_2 are called the t-dominating pair of p_1 and p_2.*

From the definition, it is clear that the positions of q_1 and q_2 are constrained by p_1 and p_2. If we fix p_1, p_2, and one Steiner point q_1, then all possible positions of the other Steiner point q_2 form a (possibly empty) region denoted as $R(p_1, p_2, q_1)$ (which is a function of p_1, p_2 and q_1) in O_2 (see Figure 2). When q_1 moves in O_1, the region changes accordingly. Similarly, if we fix the two Steiner points q_2, q_1, together with p_2, all points in O_1 t-dominated by q_2 and q_1, with respect to p_2, also form an region $R(q_2, q_1, p_2)$ in O_1.

Since the spanner G_S needs to guarantee that there exists a spanner path (or equivalently a t-dominating pair of Steiner points) from p_2 to every point in O_1, from p_2's point of view, it expects q_2 to be in some position such that O_1 can be covered by a minimum number of q_1's. , i.e. the union of $R(q_2, q_1, p_2)$ covers O_1. Thus, to determine Steiner points in O_1, we need to (1) identify a minimum set of Steiner points to cover all points in O_1 and (2) find a way to deal with the influence of the Steiner points (e.g., q_2) in O_2 and other objects.

To overcome these two difficulties, we relax the constraints in the definition of t-domination.

Definition 2 (t-Weak Domination). *Steiner point q_1 t-weakly dominates p_1 and p_2 if q_1 and p_2 are the t-dominating pair of p_1 and p_2. q_1 t-weakly dominates p_1 if for any $p_2 \in O_2$, q_1 t-weakly dominates p_1 and p_2.*

In the above definition, we assume that q_2 can be placed at arbitrary position in O_2 (or equivalently every point in O_2 is a Steiner point), when placing Steiner points in O_1. With this relaxation, we only need to consider the relation between q_1 and p_1, p_2. More specifically, we only need to find a minimum number of points in O_1 so that every point p_1 in O_1 is t-weakly dominated by some selected Steiner point. We call such a set of points as a *t-weakly dominating set* of O_1. We will show in following sections how to select t-weakly dominating set for each object (i.e., overcoming difficulty (1)).

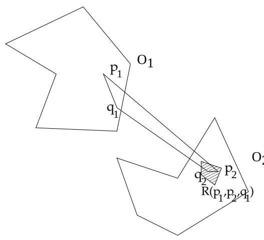

Fig. 2. The Region Dominating p_2 with p_1 and q_1 fixed

The concept of weakly dominating sets helps us to avoid the influence of Steiner points from other objects (i.e., difficulty (2)). However, t-weakly dominating sets alone do not guarantee the existence of t-dominating pair for each pair points $p_1 \in O_1$ and $p_2 \in O_2$. To overcome this difficulty, we use the concept of *imaginary Steiner points*. More specifically, let p_m be the median point of the segment $\overline{p_1 p_2}$. When we determine the position of q_1 for p_1, we assume that there is an imaginary Steiner point at p_m and find q_1 so that q_1 t-weakly dominates p_1 and p_m. Similarly we can find q_2 to t-weakly dominate p_2 and p_m. As shown in [13], such pair of q_1 and q_2 is a t-dominating pair for p_1 and p_2. All Steiner points in O_1 computed using imaginary Steiner points are called the t-dominating set of O_1 (with respect to O_2).

For the case of more than two objects, we first compute weak visibility graph for each object $O_i \in S$ and consider the Steiner-point-determination problem for O_i and each object weakly visible to O_i. The set of Steiner points in O_i computed from its weakly visible objects is called the *t-strongly dominating set* of O_i.

3 Constructing t-Spanner for Rectilinear Segments Under L_1 Distance

In [13], an $O(1)$-approximation algorithm was designed for constructing a spanner of segments under L_2 distance. In this section, we consider a special case of the segment spanner problem in which the input is a set S of rectilinear segments, and the distance function is the L_1 norm (i.e., the Manhattan distance). We show that for this special case, a much better performance ratio (i.e., 2) can be achieved.

Let s_1 and s_2 be two rectilinear segments in S, and p_1 and p_2 be two arbitrary points on s_1 and s_2 respectively. Let $q_1 \in s_1$ and $q_2 \in s_2$ be two Steiner points of p_1 and p_2.

It is easy to see that when the two segments have different orientations (i.e., one horizontal and the other vertical, say s_1 is horizontal and s_2 is vertical), one Steiner point on each segment (i.e., the point closest to the other segment) is sufficient to t-dominate the corresponding segment. Thus we only focus on the case in which s_1 and s_2 have the same orientation. Without loss of generality, we assume that s_1 and s_2 are all horizontal segments.

Let q_1 be a Steiner point in s_1 t-weakly dominating p_1 and e_{1l} and e_{1r} be the two endpoints of the region $R(p_1, p_2, p_2)$ (i.e., the interval of all possible positions of the Steiner point q_1 when q_2 coincides with p_2). Then we have the following lemma.

Lemma 1. *Let s_1 and s_2 be defined as above. Then the two endpoints e_{1l} and e_{1r} of $R(p_1, p_2, p_2)$ locate on different sides of p_1 with one of them equal to $\min\{|p_1a_1|, |p_1b_1|, \frac{t-1}{2}|p_1p_2| + |p_1p_2|_x\}$ and the other equal to $\min\{|p_1a_1|, |p_1b_1|, \frac{t-1}{2}|p_1p_2|\}$, where $|p_1p_2|_x$ is the distance along x-axis between p_1 and p_2 and a_1 and b_1 are the two endpoints of s_1.*

Proof. Assume without loss of generality that p_2 is to the left of p_1, then i) if q_1 is placed to the left of p_1, it is easy to see that q_1 is also to the left of p_2, otherwise there is no need to use q_1 as the Steiner point for p_1 due to the property of L_1 distance, therefore we have $|q_1p_2| = |p_1p_2|_y + |p_1q_1| - |p_1p_2|_x$; ii) if q_1 is placed to the right of p_1, we have $|q_1p_2| = |p_1q_1| + |p_1p_2|$. By the spanner property, we have $|p_1q_1| + |q_1p_2| \le t|p_1p_2|$. Solving the system of the equations, we get i) if q_1 is placed to the left of p_1, $|p_1q_1| \le \frac{t-1}{2}|p_1p_2| + |p_1p_2|_x$; ii) if q_1 is placed to the right of p_1, $|p_1q_1| \le \frac{t-1}{2}|p_1p_2|$. The lemma follows since q_1 has to be placed within s_1. □

Lemma 2. *The minimum of both $|p_1e_{1l}|$ and $|p_1e_{1r}|$ is $\min\{|p_1a_1|, |p_1b_1|, \frac{t-1}{2}|p_1p_2|\}$. $|p_1e_{1l}|$ (or $|p_1e_{1r}|$) achieves its minimum either when e_{1l} coincides with a_1 (or e_{1r} coincides with b_1), or p_2 is at the endpoints of s_2, or $|p_1p_2|$ is a constant that only depends on s_1 and s_2.*

Proof. In the proof of Lemma 1, it is clear that $|p_1e_{1l}|$ (or $|p_1e_{1r}|$) achieves its minimum when e_{1l} (or e_{1r}) and p_2 are on the different sides of p_1, and the minimum is $\min\{|p_1a_1|, |p_1b_1|, \frac{t-1}{2}|p_1p_2|\}$. To minimize the value of $|p_1p_2|$, p_2 should be picked as the nearest point to p_1 on s_2. Thus, p_2 is either an endpoint of s_2, or the point on s_2 that has the same x-coordinate or y-coordinate with p_1. In the latter case, $|p_1p_2|$ is the distance between s_1 and s_2, which is a constant when both segments are given. □

Let m be the parameter of p_1 in its convex combination of the two endpoints of s_1, i.e. $p_1 = (1-m)a_1 + mb_1$, for some $m \in [0, 1]$. Let $L_{1,2}(m)$ and $R_{1,2}(m)$ be the functions defining the positions of e_{1l} and e_{1r} (respectively) on s_1, i.e. $L_{1,2}(m) = m - |p_1e_{1l}|/|a_1b_1|$ and $R_{1,2}(m) = m + |p_1e_{1r}|/|a_1b_1|$.

Lemma 3. *$L_{1,2}(m)$ and $R_{1,2}(m)$ are piecewise linear functions of m.*

Proof. When p_2 is an endpoint of s_2, $\frac{t-1}{2}|p_1p_2|$ is linear in $|p_1p_2|_x = |a_1p_2|_x \mp |a_1p_1|$. Since $|p_1p_2|_y$ and $|a_1p_2|_x$ are both constants when p_2 is fixed at an endpoint of s_2, $\frac{t-1}{2}|p_1p_2|$ is linear in m by the definition of m. When p_2 is the point in s_2 that has the same x-coordinate or y-coordinate as p_1, $\frac{t-1}{2}|p_1p_2|$ is a constant. Thus $L_{1,2}(m)$ and $R_{1,2}(m)$ are (piecewise) linear in m. □

To efficiently compute a set of t-dominating set for each segment in S, we first introduce the concept of *wall*. Let s_1 and s_2 be two weakly visible segments in S, and p_1 and p_2 be their respective points. p_1 and p_2 are *horizontally (or vertically) visible pair* if p_1 and p_2 have the same y (or x) coordinate and the horizontal (or vertical) segment $\overline{p_1p_2}$ does not intersect the interior of any other segment in S. The union of all horizontally (or vertically) visible pairs forms one or more vertical (or horizontal) subsegments on each of s_1 and s_2. The corresponding subsegments on s_1 and s_2 have the same length and are called *wall* to each other. The set of such subsegments in each $s_i, i \in \{1, 2\}$, is called the *wall portion* of s_i. See Figure 3 for an example. We have the following lemmas regarding the positions of the Steiner points.

Lemma 4. *Given a set of rectilinear segments S in L_1 space, to determine the position of the set Q of Steiner points, it is sufficient to consider only those wall portions in each segment and the endpoints of S to guarantee a 2-approximation of Q (with respect to its size).*

Proof. We prove the lemma by contradiction. First, by Lemma 13 in [13], we know that to compute a t-strongly dominating set for an arbitrary segment $s_1 \in S$, it is sufficient to only consider those segments weakly visible to it (this lemma can be easily extended to the L_1 distance case). Hence the set of weakly

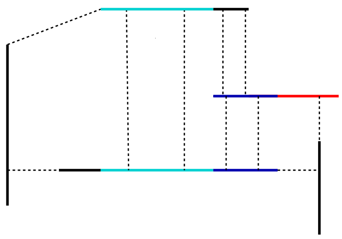

Fig. 3. Example spanner of rectilinear segments in L_1 space. Solid lines are segments. Dashed lines are spanner. Different colors represent the wall portions.

visible segments is sufficient to determine the Steiner points in s_1. Let OPT be any optimal solution whose Steiner points are determined by those weakly visible segments. Assume that in OPT there is a Steiner point q on segment s such that (i) q is not an endpoint of s; (ii) q is not in the wall portion of s. Without loss of generality, we assume that s is horizontal.

This means that q is visible to some points on other segments. Consider the nearest one of such points on the left side of q (assuming without loss of generality that there exists one), say p'. Since q is not the left endpoint, say a, of s, there is a non-empty portion of s to the left of q. Therefore i) if a is to the left of p', p' is within the wall of some subsegment, say ss, of s, and ss is a wall portion of s; ii) if a is not to the left of p', p' is visible to a. Slide q to the left along s until it reaches the endpoint a or enters the wall portion ss. Let q_l be the new position of the original q. If q is also visible to some point on its right side, we can similarly slide q to the right and select another position q_r. It is easy to see that any point of s that originally use q as its Steiner point can now use either q_l or q_r as Steiner point to meet the spanner requirement. Therefore, any Steiner point that is neither within the wall portions nor one of the endpoints can be replaced by at most two Steiner points without destroying the spanner property. This implies a 2-approximation and hence the lemma follows. □

Lemma 5. *Given a set of rectilinear segments S in L_1 space, the t-dominating set between two subsegments that are wall to each other can be computed optimally.*

Proof. Let $ss_1 \in s_1$ and $ss_2 \in s_2$ be two subsegments that are wall to each other. Notice that they have the same length. By Lemma 1 and Lemma 2, we know that $|p_1e_{1l}|$, $|p_1e_{1r}|$, $|p_2e_{2l}|$ and $|p_2e_{2r}|$ all have the same minimum value $\frac{t-1}{2}|p_1p_2|$. This implies the following two properties: i) $L_{1,2}(m)$ and $R_{1,2}(m)$ of these wall portions are straight line segments and parallel to each other; ii) they form the same $B_{1,2}$ and $B_{2,1}$ bands (i.e., the region bounded by the $L_{1,2}(m)$ and $R_{1,2}(m)$ functions in the coordinate system; see [13] for more details). Property i) means that the t-weakly dominating points can therefore be determined by a horizontal interval cover in $B_{1,2}$ (see [13]). Property ii) means that the t-weakly dominating points are chosen as pairs on ss_1 and ss_2, i.e. if ss_1 and ss_2 are both horizontal, for each t-weakly dominating Steiner point q_1 on ss_1, there exists a t-weakly dominating Steiner point q_2 on ss_2 with the same x-coordinate as q_1. Together with the property of L_1 distance ($|p_1q_2| = |p_1q_1| + |q_1q_2|$), this guarantees that the minimum set of t-weakly dominating set computed using interval cover is also a minimum set of t-dominating points (i.e. no imaginary point is needed here). □

Lemma 6. *Given a set of rectilinear segments S in L_1 space, the t-strongly dominating set for a subsegment that is the wall portion of an input segment can be computed optimally.*

Proof. Let ss_i be such a subsegment that is the wall portion of $s_i \in S$. Assume without loss of generality that s_i is horizontal. First, there are at most two walls of ss_i, say ss_j and ss_k, that are parallel to ss_i, one above and the other under ss_1. Assume without loss of generality that ss_i is closer to ss_j than to ss_k. Then the upper envelope function $L_i(m)$ of all $L_{i,r}(m), r \neq i$, and the lower envelope function $R_i(m)$ of all $R_{i,r}(m), r \neq i$, are determined by $L_{i,j}(m)$ and $R_{i,j}(m)$ because of the smaller value of $\frac{t-1}{2}|p_1p_2|$ introduced by ss_j. Together with the property of L_1 distance, this guarantees that the minimum set of t-dominating set for ss_i computed from ss_j is also a minimum set of t-strongly dominating set (i.e. no imaginary point is needed). Note that if ss_i is not a "proper" subsegment of s_i, there could be at most two other walls (actually degenerated as points) to the left and right of ss_i. One t-strongly dominating Steiner point (i.e., the endpoint of s_i) is sufficient for each of them. □

Once the t-dominating sets are determined, the bridges can be built in a way similar to the construction of segment spanner in [13]. As discussed in the proof of Lemma 5, the pairs of t-strongly dominating Steiner points are determined in such a way that the bridge connecting each pair is rectilinear. Figure 3 shows examples on the wall portions and the bridges built between them.

Lemma 4 tells us that besides the wall portions, we also need to consider the endpoints as candidates for possible Steiner points. This happens when two input segments are weakly visible to each other, but they do not have subsegments that are walls of each other. The last part in the proof of Lemma 6 also shows one case where the endpoint is selected. At least one of the endpoints of such bridges is at an endpoint of an input segment. Some of them could also be non-rectilinear if both endpoints are at the endpoints of input segments. See Figure 3 for examples.

Putting everything together we have the following theorem.

Theorem 1. *Given a set of n rectilinear segments in L_1^2 space, a set of Steiner points with size no more than $2 \times |OPT|$ can be computed in $O(|Q| + n^2 \log n)$ time.*

Proof. By Lemma 6, the set of t-strongly dominating points calculated by considering only wall portions and endpoints is optimal. Hence the approximation ratio is 2 by Lemma 4. The running time is mainly spent on finding wall portions, which can be achieved after computing all pairs of weakly visible segments. This takes $O(|Q| + n^2 \log n)$ time according to [13]. □

4 Constructing t-Spanner for Axis Aligned Rectangles Under L_1 Distance

In this section, we consider the problem of constructing a t-spanner for a set of rectangles in L_1^2 space. Let $S = \{R_1, R_2, \cdots, R_n\}$ be a set of disjoint axis aligned rectangles, and $t > 1$ be the stretch factor.

Definition 3. *Two rectangles R_i and R_j in S are* doubly separated *if their orthogonal projections on the x and y-axes do not overlap (see Figure 4).*

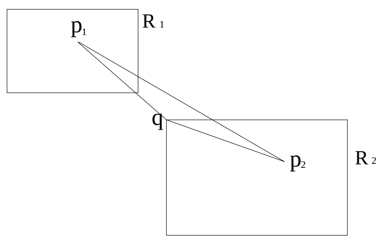

Fig. 4. Doubly Separated Rectangles

Lemma 7. *Let R_1 and R_2 be two doubly separated axis aligned rectangles, and q be the closest point of R_2 to R_1. Then for any point p_2 in R_2, q t-weakly dominates p_2*

Proof. Without loss of generality, we assume that R_2 is at the southeast corner of R_1 (see Figure 4). Let (x_1, y_1), (x_2, y_2) and (x_q, y_q) be the coordinates of p_1,p_2 and q respectively. Then we have $x_2 \geq x_q \geq x1$ and $y_2 \leq y_q \leq y_1$. Since $|p_1q| = |x_q - x_1| + |y_q - y_1| = (x_q - x_1) + (y_1 - y_q)$, $|qp_2| = |x_2 - x_q| + |y_2 - y_q| = (x_2 - x_q) + (y_q - y_2)$ and $|p_1p_2| = |x_2 - x_1| + |y_2 - y_1| = (x_2 - x_1) + (y_1 - y_2)$, we have $|p_1q| + |qp_2| = |p_1p_2| \leq t * |p_1p_2|$ for $t > 1$. This means that q t-weakly dominates p_2. □

Definition 4. *Let R_1 and R_2 be two disjoint axis aligned rectangles. R_1 is* totally below *R_2 if R_1 is below R_2 and between the leftmost and rightmost points of R_2 (see Figure 5); R_1 is* totally above *R_2 if R_1 is above R_2 and between the leftmost and rightmost points of R_2; R_1 is* totally to the left side of *R_2 if R_1 is at the left side of R_2 and between the highest and lowest t points of R_2; R_1 is* totally to the right side of *R_2 if R_1 is at the right side of R_2 and between the highest and lowest points of R_2; if R_1 and R_2 have one of the above four relationships, R_1 is* totally on one side of *R_2.*

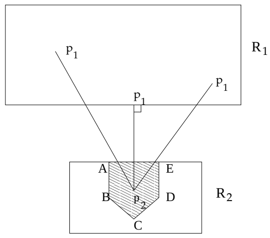

Fig. 5. R_2 is totally below R_1

Lemma 8. *Let R_1 and R_2 be two disjoint axis aligned rectangles with R_2 being totally below R_1, and p_2 be an arbitrary point in R_2 with coordinate (x_2, y_2). Let l be the L_1 distance from p_2 to R_1 and y be the y-coordinate of the top edge of R_2. Then the region t-weakly dominating p_2 is the intersection of R_2 and a pentagon $ABCDE$ with coordinates $(x_2 - \frac{t-1}{2} * l, y)$,$(x_2 - \frac{t-1}{2} * l, y_2)$,$(x_2, y_2 - \frac{t-1}{2} * l)$,$(x_2 + \frac{t-1}{2} * l, y_2)$,$(x_2 + \frac{t-1}{2} * l, y)$ respectively.*

Proof. To simplify our proof, we assume that p_2 coincides with the origin of the coordinate system. Let p_1 be an arbitrary point in R_1, q be a point in R_2 that t-weakly dominates p_2, and p be the point in R_1 that is the closest to p_2. Then the coordinate of p is $(0, l)$ (see Figure 5). Let (x_1, y_1), (x_2, y_2) and (x_q, y_q) be the coordinates of p_1,p_2 and q respectively. Since $|p_1q| + |qp_2| \leq t \times |p_1p_2|$ with $|p_1q| = |x_1 - x_q| + |y_1 - y_q|$, $|qp_2| = |x_2 - x_q| + |y_2 - y_q|$ and $|p_1p_2| = |x_2 - x_1| + |y_2 - y_1|$, we have $|x_1 - x_q| + |y_1 - y_q| + |x_2 - x_q| + |y_2 - y_q| \leq t \times (|x_2 - x_1| + |y_2 - y_1|)$, i.e.,

$$(|x_1 - x_q| + |x_2 - x_q| - |x_2 - x_1|) + (|y_1 - y_q| + |y_2 - y_q| - |y_2 - y_1|) \leq (t-1) \times |p_1p_2|. \qquad (1)$$

Based on the position of q, we have the following cases.

1. q is in the second quadrant, i.e., $x_q \leq x_2$ and $y_q \geq y_2$. For this case, we have three subcases, depending on the position of p_1.
   (a) p_1 is to the left of q (i.e., $y_2 \leq y_q \leq y_1$ and $x_1 \leq x_q \leq x_2$). For this case, Inequality (1) can be rewritten as $[(x_q - x_1) + (x_2 - x_q) - (x_2 - x_1)] + [(y_1 - y_q) + (y_q - y_2) - (y_1 - y_2)] \leq (t-1) \times |p_1p_2|$ and $0 \leq (t-1) \times |p_1p_2|$. This means that for such p_1, Inequality (1) is trivially true and does not define the boundary for q.
   (b) p_1 is between q and p_2 in the x dimension (i.e., $y_2 \leq y_q \leq y_1$ and $x_q \leq x_1 \leq x_2$). Then Inequality (1) can be rewritten as $2(x_1 - x_q) \leq (t-1) \times |p_1p_2|$. Thus $(x_1 - x_q) \leq \frac{(t-1)|p_1p_2|}{2}$, i.e. $|x_q| \leq \frac{(t-1)|p_1p_2|}{2} - x_1$, since x_q is negative. $\frac{(t-1)|p_1p_2|}{2} - x_1$ achieves its minimum $\frac{(t-1)l}{2}$ when $p_1 = p$.
   (c) p_1 is to the right of p_2 in the x dimension (i.e., $y_2 \leq y_q \leq y_1$ and $x_q \leq x_2 \leq x_1$). For this case, Inequality (1) can be rewritten as $(x_2 - x_q) \leq \frac{(t-1)|p_1p_2|}{2}$ and $x_q \leq \frac{(t-1)|p_1p_2|}{2}$, since x_q is negative and $x_2 = 0$. $\frac{(t-1)|p_1p_2|}{2}$ achieves its minimum $\frac{(t-1)l}{2}$ when $|p_1p_2| = l$ (i.e., $p_1 = p$).

   Combine case (a), (b) and (c), we know that the t-weakly dominating region in the second quadrant of p_2 is $\{(x_q, y_q) : |x_q| \leq \frac{(t-1)l}{2}, q \text{ in the second quadrant } \}$.
2. q is in the first quadrant. Similar to case 1, we have the t-weakly dominating region in the first quadrant of p_2 to be $\{(x_q, y_q) : |x_q| \leq \frac{(t-1)l}{2}, q \text{ in the first quadrant } \}$.
3. q is in the third quadrant (i.e., $x_q \leq x_2$ and $y_q \leq y_2$). For this case we have three subcases based on the position of p_1.
   (a) p_1 is to the left of q (i.e., $y_q \leq y_2 \leq y_1$ and $x_1 \leq x_q \leq x_2$). For this case, Inequality (1) can be rewritten as $[(x_q - x_1) + (x_2 - x_q) - (x_2 - x_1)] + [(y_1 - y_q) + (y_2 - y_q) - (y_1 - y_2)] \leq (t-1) \times |p_1p_2|$ and $y_2 - y_q \leq \frac{(t-1)|p_1p_2|}{2}$. Thus $|y_q| \leq \frac{(t-1)|p_1p_2|}{2} - y_2$. $\frac{(t-1)|p_1p_2|}{2} - y_2$ achieves its minimum $\frac{(t-1)l}{2}$ when $p_1 = p$.
   (b) p_1 is between q and p_2 (i.e, $y_q \leq y_2 \leq y_1$ and $x_q \leq x_1 \leq x_2$). For this case, Inequality (1) has the following form. $[(x_1 - x_q) + (x_2 - x_q) - (x_2 - x_1)] + [(y_1 - y_q) + (y_2 - y_q) - (y_1 - y_2)] \leq (t-1) \times |p_1p_2|$ or $(x_1 - x_q) + (0 - y_q) \leq \frac{(t-1)|p_1p_2|}{2}$. Thus $(|x_q| + |y_q|) \leq \frac{(t-1)|p_1p_2|}{2} - x_1$. $\frac{(t-1)|p_1p_2|}{2} - x_1$ achieves its minimum $\frac{(t-1)l}{2}$ at $p_1 = p$.
   (c) p_1 is to the right of p_2 (i.e., $y_q \leq y_2 \leq y_1$ and $x_q \leq x_2 \leq x_1$). For this case, Inequality (1) can be simplified to $[(x_1 - x_q) + (x_2 - x_q) - (x_2 - x_1)] + [(y_1 - y_q) + (y_2 - y_q) - (y_1 - y_2)] \leq (t-1) \times |p_1p_2|$,

and $(0 - x_q) + (0 - y_q) \leq \frac{(t-1)|p_1p_2|}{2}$. This is equivalent to $(|x_q| + |y_q|) \leq \frac{(t-1)|p_1p_2|}{2}$. $\frac{(t-1)|p_1p_2|}{2}$ achieves its minimum $\frac{(t-1)l}{2}$ at $p_1 = p$.

Combine case (a) (b) and (c), we know the t-weakly dominating region in the third quadrant is $\{(x_q, y_q) : |x_q| + |y_q| \leq \frac{(t-1)l}{2}, q \text{ in the third quadrant }\}$.

4. q is in the fourth quadrant. Similar to cases 2 and 3, we have the t-weakly dominating region in the fourth quadrant to be $\{(x_q, y_q) : |x_q| + |y_q| \leq \frac{(t-1)l}{2}, q \text{ in the fourth quadrant }\}$.

Combining cases 1,2,3, and 4, we know that the whole t-weakly dominating region is actually the region bounded by the pentagon $ABCDE$ with coordinates $(-\frac{t-1}{2}*l, y)$,$(-\frac{t-1}{2}*l, 0)$,$(0, 0-\frac{t-1}{2}*l)$,$(\frac{t-1}{2}*l, 0)$,$(\frac{t-1}{2}*l, y)$ respectively. □

Let R_1 and R_2 be two disjoint axis aligned rectangles with R_2 being totally below R_1, and d be the minimum distance between R_1 and R_2. By lemma 8, we know the t-weakly dominating set of R_2 can be selected only from the upper edge of R_2. This is because for any point p_2 in R_2, the region that t-weakly dominates p_2 intersects the upper edge of R_2 at segment $\overline{AE}$.

Let p_2' be another point in R_2 and (x_2, y_2) and (x_2', y_2') be the coordinates of p_2 and p_2' respectively with $y_2 \geq y_2'$ and $x_2 = x_2'$. Let the regions that t-weakly dominate p_2 and p_2' be the intersections of R_2 and pentagons $ABCDE$ and $A'B'C'D'E'$ respectively. Since the distance from p_2' to R_1 (say l') is larger than that from p_2 to R_1 (say l). Thus, by lemma 8, $\overline{AE} \subset \overline{A'E'}$. This implies that to obtain a size-minimized t-weakly dominating set for R_2, we just need to pick the t-weakly dominating Steiner points from the upper edge of R_2. Below is an algorithm for determining the t-weakly dominating set.

Input: Two disjoint axis aligned rectangles R_1 and R_2 with R_2 being totally below R_1, d being the minimum distance between R_1 and R_2, and (x_0, y_0) and (x_0', y_0) being the coordinates of the leftmost and rightmost points of the upper edge of R_2

Output: A t-weakly dominating set Q of R_2 with respect to R_1

Q=ϕ;
i=1;
if $x_0 + \frac{t-1}{2} * d < x_0'$ **then**
  put $(x_0 + \frac{t-1}{2} * d, y_0)$ in Q;
  while $x_0 + \frac{t-1}{2} * d * (2i+1) < x_0'$ **do**
    put $(x_0 + \frac{t-1}{2} * d * (2i+1), y_0)$ in Q;
    i++;
  end
  if $x_0 + \frac{t-1}{2} * d * (2i) < x_0'$ **then**
    Put(x_0', y_0) in Q;
  end
end
if $Q = \{\}$ **then**
  $Q = \{(x_0, y_0')\}$;
end
return Q;

Lemma 9. *The set of t-weakly dominating Steiner points selected by the above algorithm has the minimum size among all sets of points t-weakly dominating R_2.*

Proof. We prove by contradiction. Let Q be the set of t-weakly dominating set chosen by the above algorithm. Suppose that there exists another set Q' of smaller size. Let $Q = \{g_1, g_2, ..., g_m\}$ and $Q' = \{g_1', g_2', ..., g_k'\}, k < m$, be the two sets sorted by their x-coordinates in increasing order. The x-coordinate of g_1', say $x_{g_1'}$, must be less or equal to that of g_1, say x_{g_1}, since if $x_{g_1'} > x_{g_1}$, g_1' can't t-weakly dominate (x_0, y_0) by Lemma 8 and the above algorithm. Similarly, for any i, we have $x_{g_i'} \leq x_{g_i}$. Thus $k \geq m$. A contradiction. □

Lemma 10. *Let R_1 and R_2 be two disjoint axis aligned rectangles with R_2 being totally below R_1, d be the minimum distance between R_1 and R_2, and w be the width of R_2. Then the total number of points in the t-weakly dominating set of R_2 is at most $\lfloor \frac{w}{(t-1)d} \rfloor + 1$.*

Proof. By the above algorithm, we know that (1) if $w \leq \frac{(t-1)}{2} * d$, then there is only one t-weakly dominating Steiner point. Thus $1 \leq \lfloor \frac{w}{(t-1)d} \rfloor + 1$ (i.e., the lemma holds). (2) If $x_0' - x_0 > w > \frac{(t-1)}{2} * d$, then the total number m of t-weakly dominating Steiner points satisfies $\frac{t-1}{2} * d * (2m+1) < w$ or $\frac{t-1}{2} * d * 2(m-1) < w$. Thus, $m \leq \lfloor \frac{w}{(t-1)d} \rfloor + 1$. □

It is easy to see that when R_1 and R_2 have one of the other three relations in Definition 4, similar results can be proved as in Lemmas 8, 9, and 10.

For any pair of disjoint axis aligned rectangles R_1 and R_2, one of the following three cases holds.

1. R_1 and R_2 are doubly separated.
2. R_1 (or R_2) is totally on one side of R_2 (or R_1).
3. Neither 1 or 2 is true (see Figure 6).

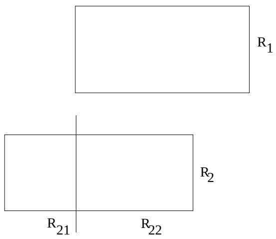

Fig. 6. R_2 is partitioned into two subrectangles.

For case 3, R_2 (or R_1) can be partitioned into two axis aligned rectangles R_{21} and R_{22} with one of them being doubly separated with R_1 and the other being totally on one side of R_1 (see Figure 6). The t-weakly dominating sets for R_{21} and R_{22} can be selected by using Lemma 7 and Algorithm 1, and the t-weakly dominating set for R_2 is just the union of the two t-weakly dominating sets.

From the above discussion, we know that for any pair of axis aligned rectangles R_1 and R_2, the t-weakly dominating set of R_2 can be selected from one edge e of R_2 and its total number is no more than $\lfloor \frac{w}{(t-1)d} \rfloor + 2$, where d is the distance between R_1 and R_2 and w is the length of e or the edge in the subrectangle of R_2 which is totally on one side of R_1.

In the two-rectangle case, the t-weakly dominating sets for each rectangle is determined by using one of its edges. For a set S of axis aligned rectangles $S = \{R_1, R_2, ..., R_n\}$, we consider the set S' of all boundary edges of S, i.e., $S' = \{E_{11}, E_{12}, E_{13}, E_{14}, E_{21}, E_{22}, E_{23}, E_{24}, \cdots, E_{n1}, E_{n2}, E_{n3}, E_{n4}\}$, where $E_{i1}, E_{i2}, E_{i3}, E_{i4}$ are the four edges of rectangle R_i. To compute t-weakly dominating sets (or t-dominating sets) of S, we reduce it to the problem of constructing spanners for rectilinear segments under L_1 distance. Below is the main idea of the reduction.

Let s_1 and s_2 be two segments in S' and ss_1 and ss_2 be subsegments of them respectively. ss_1 and ss_2 are *wall* to each other if they are weakly visible to each other, have the same vertical (or horizontal) projection, and are from the same rectangle. ss_1 is called a *wall portion* of s_1. Note that the definition of *wall* is slightly different from that in Section 3. Here we require that the two subsegments should not be part of the same rectangle.

Since Lemma 4, Lemma 5 and Lemma 6 can be easily extended to S' (details are left for the full paper) and the strongly dominating set of a rectangle is the union of the four strongly dominating sets of its four edges, we have the following theorem.

Theorem 2. *Given a set of n disjoint axis aligned rectangles in L_1^2 space, a set Q of Steiner points with size no more than $2 \times |OPT|$ can be computed in $O(|Q| + n^2 \log n)$ time.*

Proof. By Lemma 6, the t-strongly dominating set of each edge in S' calculated by considering only its wall portion and its endpoints in S' is optimal. Hence the approximation ratio is bounded by 2 according to Lemma 4. The running time is mainly spent on finding wall portions, which can be obtained after computing the weakly visible segments. This takes $O(|Q| + n^2 \log n)$ time by [13]. □

5 Constructing t-Spanner for Rectilinear Polygons Under L_1 Distance

In this section, we consider the problem of constructing spanner for a set $S = \{P_1, P_2, \cdots, P_n\}$ of rectilinear polygons in L_1^2 space.

Let P_1 and P_2 be two rectilinear polygons in S, and R_1 and R_2 be two axis aligned rectangles in P_1 and P_2 respectively. R_1 and R_2 are *wall* to each other if they are weakly visible to each other, and have the same vertical (or horizontal) projection. $R_i, i \in \{1, 2\}$, is called a *wall portion* of P_i.

To determine the set of Steiner points for S, our main idea is to partition each rectilinear polygon P_i into a set of axis aligned rectangles. Each such rectangle has at least one edge which is part of a boundary edge of P_i. It is easy to see that the partition can be done in linear time by using a plane sweeping algorithm on P_i. With this partition, we can compute the weak visibility of each rectangle and determine the wall portions of its edges.

Lemma 11. *For a set of rectilinear polygons S in L_1^2 space, to determine the set of Steiner points Q it is sufficient to consider only the wall portions and the vertices of S to guarantee a 2-approximation (with respect to the size of Q).*

Proof. We prove the lemma by contradiction. Assume in an optimal solution there is a Steiner point q in a polygon P_1, which is neither in some wall portion of P_1 nor a vertex. Then q must be in a rectangle, say R, which is doubly separated from all the weakly visible portions of some other polygons. By Lemma 7, the two closest points on the boundary of neighboring rectangles partitioned from the same polygon as R are sufficient to dominate the whole rectangle R and the region that q dominates. If both neighboring rectangles are wall portions of P_1, then we can replace q by the two points on the boundary of the polygon. Otherwise, we continue considering the neighboring rectangles along the boundary of P_1 until the rectangles have wall portions. Since all the non-wall portions are doubly separated from those weakly visible rectangles. It's sufficient to consider the two adjacent points on the boundary with the two wall portions. Since q can be replaced by two points on the boundary, a 2-approximation is guaranteed. The lemma follows. □

Lemma 12. *For a set of rectilinear polygons S in L_1^2 space, the t-dominating set between two axis aligned rectangles R_1 and R_2 that are wall to each other can be computed optimally.*

Proof. By Lemma 8 and the analysis in Section 4, we know that R_1 and R_2 have the relations that R_1 is totally on one side of R_2, and R_2 is totally on one side of R_1. So the dominating set can be selected from the boundaries of the corresponding polygons and computed optimally. □

Lemma 13. *For a set of rectilinear polygons S in L_1^2 space, the t-strongly dominating set of a rectangle R that is a wall portion of an input rectilinear polygon P_i can be computed optimally.*

Proof. For the portion of S that is weakly visible to R but not a wall to R, it's sufficient to t-strongly dominate R by two points on the boundary of R. Combining Lemma 12, the lemma follows. □

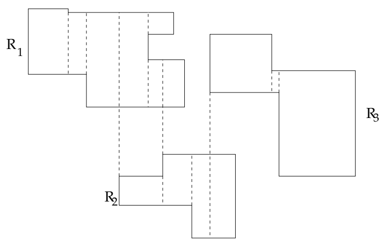

Fig. 7. Two disjoint rectilinear polygons and their rectangular partitions.

Theorem 3. *For a set S of n disjoint rectilinear polygons in L_1^2 space, a set of t-strongly dominating Steiner points with size no more than $2 \times |OPT|$ can be computed in $O(|Q| + N^2 \log N)$ time, where N is the total number of vertices in S.*

Proof. By Lemma 13, the set of t-strongly dominating Steiner points calculated by considering only wall portions and the vertices is optimal. Hence the approximation ratio is bounded by 2 according to Lemma 11. The running time is mainly spent on finding wall portions, which can be achieved after the partitioning process and computing the weak visibility of the rectangles. This takes $O(|Q| + N^2 \log N)$ time according to [13]. □

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