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Kyoto University
LECTURES IN MATHEMATICS

Department of Mathematics
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6

TYPICAL FORMAL GROUPS
IN
COMPLEX COBORDISM
AND K-THEORY

BY
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in
Complex Cobordism and K-Theory

BY
Shôrô Araki

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Introduction

The present lecture note consists of two parts.

Part I contains an exposition of Quillen's theory [18] of decompositions of complex cobordism theory localized at a prime $p$. Quillen's note [18] itself consists of two parts: the first part is connected with the proof of universality of the formal group of complex cobordism, of which detailed expositions are now available in literatures such as Adams [2], §§1-8, and Quillen [19], so I assumed these materials are known in the present lecture; the second part is the main subject of our Part I. The contents have much overlap with [2], but our exposition is given along original line of Quillen so it differs from the corresponding treatment of Adams [2] in its philosophy at least. We start with an exposition of Cartier's note [6] on the theory of (typical) curves over formal groups. This is done in §§2 and 3 in a form suitable for our purpose and restricting to one-dimensional case only. In §4 we discuss a typical formal group which is universal for typical formal groups, which turns out to be the formal group of Brown-Peterson cohomology (in §5). In §5 we prove Quillen decompositions. In §6 we discuss generators of $U^*(pt)$ and $BP^*(pt)$ in a form related with formal group. I believe this section contains some new results. Finally in §7 we discuss Landweber-Novikov type operations in Brown-Peterson cohomology.

In Part II we treat typical formal groups in (complex) K-theory and
their relation to Adams' idempotent decomposition of $K$-theory localized at a prime $p$ [1]. The results here were announced in [5].

These lecture notes came out of my lectures in Kyushu University, December 1972, Osaka City University, February and May 1973, and Kyoto University, July 1973. I acknowledge to Professors T.Kudo and H.Toda for their organizing my lectures in Kyushu University and Kyoto University, particularly to the latter for his arrangement to publish the present lecture notes as a part of "Lectures in mathematics, Department of Mathematics, Kyoto University".
§1. Formal groups

1.1. Let $R$ be a commutative ring with unity. By a (one-dimensional commutative) formal group, or a group law, we understand a formal power series $F$ in two variables over $R$ satisfying

(1.1) $F(0, X) = X,$
(1.2) $F(X, Y) = F(Y, X),$
(1.3) $F(X, F(Y, Z)) = F(F(X, Y), Z).$

Then $F$ can be expressed as

(1.4) $F(X, Y) = X + Y + XYF(X, Y)$

with $F(X, Y) \in R[[X, Y]].$

We are mainly interested in formal groups associated with cohomology theories which are complex oriented in the sense of [8], [19] (cf., §1). In such a case $R$ is graded, i.e., $R = \bigoplus R^i,$ and $F$ satisfies

(1.5) $\dim F(X, Y) = 2$ if $\dim X = \dim Y = 2,$ i.e., if we put

$$F(X, Y) = \sum_{i,j} a_{ij} X^i Y^j$$

then $a_{ij} \in R^{2(1-i-j)}$ (cf., 5.2).

1.2. Let $F$ and $F'$ be formal groups over $R,$ and $\psi$ a formal power series over $R$ in one variable without constant term satisfying

(1.6) $\psi(F(X, Y)) = F'(\psi(X), \psi(Y)),$

then we call $\psi$ a homomorphism.
\[ \psi : F \rightarrow F', \]

of formal groups.

When \( \psi : F \rightarrow F' \) and \( \varphi : F' \rightarrow F'' \), then \( \varphi \circ \psi : F \rightarrow F'' \),

where \( \varphi \circ \psi \) is the composition of formal power series. Thus formal
groups over \( R \) and their homomorphisms form a category, which will be
denoted by \( \mathcal{G}(R) \).

When \( \psi : F \rightarrow F' \) and \( \psi \) is invertible with respect
to composition, then

\[ \psi^{-1} : F' \rightarrow F \]

such that \( \psi^{-1} \circ \psi = 1_F \) and \( \psi \circ \psi^{-1} = 1_{F'} \), where \( 1_F(T) = 1_{F'}(T) = T \).

Thus \( \psi \) is an isomorphism in the category \( \mathcal{G}(R) \), denoted by

\[ \psi : F \xrightarrow{\sim} F'. \]

In particular, when

\[ \psi(T) = T + \text{higher terms}, \]

we call \( \psi \) a strict isomorphism which we denote by

\[ \psi : F \xrightarrow{\text{strict}} F'. \]

We denote the set of all homomorphisms \( F \rightarrow F' \) by \( \text{Hom}_R(F, F') \)
and put \( \text{End}_R(F) = \text{Hom}_R(F, F) \).

1.3. Let \( \theta : R \rightarrow S \) be a homomorphism of commutative rings with
unity. Let \( \theta_* : R[[X, Y]] \rightarrow S[[X, Y]] \) and \( \theta_* : R[[T]] \rightarrow S[[T]] \)
be the homomorphisms of rings of formal power series induced by coefficient map \( \theta \), i.e., \( \theta_*(\sum_{i,j} a_{ij} x^i y^j) = \sum_{i,j} \theta(a_{ij}) x^i y^j \) and \( \theta_*(\sum a_i T^i) = \sum \theta(a_i) T^i \).

Since \( \theta_* \) preserves also compositions, we see that, if \( F \in \text{obj}\mathcal{J}(R) \), then \( \theta_* F \in \text{obj}\mathcal{J}(S) \), and if \( \psi \in \text{Hom}_R(F, F') \), then \( \theta_* \psi \in \text{Hom}_S(\theta_* F, \theta_* F') \) and \( \theta_*(\psi \circ \psi) = \theta_* \theta_* \psi \), i.e., \( \theta_* : \mathcal{J}(R) \rightarrow \mathcal{J}(S) \) is a covariant functor.

Thus we obtained, roughly speaking, a functor \( \mathcal{J} \) defined on the category of commutative rings with unity with values in a category whose objects are categories of formal groups and morphisms are covariant functors \( (\mathcal{J}(\theta) = \theta_*) \). Later we meet often with needs to restrict this functor either restricting the domain to a subcategory or the range, or both.

1.4. We recall some known results without proof.

A formal group \( F_U \) defined over a ring \( U \) is called universal if for any ring \( R \) and for any formal group \( F \) on \( R \) there exists a unique homomorphism \( u : U \rightarrow R \) of rings with unity such that \( u_* F_U = F \). The existence of a universal formal group and the structure of the ring \( U \) was first established by Lazard [15]. The uniqueness of \( (F_U, U) \) up to equivalence follows by the general nature of "universality". The structure theorem of \( U \) says:

[Lazard's Theorem] \( U = \mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots] \),

a polynomial ring over integers with countable indeterminates \( x_1, x_2, \ldots, \).

We call the ring \( U \) Lazard ring. For the benefit of topologists we
mention that the Lazard ring $U$ can be given as a graded ring, graded by non-positive even dimensions so that $F_U$ satisfies the condition (1.5). In this case $\dim x_n = -2n$. Cf., also [2], §§5 and 7.

1.5. A formal group $G_a$ given by

$$G_a(X, Y) = X + Y$$

is called additive. Such a formal group is defined over any ring $R$.

Let $F$ be a formal group. A strict isomorphism

$$\ell_F : F \cong G_a$$

is called a logarithm of $F$.

[existence of logarithm] Let $F$ be a formal group defined over a $\mathbb{Q}$-algebra $R$. There exists a unique logarithm $\ell_F : F \cong G_a$.

For the proofs, cf., [13], [15] and [9], p.69. The existence is essential, and the uniqueness is easy.

Let $F$ be a formal group defined over a ring $R$ and suppose that $R$ is of characteristic zero, i.e., every prime is not a zero divisor in $R$. Then $R \subset R \otimes \mathbb{Q}$ and we can regard $F$ as a formal group over $R \otimes \mathbb{Q}$ by extending the domain of coefficients. Now we have a unique logarithm

$$\ell_F : F \cong G_a$$

over $R \otimes \mathbb{Q}$.

We often denote as $\ell_F = \log_F$, and call it the logarithm of $F$ for simplicity. If we express as
\[ \log_F T = \sum_{k=0}^{\infty} m_k T^{k+1}, \quad m_0 = 1, \]
then it is known that
\[ (k + 1) m_k \in \mathbb{R}. \]

For topologists this is familiar by Mischenko series in case \( F = F_U \), and the general case follows by functoriality (cf., §2).
§2. Modules of curves

We describe here modules of curves on formal groups according to Cartier [6].

2.1. Let $R$ be a commutative ring with unity. The ring of formal power series in one variable $T$, $R[[T]]$, is filtered by degrees, i.e.,

$$R[[T]] = R[[T]]_0 \supset R[[T]]_1 \supset \cdots \supset R[[T]]_n \supset \cdots,$$

where $R[[T]]_n = \{f(T) = \sum_{i=0}^{n} f_i T^i \in R[[T]] : f_0 = \ldots = f_{n-1} = 0\}$. $R[[T]]$ is complete and Hausdorff with respect to this filtration topology. $R[[T]]_1$ is the submodule of $R[[T]]$ consisting of all power series without constant terms.

Let $F \in \text{obj} \mathcal{F}(R)$. For $\gamma, \gamma' \in R[[T]]_1$ we define their sum $\gamma +^F \gamma'$ with respect to $F$ by

$$\gamma +^F \gamma'(T) = F(\gamma(T), \gamma'(T)).$$

Proposition 2.1. $R[[T]]_1$ with the sum $+^F$ is an abelian group.

Proof. By (1.2) and (1.3) it follows the commutativity and associativity. Zero power series $0(T) = 0$ is the zero element by (1.1). There exists a unique power series

$$1_F \in R[[T]]_1$$

satisfying $F(T, 1_F(T)) = 0$.

Then, for any $\gamma \in R[[T]]_1$

$$-^F \gamma = 1_F \circ \gamma$$
is the inverse of \( \gamma \) with respect to the addition \( +_F \).

\[ \text{q. e. d.} \]

Put \( \mathcal{C}_F = (R[[T]], +_F) \), the above additive group. We call an element of \( \mathcal{C}_F \) a curve over \( F \). Thus \( \mathcal{C}_F \) is the additive group of curves over \( F \).

The curve \( \gamma_0 \), defined by \( \gamma_0(T) = T \), plays an important role and will be called the identity curve (over \( F \)).

2.2. We remark that

\[ (2.3) \quad 1_F \in \text{End}_R(F). \]

This is proved by observing that there exists a unique power series \( \gamma(X, Y) \) satisfying \( F(F(X, Y), \gamma(X, Y)) = 0 \) and that both \( 1_F(F(X, Y)) \) and \( F(1_F(X), 1_F(Y)) \) satisfy the property of \( \gamma(X, Y) \).

We embed \( \text{Hom}_R(F', F) \) into \( \mathcal{C}_F \) canonically. Then we see easily that \( \text{Hom}_R(F', F) \) is a subgroup of \( \mathcal{C}_F \). And the map

\[ \text{Hom}_R(F', F) \times \mathcal{C}_F, \longrightarrow \mathcal{C}_F \]

is bi-additive. Thus \( \text{End}_R(F) \) is an additive subgroup of \( \mathcal{C}_F \) and is a ring with composition as multiplication and with \( \gamma_0 \) as unity (non-commutative in general). Furthermore \( \mathcal{C}_F \) is a left \( \text{End}_R(F) \)-module.

2.3. There exists a unique homomorphism of additive groups

\[ [ ]_F : \mathbb{Z} \longrightarrow \mathcal{C}_F \]

such that \( [ ]_F(1) = \gamma_0 \). We write \( [ ]_F(n) = [n]_F \) for any integer \( n \).
We have

\( (2.4) \quad [1]_F = \gamma_0, \quad [-1]_F = \gamma_F \) and \([0]_F = 0,\)

\( (2.5) \quad [n]_F(T) = F(T, [n - 1]_F(T)) = F(\gamma_F(T), [n + 1]_F(T)). \)

Remark that

\[ \gamma_F(T) = -T + \text{higher terms}, \]

which follows from (1.4) and (2.2). Then we see by (2.5) that

\( (2.6) \quad [n]_F(T) = nT + \text{higher terms}. \)

By (2.3) and (2.5) we see that

\( (2.7) \quad [n]_F \in \text{End}_R(F), \)

i.e., the map \([ \cdot ]_F : \mathbb{Z} \rightarrow C_F\) factorizes as

\[ [ \cdot ]_F : \mathbb{Z} \rightarrow \text{End}_R(F) \subseteq C_F. \]

In fact, the first factor of \([ \cdot ]_F\) is a ring homomorphism because

\[ [nm]_F = [n]_F \circ [m]_F \] as is easily seen; and the \(\mathbb{Z}\)-module structure of \(C_F\) is the same as that given by this ring homomorphism, i.e.,

\( (2.7') \quad [n]_F \circ \gamma = n \cdot \gamma, \) \( n \) times of \( \gamma \) in \( C_F. \)

2.4. Let \( d \) be an integer which is a unit in \( R. \) By (2.6) \([d]_F\)

is invertible. We define as

\[ [1/d]_F = [d]_F^{-1} \in \text{End}_R(F). \]
Suppose that $R$ is a $\Lambda$-algebra, where $\Lambda$ is a ring such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$. For any $\lambda \in \Lambda$ express $\lambda$ as a fraction $\lambda = a/b$ such that $(a, b) = 1$, then $b$ is a unit in $R$ and we define as 

$$[\lambda]_F = [a]_F \cdot [b]_F^{-1} \in \text{End}_R(F).$$

This extends the ring homomorphism $[\_]_F : \mathcal{Z} \to \text{End}_R(F)$ to the ring homomorphism

$$[\_]_F : \Lambda \to \text{End}_R(F).$$

And we obtain

**Proposition 2.2.** When $R$ is a $\Lambda$-algebra with a ring $\Lambda$ such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$, then $C_F$ is a left $\Lambda$-module by

$$\lambda \cdot \gamma = [\lambda]_F \circ \gamma$$

for $\lambda \in \Lambda$ and $\gamma \in C_F$.

2.5. Let $F$ be a formal group over a ring $R$. We define three kinds of operators on $C_F$.

i) $([a] \gamma)(T) = \gamma(aT), \ a \in R$,

ii) $(\gamma^n)(T) = \gamma(T^n), \ n \geq 1$,

iii) $(\hat{f} \gamma)(T) = \sum_{1 \leq i \leq n}^F \gamma(\zeta_{1/n}^i T^{1/n}), \ n \geq 1$,

where $\sum_{1 \leq i \leq n}^F$ is the summation in $C_F$ and $\zeta_1, \ldots, \zeta_n$ are $n$-th roots of unity. $\hat{f}_n \gamma$ lies in $R[\zeta_1, \ldots, \zeta_n][[T^{1/n}]]$ in first glance. Since $F$ is commutative, each coefficient of $\hat{f}_n \gamma$ is a symmetric polynomial of
hence a polynomial of elementary symmetric polynomials \( \sigma_1(\zeta), \ldots, \sigma_n(\zeta) \) of \( \zeta_1, \ldots, \zeta_n \). Put

\[
(f_n^\gamma)(T) = \sum_{d \geq 1} g_d(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)) T^{d/n},
\]

then \( g_d(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)) \) is a polynomial of homogeneous degree \( d \) with \( \deg \sigma_i(\zeta) = i \). Now \( \zeta_1, \ldots, \zeta_n \) are \( n \)-th roots of unity, whence

\[
\sigma_1(\zeta) = \ldots = \sigma_{n-1}(\zeta) = 0, \quad \sigma_n(\zeta) = (-1)^{n-1}.
\]

Thus

\[
g_d(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)) = 0 \quad \text{if} \quad d \not\equiv 0 \pmod{n},
\]

\[
g_{nk}(\sigma_1(\zeta), \ldots, \sigma_n(\zeta)) = g_{nk}(0, \ldots, 0, (-1)^{n-1}) \in \mathbb{R},
\]

and \( f_n^\gamma \) is a well-defined curve in \( \mathbb{C}_F \).

Operators \([a]\) are called \underline{homotheties}, \( v_n \) are called \underline{shifting operators} and \( f_n \) are called \underline{Frobenius operators}. Among three kinds of operators Frobenius operators may be regarded as the most important ones and are the only ones defined essentially depending on the formal group \( F \), so we write sometimes as \( f_n = f_{n,F} \) to clarify on what formal group they are considered.

We used notations \([\ ]\) and \([\ ]_F\) to mean entirely different objects (with or without suffix \( ! \)). I hope there arises no confusion.

Proposition 2.3. \underline{Operators} \([a]\), \( v_n \) and \( f_n \) are \underline{additive}.

Proof follows from routine calculations.
Thus $C_F$ is an operator-module. These operators satisfy certain universal relations (cf., Proposition 2.9 below).

2.6. Let $F, G \in \text{obj } \mathcal{F}(R)$ and $\psi : F \to G$ in $\mathcal{F}(R)$. We define $
abla \# : C_F \to C_G$

by $(\psi \# \gamma)(T) = (\psi \circ \gamma)(T)$.

Proposition 2.4. $\psi \#$ is linear and commutes with operators $[a]$,

$w_n$ and $f_n$, i.e., a homomorphism of operator-modules.

Proof follows by routine calculations.

In particular, operators $[a]$, $w_n$ and $f_n$ commute with operations of $\text{End}_R(F)$ on $C_F$, and we obtain

Proposition 2.5. When $R$ is a $\Lambda$-algebra such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$,

then operators $[a]$, $w_n$ and $f_n$ are endomorphisms of $\Lambda$-module $C_F$, i.e., $C_F$ is an operator-$\Lambda$-module, and $\psi : C_F \to C_G$, $\psi \in \text{Hom}_R(F, G)$.

is a homomorphism of operator-$\Lambda$-modules.

Now it is clear that "$F \mapsto C_F$, $\psi \mapsto \psi \#$" is a covariant functor on $\mathcal{F}(R)$ with values in the category of operator-modules. We denote this functor by $C(R)$.

2.7. Let $\theta : R \to S$ be a homomorphism of commutative rings with unity, and $F \in \text{obj } \mathcal{F}(R)$.

Proposition 2.6. $\theta_* : C_F \to C_{\theta_* F}$ is linear and commutes with
operators $[a], w_n$ and $f_n$ in the sense that $\theta_* \circ [a] = [\theta(a)] \circ \theta_*$
and $\theta_* \circ f_{n,F} = f_{n, \theta_* F} \circ \theta_*$, i.e., $\theta_*$ is a homomorphism of operator-modules. When $R$ is a $\Lambda$-algebra such that $\mathbb{Z} \subset \Lambda \subset \mathbb{Q}$, then $\theta_*$ is a homomorphism of operator-$\Lambda$-modules.

Proof follows again by routine calculations.

Remark also the commutativity

$$
\begin{array}{c}
\mathcal{C}_F \\
\downarrow \theta_* \\
\mathcal{C}_{\theta_* F} \\
\downarrow (\theta_* \psi)_# \\
\mathcal{C}_{\theta_* \psi}
\end{array} 
\xrightarrow{\psi_#} 
\begin{array}{c}
\mathcal{C}_G \\
\downarrow \theta_* \\
\mathcal{C}_{\theta_* G}
\end{array}
$$

for $\theta : R \rightarrow S$ and $\psi : F \rightarrow G$. Thus $\theta_*$ is a natural homomorphism of functors $\mathcal{C}(R) \rightarrow \mathcal{C}(S) \circ \theta_*$.

2.8. Let $R$ be a commutative ring with unity and $F \in \text{obj} \mathcal{F}(R)$. Put $C_n = R[[T]]_n$ for $n \geq 1$. By definition we see immediately that $C_n$ are subgroups of $C_F = C_1$. Thus we have a filtration of $C_F$:

$$
C_F = C_1 \supset C_2 \supset \ldots \supset C_n \supset \ldots
$$

We say that, for two power series $f, g \in R[[T]]$, $f \equiv g \mod \deg n$ iff $f - g \in R[[T]]_n$, i.e., they have the same terms of degree $< n$.

Lemma 2.7. Let $\gamma_1, \gamma_2 \in C_F$. $\gamma_1 \equiv \gamma_2 \mod \deg n$ iff $\gamma_1 \equiv \gamma_2 \mod \deg n$.

Proof. Suppose that $\gamma_1 \equiv \gamma_2 \mod C_n$. There exists a curve $\gamma' \in C_n$
such that $\gamma_1 + F\gamma' = \gamma_2$. Then

$$\gamma_2(T) = F(\gamma_1(T), \gamma'(T))$$

$$\equiv \gamma_1(T) + \gamma'(T) \mod \deg n+1$$

$$\equiv \gamma_1(T) \mod \deg n.$$

The converse will be proved by induction. The case $n = 1$ is trivial.

Assume it is true for $n - 1$, and suppose $\gamma_1 \equiv \gamma_2 \mod \deg n$. Then

$$\gamma_1 \equiv \gamma_2 \mod \deg n - 1,$$

hence $\gamma'_0 = \gamma_1 - F\gamma_2 \in C_{n-1}$ by assumption. Now

$$\gamma_1(T) = F(\gamma'(T), \gamma_2(T))$$

$$\equiv \gamma'(T) + \gamma_2(T) \mod \deg n$$

$$\equiv a_{n-1} T^{n-1} + \gamma_2(T) \mod \deg n,$$

where $\gamma'(T) = a_{n-1} T^{n-1} + \text{higher terms}$. Since $\gamma_1$ and $\gamma_2$ have the same terms of degree $n - 1$, we conclude that $a_{n-1} = 0$ and $\gamma' \in C_n$.

q. e. d.

By the above lemma we conclude the following

Proposition 2.8. $\mathcal{C}_F$ is complete and Hausdorff with respect to the above filtration topology.

By definition we have

$$[a] (\mathcal{C}_m) \subset \mathcal{C}_m, \quad \mathcal{V}_n (\mathcal{C}_m) \subset \mathcal{C}_{nm}, \quad f_n (\mathcal{C}_m) \subset \mathcal{C}_{[m-1/n]+1}.$$

Thus all three kinds of operators are continuous with respect to the filtration topology of $\mathcal{C}_F$. 
2.9. Let \( R \) be a commutative ring with unity and \( F \in \text{obj} \mathcal{J}(R) \).

Proposition 2.9. Among three kinds of operators on \( \mathcal{C}_F \) there hold the following relations.

i) \([a][b] = [ab], \quad a, b \in R,\]

ii) \(v_n v_m = v_{nm}, \quad n > 1, m > 1,\)

iii) \(f_n f_m = f_{nm}, \quad n > 1, m > 1,\)

iv) \(v_n [a^n] = [a] v_n, \quad n \geq 1, a \in R,\)

v) \(f_n [a] = [a^n] f_n, \quad n \geq 1, a \in R,\)

vi) \(f_n v_n = n \cdot \text{id} \mathcal{C}_F, \quad [1] = v_1 = f_1 = \text{id} \mathcal{C}_F,\)

vii) if \((n, m) = 1\) then \(f_n v_m = v_m f_n,\)

viii) \([a] + [b] = \sum_{n \geq 1} v_n [s_n(a, b)] f_n.\)

In the relation viii) of this proposition \( s_n(X, Y) \) are symmetric polynomials of degree \( n \) over integers, which are defined recursively by the formula

\[x^n + y^n = \sum_{d|n} d \cdot s_d(X, Y)^{n/d}.\]

The right hand side of viii) means an operator which sends each curve \( \gamma \in \mathcal{C}_F \) to

\[
\left( \sum_{n \geq 1} v_n [s_n(a, b)] f_n \right) \gamma = \sum_{n \geq 1} f_n v_n [s_n(a, b)] f_n \gamma,
\]

which is a Cauchy series in \( \mathcal{C}_F \), hence convergent to a curve in \( \mathcal{C}_F \) by
Proposition 2.8.

Proof. Relations i), ii), iii), iv) and v) follow by routine calculations.

For any \( \gamma \in \mathcal{C}_F \), we have

\[
f_n v_n \gamma = [n]_F \circ \gamma,
\]
then by (2.7)' it follows the relation vi).

Suppose \((m, n) = 1\) and \(\{\zeta_1, \ldots, \zeta_n\}\) be \(n\)-th roots of unity. Then \(\{\zeta_1, \ldots, \zeta_n\} = \{\zeta_1^m, \ldots, \zeta_n^m\}\), by which follows the relation vii).

It remains only the proof of the relation viii). First we prove the relation for additive group laws, i.e., suppose \(F = G_a\), an additive formal group. Remark that for \(\gamma(T) = \sum_{i \geq 1} c_i a^i T^i \in \mathcal{C}_{G_a}\) we have

\[
([a]\gamma)(T) = \sum_{i \geq 1} c_i a^i T^i, \quad a \in \mathbb{R},
\]

(2.8)

\[
(v_n \gamma)(T) = \sum_{i \geq 1} c_i T^{ni}, \quad n \geq 1,
\]

\[
(f_n, G_a \gamma)(T) = n \sum_{i \geq 1} c_{ni} T^{i}, \quad n \geq 1,
\]

and the addition in \(\mathcal{C}_{G_a}\) is the ordinary addition of formal power series.

Then

\[
([a] \gamma + [b] \gamma)(T) = \sum_{n \geq 1} c_n (a^n + b^n) T^n
\]

\[
= \sum_{n \geq 1} c_n (\sum_{n = \text{dm}} d \cdot s_d(a, b)^m) T^n
\]

\[
= \sum_{d \geq 1, m \geq 1} c_{dm} d \cdot s_d(a, b)^m T^{md}
\]

\[
= \sum_{d \geq 1} v_d (\sum_{m \geq 1} d c_{dm} \cdot s_d(a, b)^m T^m)
\]

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Next suppose that $R$ is of characteristic zero. Then there exists a unique logarithm

$$L_F : F \cong G_a$$

over $R \otimes \mathbb{Q}$ by 1.5. $L_F$ is a topological isomorphism of operator-modules by Proposition 2.4. Thus $L_F^{-1}$ preserves the relations among operators and the relation (viii) holds in $C_F$ over $R \otimes \mathbb{Q}$. Since coefficients extension $R \subset R \otimes \mathbb{Q}$ embeds $C_F$ over $R$ into $C_F$ over $R \otimes \mathbb{Q}$ as operator-module, the relation (viii) is true in $C_F$ when $R$ is of characteristic zero.

The Lazard ring $U$ is of characteristic zero (by Lazard's Theorem). We consider the universal formal group $F_U$ over $U \otimes \mathbb{Z}[t, u]$ extending coefficients domain, where $t$ and $u$ are indeterminates. The relation (viii) is true in $C_{F_U}$ over $U \otimes \mathbb{Z}[t, u]$ by the above arguments. Let $F$ be a formal group over an arbitrary ring $R$. By universality there exists a homomorphism $\theta : U \longrightarrow R$ such that $\theta_* F_U = F$. Let $a, b \in R$. Extend $\theta$ to a homomorphism $\theta : U \otimes \mathbb{Z}[t, u] \longrightarrow R$ by $\theta(t) = a$ and $\theta(u) = b$. Clearly $\theta_* F_U = F$. Now
\( \theta_* : \mathcal{C}_{F_U} \) over \( U \otimes \mathbb{Z}[t, u] \rightarrow \mathcal{C}_F \)

is a homomorphism of operator-modules by Proposition 2.6, and hence sends the relation viii) in \( \mathcal{C}_{F_U} \) for the pair \((t, u)\) to the relation viii) in \( \mathcal{C}_F \) for the pair \((a, b)\). Thus the proof is complete.

2.10. Let \( F \in \text{obj} \mathcal{F}(R) \).

Proposition 2.10. Every curve \( \gamma \) over \( F \) can be expressed uniquely as a Cauchy series

\[ \gamma = \sum_{n \geq 1}^F v_n [c_{n-1}] \gamma_0, \quad c_{n-1} \in \mathbb{R}, \quad \text{ (i.e., } \gamma(T) = \sum_{n \geq 1}^F (c_{n-1} T^n)) \].

Proof. Let \( \gamma(T) = c_0 T + \text{higher terms} \).

and put

\[ \gamma_1 = \gamma - \sum_{n \geq 1}^F [c_0] \gamma_0. \]

Then by definitions we see easily that \( \gamma_1 \in \mathcal{C}_2 \). Now let

\[ \gamma_1(T) = c_1 T + \text{higher terms} \]

and put

\[ \gamma_2 = \gamma_1 - \sum_{n \geq 1}^F v_2 [c_1] \gamma_0. \]

Then we obtain that \( \gamma_2 \in \mathcal{C}_3 \). By a recursive construction we obtain \( \gamma_n \in \mathcal{C}_{n+1} \) and \( \gamma_{n+1} = \gamma_n - \sum_{n \geq 1}^F v_{n+1} [c_n] \gamma_0 \in \mathcal{C}_{n+2} \), and so on. Thereby we obtain a Cauchy series \( \sum_{n \geq 1}^F v_n [c_{n-1}] \gamma_0 \) which converges to \( \gamma \). The
uniqueness is obvious by construction. q. e. d.

2.11. Let $F \in \text{obj} \mathcal{F}(R)$, $\gamma \in R[[T]]_1$ and invertible. We put

$$F^\gamma(x, y) = \gamma^{-1} \circ F(\gamma(x), \gamma(y)).$$

Then we see easily that $F^\gamma \in \text{obj} \mathcal{F}(R)$ and

$$\gamma : F^\gamma \not\simeq F.$$

We call $F^\gamma$ the transpose of $F$ by $\gamma$. Since $\gamma \in \text{Hom}_R(F^\gamma, F) \subset \mathcal{C}_F$, it is natural to regard $\gamma$ as an (invertible) curve over $F$ when we consider the transpose of $F$ by $\gamma$. 
§3. Typical curves and formal groups

Let $I$ be a set of primes. We use the notation $I$ only to denote such a set of primes. The following special cases are the most important: $I = (p)$, the set of all primes except $p$; $I = [p]$, the set consisting of the single prime $p$.

We denote by $\mathcal{Z}_I$ the following subring of $\mathbb{Q}$:

$$\mathcal{Z}_I = \mathbb{Z}\left[\frac{1}{q} : q \in I\right].$$

Thus $\mathcal{Z}_{(p)} = \text{integers localized at the prime } p,$

$$\mathcal{Z}_{[p]} = \text{the ring consisting of rationals of the form } a/p^k.$$

3.1. Let $R$ be a commutative ring with unity and $F$ a formal group over $R$.

A curve $\gamma$ over $F$ is called $I$-typical iff $f_q \gamma = 0$ for all $q \in I$. $F$ is called $I$-typical iff the identity curve $\gamma_0$ over $F$ is $I$-typical.

When $I = (p)$, we call simply typical in place of $(p)$-typical. Typical curves or formal groups are usually observed when $R$ is a $\mathcal{Z}_{(p)}$-algebra.

Denote by $\mathcal{C}_{TF, I}$ the set of all $I$-typical curves over $F$. Clearly it is a subgroup of $\mathcal{C}_F$ and stable under operators $[a], a \in R, w_n$ and $f_n$ such that $(n,q) = 1$ for all $q \in I$ by Proposition 2.9. We regard these operators as allowable operators on $\mathcal{C}_{TF, I}$. Then $\mathcal{C}_{TF, I}$ is an operator-module over allowable operators.
When \( I = (p) \) we write simply \( C_{TF,(p)} = C_{TF} \). In this case allowable operators are generated by \([a], a \in R, \nu_p \) and \( f_p \).

3.2. Suppose \( R \) is a \( \mathbb{Z}_I \)-algebra and define operators

\[
e_q = e_q, F : C_F \longrightarrow C_F, \quad q \in I,
\]

by \( e_q(\gamma) = \gamma - F \left( \frac{1}{q} \right) \psi_q f_q \gamma \). By Propositions 2.5 and 2.9 we see easily that \( e_q \)'s are idempotents and mutually commutative. Moreover \( e_q \gamma \equiv \gamma \mod C_q \).

Thus the product

\[
(3.1) \quad \epsilon_I = \epsilon_{I,F} = \prod_{q \in I} e_q
\]

is convergent and well-defined operator on \( C_F \). We have also a Cauchy sum expansion

\[
(3.2) \quad \epsilon_I \gamma = \sum_{n \text{ rel } I} F \left( \frac{\mu(n)}{n} \right) \psi_n f_n \gamma
\]

for \( \gamma \in C_F \), where \( \mu(n) \) is the Möbius function and the summation runs over all natural numbers \( n \) of which every prime factor belongs to \( I \) (including \( n = 1 \)).

Proposition 3.1. \( \epsilon_I \) is an idempotent and projects \( C_F \) onto the subgroup \( C_{TF,I} \).

The proof is straightforward if we remark that \( f_q e_q = 0 \).

We call the operator \( \epsilon_I \) Cartier operator over \( F \). In particular the curve

\[
(3.3) \quad \epsilon_I = \epsilon_{I,F} = \epsilon_I \gamma_0
\]
is I-typical, which we regard as the canonical I-typical curve over $F$.

By (3.2) we have

$$\xi_1(T) = T + \text{higher terms}.$$ 

Thus

$$(3.4) \quad \xi_1 : F_1 \to F.$$ 

(cf., 2.11 for definition). By Proposition 2.4 $\xi_1$ maps I-typical curves to I-typical curves and vice-versa. Then, since $\xi_1 \gamma_0 = \xi_1$, I-typical over $F$, we obtain

Proposition 3.2. $F_1$ is an I-typical formal group which is strictly isomorphic to $F$.

We regard $F_1$ as the I-typical formal group canonically associated to $F$.

3.3. Let $R$ be a $\mathbb{Z}_1$-algebra. Let $\mathbb{N}$ be the set of all natural numbers 1, 2, ..., and put

$$\mathbb{N}_1'' = \{k \in \mathbb{N} : (k, q) = 1 \text{ for all } q \in I\},$$

$$\mathbb{N}_1' = \mathbb{N} - \mathbb{N}_1''.$$ 

First consider a curve

$$\gamma(T) = \sum_{k \geq 1} \gamma_{k-1} T^k$$

over $\mathbb{G}_a$, the additive group law over $R$. By (2.8) we obtain
\[(\mathfrak{e}_q, G_a)(T) = \sum_{(k,q)=1} \gamma_{k-1} T^k\]

for \( q \in I \). Thus

\[(3.5) \quad (\mathfrak{e}_I, G_a)(T) = \sum_{k \in \mathbb{N}^*} \gamma_{k-1} T^k.\]

Next assume that \( R \) is of characteristic zero and \( F \in \text{obj} \mathcal{F}(R) \).

Proposition 3.3. Let \( \mathfrak{l} \) and \( \mathfrak{l}_I \) be the logarithms of \( F \) and \( F \) respectively over \( R \otimes Q \). Put

\[\mathfrak{l}(T) = \sum_{k \geq 1} m_{k-1} T^k,\]

then

\[\mathfrak{l}_I(T) = \sum_{k \in \mathbb{N}^*} m_{k-1} T^k.\]

Proof. By (3.4)

\[\mathfrak{l} \circ \mathfrak{e}_I : F \rightarrow G_a.\]

Then, by the uniqueness of logarithm we have

\[\mathfrak{l}_I = \mathfrak{l} \circ \mathfrak{e}_I,\]

and

\[\mathfrak{l}_I(T) = \mathfrak{l} \circ \mathfrak{e}_I(T) = \mathfrak{l} \# (\mathfrak{e}_I, F Y_0)(T)\]

\[= (\mathfrak{e}_I, G_a)(T) = (\mathfrak{e}_I, G_a)(T).\]

Now by (3.5) the proof follows.

3.4. Let \( F \in \text{obj} \mathcal{F}(R) \) and consider \( C_{T,F,I} \). Since Frobenius
operators are linear and continuous, we see easily that $C_{T_F, I}$ is closed in $C_F$. Thus $C_{T_F, I}$ is complete and Hausdorff with respect to the induced filtrations $C_{T_F, I} \cap C_n$.

Now suppose that $R$ is a $Z_I$-algebra.

Lemma 3.4. Let $\gamma \in C_F$ such that

$$\gamma(T) = a \cdot T^k + \text{higher terms}, \quad a \neq 0.$$ 

If $\gamma$ is $I$-typical, then $k \in \mathbb{N}_I$.

Proof. For any $q \in I$ we obtain

$$(f \gamma)(T) = a \cdot (\zeta_1^k + \cdots + \zeta_q^k) T^{k/q} + \text{higher terms},$$

where $\zeta_1, \ldots, \zeta_q$ are $q$-th roots of 1. Since $\gamma$ is $I$-typical, we have

$$a \cdot (\zeta_1^k + \cdots + \zeta_q^k) = 0.$$ 

If $q \mid k$, then

$$\zeta_1^k + \cdots + \zeta_q^k = q,$$

which is invertible and contradicts to the assumption. Thus $(q, k) = 1$.

q. e. d.

Lemma 3.5. Let $F$ be $I$-typical. Then, for any $k \in \mathbb{N}_I$, $a \neq 0$ in $R$, we have

$$\psi_k[a] \gamma_0 \in C_{T_F, I}.$$ 

Proof. For any $q \in I$, $(q, k) = 1$. Thus $f_q \psi_k = \psi_k f_q$ by
Proposition 2.9, vii). And

$$f_{q_k}[a]\gamma_0 = v_k[a^q]f_{q_0} = 0.$$  

q. e. d.

Theorem 3.6. Let $R$ be a $\mathbb{Z}_I$-algebra and $F$ an $I$-typical formal group over $R$. A curve $\gamma$ over $F$ is $I$-typical iff it can be expressed as

$$\gamma = \sum_{k \in N_I} v_k[c_{k-1}]\gamma_0$$

(or $\gamma(T) = \sum_{k \in N_I} (c_{k-1}T^k)$)

with $c_{k-1} \in R$. The expression is unique.

Proof. Suppose $\gamma$ is $I$-typical and express $\gamma$ as a Cauch series

$$\gamma = \sum_{k \geq 1} v_k[c_{k-1}]\gamma_0$$

in $C_F$ by Proposition 2.10. Let $c_{n-1}$ be the first non-zero coefficient in this expression. Then

$$\gamma(T) = c_{n-1}T^n + \text{higher terms},$$

and $n \in N_I$ by Lemma 3.4. Since $v_n[c_{n-1}]\gamma_0$ is $I$-typical by Lemma 3.5 we see that

$$\gamma_1 = \gamma - F v_n[c_{n-1}]\gamma_0 = \sum_{k>n} v_k[c_{k-1}]\gamma_0$$

is $I$-typical. Now apply the same argument to $\gamma_1$ and repeat. We see that $c_{k-1} = 0$ unless $k \in N_I$. Thus we obtain the desired expression.

The converse follows by Lemma 3.5 and the completeness of $C_{F,I}$.  

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The uniqueness of the expression follows by the uniqueness of Proposition 2.10.
§4. Universal typical formal groups

4.1. Let $U$ be the Lazard ring and $F_U$ the universal formal group over $U$. We regard $U$ as the graded ring by non-positive even dimensions so that $F_U$ satisfies the condition (1.5).

Let $I$ be a set of primes and put $U_I = U \otimes \mathbb{Z}_I$, $F_{U,I} = F_U$ over $U_I$ by coefficients extension. By the universality of $F_U$ it follows immediately the universality of $F_{U,I}$ for formal groups over $\mathbb{Z}_I$-algebras.

Now we want to construct an $I$-typical formal group which is universal for $I$-typical formal groups over $\mathbb{Z}_I$-algebras (by restricting the range of the functor $\mathcal{G}$).

Let $R$ be a $\mathbb{Z}_I$-algebras and $F \in \text{obj } \mathcal{G}(R)$. There exists a unique homomorphism

$$\theta : U_I \rightarrow R$$

of $\mathbb{Z}_I$-algebras such that $\theta_* F_{U,I} = F$. By Proposition 2.6 and the definition of Cartier operators we see that

$$\theta_* \varepsilon_{I,U} = \varepsilon_{I,F} \theta_*$$

where $\varepsilon_{I,U}$ denotes the Cartier operator over $F_{U,I}$. (Similar conventions apply also for other notations).

Put

$$\xi_{I,U}(T) = (\varepsilon_{I,U}^0)(T) = \sum_{k \geq 1} F_U(\varepsilon_{k-1}^0 T^k).$$
By definitions we see that

\[(4.1) \quad \varepsilon_s \in U^{-2s} \quad \text{and} \quad \varepsilon_0 = 1.\]

Since \( F \) is \( I \)-typical iff \( \varepsilon_{I,F} \gamma_0 = \gamma_0 \), and since

\[
\theta_{*} \varepsilon_{I,U} = \theta_{*} \varepsilon_{I,UY_0} = \varepsilon_{I,F} \gamma_0,
\]

we obtain

**Proposition 4.1.** \( F \) is \( I \)-typical iff

\[
\{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}, \ldots \} \subseteq \text{Ker} \ \theta.
\]

4.2. Put \( \tilde{U} = U \otimes \mathbb{Q} \), then

\[
U \subseteq U_I \subseteq \tilde{U}.
\]

Put

\[
\log_U T = \sum_{k \geq 1} m_{k-1} T^k, \quad m_0 = 1,
\]

where \( \log_U \) is the logarithm of \( F_U \) (and of course of \( F_{U,I} \)) over \( \tilde{U} \).

Then

\[(4.2) \quad \tilde{U} = \mathbb{Q}[m_1, m_2, \ldots, m_k, \ldots ], \quad \text{dim} \ m_k = -2k,
\]

as is well known.

Put

\[
\mu_{U,I} = \text{the transpose of } F_{U,I} \text{ by } \varepsilon_{I,U}.
\]

\( \mu_{U,I} \) is \( I \)-typical by Proposition 3.2. (Hereafter we use the letter " \( \mu \)"
to denote typical (or I-typical) formal groups in general. Let

\[ u_I : U_I \rightarrow U_I \]

be the unique homomorphism of \( \mathbb{Z}_I \)-algebras such that

\[ u_I \ast F_{U,I} = \nu_{U,I} \]

and

\[ \tilde{u}_I : \tilde{U} \rightarrow \tilde{U} \]

the homomorphism of \( \tilde{U} \) obtained from \( u_I \) by coefficient extension.

Apply \( \tilde{u}_I \ast \) to the strict isomorphism

\[ \log_U : F_U \cong G_a \quad \text{over } \tilde{U} \]

and obtain

\[ \tilde{u}_I \ast (\log_U) : \nu_{U,I} \cong G_a \]

by the functoriality. Thus

\[ (4.3) \quad \tilde{u}_I \ast (\log_U) = \log_{\nu_{U,I}} \]

by the uniqueness of logarithm. Then, by Proposition 3.3 we obtain

Proposition 4.2. \( \tilde{u}_I(m_{k-1}) = 0 \) if \( k \in N'_I \)

\[ = m_{k-1} \quad \text{if } k \in N''_I. \]

Corollary 4.3. \( u_I \) and \( \tilde{u}_I \) are idempotents,

\[ \text{Ker } \tilde{u}_I = \text{the ideal } (m_{k-1} ; \ell \in N'_I) \]
and

\[ \text{Im } \tilde{\mu}_1 = \mathbb{Q}[m_{k-1} : k \in \mathbb{N}_1^*, k \neq 1]. \]

4.3. Put

\[ \hat{U} = 2[m_1, m_2, \ldots, m_k, \ldots]. \]

As is well known

\[ U \supset \hat{U} \supset \tilde{U} \]

and \( \hat{U} \) is the minimal extension of \( U \) over which \( F_U \) becomes isomorphic to \( G_a \).

Since

\[ \log_{\hat{U}} \circ \xi_{I, U} : \mu_{U, I} \rightarrow G_a \]

we obtain

(4.4) \[ \log_{\mu_{U, I}} = \log_{U} \circ \xi_{I, U} \]

by the uniqueness of logarithm.

Now we compute

\[
\log_{U} \circ \xi_{I, U}(T) = \log_{U}(\sum_{i \geq 1} F_{U}(\xi_{i-1}T^{i})) \\
= \sum_{i \geq 1} \sum_{j \geq 1} m_{j-1}^i \xi_{i-1}^j T^{ij} \\
= \sum_{k \geq 1} \left( \sum_{i \geq 1, j \geq 1} m_{j-1}^i \xi_{i-1}^j \right) T^{k}.
\]

On the other hand, by Proposition 3.3 we have

(4.5) \[ \log_{\mu_{U, I}} T = \sum_{k \in \mathbb{N}_1^*} m_{k-1} T^{k}. \]
Thus, by (4.4), comparing the coefficients of $T^k$ we see the relations

\begin{equation}
\xi_{k-1} + \sum_{i,j=k} m_{j-1} \xi_i^{j-1} = 0 \quad \text{for } k \in N_I' - \{1\},
\end{equation}

(4.6)

\begin{equation}
m_{k-1} + \xi_{k-1} + \sum_{i,j=k} m_{j-1} \xi_i^{j-1} = 0 \quad \text{for } k \in N_I'.
\end{equation}

(4.7)

By (4.6), inductively on $k$, we see that

\begin{equation}
\xi_{k-1} = 0 \quad \text{for } k \in N_I'' - \{1\},
\end{equation}

(4.8)

and

\begin{equation}
F_{I,U}(T) = \sum_{k \in N_I'' \cup \{1\}} F_{U}(\xi_{k-1} T^k).
\end{equation}

(4.9)

By (4.7), again inductively on $k$ we obtain

\begin{equation}
\xi_{k-1} \in \hat{U} \cap U_I \quad \text{for } k \in N_I'
\end{equation}

(4.10)

and

\begin{equation}
\xi_{k-1} \equiv - m_{k-1} \mod \text{decomposables in } \hat{U} \quad \text{for } k \in N_I'.
\end{equation}

(4.11)

Thus

\begin{equation}
\hat{U} = \mathbb{Z}[\xi_{k-1}, k \in N_I'] \otimes \mathbb{Z}[m_{k-1}, k \in N_I'' - \{1\}].
\end{equation}

(4.12)

4.4. Let $x_1, x_2, \ldots, x_n, \ldots$ be a polynomial basis of $U$, dim $x_n = -2n$. Then

\[ U_I = \mathbb{Z}_I[x_1, x_2, \ldots, x_n, \ldots]. \]

Observe the inclusion
\( U_I \subset \hat{U}_I = \hat{U} \otimes Z_I \).

For \( k \in N' \) by (4.11)

\[ \xi_{k-1} \not\equiv 0 \mod \text{decomposables and mod } q \]

for all \( q \not\in I \) in \( \hat{U}_I \). The same must be true also in \( U_I \). Hence we can use \( \xi_{k-1}, k \in N' \), as a part of the polynomial basis of \( U_I \)

and we obtain

**Proposition 4.4.**

\[ U_I = \mathbb{Z}_I[\xi_{k-1}, k \in N' \] \( \otimes \mathbb{Z}_I[x_{k-1}, k \in N' - \{1\}] \).

By Proposition 4.1 we have

\[ \{\xi_{k-1}, k \in N' \} \subset \text{Ker } u_I. \]

On the other hand, putting \( \overline{x}_{k-1} = u_I(x_{k-1}) \) for \( k \in N' - \{1\} \) we have

\[ x_{k-1} \equiv \overline{x}_{k-1} \equiv c_{k-1} m_{k-1} \mod \text{decomposables} \]

in \( \hat{U} \) with \( c_{k-1} \neq 0 \). Thus \( \overline{x}_{k-1}, k \in N' - \{1\}, \) are algebraically independent and \( u_I \) maps \( \mathbb{Z}_I[x_{k-1}, k \in N' - \{1\}] \) isomorphically onto \( \mathbb{Z}_I[\overline{x}_{k-1}, k \in N' - \{1\}] \). Hence we obtain

**Proposition 4.5.** i) \( \text{Ker } u_I = (\xi_{k-1}, k \in N' \),

ii) \( U_I/\text{Ker } u_I \cong \mathbb{Z}_I[\overline{x}_{k-1}, k \in N' - \{1\}] \subset U_I,

iii) \( U_I = \mathbb{Z}_I[\xi_{k-1}, k \in N'] \otimes \mathbb{Z}_I[\overline{x}_{k-1}, k \in N' - \{1\}] \),

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where $\tilde{x}_{k-1} = u_I(x_{k-1})$.

4.5. Put

$$B_{I} = \text{Im } u_I \subseteq U_I.$$ 

Then

$$B_{I} = \mathbb{Z}_I[\tilde{x}_{k-1}, \ k \in \mathbb{N}_I \setminus \{1\}] \cong U_I/\ker u_I$$

by Proposition 4.5. Since $p_{U,I} = u_I^*F_U$ we see that all coefficients of $\mu_{U,I}(X, Y)$ belong to $B_{I}$, and $\mu_{U,I}$ determines a formal group

$$\mu_{BP,I} \in \text{obj} \mathcal{F}(B_P)$$

which extends to $\mu_{U,I}$ by extension of the domain of coefficients $B_{I} \subseteq U_I$.

Theorem 4.6. $\mu_{BP,I}$ is I-typical and universal for I-typical formal groups over $\mathbb{Z}_I$-algebras.

Proof. Clearly $\mu_{BP,I}$ is I-typical by definition because $\mu_{U,I}$ is I-typical.

Let $R$ be a $\mathbb{Z}_I$-algebra and $\mu$ an I-typical formal group over $R$.

There exists a unique homomorphism $\theta : U_I \rightarrow R$ such that $\theta_{*}F_{U,I} = \mu$.

By Proposition 4.1 $\ker \theta \supseteq \ker u_I$. Thus $\theta$ factorizes to

$$U_I \xrightarrow{u_I} B_P \xrightarrow{\theta_I} R.$$ 

Since $u_I^*F_{U,I} = \mu_{BP,I}$, we have
The uniqueness of $\theta_I$ follows by the uniqueness of $\theta$.

q. e. d.

Thereby we obtained also the following

**Corollary 4.7.** Let $R$ be a $\mathbb{Z}_I$-algebra and $\mu$ be an $I$-typical formal group over $R$. The homomorphism $\theta : U \rightarrow R$ such that $\theta_* F_U = \mu$

factorizes to $\theta = \theta_I \circ u_I$, $\theta_I : BP_I \rightarrow R$ such that $\theta_I*\mu_{BP,I} = \mu$.

4.6. Let $I$ and $J$ be sets of primes such that $I \subseteq J$. Let $\xi_{I,J}$ be the canonical $J$-typical curve over $\mu_{U,I}(\text{over } U_J)$. Since $\mu_{U,I}$ is $I$-typical we have

$$
\xi_{I,J} = \xi_{J,\mu_0} = \prod_{q \in J-I} \xi_{q,\mu_0}
$$

where $\mu = \mu_{U,I}$. Then

$$
\xi_{I,U} \circ \xi_{I,J} = \xi_{I,U#} \left( \prod_{q \in J-I} \xi_{q,\mu_0} \right)
= \prod_{q \in J-I} \xi_{q,U\xi_{I,U}}
$$

i.e.,

(4.14) \hspace{1cm} \xi_{I,U} \circ \xi_{I,J} = \xi_{J,U}

Since $\xi_{I,J}$ is of course $I$-typical, we have the homomorphism

$$
u_{I,J} : BP_I \otimes \mathbb{Z}_J \rightarrow BP_I \otimes \mathbb{Z}_J$$

of $\mathbb{Z}_J$-algebras such that $\nu_{I,J}*\mu_{BP,I} = \mu_{BP,I}$. Using $\log_{BP,I}$ instead of
log \( u_j \), by the same arguments as Proposition 4.2 we see that

\[
\tag{4.15} \hat{u}_{I,J}(m_{k-1}) = 0 \quad \text{if} \quad k \in \mathbb{N}_I^u - \mathbb{N}_J^u \\
\quad = m_{k-1} \quad \text{if} \quad k \in \mathbb{N}_J^u,
\]

where \( \hat{u}_{I,J} \) is the \( \mathbb{Q} \)-extension of \( u_{I,J} \). In particular \( u_{I,J} \) is an idempotent of \( BP_I \otimes \mathcal{Z}_J \) and we can expect a decomposition of \( BP_I \otimes \mathcal{Z}_J \).

By Proposition 4.2 and (4.15) we see that

\[
\tag{4.16} u_j = u_{I,J} \circ u_I
\]

regarded as the map : \( U_I \longrightarrow U_J \). Thus

\[
\tag{4.17} \text{Im } u_{I,J} = BP_J \subseteq BP_I \otimes \mathcal{Z}_J.
\]

Next we express \( \xi_{I,J} \) as

\[
\xi_{I,J}(T) = \sum_{k \in \mathbb{N}_I^u}^{BP_I} (\xi'_{k-1} T^k)
\]

by Theorem 3.6. By (4.4) and (4.14) we see that

\[
\log_{BP_I} \circ \xi_{I,J} = \log_{BP_J}
\]

then by parallel arguments to (4.7) and (4.8) we see that

\[
\xi'_{k-1} = 0 \quad \text{for} \quad k \in \mathbb{N}_J^u - \{1\}
\]

and

\[
\xi'_{k-1} \in \hat{BP}_I \cap BP_I \otimes \mathcal{Z}_J,
\]

\[
\xi'_{k-1} + m_{k-1} \equiv 0 \mod \text{decomposables in } \hat{BP}_I
\]
for $k \in \mathbb{N}''_I - \mathbb{N}''_J$. Thus

\begin{equation}
\xi_{I,J}(T) = \sum_{k \in (\mathbb{N}''_I - \mathbb{N}''_J) \cup \{1\}}^{B_{P,I}} (\xi_{k-1}^{J})^k,
\end{equation}

and by the same arguments as Propositions 4.4 and 4.5 we obtain

Proposition 4.8.

i) $B_{P,I} \otimes z_J = z_J [\xi_{k-1}^{J}, k \in \mathbb{N}''_I - \mathbb{N}''_J] \otimes B_{P,J}$,

ii) $\text{Ker } u_{I,J} = (\xi_{k-1}^{J}, k \in \mathbb{N}''_I - \mathbb{N}''_J)$. 

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§5. Quillen decomposition

5.1. Let $h^*$ be a multiplicative cohomology theory defined on finite CW-complexes. We assume that the multiplication in $h^*$ is commutative (in graded sense) and associative, and that the Euler class $e^h(L)$ is defined for any complex line bundle $L$ over a complex $X$ such that i) it is natural for bundle maps, ii) $e^h(L) \in h^2(X)$, and iii) $h^*(\mathbb{C}P^n)$ is the truncated polynomial algebra over $h^*(pt)$ generated by the Euler class $x$ of the canonical line bundle over $\mathbb{C}P^n$, truncated by $x^{n+1}$.

Then we can define Chern classes and multiplicative Thom classes in $h^*$ for complex vector bundles. Cf., Dold [8] for details. We call such a cohomology theory $h^*$ complex oriented by a terminology of Quillen [19].

In complex cobordism Thom classes and hence Euler classes for complex vector bundles are canonically defined [7]. Hence complex cobordism is one of the typical examples of complex oriented cohomology theories. We denote by $e^U(L)$ the Euler classes of line bundles in complex cobordism.

We recall the following well-known universality of complex cobordism for complex oriented cohomology theories.

[Universality of complex cobordism] Let $h^*$ be a complex oriented cohomology theory defined on finite CW-complexes. There exists a unique cohomology transformation

$$\theta: U^* \longrightarrow h^*$$
which is i) linear, ii) degree-preserving, iii) multiplicative \(\Theta(1) = 1\) for \(1 \in U^0(\text{pt})\), and iv) \(\Theta(e^L) = e^\theta(L)\) for complex line bundle \(L\).

For proofs we refer to [8], [19].

This universality is actually true also for complex oriented \(h^*\) defined on "arbitrary" CW-complexes if we assume \(h^*\) to be "additive" [8]. And we can expect to develop Quillen decomposition theory for arbitrary CW-complexes. But in that case we need in certain places to discuss convergences with respect to filtrations by finite subcomplexes. To avoid this complexity we shall be content with limiting our discussions only to finite CW-complexes.

5.2. Let \(h^*\) be a complex oriented cohomology theory. For complex line bundles \(L_1\) and \(L_2\) we have

\[e^h(L_1 \otimes L_2) = \sum a_{ij} e^h(L_1)^i e^h(L_2)^j\]

with \(a_{ij} \in h^2(1-i-j)(\text{pt})\). By naturality the coefficients \(a_{ij}\) do not depend on the choices of \(L_1\) and \(L_2\) and we have a well-determined formal power series

\[F_h(X, Y) = \sum a_{ij} x^i y^j\]

of two variables over \(h^*(\text{pt})\). By commutativity and associativity of tensor products, and naturality of Euler classes, we see that \(F_h\) satisfies (1.1), (1.2) and (1.3), i.e., is a formal group. Moreover \(F_h\) satisfies
the condition (1.5) by our choice of dimension of Euler classes.

Of course this formal group \( F_h \) depends on the complex orientation of \( h^* \) (i.e., the choice of Euler classes). So that we may have several formal groups associated with the same cohomology theory \( h^* \) depending on various choices of Euler classes.

Here we recall that Quillen identified \( U^*(pt) \) with the Lazard ring \( U \), whereby he identified the formal group of complex cobordism with the universal formal group \( F_U \), i.e., we have

\[
e^U(L_1 \otimes L_2) = F_U(e^U(L_1), e^U(L_2))
\]

for complex line bundles \( L_1 \) and \( L_2 \). Cf., [18], [19] and [2], §8.

Now let \( h^* \) be complex oriented and

\[
\Theta : U^* \longrightarrow h^*
\]

the unique cohomology transformation by the universality of complex cobordism. Since \( \Theta \) is linear, multiplicative and preserves Euler classes we see readily that

\[
(5.1) \quad \Theta(pt)_* F_U = F_h.
\]

5.3. Let \( I \) be a set of primes. The assignment

\[
(X, A) \mapsto U^*(X, A)_I = U^*(X, A) \otimes \mathbb{Z}_I
\]

is a multiplicative cohomology, denoted by \( U^*( )_I \).
Using the power series $\xi_{I,U}$ we define

$$\xi_{I,U}^{-1}(e^U(L)) \in U^2(X)_I$$

as Euler class of a line bundle $L$ over $X$ for $U^*(\cdot)_I$. Thus $U^*(\cdot)_I$ is complex oriented. Since

$$\xi_{I,U}^{-1}(e^U(L_1 \otimes L_2)) = \mu_{U,I}(\xi_{I,U}^{-1}(e^U(L_1)), \xi_{I,U}^{-1}(e^U(L_2)))$$

the corresponding formal group is $\mathcal{U}_{U,I}$.

By the universality of complex cobordism we have a cohomology transformation

$$U^* \longrightarrow U^*(\cdot)_I$$

which sends $e^U(L)$ to $\xi_{I,U}^{-1}(e^U(L))$. Extending this $\mathbb{Z}_I$-linearly we obtain the cohomology transformation

$$(5.2) \quad \bar{F}_I : U^*(\cdot)_I \longrightarrow U^*(\cdot)_I$$

which is $\mathbb{Z}_I$-linear, degree-preserving, multiplicative and $\bar{F}_I(e^U(L)) = \xi_{I,U}^{-1}(e^U(L))$. Then

$$\bar{F}_I(pt)_* F_{U,I} = \mu_{U,I}$$

by (5.1), i.e.,

$$(5.3) \quad \xi_I(pt) = u_I : U^*(pt)_I \longrightarrow U^*(pt)_I.$$

In particular $\bar{F}_I(pt)$ is an idempotent of $U^*(pt)_I$ by Corollary 4.3.
5.4. We want to show that $\tilde{\xi}_I$ is an idempotent of the cohomology theory $U^*(\ )_I$. To this end we use Landweber-Novikov operations [14], [17] in a modified form.

Let $\xi = (t_1, t_2, \ldots, t_n, \ldots)$ be a sequence of indeterminates $t_n$ with $\dim t_n = -2n$. For each finite CW-pair $(X, A)$ we put

$$U^*(X, A)[\xi] = U^*(X, A) \otimes \mathbb{Z}[t_1, t_2, \ldots, t_n, \ldots].$$

Obviously $U^*(\ )[\xi]$ is a multiplicative cohomology theory.

Put

$$\phi_\xi(T) = \sum_{k \geq 1} F_U(t_{k-1}T^k), \quad t_0 = 1,$$

and assign

$$\phi_\xi^{-1}(e_U(L)) \in U^2(X)[\xi]$$

as Euler class of a line bundle $L$ over $X$. Thus $U^*(\ )[\xi]$ is complex oriented and its formal group is $F_U$. Then by the universality of complex cobordism we have a cohomology transformation

$$\xi : U^* \longrightarrow U^*(\ )[\xi]$$

which is linear, degree-preserving, multiplicative and

$$\xi(e_U(L)) = \phi_\xi^{-1}(e_U(L))$$

for a line bundle $L$. And

$$\xi(pt)_*F_U = F_U.$$
This is parallel to Quillen's presentation [19] of Landweber-Novikov operations but not the same. After certain polynomial changes of indeterminates over $U^*(pt)$ our $s_1$ could be identified with Quillen's $s_1$.

Put

$$s_1(x) = \sum_{\alpha} s_\alpha(x) t^\alpha$$

for any $x \in U^*(X)$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots)$ is a sequence of non-negative integers such that all $\alpha_n$ but finite are zero, and $t^\alpha$ denotes the monomial

$$t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_n^{\alpha_n} \ldots$$

Then we get linear cohomology operations

$$s_\alpha : U^* \rightarrow U^*$$

of degree $2|\alpha|$ for each sequence $\alpha$, where $|\alpha| = \sum_{n=1}^{\infty} \alpha_n$. These are our modified Landweber-Novikov operations and can be expressed as linear combinations of Landweber-Novikov operations over $U^*(pt)$.

By the property of $s_1$, it follows that

(5.7) $s_0 = \text{id}$, where $0 = (0, 0, \ldots, 0, \ldots)$,

(5.8) $s_\alpha(x, y) = \sum_{\beta + \gamma = \alpha} s_\beta(x) \cdot s_\gamma(y)$

for internal and external multiplications.
5.5. Let

\[ \rho : U^*(\_)[t] \longrightarrow U^*(\_)_I \]

be a cohomology transformation defined by \( \rho(t_j) = \xi_j, \ j \geq 1 \), and \( \rho(x) = x \) for \( x \in U^*(X) \), where \( \xi_j \) are coefficients of \( \xi^I_*U(T) \) in the expression (4.9), whence \( \xi_{k-1} = 0 \) if \( k \in \mathbb{N}_I^* \) and \( k \neq 1 \).

Clearly \( \rho \) is linear, degree preserving and multiplicative, and \( \rho \circ \xi_1 \) sends \( e^U(L) \) to \( \xi_1^{-1}(e^U(L)) \). Hence, by the uniqueness of cohomology transformation obtained by the universality of complex cobordism we see that

(5.9) \[ \xi_I = \rho \circ \xi_1. \]

Theorem 5.1. \( \xi_I \) is an idempotent of \( U^*(\_)_I \).

Proof. By (5.3) and Proposition 4.5 it follows that

\[ \xi_I(\xi^\alpha) = 0 \quad \text{if} \quad \alpha \neq 0, \]

where \( \xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \ldots \). Now for any \( x \in U^*(X, A)_I \) we have

\[ \xi_I(x) = \sum_\alpha g^\alpha(x) \xi^\alpha \]

by (5.9). Then

\[ \xi_I(\xi_I(x)) = \sum_\alpha \xi_1(g^\alpha(x)) \cdot \xi_I(\xi^\alpha) = \xi_1(g^0(x)) = \xi_I(x), \]

i.e., \( \xi_I \) is an idempotent.
Corollary 5.2. There holds natural stable direct sum decomposition

$$U^*(X, A)_I = \text{Im} \tilde{e}_I(X, A) \oplus \text{Ker} \tilde{e}_I(X, A).$$

For any $x \in \text{Ker} \tilde{e}_I(X, A)$ we have

$$x + \sum_{\alpha \neq 0} \bar{g}_\alpha(x)x^\alpha = \tilde{e}_I(x) = 0,$$

i.e.,

$$x = -\sum_{\alpha \neq 0} \bar{g}_\alpha(x)x^\alpha$$

and we obtain

Corollary 5.3. $\text{Ker} \tilde{e}_I(X, A) = (\text{Ker} \tilde{e}_I(pt)) \cdot U^*(X, A)_I.$

5.6. Put

$$(5.10) \quad BP^*_I(X, A) = \text{Im} \tilde{e}_I(X, A)$$

for any finite CW-pair $(X, A)$. By (5.5) and (4.13)

$$BP^*_I(pt) = BP^*_I,$$

and by Corollary 5.2 the assignment

$$(X, A) \longmapsto BP^*_I(X, A)$$

is a cohomology theory. Moreover it is multiplicative because $\tilde{e}_I$ is multiplicative, that is,

Proposition 5.4. $BP^*_I$ is a multiplicative cohomology theory such that $BP^*_I(pt) = BP^*_I$, a $\mathbb{Z}_I$-algebra.

By definition (5.10) we have canonical cohomology transformations
\[(5.11) \quad \pi_I : U^* (\_)_I \rightarrow BP^*_I, \text{ natural surjection,} \]
\[(5.12) \quad i_I : BP^*_I \rightarrow U^* (\_)_I, \text{ natural injection,} \]
such that
\[(5.13) \quad i_I \circ \pi_I = \tilde{\varepsilon}_I. \]

Using coefficients $\xi_{2^{-1}}$ of $\xi_{I, U(T)}$ in the expression (4.9) we put
\[
V^*_I (X, A) = \mathbb{Z}_I [\xi_{2^{-1}}, \, \& \, \in \mathbb{N}^+_I] \otimes BP^*_I (X, A)
\]
for finite CW-pairs $(X, A)$. Then $V^*_I$ is a multiplicative cohomology theory. Define
\[
\Theta_I : V^*_I \rightarrow U^* (\_)_I
\]
by $\Theta_I (\xi^\alpha \otimes x) = \xi^\alpha \cdot i_I (x)$ for $x \in BP^*_I (X, A)$. As is easily seen $\Theta_I$ is a linear multiplicative cohomology transformation, and $\Theta_I (pt)$ is an isomorphism by Proposition 4.5. Hence $\Theta_I$ is an isomorphism of cohomology theories by the comparison theorem of cohomology theories over finite CW-pairs. Thus

Theorem 5.5. $i_I$ induces the natural isomorphism
\[
U^* (\_)_I \approx \mathbb{Z}_I [\xi_{2^{-1}}, \, \& \, \in \mathbb{N}^+_I] \otimes BP^*_I
\]
of cohomology theories.

5.7. Let $p$ be a specified prime and put
\[
(5.14) \quad BP^*_p = BP^*_I (p)
\]
for $I = (p)$. This is called Brown-Peterson cohomology theory.

In this case the isomorphism of Theorem 5.5 takes the form

$$U^*(\ )|_p = \mathbb{Z}_p[\xi_{p-1}^w; \ell \neq p^s] \otimes \mathbb{B}^p.$$  

This is the Quillen decomposition of complex cobordism localized at the prime $p$.

5.8. Let $I$ be a set of primes and put

$$e_{\mathbb{B}^p, I}(L) = \pi^U_1(e^U(L))$$

for a line bundle $L$. By the decomposition Theorem 5.5 we see easily that $e_{\mathbb{B}^p, I}(L)$ satisfies the required properties of Euler classes. Hence $\mathbb{B}^p$ is complex oriented. By definition of $\tilde{e}_I$ and (5.13) we have

$$i_I(e_{\mathbb{B}^p, I}(L)) = \tilde{e}_I^{-1}(e^U(L)).$$

Then, by (5.1), (5.3) and the definition of $\nu_{\mathbb{B}^p, I}$ we obtain

**Theorem 5.6.** $\mathbb{B}^p$ is complex oriented and its associated formal group is $\nu_{\mathbb{B}^p, I}$.

5.9. Our next purpose is to give a decomposition of $\mathbb{B}^p(I) \otimes \mathbb{Z}_J$ into $\mathbb{B}^p(J \subset J)$ which extends the decomposition of Proposition 4.8 to cohomology theory. For this purpose we start with introducing Landweber-Novikov type operations in $\mathbb{B}^p$.

Let $I$ be a set of primes and

$$t_I = \{t_{k-1}; k \in \mathbb{N}_I - \{1\}\}$$
the subsequence of $t$. We consider multiplicative cohomology

$$BP_I^*(\mathbb{Z}[t]) = BP_I^*(\mathbb{Z}) \otimes \mathbb{Z}[t_{k-1}; k \in \mathbb{N}_I - \{1\}] .$$

Putting

$$(5.18) \quad \phi_{t,I}(T) = \sum_{k \in \mathbb{N}_I} \mu_{BP,I}(t_{k-1})^k, \quad t_0 = 1,$$

we assign

$$\phi_{t,I}^{-1}(e_{BP,I}(L)) \in BP_I^2(X)[q_t]$$

as Euler class of a line bundle $L$ over $X$. Thus $BP_I^*(\mathbb{Z}[t])$ is complex oriented and its associated formal group is $\phi_{t,I}$. By the universality of complex cobordism we have a cohomology transformation

$$\mathcal{S}_{t,I} : U^* \longrightarrow BP_I^*(\mathbb{Z}[t])$$

which is linear, degree-preserving, multiplicative and

$$\mathcal{S}_{t,I}(e_U(L)) = \phi_{t,I}^{-1}(e_{BP,I}(L))$$

for a line bundle $L$. Then

$$\mathcal{S}_{t,I}(pt)^*F_U = \mu_{BP,I}.$$

Here we remark that $\mu_{BP,I}$ is $I$-typical and $\phi_{t,I}$ is an $I$-typical curve over $\mu_{BP,I}$ (extending the domain of coefficients to $BP_I \otimes \mathbb{Z}[t]$) by Theorem 3.6. Thus $\mu_{BP,I}$ is $I$-typical. Then, by Proposition 4.1

$$\{\xi_{k-1}; k \in \mathbb{N}_I\} \subseteq \text{Ker} \mathcal{S}_{t,I}(pt)$$

(extendng $\mathcal{S}_{t,I}$ over $U_I^*$ by $\mathbb{Z}_I$-linearity), where $\xi_{k-1}$ are coefficients...
of $\xi_{I,U}(T)$ in expression (4.9). Now by Corollary 5.3 we have a factorization of $S_{t,I}$:

$$
\begin{array}{ccc}
U^*(t)_I & \xrightarrow{t_I} & BP^*_I(t) \\
& & \xrightarrow{r_{t,I}} BP^*_I(t)[t_I] \\
& & \xrightarrow{S_{t,I}} \\
\end{array}
$$

(5.19)

By construction it is clear that $r_{t,I}$ is a linear cohomology transformation which is degree-preserving, multiplicative and

$$
r_{t,I}(e_{BP,I}(L)) = \phi_{t,I}^{-1}(e_{BP,I}(L))
$$

(5.20)

for a line bundle $L$. Then

$$
r_{t,I}(pt)_* \mu_{BP,I} = \phi_{t,I}.
$$

(5.21)

If we take the coefficients of monomials (of $t_I$) in $r_{t,I}$ we get Landweber-Novikov type operations in $BP^*_I$. We discuss their properties for $I = \{p\}$ later in §7.

5.10. Let $I$ and $J$ be sets of primes such that $I \subseteq J$. Take the canonical $J$-typical curve $\xi_{I,J}$ over $\mu_{BP,I}$ (over $BP^*_I \otimes \mathbb{Z}_J$). Let $\xi_{k-1}^I$, $k \in \mathbb{N}_I''$, be the coefficients of $\xi_{I,J}(T)$ in expression (4.18) and define cohomology transformation

$$
\rho' : BP^*_I(t) \rightarrow BP^*_I(t) \otimes \mathbb{Z}_J
$$

by $\rho'(t_{k-1}) = \xi_{k-1}^I$, $k \in \mathbb{N}_I''$, and $\rho'(x) = x$ for $x \in BP^*_I(X)$. Put

$$
\xi_{I,J} = \rho' \circ S_{t,I} : BP^*_I(t) \otimes \mathbb{Z}_J \rightarrow BP^*_I(t) \otimes \mathbb{Z}_J.
$$

(5.22)
This is a linear cohomology transformation which is degree-preserving, multiplicative and
\[ e_{I,J}(\text{BP}_I(L)) = \xi_{I,J}^{-1}(e_{I,J}(\text{BP}_I(L))). \]
Thus

\[ (5.23) \quad e_{I,J}(\text{pt})_\ast \mu_{\text{BP}_I} = e_{I,J}_\ast \mu_{\text{BP}_I} = e_{\text{BP}_J}. \]

Therefore

\[ (5.24) \quad e_{I,J}(\text{pt}) : \text{BP}_I \otimes \mathbb{Z}_J \rightarrow \text{BP}_I \otimes \mathbb{Z}_J. \]

In particular \( e_{I,J}(\text{pt}) \) is an idempotent of \( \text{BP}_I \otimes \mathbb{Z}_J \). Now by a parallel argument to Theorem 5.1 we obtain

Proposition 5.7. \( e_{I,J} \) is an idempotent of \( \text{BP}_I(\ ) \otimes \mathbb{Z}_J \).

Corollary 5.8. i) \( \text{Im } e_{I,J} = \text{BP}_J^\ast \).

ii) \( \text{Ker } e_{I,J}(X,A) = \text{(Ker } e_{I,J}(\text{pt})). \text{BP}_I^\ast (X,A) \otimes \mathbb{Z}_J \).

iii) \( \text{BP}_I^\ast ( ) \otimes \mathbb{Z}_J = \mathbb{Z}_J[e_{k-1}^\prime, k \in N'_I - N''_J] \otimes \text{BP}_J^\ast \).

In particular, when \( p \not\in I \) and \( J = (p) \) we have the decomposition

\[ (5.25) \quad \text{BP}_I^\ast ( ) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[e_{k-1}^\prime, k \in N'_I, k \neq p^S] \otimes \text{BP}_I^\ast , \]

which we call the Quillen decomposition of \( \text{BP}_I^\ast \) localized at the prime \( p \).

5.11. Let \( \Omega^\ast ( ) \) be the oriented cobordism theory. Here we consider the Quillen decomposition of \( \Omega^\ast ( )[2] = \Omega^\ast ( ) \otimes \mathbb{Z}_{[\frac{1}{2}]} \). Let
be the forgetful functor of complex structures. Clearly $S$ is a multiplicative cohomology transformation.

In $\Omega^*(\ )$ Euler classes are canonically defined for real oriented vector bundles. Every complex line bundle $L$ determines canonically an oriented 2-plane bundle $L_{\mathbb{R}}$. We define

$$e^{SO}(L) = \Omega^*\text{-Euler class of } L_{\mathbb{R}}.$$

Thus $\Omega^*(\ )$ is complex oriented. We denote its associated formal group by $F_{SO}$. We see easily that

$$S(pt)_*F_{\mathbb{U}} = F_{SO}.$$

Remark that, if we change the orientation of a real oriented vector bundle, then $\Omega^*$-Euler class changes to its negative. Thus

$$e^{SO}(L) = - e^{SO}(L)$$

and

$$F_{SO}( - T, T) = 0.$$

Now we have

$$(f_2, SO\gamma_0)(T) = F_{SO}( - T^{1/2}, T^{1/2}) = 0,$$

i.e.,

**Proposition 5.9.** $F_{SO}$ is $[2]$-typical.
Next we observe the cohomology transformation

\[ S_2 = S \otimes \mathbb{Z}_2 : U^*(\_)[2] \longrightarrow \Omega^*(\_)[2]. \]

By Propositions 4.1 and 5.9 we see that

\[ \text{Ker } S_2(\text{pt}) \supset \text{Ker } \xi_2(\text{pt}). \]

Then, by Corollary 5.3 and multiplicativity of \( S_2 \)
we see that

\[ \text{Ker } S_2(X, A) \supset \text{Ker } \xi_2(X, A) \]

for any \((X, A)\), i.e., \( S_2 \) factorizes to

\[ \begin{array}{ccc}
U^*(\_)[2] & \xrightarrow{\pi[2]} & \text{BP}^*[2] \\
\downarrow & & \downarrow \phi \\
S[2] & & \Omega^*(\_)[2]
\end{array} \]

By Proposition 4.5 we know that \( \text{Ker } \pi_2[2](\text{pt}) \) is the ideal generated by all elements of dimensions \( \equiv 2 \ (\text{mod } 4) \). But \( \text{Ker } S_2[2](\text{pt}) \) is also the same by Stong [21], p.178. Thus \( \phi(\text{pt}) \) is injective. On the other hand \( S_2[2](\text{pt}) \) is surjective by [21], p.180. Hence \( \phi(\text{pt}) \) is isomorphic and by the comparison theorem of cohomology theories we obtain

Theorem 5.10. The forgetful functor \( S \) of complex structures induces an isomorphism

\[ \Phi : \text{BP}^*[2] \cong \Omega^*(\_)[2] \]

of cohomology theories.
Let $p$ be an odd prime. By Quillen decomposition (5.25) we obtain the following decomposition of $\Omega^*(\mathbb{A})_p$:

\begin{equation}
\Omega^*(\mathbb{A})_p \cong \mathbb{Z}_p[\mathbb{F}_k^{-1}, \text{k odd and } k \neq p^s] \otimes BP^*.
\end{equation}
§6. Generators of U*(pt) and BP*(pt)

6.1. Let \( p \) be a prime. Putting

\[(f_{p,U}U_0)(T) = \sum_{n \geq 1} \psi(p)_{p^n-1} T^n,\]

we see that \( \psi_s = \psi_s(p) \in U^{-2s} \) by definitions. Compute \( \log_{U} f_{p,U}U_0 \) in two ways:

\[
(\log_{U} f_{p,U}U_0)(T) = \log_{U}(\sum_{n \geq 1} \psi(p)_{p^n-1} T^n)) = \sum_{i \geq 1} \sum_{j \geq 1} m_{ij} v^i_{p^n-1} T^{ij} = \sum_{n \geq 1} (\sum_{i=j} m_{ij} v^i_{p^n-1}) T^n
\]

and

\[
(\log_{U} f_{p,U}U_0)(T) = (f_{p,G} \log_U)(T) = p^* \sum_{n \geq 1} m_{p^n-1} T^n,
\]

(cf., (2.8)). Then compare the coefficients of \( T^n \), and we obtain

\[(6.2) \quad p^* m_{p^n-1} = \sum_{i=j} m_{ij} v^i_{p^n-1} = \psi(p)_{p^n-1} + \sum_{1 \leq i < n} m_{ij} v^i_{p^n-1} \]

for all \( n \geq 1 \).

Let \( s_n \) denote the Chern number corresponding to the power sum \( \sum t_i^n \).

Remark that

\[
s_n : U^{-2n}(pt) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}
\]

is a linear map such that

\[
s_n \text{ (decomposable element)} = 0.
\]
By Mischenko series we have

\[ m_{k-1} = \frac{[CP_{k-1}]}{k}. \]

Thus

\[ s_{k-1}(m_{k-1}) = 1 \]

for all \( k > 1 \). Now apply \( s_{pn-1} \) to (6.2) we obtain

\[ s_{pn-1}(v^{(p)}) = p. \]  

(6.3)

By a well known theorem of Milnor, if a sequence \( \{ x_1, x_2, \ldots, x_n, \ldots \} \) of elements of U*(pt), dim \( x_n = -2n \), satisfies

\[ s_n(x_n) = p \quad \text{if} \quad n = p^s - 1 \quad \text{for some prime} \quad p, \]

(6.4)

\[ = 1 \quad \text{otherwise}, \]

then it is a polynomial basis of U*(pt). Such a sequence is called a Milnor basis of U*(pt). Then (6.3) shows that

\[ \{ v^{(p)}_{p-1}, v^{(p)}_{p^2-1}, \ldots, v^{(p)}_{p^{k-1}}, \ldots \} \]  

forms a part of a Milnor basis.

(6.5)

Let \( k \) be an integer > 1 which is not a prime power. Let \( p \) and \( q \) be different prime factors of \( k \). Since \( s_{k-1}(v^{(p)}_{k-1}) = p \) and \( s_{k-1}(v^{(q)}_{k-1}) = q \) we find integers \( a \) and \( b \) such that

\[ s_{k-1}(a v^{(p)}_{k-1} + b v^{(q)}_{k-1}) = 1. \]

Thus for each dimension - 2n we can find an element \( x_n \) satisfying (6.4) as a linear combination of our elements \( v^{(p)}_n \)'s, and we obtain
Proposition 6.1. The elements $v^{(p)}_{pn-1}$ defined by (6.1), for all $n > 1$ and all primes $p$, generate $U^*(pt)$.

6.2. Let $p$ be a fixed prime. In (6.2) we put $n = p^{k-1}$, then

(6.6) $\frac{p^m k}{p^k - 1} = v^{(p)}_{p^{k-1}} + \sum_{i=1}^{k-1} \frac{m^{p^{k-1}}}{p^{k-1}} (v^{(p)}_{p^{k-1}})^{p^{k-i}}$ for all $k \geq 1$.

Apply $\bar{u}_{(p)}$ on both sides of (6.6). By Proposition 4.2 we have

(6.7) $\frac{p^m k}{p^k - 1} = \bar{v}^{(p)}_{p^{k-1}} + \sum_{i=1}^{k-1} \frac{m^{p^{k-1}}}{p^{k-1}} \bar{v}^{(p)}_{p^{i-1}}$ for all $k \geq 1$.

where $\bar{v}^{(p)}_{p^{i-1}} = u_{(p)}(v^{(p)}_{p^{i-1}})$. Comparing the two recursive formulas (6.6) and (6.7) we obtain

(6.8) $u_{(p)}(v^{(p)}_{p^{k-1}}) = \frac{v^{(p)}_{p^{k-1}}}{p^{k-1}}$ for all $k \geq 1$.

Now by Proposition 4.5, (6.5) and (6.8) we obtain

Theorem 6.2. $BP^*(pt) = \mathbb{Z}_{(p)}[v^{(p)}_{p^{2-1}}, v^{(p)}_{p^{3-1}}, \ldots, v^{(p)}_{p^{k-1}}, \ldots]$.

Apply $\pi_{(p)}^*$ to both sides of (6.1) and obtain

$\left(\mathbb{F}_{p, BP} Y_0(T)\right) = \sum_{n \geq 1} \mu_{BP} \left(\bar{v}^{(p)}_{pn-1} T^n\right)$,

where $\bar{v}^{(p)}_{pn-1} = \pi_{(p)}(v^{(p)}_{pn-1})$. Since $\mathbb{F}_{p, BP} Y_0$ is a typical curve over $\mu_{BP}$, we see by Theorem 3.6 that

(6.9) $u_{(p)}(v^{(p)}_{pn-1}) = 0$ for $n \neq p^q$,

and using (6.8) we obtain

Theorem 6.3. $\mathbb{F}_{p, BP} Y_0(T) = \sum_{i \geq 1} \mu_{BP} \left(v^{(p)}_{p^{i-1}} T^{p^{i+1}}\right)$,

where the coefficients are the polynomial basis of $BP^*(pt)$ of Theorem 6.2.
Remark. i) Our polynomial basis of $BP^*(pt)$ of Theorem 6.2 satisfies the recursive formula (6.6) which is the same as the corresponding formula of Hazewinkel [12]. Hence our generators are the same as those of Hazewinkel. In case $p = 2$ a similar recursive formula is obtained also by Liulevicius [16].

ii) By our method it is already clear that the generators $\{v_k(p)\}_{p^{k-1}}$ of $BP^*(pt)$ are integrable, i.e., elements of $U^*(pt)$. This fact was observed also by Alexander [4].

6.3. Let $p$ be a fixed prime and $q$ be another prime. Since $f_q, BP^0 = 0$, applying $\pi(p)(pt)_*^*$ to $f_q, u^0$ expressed in the form (6.1), we see that

$$u(p)^{v(q)}_{q^{n-1}} = 0$$

for all $n \geq 1$. By (6.8), (6.9), (6.10) and Proposition 6.1 we see

Proposition 6.4 (Integrity of $u(p)$). $u(p)^{(U)} = 2[v_{p^{i-1}}, i \geq 1] \subseteq U$, i.e., $U$ is stable under the idempotent $u(p)$ and by restriction $u(p)$ determines an idempotent of $U$.

6.4. Let $p$ be a prime and put

$$[p]_U(T) = \sum_{n \geq 1} U_n \cdot (p)^{n-1}.$$

Then $w_{n-1} = w_n(p) \in U^{-2(n-1)}$ and $w_0 = p$. Now compute $log_U \circ [p]_U$ and obtain
In particular we obtain

\[ w^{(p)}(p) = (p - p^k) m^{k-1} - \sum_{i=1}^{k-1} p^{k-i} w^{p^i} \]

for \( k \geq 1 \). Then we see that

\[ s_{p^{k-1}} w^{(p)} = p^{k-1}, \]

hence \( \{w^{(p)}_{p-1}, w^{(p)}_{p^2}, \ldots, w^{(p)}_{p^k}, \ldots\} \) forms a part of a polynomial basis of \( U^*(pt)(p) \). On the other hand applying \( u^{(p)} \) to (6.12) we obtain

\[ u^{(p)} w^{(p)}_{p^k} = w^{(p)}_{p^{k-1}}. \]

Again apply \( \pi_{(p)}^* \) to \([p]_U\) and remark that \([p]_{BP}\) is a typical curve.

Thus we obtain

**Theorem 6.5.** Putting

\[ [p]_{BP}(T) = \sum_{k \geq 0}^{1} p^k w^{(p)}_{p^{k-1}}, \quad w^{(p)}_0 = p, \]

we obtain

\[ BP^*(pt) = Z_{(p)}[w^{(p)}_{p-1}, w^{(p)}_{p^2}, \ldots, w^{(p)}_{p^k}, \ldots]. \]

These generators are also integrable.

7.1. Fix a prime $p$. In 5.9 we defined the cohomology transformation

$$r_{(p)} = r_{(p)}(P): BP^* \to BP^*[r_{(p)}]$$

which is $\mathbb{Z}_{(p)}$-linear, degree-preserving, multiplicative and

$$r_{(p)}(e_{BP}(L)) = \phi_{(p)}^{-1}(e_{BP}(L)),$$

where $t_{(p)} = (t_1, t_2, \ldots, t_n, \ldots)$ is a sequence of indeterminates with $\dim t_k = 2(p^k - 1)$ (we replace here the letter $t_k$ by $t_k$ for simplicity), $\phi_{(p)}(T) = \sum_{k \geq 0} t_k^{l_{BP}}(t_k^{p^k - 1})$ with $t_0 = 1$, and $e_{BP}(L) \in BP^2(X)$, $BP^*$-Euler class of a line bundle $L$ over $X$.

Put

$$r_{(p)}(x) = \sum_{\alpha} r_{\alpha}(x) \alpha_{(p)}, \quad x \in BP^*(X, A),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots)$ is a sequence of non-negative integers such that all $\alpha_k$ but a finite are zero and

$$\alpha_{(p)} = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_k^{\alpha_k} \ldots$$

is a monomial of $t_k$'s. Then we get linear stable operations

$$r_{\alpha}: BP^*( ) \to BP^*( )$$

with

$$\deg r_{\alpha} = 2 \sum_{i} \alpha_i(p^i - 1)$$

for each sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$. These are Landweber-Novikov type operations.
operations in Brown-Peterson cohomology, and we call them Quillen operations.

After identifying

$$BP^*(x)[t_0, p] = (BP \wedge BP)^*(x)$$

by making use of Brown-Peterson spectrum, we can see that every operation in $BP^*$ can be expressed uniquely as an infinite sum

$$\sum \alpha \cdot u_\alpha \cdot r_\alpha', \quad u_\alpha \in BP^*(pt),$$

as in [2], [14], [17]. But we will not discuss this point here, but rather we observe certain properties of these Quillen operations.

First of all it is clear by definition and properties of $r_\alpha$ that

(7.5) \[ r_0 = \text{id}, \quad \alpha = (0, 0, \ldots, 0, \ldots), \]

and

(7.6) \[ r_\alpha(x \cdot y) = \sum \beta + \gamma = \alpha \cdot r_\beta(x) \cdot r_\gamma(y) \]

for internal and external multiplications.

7.2. Putting $\mu = \mu_{BP}$ and $\mu' = \mu_{q_I}(p)$, we have

(7.7) \[ r_\alpha(pt)_* \mu_{BP} = \mu', \]

(cf., (5.21)). Let

$$\log_{\mu} : \mu \cong G_a$$

$$\log_{\mu'} : \mu' \cong G_a,$$

logarithms over 0-extensions. Then

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by functionality and the uniqueness of logarithm. On the other hand, since

\[ \phi_{t}(p) : \mu' \cong \mu, \text{ by the uniqueness of logarithm we obtain} \]

\[ \log_{\mu'} = \log_{\mu} \circ \phi_{t}(p). \]

We compute \( \log_{\mu'} \) in two ways by (7.8) and (7.9):

\[
(7.8) \quad r_{t}(pt) \cdot \log_{\mu} = \log_{\mu'},
\]

since

\[
(7.9) \quad \log_{\mu'} = \log_{\mu} \circ \phi_{t}(p).
\]

We compute \( \log_{\mu'} \) in two ways by (7.8) and (7.9):

\[
(7.10) \quad \log_{\mu} T = \sum_{k \geq 0} m_{k-1} T^{p^{k}}
\]

by Proposition 3.3, where \( m_{k-1} \) is the coefficient of \( T^{k} \) in \( \log_{\mu} T \),

we see by (7.8) that

\[ \log_{\mu'} T = \sum_{k \geq 0} r_{t}(pt)(m_{k-1}) T^{p^{k}} ; \]

on the other hand by (7.9) we see that

\[ \log_{\mu'} T = \log_{\mu}(t(T)) \]

\[ = \sum_{j \geq 0} \log_{\mu} T^{p^{j}} \]

\[ = \sum_{k \geq 0} \left( \sum_{h=0}^{k} m_{h} T^{p^{h}(k-h)} T^{p^{k}} \right) . \]

Comparing the coefficients of \( T^{p^{n}} \) we obtain

\[
(7.11) \quad r_{t}(pt)(m_{n-1}) = \sum_{h=0}^{n} m_{h} T^{p^{h}(n-h)}
\]

after extending \( r_{t}(pt) \) over \( BP^{*}(pt) \otimes Q \) by \( Q \)-linearity.

Since \( m_{n-1} = \frac{[CP^{n-1}]}{p^{n-1}} \) we obtain
Theorem 7.1 (Theorem 5, (iii) of [18]).

\[ r_\alpha(pt)([CP_{p^n-1}]) = \sum_{h=0}^{n} p^{n-h} [CP_{p^h}] t_{p^h}^{n-h}. \]

This theorem describes the actions of \( r_\alpha \) on \( BP^*(pt) \) at least theoretically.

7.3. I feel it is better to formulate a general theorem of which the operation \( r_\xi \) is obtained by a specialization.

Theorem 7.2. Let \( h^* \) be a complex-oriented cohomology theory such that \( h^*(pt) \) is a \( \mathbb{Z}_p \)-algebra and its associated formal group is typical. Then there exists a unique cohomology transformation

\[ \theta_{(p)} : BP^* \longrightarrow h^* \]

which is \( \mathbb{Z}_p \)-linear, degree-preserving, multiplicative and

\[ \theta_{(p)}(e^{BP}(L)) = e^{h}(L) \]

for a complex line bundle \( L \). It results also

\[ \theta_{(p)}(pt)^* \mu_{BP} = \mu_{h}, \]

the typical formal group of \( h^* \).

Thus we may say that \( BP^* \) is universal for cohomology theories with typical formal groups. \( \theta_{(p)} \) can be obtained by factorizing the unique cohomology transformation

\[ \theta : U^* \longrightarrow h^* \]
which follows by the universality of complex bordism. Our necessary arguments for the proof are quite parallel to those in 5.9 to establish $r_t$, so I will omit them.

7.4. Next we discuss compositions $r_\alpha \circ r_\beta$ of Quillen operations. Consider the diagram

$$
\begin{array}{ccc}
BP^*() & \xrightarrow{r_\phi} & BP^*[t_{(p)}] \\
& \overrightarrow{r_\phi} & \downarrow \ \\
& & BP^*[s_{(p)}, \eta_{(p)}],
\end{array}
$$

where $s_{(p)} = (s_1, s_2, \ldots, s_k, \ldots)$ is another sequence of indeterminates such that $\dim s_k = -2(p^2 - 1)$ and $r_\phi \otimes 1$ is an extension of $r_\phi : BP^*() \rightarrow BP^*[s_{(p)}]$ such that $(r_\phi \otimes 1)(t_j) = t_j$, $j \geq 1$.

The composition $r_\phi \otimes 1 \circ r_t$ is a cohomology transformation which is linear, degree-preserving and multiplicative. Moreover, putting $\mu_\phi^i = \mu_{s_{(p)}}$ and $\psi_{s_{(p)}}(T) = \sum_{i \geq 0} \mu_\phi^i (t_i T^i)$, we have

$$
(7.12) \quad (r_\phi \otimes 1 \circ r_t)(e_{BP}(L)) = \psi_{s_{(p)}}^{-1}(\phi_{s_{(p)}}^{-1}(e_{BP}(L)))
$$

for a line bundle $L$. This formula can be seen as follows: remark that $r_t(pt)^\ast \mu = \mu_\phi^i$

by (7.7). Then

$$
(r_\phi \otimes 1 \circ r_t)(e_{BP}(L)) = (r_\phi \otimes 1)(\phi_{s_{(p)}}^{-1}(e_{BP}(L)))
$$

$$
= (r_\phi(pt)^\ast \phi_{s_{(p)}}^{-1}(\phi_{s_{(p)}}^{-1}(e_{BP}(L))),
$$

and
$$\left( r_s(\text{pt}) \circ \psi_t, (p) \right)(T) = r_s(\text{pt}) \circ \gamma^\mu(t_j t^p_j)$$

$$= \sum_{j \geq 0} \gamma^\mu(t_j t^p_j) = \psi_t, (p)(T).$$

We give complex-orientation of the cohomology theory $BP^*[s_p, t_p]$ by assigning $\left( \phi_s, (p) \circ \psi_t, (p) \right)^{-1}(e^\text{BP}(L))$ as Euler class of a line bundle $L$. Then the associated formal group $\mu''$ is the transpose of $\mu'_s$ by $\psi_t, (p)$. Now $\psi_t, (p)$ is a typical curve over $\mu'_s$. Hence $\mu''$ is typical. Hence

$$r_s \otimes 1 \circ r_t : BP^* \longrightarrow BP^*[s_p, t_p]$$

is the unique cohomology transformation of Theorem 7.2.

By our construction it is clear that

$$\left( r_s \otimes 1 \circ r_t \right)(x) = \sum_{\alpha, \beta} r_\beta(r_\alpha(x)) t^\alpha s^\beta,$$

where $t^\alpha$ and $s^\beta$ are monomials of $(t_1, t_2, \ldots)$ and $(s_1, s_2, \ldots)$.

7.5. Remark that $\phi_s, (p) \circ \psi_t, (p)$ is a typical curve over $\mu$, the extension of $\mu_{BP}$ over $BP^*(\text{pt})[s_p, t_p]$. Hence we have a unique expression

$$\left( \phi_s, (p) \circ \psi_t, (p) \right)(T) = \sum_{j \geq 0} \gamma^\mu(u_j t^p_j), \quad u_0 = 1,$$

where $u_j = u_j(s_1, \ldots, s_j, t_1, \ldots, t_j)$ are polynomials of $s_1, \ldots, s_j, t_1, \ldots, t_j$ over $BP^*(\text{pt})$ such that $\dim u_j = -2(p_j^2 - 1)$.

We want to obtain these polynomials if possible. Since

$$\phi_s, (p) \circ \psi_t, (p)(T) = \phi_s, (p)\#(\psi_t, (p)(T))$$
applying \( \log_{u^k} \) on both sides of (7.14) we obtain

\[
\sum_{i,j,k} m_{i-j,k} s_i t^i j p^i j p^i j = \sum_{k,l} m_{k-l} u_k p^k l p^k l.
\]

Comparing the coefficients of \( p^n \) we obtain

\[
\sum_{\ell=0}^n \sum_{i+j=n-\ell} m_{i-j,n-\ell} s_i t^i j p^i j p^i j = \sum_{\ell=0}^n \sum_{i+j=n-\ell} m_{i-j,n-\ell} u_k p^k l p^k l.
\]

or, since the terms of \( \ell = n \) of both sides are the same \( m_{n-1} \), we see that

\[
\sum_{\ell=0}^{n-1} \sum_{i+j=n-\ell} m_{i-j,n-\ell} s_i t^i j p^i j p^i j = \sum_{\ell=0}^{n-1} \sum_{i+j=n-\ell} m_{i-j,n-\ell} u_k p^k l p^k l.
\]

This is a recursive formula to determine \( u_n \) over \( \mathbb{Q} \)-extensions. Multiplying \( p^n \) to both sides of (7.15) we obtain

\[
\sum_{\ell=0}^{n-1} \sum_{i+j=n-\ell} m_{i-j,n-\ell} s_i t^i j p^i j [p^{n-\ell} t^i j p^{n-\ell}] = \sum_{\ell=0}^{n-1} \sum_{i+j=n-\ell} m_{i-j,n-\ell} u_k p^{k-1} [p^{n-\ell} t^i j p^{n-\ell}].
\]

This is a recursive formula over \( BP^*(pt) \).

By (7.16) we see easily that \( u_j \) is a polynomial of \( s_1, \ldots, s_j, t_1, \ldots, t_j \). But it seems to be very difficult to write these polynomials completely.

7.6. Let \( \mathfrak{u}_{(p)} = (u_1, u_2, \ldots, u_k, \ldots) \) be a sequence of indeterminates such that \( \dim u_j = -2(p^j - 1) \), and

\[
\lambda : BP^*(\mathfrak{s}_{(p)}) \longrightarrow BP^*(\mathfrak{t}_{(p)l})
\]

be a cohomology transformation such that \( \lambda(x) = x \) for \( x \in BP^*(X) \) and \( \lambda(u_j) = (s_1, \ldots, s_j, t_1, \ldots, t_j) \), the polynomials determined by (7.16).
\( \lambda \) is linear, degree-preserving and multiplicative; and by (7.14) we see that both \( \lambda \circ r_u \) and \( r_s \otimes 1 \circ r_t \) send \( \mathbb{e}^{BP}(L) \) to \( \mathbb{e}^{U''}(L) \). Thus, by the uniqueness of Theorem 7.2 we obtain

\[
\lambda \circ r_u = r_s \otimes 1 \circ r_t,
\]

or

\[
(7.17) \quad \sum_{\alpha, \beta} r_{\beta}(r_{\alpha}(x)) s_{\alpha}^{(p)} s_{\beta}^{(p)} = r_{\gamma}(x) u_{(p)}^{\gamma},
\]

for \( x \in \text{BP}^*(X) \), where \( u = (u_1(s_1, t_1), u_2(s_1, s_2, t_1, t_2), \ldots, u_j(s_1, \ldots, s_j, t_1, \ldots, t_j), \ldots) \) is the sequence of polynomials determined by (7.16). If we write monomials \( u_{(p)}^{\gamma} \) as polynomials

\[
(7.18) \quad u_{(p)}^{\gamma} = \sum_{\alpha, \beta} a_{\alpha, \beta}^{\gamma} t_{(p)}^{\alpha} s_{(p)}^{\beta},
\]

over \( \text{BP}^*(pt) \), then we get

\[
(7.19) \quad r_{\beta} \circ r_{\alpha} = \sum_{\gamma} a_{\alpha, \beta}^{\gamma} r_{\gamma},
\]

the formulas to express compositions \( r_{\beta} \circ r_{\alpha} \) as linear combinations of \( r_{\gamma} \) over \( \text{BP}^*(pt) \) (Theorem 5, (iv) of [18]).
Part II

§8. Typical formal groups in complex K-theory

8.1. Let $K$ be the complex $K$-functor over finite CW-complexes.

For any complex vector bundle $E$ we use

$$e^K(E) = \lambda_{-1}(E) = \sum_{i=1}^{\infty} (-1)^i \lambda_i(E)$$

as its $K$-theoretic Euler class. Thus

$$e^K(L) = 1 - L$$

for a line bundle $L$. Then

$$e^K(L_1 \otimes L_2) = e^K(L_1) + e^K(L_2) - e^K(L_1) \cdot e^K(L_2)$$

for line bundles $L_1$ and $L_2$, i.e., the associated formal group $F_K$ of $K$-functor is given by

$$F_K(X, Y) = X + Y - XY = 1 - (1 - X)(1 - Y). \tag{8.1}$$

$n$ fold multiplication with respect to $F_K$ is defined by

$$F_K(X_1, \ldots, X_n) = F_K(X_1, F_K(X_2, \ldots, X_n))$$

recursively. Then

$$F_K(X_1, \ldots, X_n) = 1 - (1 - X_1) \cdots (1 - X_n). \tag{8.2}$$

Thus

$$[n]_K(T) = 1 - (1 - T)^n$$
for any positive integer \( n \). More generally, over any ring \( A \) such that 
\( \mathbb{Z} \subset A \subset \mathbb{Q} \),

\[
[a]_K(T) = 1 - (1 - T)^a
\]  

(8.3)

for any \( a \in A \), where

\[
(1 - T)^a = 1 - aT + \frac{a(a-1)}{2} T^2 - \ldots + (-1)^k \frac{a(a-1)\ldots(a-k+1)}{k!} T^k + \ldots
\]

(cf., 2.4).

The Frobenius operator \( f_{n,K} \), \( n \geq 1 \), applied to the identity curve \( \gamma_0 \), is computed as follows.

\[
(f_{n,K} \gamma_0)(T) = (-1)^{n-1} (\zeta_1^{1/n} \ldots \zeta_n^{1/n}) = T,
\]

where \( \zeta_1, \ldots, \zeta_n \) are \( n \)-th roots of 1, i.e.,

\[
(f_{n,K} \gamma_0) = \gamma_0
\]  

(8.4)

for any \( n \geq 1 \).

Over \( \mathbb{Q} \) the logarithm

\[
\log_K : F_K \rightarrow \mathbb{G}_a
\]

is described by

\[
\log_K T = - \log(1 - T) = \sum_{n \geq 1} \frac{1}{n} T^n,
\]

(8.5)

where \( \log \) is the usual natural logarithm.
8.2. Let \( p \) be a fixed prime. Over \( \mathbb{Z}_p = \mathbb{K}(pt)(p) \) the canonical typical curve \( \xi_K = \xi_K(p) \) can be computed by (8.2), (8.3) and (8.4), and we obtain

\[
\xi_K(T) = (\xi_K(\varphi_0))(T) = \sum_{(m,p)=1}^{\mathbb{F}_K} \frac{\mu(m)}{(1 - (1 - T^m)^m)} = 1 - P(1 - T),
\]

where \( \mu(m) \) is the Möbius function and

\[
P(1 - T) = \prod_{(m,p)=1}^{\mathbb{F}_K} (1 - T^m)^m
\]

is the Artin-Hasse series. (Cf., [10].)

Let \( \xi_K = \mathbb{F}_K \), the typical formal group canonically associated to \( \mathbb{F}_K \). Then

\[
\log \xi_K = \log \circ \xi_K
\]

over \( \mathbb{Q} \), and by Proposition 3.3 and (8.5) we have

\[
\log \xi_K(T) = \sum_{k \geq 0} \frac{1}{p^k} T^p = L(1 - T)
\]

using a notation \( L(1 - T) \) of Hasse [10]. Now

\[
\log \xi_K(T) = \log \xi_K(T) \quad \text{by (8.8)}
\]

\[
= - \log \left(1 - \xi_K(T)\right) \quad \text{by (8.5)}
\]

\[
= - \log P(1 - T) \quad \text{by (8.6)}.
\]

Thus

\[
L(1 - T) = - \log P(1 - T)
\]
by (8.9). This was observed also in [10].

8.3. Next we observe formal groups of periodic $K$-cohomology $K^*(\ )$.

Its coefficient object is $K^*(pt) = 2[u, u^{-1}]$, where $u \in K^{-2}(pt)$ is the Bott periodic element. To make $K^*$ complex oriented we define $K^*$-Euler class of a line bundle $L$ over $X$ by

$$e^{K^*}(L) = u^{-1}e^{K}(L) \in K^2(X).$$

Then its associated formal group is

$$F_{K^*} = F_{K^*}^{[u]Y_0},$$

i.e.,

$$(8.1)^* F_{K^*}(X, Y) = X + Y - u^*XY = u^{-1}(1 - (1 - uX)(1 - uY)).$$

Thus $n$ fold multiplication with respect to $F_{K^*}$ is

$$(8.2)^* F_{K^*}(X_1, \ldots, X_n) = u^{-1}(1 - (1 - uX_1)\ldots(1 - uX_n))$$

and

$$(8.3)^* [a]_{K^*}(T) = u^{-1}(1 - (1 - uT)^a)$$

for any $a \in \Lambda$ over $K^*(pt) \otimes \Lambda$, where $Z \subset \Lambda \subset \mathbb{Q}$.

The Frobenius operator $f_{n, K^*}$, $n \geq 1$, applied to the identity curve $Y_0$, is

$$f_{n, K^*} Y_0 = [u^{n-1}] Y_0.$$ 

Finally, over $K^*(pt) \otimes \mathbb{Q}$ the logarithm
\[ \log_{K^*} : F_{K^*} \rightarrow G_a \]

is described by

(8.5) \[ \log_{K^*} T = -u^{-1} \log(1 - uT) = \sum_{n \geq 1} \frac{u^{n-1}}{n} \cdot T^n. \]

8.4. Let \( p \) be a fixed prime. Over \( K^*(pt)(p) = K^*(pt) \otimes \mathbb{Z}(p) \) the canonical typical curve \( \xi_{K^*} \) can be computed by (8.2)*, (8.3)* and (8.4)* and we obtain

(8.6) \[ \xi_{K^*}(T) = (\xi_{K^*} \cdot Y_0)(T) = u^{-1}(1 - P(1 - uT)). \]

Let \( \mu_{K^*} = F_{K^*} \), the typical formal group canonically associated to \( F_{K^*} \). Then

(8.8) \[ \log_{\mu_{K^*}} = \log_{K^*} \circ \xi_{K^*} \]

over \( K^*(pt) \otimes \mathbb{Q} \), and by Proposition 3.3 and (8.5)* we have

(8.9) \[ \log_{\mu_{K^*}} T = u^{-1}L(1 - uT) = \sum_{k \geq 0} \frac{u^{p^{-1}}}{k} \cdot T^p. \]

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§9. Adams decompositions and typical formal groups


Let $K( )_{(p)} = K( ) \otimes \mathbb{Z}(p)$. Adams [1] gave linear idempotents of this functor

$$E_s : K( )_{(p)} \longrightarrow K( )_{(p)}$$

for each $s \in \mathbb{Z}$ by

$$E_s(L) = \frac{1}{p-1} \sum_{m=1}^{p-1} w^m (1 - e^{K}(L))^{-1}$$

for a line bundle $L$, where $w$ is a primitive $(p-1)$-th root of $1$ as a $p$-adic integer. Even though $w \in \mathbb{Z}_p$ all coefficients of $E_s(L)$ (as a power series of $e^{K}(L)$) lie actually in $\mathbb{Z}(p)$ so that (9.1) is a well-defined formula. The formula (9.1) determines $\mathbb{Z}(p)$-linear natural transformations $E_s$ completely by splitting principle.

Following [1] we list basic properties of $E_s$ quickly.

(9.2) $E_s = E_{s'}$ if $s \equiv s' \mod p-1$.

Thus these natural transformations are defined actually for elements $\alpha = \{s\} \in \mathbb{Z}/(p-1)\mathbb{Z}$. Then

(9.3) $E_{\alpha}^2 = E_{\alpha}$ (idempotent),

(9.4) $E_{\alpha}E_{\beta} = 0$ if $\alpha \neq \beta$ in $\mathbb{Z}/(p-1)\mathbb{Z}$,
By (9.3), (9.4) and (9.5) we have a natural decomposition

\[(9.6) \quad K(\_)(p) = E_0K(\_)(p) \oplus E_1K(\_)(p) \oplus \ldots \oplus E_{p-2}K(\_)(p)\]

of the functor \(K(\_)(p)\) as a direct sum. Next

\[(9.7) \quad E_\alpha(xy) = \sum_{\beta+\gamma=\alpha} E_\beta(x)E_\gamma(y)\]

for internal and external multiplications. In particular we see that, if \(x \in E_\beta K(X)(p)\) and \(y \in E_\gamma K(X)(p)\), then \(xy \in E_{\beta+\gamma} K(X)(p)\), and

\[(9.8) \quad E_0K(\_)(p) \quad \text{is a multiplicative functor.}\]

9.2. Let \(L_1\) be a line bundle over \(S^2\). Since \(e^K(L_1)^2 = 0\), we have

\[
E_1(L_1) = \frac{1}{p-1} \sum_{m=1}^{p-1} \omega^{-m} - e^K(L_1)^m = \frac{1}{p-1} \sum_{m=1}^{p-1} \omega^{-m} - e^K(L_1) = \frac{1}{p-1} \left( \sum_{m=1}^{p-1} \omega^{-m} \right) - e^K(L_1) = - e^K(L_1).
\]

Here we used the fact that

\[
\sum_{m=1}^{p-1} \omega^m = \begin{cases} 0 & \text{if } s \not\equiv 0 \mod p-1 \\ p-1 & \text{if } s \equiv 0 \mod p-1, \end{cases}
\]

which will be used later freely. In particular

\[E_1(1) = 0.\]
Thus

\[ E_1(e^K(L_1)) = e^K(L_1). \]

Since \( K(S^2) \) (p) is generated by \( e^K(L_1) \) (by choosing \( L_1 \) as the canonical line bundle) we see that

\[ E_1K(S^2)(p) = \tilde{K}(S^2)(p), \]

(9.9)

\[ E_\alpha K(S^2)(p) = 0 \text{ if } \alpha \neq 1 \text{ in } \mathbb{Z}/(p-1)\mathbb{Z}. \]

Apply (9.7) to the smash product \( S^2 \wedge \ldots \wedge S^2 = S^{2n} \) and obtain

\[ E_nK(S^{2n})(p) = \tilde{K}(S^{2n})(p), \]

(9.10)

\[ E_sK(S^{2n})(p) = 0 \text{ if } s \neq n \text{ mod } p-1. \]

Let \( \phi : \tilde{K}(X) \cong \tilde{K}(S^2 \wedge X) \) be the Bott isomorphism. By (9.7) and (9.9) we have the commutativity

\[
\begin{array}{ccc}
\tilde{K}(X)(p) & \xrightarrow{\phi} & \tilde{K}(S^2 \wedge X)(p) \\
\downarrow E_\alpha & & \downarrow E_{\alpha+1} \\
K(X)(p) & \xrightarrow{\phi} & \tilde{K}(S^2 \wedge X)(p)
\end{array}
\]

and \( \phi \) induces an isomorphism

(9.11) \[ \phi_\alpha : E_\alpha K(X)(p) \cong E_{\alpha+1} \tilde{K}(S^2 \wedge X)(p) \]

for each \( \alpha \in \mathbb{Z}/(p-1)\mathbb{Z} \).

9.3. From the above idempotents we define linear idempotents
of $K$-cohomology localized at $p$ for each $s \in \mathbb{Z}$. Define

$$E^*_{s} : K^*(p) \longrightarrow K^*(p)$$

by requirement that the following diagram

$$\begin{array}{ccc}
K^{2i}(X)(p) & \xrightarrow{\beta^i} & K(X)(p) \\
\downarrow E^*_{s} & & \downarrow E_{s+i} \\
K^{2i}(X)(p) & \xrightarrow{\beta^i} & K(X)(p)
\end{array}$$

commutes, where $\beta$ is the Bott periodicity, i.e., the multiplication with $u \in K^{-2}(pt)$. Define

$$E^{2i+1}_{s} : K^{2i+1}(p) \longrightarrow K^{2i+1}(p)$$

by requirement that the following diagram

$$\begin{array}{ccc}
K^{2i+1}(X)(p) & \xrightarrow{\sigma} & \tilde{K}^{2i+2}(S^1 \wedge X)(p) \\
\downarrow E^{2i+1}_{s} & & \downarrow E^{2i+2}_{s} \\
K^{2i+1}(X)(p) & \xrightarrow{\sigma} & \tilde{K}^{2i+2}(S^1 \wedge X)(p)
\end{array}$$

commutes, where $\sigma$ is the suspension isomorphism. Then by (9.11) we see that $E^*_{s} = \{E^i_{s}, i \in \mathbb{Z}\}$ commutes with suspensions and is a well-defined $\mathbb{Z}(p)$-linear, degree-preserving idempotents of the cohomology theory $K^*(p)$. 

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Basic properties of these idempotents follow by the corresponding properties of $E_S's$. First of all

\[(9.2)^* \quad E_S^* = E_s^*, \quad \text{if } s \equiv s' \mod p-1.\]

Thus these idempotents are defined actually for elements $\alpha = \{s\} \in \mathbb{Z}/(p-1)\mathbb{Z}$.

Then, by (9.3), (9.4) and (9.5) we obtain

\[(9.3)^* \quad (E_\alpha^*)^2 = E_\alpha^*, \]
\[(9.4)^* \quad E_\alpha^* E_\beta^* = 0 \quad \text{if } \alpha \neq \beta \text{ in } \mathbb{Z}/(p-1)\mathbb{Z}, \]
\[(9.5)^* \quad \sum_{\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}} E_\alpha^* = 1.\]

Thus we have a natural decomposition

\[(9.6)^* \quad K^*( )_{(p)} = E_0^*K^*( )_{(p)} \oplus E_1^*K^*( )_{(p)} \oplus \ldots \oplus E_{p-2}^*K^*( )_{(p)} \]

of the cohomology $K^*( )_{(p)}$ as a direct sum and each direct factor $E_\alpha^*K^*( )_{(p)}$ itself is a cohomology theory.

Next by (9.7) and the definition of $E_\alpha^*$ we obtain

\[(9.7)^* \quad E_\alpha^*(xy) = \sum_{\beta+\gamma=\alpha} E_\beta^*(x)E_\gamma^*(y) \]

for internal and external multiplications. In particular we see that, if $x \in E_\beta^*K^*(X)_{(p)}$ and $y \in E_\gamma^*K^*(X)_{(p)}$, then $xy \in E_{\beta+\gamma}^*K^*(X)_{(p)}$, and

\[(9.8)^* \quad E_0^*K^*( )_{(p)} \text{ is a multiplicative cohomology theory.}\]

9.4. Put

\[(9.12) \quad G^*( ) = E_0^*K^*( )_{(p)}.\]
This is a multiplicative cohomology theory inheriting its multiplicative structure from $K^*(\ )$. By definitions we see that

\[(9.13) \quad G^*(pt) = \mathbb{Z}_p[u_1, u_1^{-1}], \quad u_1 = u^{p-1}, \]

i.e., $G^*(\ )$ is a periodic cohomology theory of period $2(p-1)$ with $u_1$ as the periodicity element.

We give complex orientation of $K^*(\ )_{(p)}$ by assigning $\xi_{K^*}^{-1}(e^{K^*}(L))$ as Euler class of a line bundle $L$. Then its associated formal group is the typical group law $u_{K^*}$. We denote as

\[e^{\mu_{K^*}}(L) = \xi_{K^*}^{-1}(e^{K^*}(L)).\]

**Theorem 9.1.** \[e^{\mu_{K^*}}(L) \in G^2(X)\]

for any line bundle $L$ over $X$.

**Proof.** Using the notations of [10] we put

\[1 - \tau = P(1 - T) \quad \text{and} \quad 1 - T = Q(1 - \tau).\]

Then

\[\tau = \xi_K(T)\]

by (8.6). As is well known

\[(9.14) \quad (1 - \tau)^{\omega^m} = P(1 - \omega^m T),\]

where $\omega$ is the primitive $(p-1)$-th root of 1 in $\mathbb{Z}_p$ and $m \in \mathbb{Z}$, which can be seen by taking $\log$ of both sides and by easy computations.
Putting 
\[ \xi_K(T) = \sum_{j \geq 0} \xi_j^j T^{j+1}, \]
we obtain 
\[ (1 - T)^w = 1 - \sum_{j \geq 0} \xi_j^j (\omega T)^{j+1} \]
by (9.14) and (8.6). Now compute, for \( s \in \mathbb{Z} \),
\[
\frac{1}{p-1} \sum_{m=1}^{p-1} \omega^{-ms} (1 - T)^w = \frac{1}{p-1} \sum_{m=1}^{p-1} \omega^{-ms} \left( \sum_{j \geq 0} \xi_j^j (\sum_{m=1}^{p-1} \omega^{m(j+1-s)}) T^{j+1} \right)
\]
\[
= \begin{cases} 
1 - \frac{1}{p-1} \sum_{j \geq 0} \xi_j^j T^{j+1} & \text{if } s \equiv 0 \mod p-1 \\
1 - \frac{1}{p-1} \sum_{j \geq 0} \xi_j^j T^{j+1-s} & \text{if } s \not\equiv 0 \mod p-1.
\end{cases}
\]
Putting \( T = e^L \), by (9.1) we obtain
\[
E_0(L) = 1 - \frac{1}{p-1} \sum_{k \geq 1} \xi_k^k (e^{\mu_k(L)})^{k(p-1)},
\]
(9.15)
\[
E_s(L) = - \frac{1}{p-1} \sum_{k \geq 1} \xi_k^k (e^{\mu_k(L)})^{k(p-1)+s}
\]
for \( 1 \leq s \leq p-2 \). (Remark that \( e^{\mu_k(L)} = \xi_k^{-1} (e^L)^k \).)

Put
\[ \xi_{K^*}(T) = \sum_{j \geq 0} \xi_j^{K^*} T^{j+1}, \]
then by (8.6) and (8.6)* we see that
\[ \xi_j^{K^*} = \xi_j^{K^*} u^j \]
for \( j \geq 0 \). Since \( e^{K^*}(L) = u^{-1} \cdot e^{K}(L) \) for a line bundle \( L \) over \( X \), we have

\[
e^{K^*}(L) = u^{-1} \cdot e^{K}(L).
\]

Now put

\[
A(L) = \sum_{k \geq 0} \xi^K_{k(p-1)} (e^{K^*}(L)) k^{(p-1)1}.
\]

Then

\[
A(L) = u^{-1} \sum_{k \geq 0} \xi^K_{k(p-1)} (e^{K^*}(L)) k^{(p-1)1} = u^{-1} \cdot E_1(L)
\]

by (9.15). Hence by definition we see that

\[
A(L) \in E_0^{2K^2}(X)(p) = G_1^{2}(X),
\]

Remark that \( A(L) \) is an invertible power series of \( e^{K^*}(L) \), and all possible non-zero coefficients are

\[
\xi^K_{k(p-1)} = \xi^K_{k(p-1)} (u^{p-1})^k \in G^*(pt).
\]

Thus \( e^{K^*}(L) \) can be solved as a power series of \( A(L) \) with coefficients in \( G^*(pt) \). Hence

\[
e^{K^*}(L) \in G^2(X), \quad q. e. d.
\]

9.5. The above theorem implies that all coefficients of \( \mu^*_K(X, Y) \) lie in \( G^*(pt) \) and \( \mu^*_K \) determines a typical formal group \( \mu^*_G \) over
G*(pt) by restricting the domain of coefficients to G*(pt). The corresponding complex orientation of G*( ) is given by assigning

\[ e^{G^*}(L) = e^{\mu_{K^*}}(L) \in G^2(X) \]

as Euler class of a line bundle L over X. Its logarithm

\[ \log_{\mu_{G^*}} : \mu_{G^*} \rightarrow G_a \]

over G*(pt) \( \otimes \mathbb{Q} \) is given by

(9.16) \[ \log_{\mu_{G^*}} T = \sum_{k \geq 0} \frac{1}{k} u_1 \cdots \cdots + p^{k-1} T^k. \]

(Cf., (8.9)*).

Identifying by periodic isomorphisms in G*( ) we obtain a multiplicative cohomology G#( ) graded in \( \mathbb{Z}/2(p-1)\mathbb{Z} \). (Remark that the difference of notations from the usual convention in K-theory!) G# is complex oriented by assigning

\[ e^{G}(L) = e^{G^*}(L) \]

as Euler classes. Its associated formal group is a typical formal group \( \mu_G \) over G#(pt) \( \cong \mathbb{Z}(p) \) with

(9.16') \[ \log_{\mu_G} T = \sum_{k \geq 0} \frac{1}{k} T^k. \]

9.6. By the universality of complex cobordism (cf., 5.1) we have a unique cohomology transformation...
which is linear, degree-preserving, multiplicative and

\[
\text{td}(e^U(L)) = e^{K^*}(L)
\]

for a line bundle \( L \). This is essentially the same as the \( \text{td} \)-map of Conner-Floyd [7], and we have

\[
\text{td}(x_{2n}) = \text{Td}(x_{2n}) \cdot u^n
\]

for \( x_{2n} \in U^{-2n}(pt) \), where \( \text{Td}(x_{2n}) \) denotes the Todd genus of the weakly complex manifold representing \( x_{2n} \). Remark the difference of signs from the corresponding formula of [7]. This point is adjusted by a choice of Bott-periodicity element \( u \) (among \( \pm u \)).

By (9.18) we see that

\[
\text{td}(pt)^* F_U = F_{K^*} \quad \text{and} \quad \text{td}(pt)^* \xi_{U^*(p)} = \xi_{K^*(p)}
\]

after localized at a prime \( p \). Hence

\[
\text{td}(e^{BP}(L)) = e^{W_{K^*}(L)} = e^{G^*}(L)
\]

for a line bundle \( L \). This implies that

\[
\text{td}(BP^*(X)) \subseteq G^*(X)
\]

(cf., Theorem 7.2). And we obtain

**Theorem 9.2.** By restricting \( \text{td} \) to \( BP^*(\_\_) \) we obtain a cohomology
transformation

\[ \tilde{\text{td}} : \text{BP}^*( ) \rightarrow \text{G}^*( ) \]

which is \( \mathbb{Z}_p \)-linear, degree-preserving, multiplicative and

\[ \tilde{\text{td}}(e^{\text{BP}^*(L)}) = e^{\text{G}^*(L)} \]

for a line bundle \( L \).
§10. Coefficients of curves

Practically we need some calculus of coefficients of curves modulo some ideals. Here we collect some propositions necessary for these purposes.

10.1. Let $R$ be a commutative ring with unity and $I$ an ideal of $R$. Let $f(T) = \sum f_i T^i$ and $g(T) = \sum g_i T^i$ be formal power series over $R$. We say that

$$f \equiv 0 \mod I$$

iff $f_i \in I$ for all $i > 0$, and

$$f \equiv g \mod I$$

iff $f - g \equiv 0 \mod I$.

Lemma 10.1. Let $f, f', g$ and $g'$ be formal power series over $R$. If $f \equiv f' \mod I$ and $g \equiv g' \mod I$, then

$$f + g \equiv f' + g' \mod I, \quad fg \equiv f'g' \mod I$$

and, when $g$ and $g'$ are without constant terms,

$$f \circ g \equiv f' \circ g' \mod I.$$

Proof follows by routine arguments.

Let $F$ be a formal group over $R$.

Lemma 10.2. Let $\gamma_1, \gamma_1', \gamma_2$ and $\gamma_2'$ be curves over $F$. If
\[ y_1 \equiv y'_1 \mod I \quad \text{and} \quad y_2 \equiv y'_2 \mod I, \quad \text{then} \]
\[ y_1 + F y_2 \equiv y'_1 + F y'_2 \mod I. \]

Proof follows by definition and the above Lemma.

Proposition 10.3. Let \( \{y_1, y_2, \ldots \} \) and \( \{y'_1, y'_2, \ldots \} \) be Cauchy sequences in \( C_F \) such that
\[ y_i \equiv y'_i \mod I \]
for all \( i > 1 \), then
\[ \sum_{i=1}^{\infty} y_i \equiv \sum_{i=1}^{\infty} y'_i \mod I. \]

Proof. By the above Lemma the congruence is true for every finite partial sums. Since every coefficient of infinite sums can be found as a coefficient of suitable finite sum, the Proposition is true.

Lemma 10.4. Let \( y_1 \) and \( y_2 \) be curves over \( F \) such that \( y_1 \equiv 0 \mod I \) and \( y_2 \equiv 0 \mod I \), then
\[ y_1 + F y_2 \equiv y_1 + y_2 \mod I^2. \]

Proof. By (1.4)
\[ F(X, Y) = X + Y + XYF(X, Y). \]

Thus
\[ (y_1 + F y_2)(T) = y_1(T) + y_2(T) + y_1(T)y_2(T)F(y_1(T), y_2(T)). \]
\[ \sum_{i \geq 1} F_{\gamma_i} \equiv \gamma_1 + \gamma_2 + \ldots + \gamma_n + \ldots \mod I^2. \]

Proposition 10.5. Let \( \{\gamma_1, \gamma_2, \ldots \} \) be a Cauchy sequence in \( C_F \) such that

\[ \gamma_i \equiv 0 \mod I \]

for all \( i \geq 1 \), then

\[ \sum_{i \geq 1} F_{\gamma_i} \equiv \gamma_1 + \gamma_2 + \ldots + \gamma_n + \ldots \mod I^2. \]

Proof. The congruence is true for every finite partial sums by Lemmas 10.1 and 10.4, whence true for infinite sums.

Corollary 10.6. Let \( \gamma \) be a curve over \( F \) such that \( \gamma \equiv 0 \mod I \).

Express as

\[ \gamma(T) = \sum_{k \geq 1} F(c_{k-1}T^k), \]

then

\[ c_k \in I \]

for all \( k \geq 0 \), and

\[ \gamma(T) \equiv c_0 T + c_1 T^2 + \ldots + c_{k-1} T^k + \ldots \mod I^2. \]

In particular, when \( R \) is a \( \mathbb{Z}_p \)-algebra, \( \mu \) is a typical formal group over \( R \) and \( \gamma \) is a typical curve over \( \mu \), expressing \( \gamma \) as

\[ \gamma(T) = \sum_{k \geq 0} \mu(c_k T^p)^k, \]

if \( \gamma \equiv 0 \mod I \), then \( c_k \in I \) for all \( k \geq 0 \) and
10.2. Let $R, I, F$ be as above.

Lemma 10.7. Let $\gamma$ be a curve over $F$ such that $\gamma \equiv 0 \mod I$. Then

\[
([c]\gamma_0 + F \gamma)(T) \equiv cT + \frac{\partial F}{\partial Y}(cT, 0) \cdot \gamma(T) \mod I^2
\]

for $c \in R$, where $\gamma_0$ is the identity curve over $R$.

Proof. $([c]\gamma_0 + F \gamma)(T) = F(cT, \gamma(T))$

\[
= \sum a_{ij}(cT)^i \gamma(T)^j
\]

\[
\equiv cT + \sum_{n \geq 0} a_{n1}(cT)^n \cdot \gamma(T) \mod I^2
\]

\[
\equiv cT + \frac{\partial F}{\partial Y}(cT, 0) \cdot \gamma(T) \mod I^2.
\]

q. e. d.

Next suppose $R$ is of characteristic zero. Differentiating the relation

\[
\log_F F(X, Y) = \log_F X + \log_F Y
\]

with respect to $Y$, we obtain

\[
\frac{\partial F}{\partial Y}(T, 0) \cdot \log_F' T = 1,
\]

where $\log_F'$ is the derivative of $\log_F$. This shows firstly that $\log_F'$ is a power series defined over $R$, i.e., if we put

(10.1) \[ \log_F T = \sum_{k \geq 1} \frac{n_k}{k+1} \frac{1}{T^{k+1}}, \quad n_0 = 1, \]
then all \( n_k \in R \), and secondly that all coefficients of \( \frac{\partial F}{\partial Y}(T, 0) \) are integral polynomials of coefficients \( n_k \) of \( \log F \).

Proposition 10.8. Let \( R \) be of characteristic zero, and \( I \) an ideal of \( R \) containing all coefficients \( n_k \) of positive degrees of \( \log F \). Let \( \gamma \) be a curve over \( F \), and expressing as \( \gamma(T) = \sum_{k \geq 1} F(c_{k-1} T^k) \), suppose that \( c_k \in I \) for all \( k > 0 \). Then

\[
\gamma(T) \equiv c_0 T + c_1 T^2 + \ldots + c_{k-1} T^k + \ldots \mod I^2.
\]

Proof. Put

\[
\gamma_1(T) = \sum_{k \geq 1} F(c_{k-1} T^k),
\]

then \( \gamma_1 \equiv 0 \mod I \) and by Proposition 10.5

\[
\gamma_1(T) \equiv c_1 T^2 + \ldots + c_k T^k + \ldots \mod I^2.
\]

Now by Lemma 10.7

\[
\gamma(T) = ([c_0] Y_0 + F \gamma_1)(T)
\]

\[
\equiv c_0 T + \frac{\partial F}{\partial Y}(c_0 T, 0) \cdot \gamma_1(T) \mod I^2.
\]

By the remark above the proposition we have

\[
\frac{\partial F}{\partial Y}(c_0 T, 0) - 1 \equiv 0 \mod I.
\]

Thus

\[
\gamma(T) \equiv c_0 T + \gamma_1(T) \mod I^2
\]

\[
\equiv c_0 T + c_1 T^2 + \ldots + c_k T^k + \ldots \mod I^2.
\]
10.3. Let $R$ be a $\mathbb{Z}_{(p)}$-algebra and $\mu$ a typical formal group over $R$. Let $\mathbf{t} = (t_1, t_2, \ldots)$ be a sequence of indeterminates and put
\[ k \sum_{k>0} (t_k \mathbf{t}^k), \quad t_0 = 1, \]
which is a typical curve over $\mu$ (extending the domain of coefficients to $R[\mathbf{t}] = R[t_1, t_2, \ldots]$).
\[
\mu' = \mu^\phi
\]
is a typical formal group over $R[\mathbf{t}]$. Since
\[ \phi^{-1}_\mathbf{t} : \mu \rightarrow \mu' \]
and $\mu$ is typical, $\phi^{-1}_\mathbf{t}$ is a typical curve over $\mu'$. Put
\[ k \sum_{k>0} \mu'(s_k \mathbf{t}^k), \quad s_0 = 1, \]
then $s_j = s_j(t_1, t_2 \ldots) \in R[\mathbf{t}]$. Here we put $\mathbf{t} = 0 = (0, \ldots, 0, \ldots)$, then $\phi_0 = \gamma_0$ so $\phi^{-1}_0 = \gamma_0$ and
\[ s_j(0, \ldots, 0, \ldots) = 0 \quad \text{for } j > 0, \]
i.e.,
\[ s_j \in I = (t_1, t_2, \ldots) \quad \text{for all } j > 0. \]
the ideal generated by $t_1, t_2, \ldots$, in $R[\#]$. Then

$$s_k + t_k \equiv 0 \mod I^2$$

for all $k > 0$.

Proof. 

$$T = \phi_\# \circ \phi^{-1}_\#(T)$$

$$= \phi_\# \left( \sum_{j \geq 0} \mu^j s_j T^{p_j} \right)$$

$$= \sum_{i > 0, j \geq 0} \mu^i (t_i s_j T^{p_{i+j}})$$

$$= T \mu^i \sum_{i+j \geq 0} \mu^j (t_i s_j T^{p_{i+j}}).$$

Thus

$$\sum_{i+j > 0} \mu^i (t_i s_j T^{p_{i+j}}) = 0.$$

Then, by Proposition 10.5

$$0 = \sum_{i+j > 0} \mu^i (t_i s_j T^{p_{i+j}}) \equiv \sum_{i+j > 0} t_i s_j T^{p_{i+j}} \mod I^2$$

$$\equiv \sum_{k > 0} (t_k + s_k) T^{p_k} \mod I^2.$$

Hence

$$t_k + s_k \equiv 0 \mod I^2$$

for all $k > 0$. 

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§11. Strong-Hattori Theorem

In this section we prove Strong-Hattori Theorem [11], [20] in our version based on formal group materials.

11.1. Here we put $\mu = \mu_{G^*}$. Let $t = (t_1, t_2, \ldots)$ be a sequence of indeterminates with $\dim t_j = -2(p^j - 1)$, and put

$$
\phi_t(T) = \sum_{j \geq 0} (t_j T^p)^j, \quad t_0 = 1.
$$

$\phi_t$ is a typical curve over $\mu$ by extending the domain of coefficients to $G^*(pt)[t] = G^*(pt)[t_1, t_2, \ldots]$. $\mu'$ is a typical formal group over $G^*(pt)[t]$. We give the complex orientation of the cohomology theory $G^*(\_)[t]$ by assigning $\phi_t^{-1}(e_{G^*}(L))$ as Euler class of a line bundle $L$. Then its associated formal group is the typical $\mu'$. Hence by Theorem 7.2 there exists a unique multiplicative cohomology transformation

$$
h : BP^*(\_ \rightarrow \rightarrow G^*(\_)[t]
$$

such that

$$
h(e_{BP}(L)) = \phi_t^{-1}(e_{G^*}(L))
$$

for a line bundle $L$, and

$$
(11.1) \quad h(pt) \cdot \mu_{BP} = \mu'.
$$
Put

\[ \hat{h} = h(pt) \]

for simplicity. By a standard argument (cf., [2]) we can identify \( \hat{h} \) with the Boardman map

\[ \pi_*(BP) \longrightarrow \pi_*(G \wedge BP), \]

and the Stong-Hattori map

\[ \pi_*(MU) \longrightarrow \pi_*(K \wedge MU) \]

decomposes as direct sum of copies of \( \hat{h} \) after localized at the prime \( p \). Thus we can state Stong-Hattori Theorem in our version as

**Theorem 11.1.** \( \hat{h} = h(pt) : BP^*(pt) \longrightarrow G^*(pt)[1] \) is a split monomorphism.

Stong-Hattori Theorem in this form is proved also in [3] by a different method.

**11.2.** Before going into the proof of Theorem 11.1 we compute some materials of \( \mu \). \( \log_\mu \) is already given in (9.16).

We compute \( [p]_\mu \):

\[
\log_\mu[p]_\mu(T) = p \cdot \log_\mu T \\
= pT + \sum_{k \geq 0} \frac{1}{k} u_1^{1+p+\ldots+p_k} T^{p_k+1} \\
= pT + \sum_{k \geq 0} \frac{1}{p^k} u_1^{1+p+\ldots+p_k-1} (u_1 T^p)^p \\
\]

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\[ \log(\exp(\mu T)) + \log(u_{1T}^\mu), \]

where \( \exp_\mu = \log^{-1} \). Hence

(11.2) \[ [p]_\mu(T) = \exp_\mu(pT) + \mu(u_{1T}^\mu). \]

Since

\( \exp_\mu : G_a \rightarrow \mu \) over \( G^*(pt) \otimes \mathbb{Q} \)

and \( G_a \) is additive, whence typical, we see that \( \exp_\mu \) is a typical curve over \( \mu \) (over \( G^*(pt) \otimes \mathbb{Q} \)). Put

\( \exp_\mu T = \sum_{i \geq 0} (e_i T^p)^i, \quad e_0 = 1, \)

with \( \dim e_i = -2(p^i - 1) \). Then

\[ T = \log_\mu(\exp_\mu T) \]

\[ = \log_\mu\#(\sum_{i \geq 0} (e_i T^p)^i) \]

\[ = \sum_{i \geq 0} \sum_{j \geq 0} \frac{1}{p^{i+1+p+\ldots+p^{j-1}}} e_i^{p^j} T^{p^{i+j}}. \]

Therefore

\[ \sum_{j=0}^{k} \frac{1}{p^j} u_{1+\ldots+p^{j-1}} e_k^{p^j} e_{k-j} = 0 \]

for all \( k > 0 \). Or, multiplying \( p^k \) to this formula we obtain

(11.3) \[ \sum_{j=0}^{k} \frac{1}{p^j} u_{1+\ldots+p^{j-1}} (p^{k-j})^{p^j} e_{k-j} = 0 \quad \text{for } k > 0. \]

Lemma 11.2. \( p^k e_k \in p \cdot G^*(pt) \) for all \( k > 0 \).

Proof. For \( k = 0 \): since \( e_0 = 1 \) we have
\[ p^0 e_0 = p \in p \cdot G^*(pt). \]

We prove the Lemma by induction on \( k \). Assume it is proved until \( k - 1 \).

Then by (11.3)

\[ -p^k e_k = \sum_{j=1}^{k} \frac{1}{p^j} u_1^{1+p+\ldots+p^{j-1}} (p^{p_{k-j}} e_{k-j}) p^j. \]

Here

\[ p^{p_{k-j}} e_{k-j} \in p \cdot G^*(pt) \]

for \( 1 \leq j \leq k \) by induction hypothesis. Then, since \( j < p^j \) for \( 1 \leq j \leq k \) we have

\[ \frac{1}{p^j} (p^{p_{k-j}} e_{k-j}) p^j \in p \cdot G^*(pt) \]

for \( 1 \leq j \leq k \). Hence

\[ -p^k e_k \in p \cdot G^*(pt). \]

q. e. d.

Now

\[ \exp_\mu (pt) = \sum_{k=0}^{\infty} (p^k e_k p^k). \]

Then by Proposition 10.5 and the above Lemma, putting \( I = p \cdot G^*(pt) \) we obtain

Lemma 11.3. \( \exp_\mu (pt) \) is a power series over \( G^*(pt) \) and

\[ \exp_\mu (pt) \equiv 0 \mod p \cdot G^*(pt). \]

Corollary 11.4. \( [p]_\mu (T) \equiv u_1 T^p \mod p \cdot G^*(pt) \).
This follows by Lemma 10.2, (11.2) and Lemma 11.3.

11.3. We compute \( \mathcal{f}_{p, \mu} \gamma_0 \):

\[
\xi_{K^*}(\mathcal{f}_{p, \mu} \gamma_0) = \mathcal{f}_{p, K^*}(\xi_{K^*}) = \mathcal{f}_{p, K^*} \varepsilon_{K^*} \gamma_0 = \varepsilon_{K^*} \mathcal{f}_{p, K^*} \gamma_0 = \varepsilon_{K^*}([u^{p-1}] \gamma_0) \quad \text{by (8.4)}^* \\
= [u^{p-1}] \varepsilon_{K^*} \gamma_0 \quad \text{by Proposition 2.9} \\
= [u^{p-1}] \varepsilon_{K^*} \gamma_0 \quad \text{by Proposition 2.4.}
\]

Since \( \xi_{K^*} : \mathcal{C}_K^* \rightarrow \mathcal{C}_{F^*} \), an isomorphism, we obtain

\[
\mathcal{f}_{p, \mu} \gamma_0 = [u^{p-1}] \gamma_0.
\]

Hence

\[
(11.4) \quad \mathcal{f}_{p, \mu} \gamma_0 = [u_1] \gamma_0.
\]

Then we compute \( \mathcal{f}_{p, \mu} (t_j T^j) \):

for \( j = 0 \), since \( t_0 = 1 \) we have

\[
\mathcal{f}_{p, \mu} (t_0 T) = (\mathcal{f}_{p, \mu} \gamma_0)(T) = u_1 T \quad \text{by (11.4)} ;
\]

for \( j > 0 \),

\[
\mathcal{f}_{p, \mu} (t_j T^j) = (\mathcal{f}_{p, \mu} \gamma_0)(T) = [p \mu (w_{p_j-1} [t_j] \gamma_0)(T) \quad \text{by Proposition 2.9}
\]

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by Corollary 11.4. Thus we obtained

Lemma 11.5. i) \( f_p, u(t_0T) = u_1T \),

ii) \( f_{p, \mu}(t_jT^j) = u_1t_jT^j \mod p \cdot G^*(pt)[t] \)

for \( j > 0 \).

11.4. Proof of Theorem 11.1. Put

\[ v_i = \overline{v}_i \]

for \( i > 1 \), where \( v^{(p)}_i \)'s are the polynomial basis of \( BP^*(pt) \) (Theorem 6.2). Then by Theorem 6.3 we obtain

\[ (f^\#_{p, \mu}, Y_0)(T) = \overline{v}_1 \]

Since \( \phi^\#_\mu : \mu' \cong \mu \), we have

\[ \phi^\#_\mu \circ (f_{p, \mu}, Y_0) = \phi^\#_\mu (f_{p, \mu}, Y_0) \]

\[ = f_{p, \mu}(\phi^\#_\mu Y_0) \]

\[ = f_{p, \mu} \phi^\#_\mu. \]

Thus

\[ (f_{p, \mu}, Y_0)(T) = \phi^{-1}_\mu \left( \sum_{j \geq 0} f_{p, \mu}(t_jT^j) \right) \]

\[ \equiv \phi^{-1}_\mu \left( \sum_{j \geq 0} (u_1t_jT^j) \right) \mod p \cdot G^*(pt)[t] \]
by Lemma 10.1, Proposition 10.3 and Lemma 11.5. Comparing the lowest terms (deg 1) of both sides of (11.6) we obtain

\[ \overline{v}_1 \equiv u_1 \mod p \cdot G^*(pt)[t]. \]  

Put

\[ \phi^{-1}_t(T) = \sum_{j \geq 0} (s_j t^p)^j, \quad s_0 = 1, \]

as in 10.3, and put

\[ I = (t_1, t_2, \ldots), \]

the ideal generated by \( t_1, t_2, \ldots \) in \( G^*(pt)[t] \). Then

\[ s_j + t_j \equiv 0 \mod I^2 \]

for \( j > 0 \) by Proposition 10.9. Hence we can use \( s_1, s_2, \ldots \) as a polynomial basis of \( G^*(pt)[t] \), i.e.,

\[ G^*(pt)[t] = G^*(pt)[s_1, s_2, \ldots]. \]

By (11.6), using Lemma 10.1 and Proposition 10.3, we obtain

\[ (\mathcal{F}_{p, u, \gamma_0}(T)) \equiv \sum_{j \geq 0} (u_1^j s_j t^p)^j \mod p \cdot G^*(pt)[t] + I^2, \]

whence by (11.5) we see that

\[ \overline{v}_j \equiv u_1^{p^j-1} s_{j-1} \mod p \cdot G^*(pt)[t] + I^2 \]

for \( j > 1 \).

To prove Theorem 11.1, it is sufficient to prove that \( \overline{w} \mod p \) is
injective. Since \( u_1 \) is invertible we can use \( \{u_1^j s_j, j \geq 1\} \) as a polynomial basis of \( G^*(pt)[t] \). Then by (11.8)

\[
G^*(pt)[t] \otimes F_p = G^*(pt)[v_2, v_3, \ldots] \otimes F_p,
\]

where \( F_p = \mathbb{Z}/p \cdot \mathbb{Z} \), which contains \( F_p[u_1, v_2, v_3, \ldots] \) as a subalgebra.

Finally by (11.7) we see that \( \overline{v}_1, \overline{v}_2, \ldots \) are algebraically independent over \( F_p \). Therefore \( \overline{h} \mod p \) is injective.
§12. Conner-Floyd Theorem

Conner-Floyd [7] proved the natural isomorphism

$$ U^*(X) \otimes U^*(pt) \cong K^*(X) $$

regarding both sides as $\mathbb{Z}_2$-graded. Here we shall see a corresponding relation holds between $BP^*$ and $G^*$.

12.1. Here we write the polynomial basis $v^{(p)}_k$ of $BP^*(pt)$ (Theorem 6.2) by $v_k$ for simplicity. Compute $Td(v_k)$ by the recursive formula (6.6). Remarking that

$$ Td(m_{p^{-1}}) = \frac{1}{p^k}, $$

by an induction on $k$ we obtain

$$ Td(v_1) = 1 $$

(12.1)

$$ Td(v_k) = 0 \quad \text{for} \quad k > 1. $$

12.2. Using notations of 9.4 we map a line bundle $L$ over $X$ to

$$ 1 - \sum_{k \geq 1} \xi_k^{K(p-1)} v_k^{e_{BP}(L)k(p-1)} \in BP^0(X) $$

where we regard as $\xi_j^{K(p-1)} \in \mathbb{Z}(p)$. By splitting principle this extends to a natural map

$$ \chi' : K(X) \longrightarrow BP^0(X). $$

For a line bundle $L$ we have
\[
\text{td} \circ \chi'(L) = 1 - \sum_{k \geq 1} \xi_k^{K(p-1)-1} \mu_k^{(L)} \left( e^{\mu_k(L)} \right)^{k(p-1)} \\
= 1 - \sum_{k \geq 1} \xi_k^{K(p-1)-1} u^{k(p-1)} \left( u^{-1} e^{\mu_k(L)} \right)^{k(p-1)} \\
= 1 - \sum_{k \geq 1} \xi_k^{K(p-1)-1} e^{\mu_k(L)}^{k(p-1)} \\
= E_0(L)
\]

by (9.15). Thus

\[(12.2) \quad \text{td} \circ \chi' = E_0.\]

Remark that

\[G^0(X) = E_0 K(X) \subset K(X)\]

and define

\[
\chi^0 : G^0(X) \longrightarrow BP^0(X)
\]

by a restriction of \(\chi'\). Since \(E_0\) is an idempotent, by (12.2) we see that

\[\widehat{\text{td}} \circ \chi^0 = 1.\]

For negative integers \(s\) such that \(-2(p-1) < s < 0\) we define

\[
\chi^s : G^s(X) \longrightarrow BP^s(X)
\]

by requiring they commute with suspensions and \(\chi^0, \chi^s\) is uniquely defined by this requirement. Since \(\widehat{\text{td}}\) also commutes with suspensions we see that

\[(12.3) \quad \widehat{\text{td}} \circ \chi^s = 1\]

for \(-2(p-1) < s \leq 0\).
12.3. Make \( BP^* \) \( \mathbb{Z}/2(p-1)\mathbb{Z} \)-graded by

\[
BP^\alpha(X) = \sum_{s \equiv \alpha \mod 2(p-1)} BP^S(X)
\]

for \( \alpha \in \mathbb{Z}/2(p-1)\mathbb{Z} \). We denote this cohomology by \( BP^# \). Then \( \text{td} \) induces multiplicative cohomology transformation

\[
\widetilde{\text{td}}^# : BP^#( ) \longrightarrow G^#( )
\]

such that

\[
\widetilde{\text{td}}^#(pt) = Td : BP^#(pt) \longrightarrow \mathbb{Z}(p).
\]

Thus \( \mathbb{Z}(p) \) is a \( BP^#(pt) \)-module. Now we can state

Theorem 12.1. There exists natural isomorphism

\[
BP^#(X) \otimes_{BP^#(pt)} \mathbb{Z}(p) = G^#(X).
\]

For the proof of this theorem the most basic thing is the existence of natural degree-preserving map

\[
\chi^# : G^#(X) \longrightarrow BP^#(X)
\]

such that

(12.4) \( \widetilde{\text{td}}^# \circ \chi^# = 1 \).

This is defined by \( \chi^# = \{\chi^S : -2(p-1) < s \leq 0\} \) and proved by (12.3).

The rest of the proof is completely parallel to the proof of [7], Theorem (10.1), p.60. The proof is divided into three steps as in [7].
At each step Quillen decomposition and the use of corresponding facts of complex cobordism are helpful. Details are left to readers.
References


2. J.F. Adams, Quillen's work on formal groups and complex cobordism, University of Chicago, 1970.


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