LECTURES IN MATHEMATICS

Department of Mathematics
KYOTO UNIVERSITY

7

SPHERICAL FUNCTIONS AND SPHERICAL MATRIX FUNCTIONS ON LOCALLY COMPACT GROUPS

BY

HITOSHI SHIN'YA

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Hitoshi SHIN'YA

Ehime University

Introduction

In 1950, I.M.Gel'fand defined the generalized "spherical functions" and studied the connection with irreducible unitary representations in [4]. He studied only the case when the irreducible representation of the given compact subgroup K is $k \longrightarrow 1$.

After that, R.Godement defined the still more generalized spherical functions, and studied the connection with representations on Banach spaces. For a representation $\{\mathfrak{H}, T_x\}$ of a locally compact unimodular group G, we can define a representation $\{\mathfrak{H}, T_f\}$ of the algebra L(G), which is the algebra of all continuous functions on G with compact supports. Then he said that $\{\mathfrak{H}, T_x\}$ is algebraically irreducible when non trivial T_f-invariant subspaces of $\mathfrak{H}$ do not exist, completely irreducible when every continuous linear operator T on $\mathfrak{H}$ can be strongly approximated by T_f, and topologically irreducible when non trivial closed T_f-invariant subspaces of $\mathfrak{H}$ do not exist.

Let K be a compact subgroup of G, and δ an equivalence class of irreducible representations of K. We shall denote by $\mathfrak{H}(\delta)$ the space of all vectors in $\mathfrak{H}$ transformed according to δ under $k \longrightarrow T_k$, and by $E(\delta)$ the usual continuous projection on $\mathfrak{H}(\delta)$. If a completely irreducible representation $\{\mathfrak{H}, T_x\}$ of G on a Banach space $\mathfrak{H}$ satisfies $0 < \dim \mathfrak{H}(\delta) =$

$p \cdot d(\delta) < +\infty$ where $d(\delta)$ is the degree of δ, he called the function

$$\phi(x) = \text{trace}[E(\delta)T_x]$$

a spherical function of type δ of height p. However, he studied only the case of completely irreducible representations on Banach spaces, and moreover he assumed on G that every δ is contained at most finite times in every completely irreducible representation of G. This assumption is automatically satisfied for semi-simple Lie groups with finite center and motion groups where K are maximal compact subgroups. But the author feels it is a rather restrictive assumption to study spherical functions as a whole.

The author generalizes the theory for an arbitrary locally compact unimodular group and its representation on a Hausdorff, complete, locally convex topological vector space which is not completely irreducible in general but topologically irreducible. A representation $\{\mathfrak{H}, T_x\}$ of G is called "nice", if there exists a compact subgroup K' and an equivalence class δ' of irreducible representations of K' such that $0 < \dim \mathfrak{H}(\delta') < +\infty$. We study topologically irreducible nice representations and generalize the propositions of R.Godement for completely irreducible representations on Banach spaces.

We define spherical functions for topologically irreducible nice representations. We take an irreducible repre-

sentation $k \to D(k)$ of K belonging to δ, and put $\chi_\delta(k) = d(\delta)\cdot\mathrm{trace}[D(k)]$. Then $L^\circ(\delta) = \{f \in L(G)\ ;\ \overline{\chi_\delta} * f = f,\ f(x) = f^\circ(x) = \int_K f(kxk^{-1})dk\}$ is a subalgebra of $L(G)$. If G is σ-compact, we can obtain an explicit one-to-one correspondence between the set $\mathcal{T}(\delta)$ of all equivalence classes of finite-dimensional irreducible representations of $L^\circ(\delta)$ and the set $\Phi(\delta)$ of all spherical functions of type δ on G. Namely, if we take an irreducible representation $f \to U(f)$ of $L^\circ(\delta)$ belonging to a class $\tau \in \mathcal{T}(\delta)$, then the linear functional

$$\phi_\tau(f) = d(\delta)\cdot\mathrm{trace}[U(\overline{\chi_\delta} * f^\circ)]$$

on $L(G)$ is a function on G, and $\phi_\tau \in \Phi(\delta)$. If G is not σ-compact, we obtain a one-to-one correspondence between $\mathcal{T}(\delta)$ and a set $\Phi_g(\delta)$ including $\Phi(\delta)$ (the author does not know whether $\Phi_g(\delta)$ is actually larger than $\Phi(\delta)$ or not).

In addition to spherical functions, we define spherical matrix functions. A spherical matrix function of type δ means a matrix-valued continuous function $U = U(x)$ on G such that

(1) $\{U(x)\ ;\ x \in G\}$ is an irreducible family of matrices,

(2) $\chi_\delta * U = U$,

(3) $\int_K U(kxk^{-1}y)dk = U(x)U(y)$ for all $x, y \in G$.

If $U = U(x)$ is a spherical matrix function of type δ, then $f \to U(f) = \int_G U(x)f(x)dx$ is an irreducible representation of $L^\circ(\delta)$. We shall say that two spherical matrix functions U_1

and U_2 are equivalent if two irreducible representations $f \to U_1(f)$ and $f \to U_2(f)$ of $L^\circ(\delta)$ are equivalent. Then we obtain a one-to-one correspondence between $\mathcal{T}(\delta)$ and the set of all equivalence classes of spherical matrix functions of type δ. On the other hand, if G is σ-compact, the function

$$\phi(x) = d(\delta) \cdot \mathrm{trace}[U(x)]$$

is a spherical function of type δ for every spherical matrix function U of the same type. If G is not σ-compact, $\phi(x)$ is an element in $\Phi_g(\delta)$. Therefore we may consider spherical matrix functions instead of spherical functions, and the author feels it is rather natural to study spherical matrix functions. For instance, the functional equation (3) cannot be modified in a simple form for spherical functions. In general, it may be difficult to find all of the spherical matrix functions. But, for a group G which can be decomposed into the form $G = KS$, $K \cap S = \{e\}$, where S is a closed subgroup of G, we can construct a family of matrix-valued functions satisfying the equations (2) and (3). Especially, if G is a motion group or a connected semi-simple Lie group with finite center, we know that all spherical matrix functions are obtained as "irreducible components" of these matrix-valued functions.

Some lemmas in this paper are very similar to those in [5] but proved somewhat weaker assumptions, and for the sake of completeness the author does not omit them.

The outline of each section in this paper is as follows.

In §1, we give some definitions and prove some general lemmas on the irreducibilities.

In §2, we study a canonical irreducible subspace $\mathfrak{H}_0$ of $\mathfrak{H}$ for a topologically irreducible nice representation $\{\mathfrak{H}, T_x\}$. In the definition and study of spherical functions or spherical matrix functions, this subspace is more essential than $\mathfrak{H}$.

In §3, we study the multiplicity of δ in completely irreducible representations or in topologically irreducible nice representations.

In §4, we define spherical functions and obtain a one-to-one correspondence between the set of all spherical functions of type δ and the set of all equivalence classes of finite-dimensional irreducible representations of $L^\circ(\delta)$. But, if G is not σ-compact, our result is rather incomplete.

In §5, analyzing the method of the construction of representations in §4, we obtain a connection between spherical functions and representations.

In §6, we study the correspondence between finite-dimensional irreducible representations of $L^\circ(\delta)$ and those of the algebra $L(\delta) = \{f \in L(G) ; \overline{\chi_\delta} * f = f * \overline{\chi_\delta} = f\}$, and we obtain another theorem on the multiplicity of δ in completely irreducible representations or in topologically irreducible nice representations. It is suitable that this theorem is placed

in §3, but, for the proof, we must use the results obtained in §§4 and 5.

In §7, we define spherical matrix functions and study the connection with spherical functions or finite-dimensional irreducible representations of $L^\circ(\delta)$. Moreover we study the connection between the "irreducible components" of matrix-valued functions satisfying the equations (2),(3) and those of representations which are not topologically irreducible.

In §8, we study spherical matrix functions on a connected unimodular Lie group G. Here we assume that K is a compact analytic subgroup of G. For two K-finite topologically irreducible representations of G, they are "infinitesimally equivalent" if and only if they define the same spherical functions. On the characterization of spherical matrix functions, we obtain some results which are generalizations of those by R.Godement for spherical functions of height one. If G is a connected semi-simple Lie group with finite center, we show that a topologically irreducible representation is quasi-simple in the sense of Harish-Chandra if it is nice.

In §9, we assume $G = KS$, $K \cap S = \{e\}$, where S is a closed subgroup of G. For every finite-dimensional irreducible representation of S, we construct matrix-valued functions satisfying (2) and (3). If G is a motion group or a connected semi-simple Lie group with finite center, we prove that all

spherical matrix functions with respect to a maximal compact subgroup K are obtained as "irreducible components" of these matrix-valued functions.

§1. Representations and their irreducibilities

Let G be a locally compact unimodular group, and $\mathfrak{H}$ a complete locally convex topological vector space (we always assume that locally convex topology is Hausdorff). A representation of G on $\mathfrak{H}$ is a homomrphism $x \to T_x$ of G in a group of non-singular continuous linear operators on $\mathfrak{H}$ such that

(a) for $v \in \mathfrak{H}$, $G \ni x \to T_x v \in \mathfrak{H}$ is continuous,

(b) for every compact subset F of G, $\{T_x ; x \in F\}$ is equicontinuous.

If $\mathfrak{H}$ is "tonnelé", (a) implies (b) [1], hence in the case of a Banach space or a Fréchet space, the condition (b) is not necessary. And in general, the pair of conditions (a) and (b) is equivalent to the following condition

(c) $G \times \mathfrak{H} \ni (x , v) \to T_x v \in \mathfrak{H}$ is continuous.

Let L(G) be the algebra of all continuous functions on G with compact supports (the product is convolution product). For every compact subset F of G, denote by $L_F(G)$ the space of all continuous functions on G whose supports are contained in F, then $L_F(G)$ is a normed space with supremum norm. We shall topologize L(G) as the inductive limit of $L_F(G)$. On the other hand, we shall denote by $L_s(\mathfrak{H} , \mathfrak{H})$ the space of all continuous linear operators on $\mathfrak{H}$, topologized by the strong convergence, and by $L_b(\mathfrak{H} , \mathfrak{H})$ the same space, topologized by the uniform convergence on every bounded sub-

set of $\mathfrak{H}$.

For a representation $\{\mathfrak{H}, T_x\}$ of G, we can define a representation T_f of the algebra L(G) by

$$T_f = \int_G T_x f(x)dx,$$

where dx is a Haar measure on G. Then the following facts are known [3]; the representation $\{\mathfrak{H}, T_f\}$ of L(G) satisfies

(i) $L(G) \ni f \to T_f \in L_b(\mathfrak{H}, \mathfrak{H})$ is a continuous homomorphism,

(ii) $\{T_f v ; f \in L(G), v \in \mathfrak{H}\}$ spans a dense subspace of $\mathfrak{H}$,

(iii) for every compact subset F of G, the family $\{T_f ; f \in L(G), \mathrm{supp}[f] \subset F, \|f\|_{L'} \leqq 1\}$ is equicontinuous.

Conversely, a representation of L(G) which satisfies (i), (ii), and (iii) is deduced by the above method from a representation of G.

Let A be an associative algebra over the complex number field $\mathbb{C}$, and $\mathfrak{H}$ a vector space over $\mathbb{C}$. A representation $\{\mathfrak{H}, T_a\}$ of A on $\mathfrak{H}$ is called algebraically irreducible if its invariant subspaces are only $\{0\}$ and $\mathfrak{H}$. And in the case that $\mathfrak{H}$ is a locally convex topological vector space, a representation $\{\mathfrak{H}, T_a\}$ of A on $\mathfrak{H}$ is called completely irreducible if $\{T_a ; a \in A\}$ is dense in $L_s(\mathfrak{H}, \mathfrak{H})$, and called topologically irreducible if its closed invariant subspaces are only $\{0\}$ and $\mathfrak{H}$. These definitions were given by R.Godement in [5].

If the representation space $\mathfrak{H}$ is finite-dimensional,

these three irreducibilities are equivalent by the Burnside's theorem [8]. And using a theorem on the extension of a continuous linear functional [10], we know that the complete irreducibility implies topological irreducibility. If $\mathfrak{H}$ is a Banach space, algebraic irreducibility implies complete irreducibility [5]. We shall define the irreducibility of a representation of G by that of the corresponding representation of L(G). The following lemma plays an important role in this paper.

LEMMA 1. Let A be an associative algebra over $\mathbb{C}$, and $\mathfrak{H}$ a locally convex topological vector space. An algebraically irreducible representation $\{\mathfrak{H}, T_a\}$ of A is completely irreducible, if every continuous linear operator which commutes with all T_a ($a \in A$) is a scalar multiple of the identity operator.

PROOF. Let's show more strong fact that

(a) for arbitrarily given vectors $v_1, v_2, \cdots, v_n \in \mathfrak{H}$ and a continuous linear operator T on $\mathfrak{H}$, there exists an element $a \in A$ such that $T_a v_i = Tv_i$ for $1 \leqq i \leqq n$.

We prove this by induction on n. For $n = 1$, this is clearly true. Suppose (a) is true for n-1, and let's prove it for n. Clearly we may assume that $v_1, v_2, \cdots, v_n$ are linearly independent. By the assumption of induction,

(b) for every n-1 vectors $w_1, w_2, \cdots, w_{n-1} \in \mathfrak{H}$, there

exists an element $a' \in A$ such that $T_{a'} v_i = w_i$ for $1 \leqq i \leqq n-1$.

Denote by $\mathfrak{G}$ the subspace of $\mathfrak{H}$ spanned by $v_1, v_2, \cdots, v_{n-1}$. Let's show the following fact :

(c) suppose $T_a w = 0$ for every $a \in N = \{a \in A ; T_a = 0 \text{ on } \mathfrak{G}\}$, then w is in $\mathfrak{G}$.

For every $(w_1, w_2, \cdots, w_{n-1}) \in \mathfrak{H}^{n-1}$, take $a' \in A$ such that $T_{a'} v_i = w_i$ for $1 \leqq i \leqq n-1$, and define a linear mapping Φ from $\mathfrak{H}^{n-1}$ to $\mathfrak{H}$ by

$$\Phi(w_1, w_2, \cdots, w_{n-1}) = T_{a'} w.$$

This is well-defined. Let I_i be the imbedding of $\mathfrak{H}$ to the ith component of $\mathfrak{H}^{n-1}$, and put $F_i = \Phi I_i$ $(1 \leqq i \leqq n-1)$. Now we fix i and take arbitrary $v \in \mathfrak{H}$ and $a \in A$, then we can find elements a_1, a_2 in A such that

$$T_{a_1} v_j = \delta_{ij} T_a v \qquad (1 \leqq j \leqq n-1),$$

$$T_{a_2} v_j = \delta_{ij} v \qquad (1 \leqq j \leqq n-1),$$

where δ_{ij} is the Kronecker's delta. Then we have

$$F_i T_a v = \Phi I_i T_a v = T_{a_1} w,$$

$$T_a F_i v = T_a \Phi I_i v = T_a T_{a_2} w.$$

Since $T_{aa_2} v_j = \delta_{ij} T_a v = T_{a_1} v_j$ $(1 \leqq j \leqq n-1)$, $aa_2 - a_1$ is in N and $T_{a_1} w = T_{aa_2} w$. Thus $F_i T_a = T_a F_i$ for all $a \in A$. Hence we have $F_i = \lambda_i \cdot 1$ $(\lambda_i \in \mathbb{C}, 1 \leqq i \leqq n-1)$ where 1 is the identity operator on $\mathfrak{H}$, and

$$T_{a'} w = \Phi(w_1, w_2, \cdots, w_{n-1}) = \sum_{i=1}^{n-1} \lambda_i w_i$$

$$= \sum_{i=1}^{n-1} \lambda_i T_{a'} v_i = T_{a'}(\sum_{i=1}^{n-1} \lambda_i v_i).$$

Since $a' \in A$ in (b) can be arbitrarily chosen,

$$T_{a'} w = T_{a'}(\sum_{i=1}^{n-1} \lambda_i v_i)$$

is true for all $a' \in A$. Therefore we obtain $w = \sum_{i=1}^{n-1} \lambda_i v_i \in \mathfrak{S}$. Thus (c) is proved.

Since $v_n \notin \mathfrak{S}$, it follows from (c) that there exists an element $a \in N$ such that $T_a v_n \neq 0$. Therefore $\{T_a v_n ; a \in N\} = \mathfrak{H}$ by the algebraic irreducibility of $\{\mathfrak{H}, T_a\}$. Let $a_o \in A$ be an element such that $T_{a_o} v_i = v_i$ for $1 \leqq i \leqq n-1$. Then there exists an element $a_1 \in N$ such that $T_{a_1} v_n = Tv_n - T_{a_o} v_n$. Now the element $a = a_1 + a_o$ satisfies $T_a v_i = Tv_i$ $(1 \leqq i \leqq n)$.

q.e.d.

Let K be a compact subgroup of G and Ω_K the set of all equivalence classes of irreducible representations of K. For a class $\delta \in \Omega_K$, we choose an irreducible representation $k \to D(k)$ of K in δ and put

$$\chi_\delta(k) = d(\delta) \cdot \mathrm{trace}[D(k)]$$

where $d(\delta)$ denotes the degree of δ. We define the convolutions of $\overline{\chi}_\delta$ and $f \in L(G)$ by

$$\overline{\chi}_\delta * f(x) = \int_K f(k^{-1}x) \overline{\chi}_\delta(k) dk,$$

$$f*\overline{\chi}_\delta(x) = \int_K f(xk^{-1})\overline{\chi}_\delta(k)dk,$$

and put $L(\delta) = \{f \in L(G) ; \overline{\chi}_\delta * f = f * \overline{\chi}_\delta = f\}$. This is a subalgebra of $L(G)$.

For a given representation $\{\mathfrak{H}, T_x\}$ of G on a complete locally convex topological vector space $\mathfrak{H}$, we shall denote by $\mathfrak{H}(\delta)$ the space of all vectors in $\mathfrak{H}$ transformed according to δ under $k \to T_k$. Then

$$E(\delta) = \int_K T_k \overline{\chi}_\delta(k)dk$$

is a continuous projection onto the subspace $\mathfrak{H}(\delta)$. Since $\mathfrak{H}(\delta)$ is invariant under T_k for all $k \in K$ and T_f for all $f \in L(\delta)$, we shall denote by $\tilde{T}_k$ and $\tilde{T}_f$ the restrictions of T_k and T_f on $\mathfrak{H}(\delta)$ respectively. The following lemma is essentially due to R.Godement.

LEMMA 2. If a representation $\{\mathfrak{H}, T_x\}$ of G on a complete locally convex topological vector space $\mathfrak{H}$ is algebraically, completely, or topologically irreducible, the corresponding representation $\{\mathfrak{H}(\delta), T_f\}$ of $L(\delta)$ is respectively algebraically, completely, or topologically irreducible too.

§2. Maximal ideals in L(G) and topologically irreducible nice representations

Let A be an associative algebra, and $\mathfrak{m}$ a left ideal in A. $\mathfrak{m}$ is called regular if there exists an element $u \in A$ such that $xu \equiv x \pmod{\mathfrak{m}}$ for all $x \in A$. Similar definitions apply to right ideals and two-sided ideals.

Let G be a locally compact unimodular group, and K a compact subgroup of G. Now we fix a class $\delta \in \Omega_K$ and consider an associative algebra over $\mathbb{C}$ such that

(a) for every element $f \in A$ both $f\overline{\chi}_\delta$ and $\overline{\chi}_\delta f$ are in A,

(b) $\overline{\chi}_\delta(f\overline{\chi}_\delta) = (\overline{\chi}_\delta f)\overline{\chi}_\delta$ for all $f \in A$,

(c) $(f\overline{\chi}_\delta)g = f(\overline{\chi}_\delta g)$ for all $f, g \in A$,

(d) $\overline{\chi}_\delta(\overline{\chi}_\delta f) = \overline{\chi}_\delta f$, $(f\overline{\chi}_\delta)\overline{\chi}_\delta = f\overline{\chi}_\delta$ for all $f \in A$.

Then the following lemma is proved in [5].

LEMMA 3. Let $\mathfrak{o}$ be a regular maximal left ideal in the subalgebra $A(\delta) = \{f \in A ; \overline{\chi}_\delta f = f\overline{\chi}_\delta = f\}$ of A, and put

$$\mathfrak{m} = \{f \in A ; \overline{\chi}_\delta g f \overline{\chi}_\delta \in \mathfrak{o} \quad \text{for all } g \in A\},$$

then $\mathfrak{m}$ is a regular maximal left ideal in A such that $\mathfrak{o} = \mathfrak{m} \cap A(\delta)$, and we have $f\overline{\chi}_\delta \equiv f \pmod{\mathfrak{m}}$ for all $f \in A$.

The following theorem is essentially due to R.Godement [5], but for the sake of completeness we give it here.

THEOREM 1. Let G be a locally compact unimodular group, K a compact subgroup of G, and $\{\mathfrak{H}, T_x\}$ a topologically irre-

ducible representation of G on a complete locally convex topological vector space $\mathfrak{H}$. If we have $0 < \dim \mathfrak{H}(\delta) < +\infty$ for a class $\delta \in \Omega_K$,

$$\mathfrak{m}_v = \{f \in L(G) ; T_f v = 0\} \quad (v \in \mathfrak{H}(\delta),\ v \neq 0)$$

is a closed regular maximal left ideal in L(G).

PROOF. By Lemma 2 and the Burnside's theorem, there exists an element $u \in L(\delta)$ such that $\widetilde{T}_u = 1$ where 1 is the identity operator on $\mathfrak{H}(\delta)$. As is easily seen, $\mathfrak{l} = \mathfrak{m}_v \cap L(\delta)$ is a closed regular maximal left ideal in $L(\delta)$ with right identity u. Thus, from Lemma 3, it follows that

$$\mathfrak{m} = \{f \in L(G) ;\ \overline{\chi}_\delta * g * f * \overline{\chi}_\delta \in \mathfrak{l} \quad \text{for all } g \in L(G)\}$$

is a regular maximal left ideal in L(G), so we have only to prove $\mathfrak{m} \subset \mathfrak{m}_v$. If f is in $\mathfrak{m}$, we have

$$E(\delta)T_f v = T_u T_f v = T_h v = \widetilde{T}_h v = 0 \quad (h = u * f * \overline{\chi}_\delta \in \mathfrak{l}).$$

This implies $E(\delta)\{T_f v ; f \in \mathfrak{m}\} = \{0\}$. If the invariant subspace $\{T_f v ; f \in \mathfrak{m}\}$ of $\mathfrak{H}$ is not equal to $\{0\}$, it is dense in $\mathfrak{H}$ and hence $E(\delta)\{T_f v ; f \in \mathfrak{m}\} = \mathfrak{H}(\delta) \neq \{0\}$. Therefore $\{T_f v ; f \in \mathfrak{m}\} = \{0\}$ and this implies $\mathfrak{m} \subset \mathfrak{m}_v$. q.e.d.

Let $\{\mathfrak{H}, T_x\}$ be a representation of G. We shall call it nice if there exists a compact subgroup K such that

$$0 < \dim \mathfrak{H}(\delta) < +\infty$$

for some class $\delta \in \Omega_K$. If $\{\mathfrak{H}, T_x\}$ is a topologically irreducible nice representation of G on a complete locally convex topological vector space $\mathfrak{H}$, we take such a compact subgroup

K of G and a class $\delta \in \Omega_K$, and put

$$\mathfrak{H}_0 = \mathfrak{H}_0[K,\delta,v] = \{T_f v \; ; \; f \in L(G)\}$$

where v is a non zero vector in $\mathfrak{H}(\delta)$.

LEMMA 4. The space $\mathfrak{H}_0$ is independent of K, $\delta \in \Omega_K$ such that $0 < \dim \mathfrak{H}(\delta) < +\infty$, and of $v \in \mathfrak{H}(\delta)$.

PROOF. Let (K',δ') be another pair of compact subgroup K' of G and δ' in $\Omega_{K'}$ such that $0 < \dim \mathfrak{H}(\delta') < +\infty$. Let $v' \in \mathfrak{H}(\delta')$, $v' \neq 0$. By the topological irreducibility of $\{\mathfrak{H}, T_x\}$, both $\mathfrak{H}_0[K,\delta,v]$ and $\mathfrak{H}_0[K',\delta',v']$ are dense in $\mathfrak{H}$. Therefore $\mathfrak{H}(\delta) = E(\delta)\,\mathfrak{H}_0[K',\delta',v'] \subset \mathfrak{H}_0[K',\delta',v']$. Hence we have $\mathfrak{H}_0[K,\delta,v] \subset \mathfrak{H}_0[K',\delta',v']$. q.e.d.

THEOREM 2. Let G be a locally compact unimodular group, and $\{\mathfrak{H}, T_x\}$ a topologically irreducible nice representation of G on a complete locally convex topological vector space $\mathfrak{H}$. Then the representation $\{\mathfrak{H}_0, T_f\}$ of L(G) is algebraically and completely irreducible.

PROOF. Let K be a compact subgroup of G such that $0 < \dim \mathfrak{H}(\delta) < +\infty$ for some class $\delta \in \Omega_K$, and v a non zero vector in $\mathfrak{H}(\delta)$. Then $\mathfrak{m}_v = \{f \in L(G) \; ; \; T_f v = 0\}$ is a closed regular maximal left ideal in L(G) (Theorem 1). Suppose $T_h v$ is an arbitrary non zero element in $\mathfrak{H}_0$, then $h \notin \mathfrak{m}_v$. For every neighborhood U of the unit e in G, we take a non negative

function $e_U \in L(G)$ such that $\|e_U\|_{L^1} = 1$ and $\mathrm{supp}[e_U] \subset U$. Since $\lim_{U \to e} e_U * h = h$ and $\mathfrak{m}_v$ is closed in $L(G)$, $e_U * h$ is not in $\mathfrak{m}_v$ for sufficiently small U. Therefore $L(G) * h \not\subset \mathfrak{m}_v$. This implies $L(G) * h + \mathfrak{m}_v = L(G)$, and hence we obtain $\{T_f(T_h v) ; f \in L(G)\} = \mathfrak{H}_0$.

Now let's show the complete irreducibility. Let T be a continuous linear operator on $\mathfrak{H}_0$ which commutes with all T_f $(f \in L(G))$. As in the proof of Theorem 1, we take $u \in L(\delta)$ such that $\tilde{T}_u = 1$ on $\mathfrak{H}(\delta)$. Since $\mathfrak{H}(\delta) \subset \mathfrak{H}_0$, we have

$$T_x T w = T_x T T_u w = T_x T_u T w = T_{L_x u} T w = T T_{L_x u} w$$
$$= T T_x w$$

for all $w \in \mathfrak{H}(\delta)$, where $(L_x u)(y) = u(x^{-1}y)$. Noting the fact that $T_k w \in \mathfrak{H}(\delta)$ for every $k \in K$, we obtain

$$E(\delta) T w = \int_K T_k (T w) \overline{\chi}_\delta(k) dk = \int_K T(T_k w) \overline{\chi}_\delta(k) dk$$
$$= T \int_K T_k w \overline{\chi}_\delta(k) dk = T(E(\delta) w) = T w,$$

i.e., Tw is a vector in $\mathfrak{H}(\delta)$. Thus $\tilde{T}_f T w = T \tilde{T}_f w$ is valid for all $f \in L(\delta)$ and $w \in \mathfrak{H}(\delta)$. Therefore the operator T is a scalar multiple of the identity operator on $\mathfrak{H}(\delta)$ and this implies that T is also a scalar multiple of the identity operator on $\mathfrak{H}_0$. Now, by Lemma 1, we have proved that the representation $\{\mathfrak{H}_0, T_f\}$ of $L(G)$ is completely irreducible.

q.e.d.

REMARK. Although the space $\mathfrak{H}_0$ is not necessarily com-

plete, it is invariant under not only T_x ($x \in G$) but also T_f ($f \in L(G)$) and $E(\delta)$ ($\delta \in \Omega_K$). Thus we can consider a "representation" $T_x|\mathfrak{H}_0$ of G on $\mathfrak{H}_0$ without much inconvenience.

As a corollary of Theorem 2, we have the following

COROLLARY. An algebraically irreducible nice representation of G is completely irreducible.

§3. The multiplicities of irreducible representations of a compact subgroup

Let G be a locally compact unimodular group, and K a compact subgroup of G. Let $\{\mathfrak{H}, T_x\}$ be a representation of G. For an arbitrary class $\delta \in \Omega_K$, we say that δ is contained p times in $\{\mathfrak{H}, T_x\}$ if $\dim \mathfrak{H}(\delta) = p \cdot d(\delta)$.

LEMMA 5. (See [5, p.503, Lemma 1]) If an associative algebra A over $\mathbb{C}$ has sufficiently many representations whose dimensions are not greater than n, the dimension of every completely irreducible representation of A on a locally convex topological vector space is also not greater than n.

Let Σ be a set of representations of G. We shall say after R.Godement that Σ is "complete", if for every $f \in L(G)$ we can choose some representation $\{\mathfrak{H}, T_x\} \in \Sigma$ such that $T_f \neq 0$.

LEMMA 6. Let Σ be a complete set of representations of G. If $\delta \in \Omega_K$ is contained at most p times in every representation in Σ, we have

(i) δ is contained at most p times in every completely irreducible representation of G,

(ii) δ is contained at most p times in every topologically irreducible nice representation of G,

(iii) δ is contained at most p times or infinitely many

times in every topologically irreducible representation of G. In the latter case, every equivalence class of irreducible representations of every compact subgroup of G is contained either no times or infinitely many times.

PROOF. For every representation $\{\mathfrak{H}', T'_x\}$ in Σ, we obtain the representation $\{\mathfrak{H}'(\delta), \tilde{T}'_f\}$ of $L(\delta)$. All such representations make a family containing sufficiently many representations of $L(\delta)$, and $\dim \mathfrak{H}'(\delta)$ is always not greater than $p \cdot d(\delta)$ by the assumption. So, by Lemma 5, we know that every completely irreducible representation of $L(\delta)$ has dimension $\leq p \cdot d(\delta)$. Now (i) is clear by Lemma 2.

Let's prove (ii). Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible nice representation of G. If $\mathfrak{H}_0 = \mathfrak{H}_0[K', \delta', v']$, $\{\mathfrak{H}_0, T_f\}$ is a completely irreducible representation of L(G). Therefore $\{E(\delta)\mathfrak{H}_0, \tilde{T}_f\}$ is a completely irreducible representation of $L(\delta)$. (Lemma 2, in which the representation space is complete, is also true in this case.) Hence $\dim E(\delta)\mathfrak{H}_0 \leq p \cdot d(\delta)$. Since $E(\delta)\mathfrak{H}_0$ is dense in $E(\delta)\mathfrak{H} = \mathfrak{H}(\delta)$, we have $\dim \mathfrak{H}(\delta) \leq p \cdot d(\delta)$.

(iii) is clear from (i) and (ii). q.e.d.

LEMMA 7. Let G_0 be the intersection of the kernels of finite-dimensional representations of G. Then the set of all finite-dimensional representations of G is complete if and only if $G_0 = \{e\}$.

PROOF. Assume $G_o \neq \{e\}$. Then G_o contains a non trivial closed abelian subgroup, say Z. If $z_o \in Z$ is not the unit e, there exists a neighborhood U of e in G such that

$$Uz_o \cap U = \phi.$$

Now we choose a non zero function $\varphi \in L(G)$ such that $\mathrm{supp}[\varphi] \subset U$, and define a non zero function $f \in L(G)$ by

$$f(x) = \begin{cases} \varphi(x) & \text{for } x \in U, \\ -\varphi(xz_o) & \text{for } x \in Uz_o, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily see that

$$\int_Z f(xz)dz = 0 \qquad \text{for all } x \in G.$$

From this, we have

$$\int_G \theta(x)f(x)dx = 0$$

for an arbitrary matrix element $\theta(x)$ of every finite-dimensional representation of G.

The converse is proved in [5, p.506, Lemma 5].

q.e.d.

LEMMA 8. Let G be a connected semi-simple Lie group. The set of all finite-dimensional irreducible representations of G is complete if and only if G has a finite-dimensional faithful representations.

PROOF. Using the notation in Lemma 7, $G_o = \{e\}$ is equivalent to the fact that G has a finite-dimensional

faithful representation [6]. On the other hand, every finite-dimensional representation of G is completely reducible. Therefore this lemma follows from Lemma 7. q.e.d.

Using Lemmas 6 and 8, we obtain the following

THEOREM 3. Let G be a connected semi-simple Lie group with a finite-dimensional faithful representation, and K a compact subgroup of G. If $\delta \in \Omega_K$ is contained at most p times in every finite-dimensional irreducible representation of G, δ is contained at most p times in every completely irreducible representation and in every topologically irreducible nice representation of G.

THEOREM 4. Let G be a connected semi-simple Lie group with finite center, and K a maximal compact subgroup of G. Then $\delta \in \Omega_K$ is contained at most $d(\delta)$ times in every completely irreducible representation and in every topologically irreducible representation of G.

PROOF. By Corollary5.5.1.6 in [9], we know that the algebra $L(\delta)$ has sufficiently many representations whose dimensions are $\leqq d(\delta)^2$. Therefore, by Lemma 5, we obtain this theorem. q.e.d.

The following two theorems are proved in [5] for

completely irreducible representations on Banach spaces.

THEOREM 5. Let G be a connected complex semi-simple Lie group, K a maximal compact subgroup of G, and Γ a maximal abelian subgroup of K. Then the multiplicity of $\delta \in \Omega_K$ in any completely irreducible representation or in any topologically irreducible nice representation of G are not greater than the maximum of the multiplicities of irreducible representations of Γ in δ.

THEOREM 6. Let G be a locally compact unimodular group, and K a compact subgroup of G. If there exists an abelian subgroup N such that $G = NK$, then every $\delta \in \Omega_K$ is contained at most $d(\delta)$ times in every completely irreducible representation and in every topologically irreducible nice representation of G.

We shall say that a representation $\{\mathfrak{H}, T_x\}$ of G is K-finite if $\dim \mathfrak{H}(\delta) < +\infty$ for all $\delta \in \Omega_K$. Then we have

THEOREM 7. Let G be a locally compact unimodular group, and K a compact subgroup of G. If a representation $\{\mathfrak{H}, T_x\}$ of G is K-finite, T_f are completely continuous for all $f \in L(G$

For the proof of Theorem 7, we use the fact that the set of all completely continuous operators on $\mathfrak{H}$ is closed

in $L_b(\mathfrak{H}, \mathfrak{H})$. The outline of the proof is similar to that of Theorem 7 in [5].

In §6, we will obtain another theorem on the multiplicity of $\delta \in \Omega_K$ in a completely irreducible or a topologically irreducible nice representation of G (Theorem 10).

§4. Spherical functions

In this section, G is a locally compact unimodular group and K is a compact subgroup of G. Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible representation of G on a complete locally convex topological vector space $\mathfrak{H}$. If $\dim \mathfrak{H}(\delta) = p \cdot d(\delta) < +\infty$ for a class $\delta \in \Omega_K$, $E(\delta)T_x$ are of finite rank for all $x \in G$. Then we define a continuous function ϕ on G by

$$\phi(x) = \mathrm{trace}[E(\delta)T_x],$$

and call it a spherical function of type δ of height p. R.Godement treated spherical functions only for the completely irreducible representations on Banach spaces.

If the projection $E(\delta)$ can be defined on $\mathfrak{H}$, the completeness of $\mathfrak{H}$ is not essential in the definition of spherical functions. Now let's define generalized spherical functions which are really spherical functions if G is σ-compact. Let $\mathfrak{H}$ be a locally convex topological vector space, and $x \longrightarrow T_x$ a homomorphism of G in a group of non-singular continuous linear operators on $\mathfrak{H}$ which satisfies the conditions (a) and (b) in §1. Also in this case, we shall call $\{\mathfrak{H}, T_x\}$ a representation of G. If the integrals, which define $E(\delta)$ and T_f for all $f \in L(G)$, converge in $\mathfrak{H}$, and if the representation $\{\mathfrak{H}, T_f\}$ of L(G) is topologically irreducible and $\dim E(\delta)\mathfrak{H} = p \cdot d(\delta)$, we call the function

$$\phi(x) = \mathrm{trace}[E(\delta)T_x]$$

a generalized spherical function of type δ of height p.

We shall denote by $\Phi(\delta)$ the set of all spherical functions of type δ and by $\Phi_g(\delta)$ that of all generalized spherical functions of type δ.

On the other hand, we denote by $L^\circ(G)$ the space of all functions f°, where f is in $L(G)$ and $f^\circ(x) = \int_K f(kxk^{-1})dk$, and put

$$L^\circ(\delta) = \{f \in L^\circ(G) \; ; \; \overline{\chi}_\delta * f = f * \overline{\chi}_\delta = f\}.$$

It is clear that $L^\circ(\delta) = \overline{\chi}_\delta * L^\circ(G) * \overline{\chi}_\delta = \overline{\chi}_\delta * L^\circ(G)$. Let $\mathcal{T}(\delta)$ be the set of all equivalence classes of finite-dimensional irreducible representations of the algebra $L^\circ(\delta)$. If a representation $f \to U(f)$ of $L^\circ(\delta)$ belongs to $\tau \in \mathcal{T}(\delta)$, we put

$$\phi_\tau(f) = d(\delta) \cdot \mathrm{trace}[U(\overline{\chi}_\delta * f^\circ)]$$

for all $f \in L(G)$. Clearly ϕ_τ is a continuous linear functional on $L(G)$.

Then our aim in this section is to prove the following

THEOREM 8. (i) ϕ_τ is a function on G for all $\tau \in \mathcal{T}(\delta)$ and $\tau \to \phi_\tau$ is a one-to-one mapping from $\mathcal{T}(\delta)$ onto $\Phi_g(\delta)$ such that τ is p-dimensional if and only if ϕ_τ is of height p.

(ii) If $\phi_\tau \in \Phi_g(\delta)$ is positive definite, then it belongs to $\Phi(\delta)$ and it is defined by an irreducible unitary representation of G.

(iii) If G is σ-compact, we have $\Phi(\delta) = \Phi_g(\delta)$, and all spherical functions are defined by topologically irreducible

representations on Fréchet spaces.

At first, we take an arbitrary generalized spherical function $\phi \in \Phi_g(\delta)$. Let ϕ be defined by a representation $\{\mathfrak{H}, T_x\}$ of G on a locally convex topological vector space $\mathfrak{H}$ which is not necessarily complete. Namely, the operators $E(\delta)$ and T_f $(f \in L(G))$ can be defined on $\mathfrak{H}$, $\{\mathfrak{H}, T_f\}$ is a topologically irreducible representation of L(G), $\mathfrak{H}(\delta) = E(\delta)\mathfrak{H}$ is a p-dimensional subspace of $\mathfrak{H}$ where p is the height of ϕ, and $\phi(x) = d(\delta)\cdot\mathrm{trace}[E(\delta)T_x]$.

*LEMMA 9. Considering only on $\mathfrak{H}(\delta)$, the set of all linear operators which commute with all T_k $(k \in K)$ is

$$\{\tilde{T}_f = T_f|\mathfrak{H}(\delta)\ ;\ f \in L^\circ(\delta)\}.$$

PROOF. For every linear operator A on $\mathfrak{H}(\delta)$, there exists at least one function $f \in L(\delta)$ such that $\tilde{T}_f = A$ by the Burnside's theorem. If A commutes with all T_k, we have

$$A = \int_K T_k T_f T_k^{-1}\, dk = \int_K \int_G T_k T_x T_k^{-1}\, f(x)dxdk = \tilde{T}_{f^\circ}.$$

The converse is clear. q.e.d.

The representation $\{\mathfrak{H}(\delta), \tilde{T}_k\}$ of K, where $\tilde{T}_k$ is the restriction of T_k on $\mathfrak{H}(\delta)$, is equivalent to the p-times direct sum of δ. Hence, by the above lemma, $\{\tilde{T}_f\ ;\ f \in L^\circ(\delta)\}$ is identified with the set $M(p,\mathbb{C})$ of all $p \times p$-complex

matrices, and we may write

$$\tilde{T}_f = U(f) \otimes I_{d(\delta)} = \begin{pmatrix} u_{11}(f)I_{d(\delta)} & \cdots & u_{1p}(f)I_{d(\delta)} \\ \vdots & & \vdots \\ u_{p1}(f)I_{d(\delta)} & \cdots & u_{pp}(f)I_{d(\delta)} \end{pmatrix},$$

where $U(f)$ is a matrix in $M(p,\mathbf{C})$ and $I_{d(\delta)}$ is the unit matrix of degree $d(\delta)$. The representation $f \longrightarrow U(f)$ of $L^\circ(\delta)$ is clearly irreducible, and for every $f \in L(G)$,

$$d(\delta)\cdot\mathrm{trace}[U(\overline{\chi}_\delta * f^\circ)] = \mathrm{trace}[\tilde{T}_{\overline{\chi}_\delta * f^\circ}] = \mathrm{trace}[E(\delta)T_{f^\circ}]$$
$$= \int_G \phi(x) f^\circ(x)dx = \int_G \phi(x) f(x)dx.$$

Therefore, if we take $\tau \in \mathcal{T}(\delta)$ to which the irreducible representation $f \longrightarrow U(f)$ belongs, we obtain $\phi_\tau = \phi$. Thus we have proved the surjectiveness of the mapping $\tau \longrightarrow \phi_\tau$ from $\mathcal{T}(\delta)$ to $\Phi_g(\delta)$.

To prove Theorem 8 completely, we must make use of some lemmas.

LEMMA 10. The set $\{L_k f\ ;\ k \in K,\ f \in L^\circ(\delta)\}$ is total in $L(\delta)$, where $(L_k f)(x) = f(k^{-1}x)$.

The proof of this lemma is essentially same as that of Lemma 11 in [5].

Let's fix an equivalence class τ in $\mathcal{T}(\delta)$ and an irreducible representation $f \longrightarrow U(f)$ of $L^\circ(\delta)$ belonging to τ until Theorem 8 is completely proved.

LEMMA 11. If f is a function in $L(\delta)$, we have

$$\phi_\tau(f*g) = \phi_\tau(g*f)$$

for all $g \in L(G)$.

PROOF. If a function f is in $L^\circ(\delta)$ and g an arbitrary function in $L(G)$, we have

$$\begin{aligned}\phi_\tau(f*g) &= d(\delta)\cdot\mathrm{trace}[U(\bar{X}_\delta * f * g^\circ)] \\ &= d(\delta)\cdot\mathrm{trace}[U(f)U(\bar{X}_\delta * g^\circ)] \\ &= d(\delta)\cdot\mathrm{trace}[U(\bar{X}_\delta * g^\circ)U(f)] \\ &= d(\delta)\cdot\mathrm{trace}[U(\bar{X}_\delta * (g*f)^\circ)] \\ &= \phi_\tau(g*f).\end{aligned}$$

We shall denote by ε_x the measure such that $\varepsilon_x(f) = f(x)$. Then, for every $k \in K$ and $f \in L^\circ(\delta)$,

$$\begin{aligned}\phi_\tau((L_k f)*g) &= \phi_\tau((\varepsilon_k * f * g)^\circ) = \phi_\tau((f*g*\varepsilon_k)^\circ) \\ &= \phi_\tau(f*(g*\varepsilon_k)) = \phi_\tau(g*\varepsilon_k*f) = \phi_\tau(g*(\varepsilon_k*f)) \\ &= \phi_\tau(g*(L_k f)).\end{aligned}$$

Therefore, by Lemma 10, this implies $\phi_\tau(f*g) = \phi_\tau(g*f)$ for every $f \in L(\delta)$. q.e.d.

Denote by V the space on which linear operators U(f) act. For every $k \in K$ and $v \in V$, we associate a V-valued continuous linear function

$$\Theta_{v,k}(f) = U((f*\varepsilon_k)^\circ)v$$

on $L(\delta)$.

LEMMA 12. The set $\{\Theta_{v,k}\,;\, k \in K,\ v \in V\}$ spans a finite-

dimensional vector space W.

PROOF. Let $k \to D(k)$ be an irreducible unitary representation of K belonging to δ, and $d_{ij}(k)$ the matrix elements of $D(k)$. We choose a base $v_1, \cdots, v_p$ in V where p is the dimension of τ. Now we define $p \cdot (d(\delta))^2$ functions on $L(\delta)$ by

$$\theta_{r,i,j}(f) = \theta_{v_r,e}(f*\bar{d}_{ij}), \quad 1 \leqq r \leqq p, \ 1 \leqq i,j \leqq d(\delta).$$

We have only to prove that all $\theta_{v,k}$ are linear combinations of $\theta_{r,i,j}$. For every $f \in L(\delta)$ and $k \in K$, we have

$$f*\varepsilon_k = f*\bar{\chi}_\delta*\varepsilon_k = f*\left\{d(\delta)\sum_{i,j=1}^{d(\delta)} d_{ij}(k)\bar{d}_{ij}\right\}$$

$$= d(\delta)\sum_{i,j=1}^{d(\delta)} d_{ij}(k)(f*\bar{d}_{ij}).$$

Therefore, for $v = c_1v_1 + \cdots + c_pv_p \in V$, we obtain

$$\theta_{v,k}(f) = U((f*\varepsilon_k)^\circ)v = \sum_{r=1}^{p} c_r U((f*\varepsilon_k)^\circ)v_r .$$

$$= \sum_{r=1}^{p}\sum_{i,j=1}^{d(\delta)} d(\delta)c_r d_{ij}(k)U((f*\bar{d}_{ij})^\circ)v_r$$

$$= \sum_{r=1}^{p}\sum_{i,j=1}^{d(\delta)} d(\delta)c_r d_{ij}(k)\theta_{v_r,e}(f*\bar{d}_{ij})$$

$$= \sum_{r=1}^{p}\sum_{i,j=1}^{d(\delta)} d(\delta)c_r d_{ij}(k)\theta_{r,i,j}(f).$$

Thus the lemma is proved. q.e.d.

For every $v \in V$ and $f \in L(\delta)$, let's define a V-valued continuous linear function $\theta_{v,f}$ on $L(\delta)$ as

$$\Theta_{v,f}(g) = U((g*f)^\circ)v.$$

Clearly we have

$$\Theta_{v+w,f} = \Theta_{v,f} + \Theta_{w,f}, \quad \Theta_{v,f+g} = \Theta_{v,f} + \Theta_{v,g},$$

$$\Theta_{\lambda v,f} = \lambda\Theta_{v,f} = \Theta_{v,\lambda f} \quad (\lambda \in \mathbb{C}).$$

LEMMA 13. $\Theta_{v,f}$ are functions in W for all $f \in L(\delta)$ and $v \in V$.

PROOF. Let X be the dense subspace of $L(\delta)$ spanned by $\{L_k f\ ;\ f \in L^\circ(\delta),\ k \in K\}$, and put

$$H_v = \{\Theta_{v,f}\ ;\ f \in L(\delta)\},$$

$$H_v' = \{\Theta_{v,f}\ ;\ f \in X\}.$$

By the pointwise convergence, H_v is a topological vector space. Since the linear mapping

$$L(\delta) \ni f \longrightarrow \Theta_{v,f} \in H_v$$

is continuous, H_v' is densely contained in H_v by Lemma 10. On the other hand, for every $L_k f \in X$, we have

$$\Theta_{v,L_k f}(g) = U((g*\varepsilon_k*f)^\circ)v = U((g*\varepsilon_k)^\circ*f)v$$

$$= U((g*\varepsilon_k)^\circ)U(f)v = \Theta_{U(f)v,k}(g),$$

i.e., $\Theta_{v,L_k f} = \Theta_{U(f)v,k} \in W$. This shows that $H_v' \subset W$, and therefore H_v' is finite-dimensional. Consequently H_v must also finite-dimensional and $H_v = H_v' \subset W$. q.e.d.

By Lemma 13, we can define linear operators R_f $(f \in L(\delta))$ on W by

$$(R_f\Theta)(g) = \Theta(g*f) \qquad (g \in L(\delta)).$$

Moreover, $f \longrightarrow R_f$ is a finite-dimensional (continuous) representation of $L(\delta)$ on W.

If we put $f'(x) = f(x^{-1})$, it is natural to denote by $f'*\phi_\tau$ $(f \in L(\delta))$ the measure

$$L(G) \ni g \longrightarrow \phi_\tau(f*g).$$

LEMMA 14. Put $\mathfrak{P} = \{f \in L(\delta) ; f'*\phi_\tau = 0\}$. Then $\mathfrak{P}$ is a closed regular maximal two-sided ideal in $L(\delta)$ such that

$$\dim(L(\delta)/\mathfrak{P}) < +\infty.$$

PROOF. It is obvious that $\mathfrak{P}$ is closed. For $f \in \mathfrak{P}$, $g \in L(\delta)$, and $h \in L(G)$, we have

$$(g*f)'*\phi_\tau(h) = \phi_\tau(g*f*h) = \phi_\tau(f*h*g) = (f'*\phi_\tau)(h*g) = 0,$$

$$(f*g)'*\phi_\tau(h) = \phi_\tau(f*g*h) = (f'*\phi_\tau)(g*h) = 0.$$

This implies that $g*f$, $f*g \in \mathfrak{P}$, i.e., $\mathfrak{P}$ is a two-sided ideal in $L(\delta)$. The regularity of $\mathfrak{P}$ follows from the existence of a function $u \in L^0(\delta)$ such that $U(u) = 1$.

To prove the fact that $\dim(L(\delta)/\mathfrak{P}) < +\infty$, it is sufficient to show that $\mathfrak{P} = \{f \in L(\delta) ; R_f = 0\}$ since R_f is a finite-dimensional representation of $L(\delta)$. Using the notation $A \Leftrightarrow B$ to denote the equivalence of statements A and B,

$$f \in \mathfrak{P} \Longleftrightarrow \phi_\tau(f*g) = 0 \text{ for every } g \in L(G)$$

$$\Longleftrightarrow \operatorname{trace}[U(\bar{\chi}_\delta*(f*g)^\circ)] = 0 \text{ for every } g \in L(G)$$

$$\Longleftrightarrow \operatorname{trace}[U((\bar{\chi}_\delta*f*g*\bar{\chi}_\delta)^\circ)] = 0 \text{ for every } g \in L(G)$$

$$\Longleftrightarrow \operatorname{trace}[U((f*g)^\circ)] = 0 \text{ for every } g \in L(\delta)$$

$\iff$ $\mathrm{trace}[U((f*g*\varepsilon_k)^\circ)] = 0$ for every $g \in L^\circ(\delta)$ and $k \in K$

$\iff$ $\mathrm{trace}[U((\varepsilon_k*f*g)^\circ)] = 0$ for every $g \in L^\circ(\delta)$ and $k \in K$

$\iff$ $\mathrm{trace}[U((\varepsilon_k*f)^\circ)U(g)] = 0$ for every $g \in L^\circ(\delta)$ and $k \in K$

$\iff$ $U((\varepsilon_k*f)^\circ) = 0$ for every $k \in K$

$\iff$ $U((\varepsilon_{k'}*\varepsilon_k*f)^\circ)U(g) = 0$ for every $k,k' \in K$ and $g \in L^\circ(\delta)$

$\iff$ $U((\varepsilon_k*f*g*\varepsilon_{k'})^\circ) = 0$ for every $k,k' \in K$ and $g \in L^\circ(\delta)$

$\iff$ $U((\varepsilon_k*f*g)^\circ) = 0$ for every $k \in K$ and $g \in L(\delta)$

$\iff$ $U((f*g*\varepsilon_k)^\circ) = 0$ for every $k \in K$ and $g \in L(\delta)$

$\iff$ $U(h)U((f*g*\varepsilon_k)^\circ) = 0$ for every $k \in K$, $h \in L^\circ(\delta)$, and $g \in L(\delta)$

$\iff$ $U((h*f*g*\varepsilon_k)^\circ) = 0$ for every $k \in K$, $h \in L^\circ(\delta)$, and $g \in L(\delta)$

$\iff$ $U((\varepsilon_k*h*f*g)^\circ) = 0$ for every $k \in K$, $h \in L^\circ(\delta)$, and $g \in L(\delta)$

$\iff$ $U((h*f*g)^\circ) = 0$ for every $h,g \in L(\delta)$

$\iff$ $U((h*f*g)^\circ)v = 0$ for every $v \in V$ and $h,g \in L(\delta)$

$\iff$ $R_f\Theta_{v,g} = 0$ for every $v \in V$ and $g \in L(\delta)$

$\iff$ $R_f = 0$. q.e.d.

LEMMA 15. Let $\mathcal{P}$ be the closed regular two-sided ideal

in $L(\delta)$ defined in Lemma 14. Then we have

$$L^{\circ}(\delta) \cap \mathfrak{P} = \{f \in L^{\circ}(\delta) ; U(f) = 0\}.$$

PROOF. Let $f \in L^{\circ}(\delta)$ be a function such that $U(f) = 0$. Then for every $g \in L(G)$,

$$\begin{aligned} f'*\phi_{\tau}(g) &= \phi_{\tau}(f*g) = d(\delta)\cdot\mathrm{trace}[U(\overline{\chi}_{\delta}*(f*g)^{\circ})] \\ &= d(\delta)\cdot\mathrm{trace}[U(\overline{\chi}_{\delta}*f*g^{\circ})] \\ &= d(\delta)\cdot\mathrm{trace}[U(f)U(\overline{\chi}_{\delta}*g^{\circ})] = 0. \end{aligned}$$

Therefore f is a function in $L^{\circ}(\delta) \cap \mathfrak{P}$.

Conversely let f be a function in $L^{\circ}(\delta) \cap \mathfrak{P}$. Then we have

$$f'*\phi_{\tau}(g) = d(\delta)\cdot\mathrm{trace}[U(f)U(\overline{\chi}_{\delta}*g^{\circ})] = 0$$

for all $g \in L(G)$. Therefore the equality

$$\mathrm{trace}[U(f)U(g)] = 0$$

is valid for all $g \in L^{\circ}(\delta)$. Since $g \longrightarrow U(g)$ is an irreducible representation of $L^{\circ}(\delta)$, we obtain $U(f) = 0$. q.e.d.

A. Proof of (i) in Theorem 8

Let's denote by $\mathfrak{a}$ the maximal left ideal in $L(\delta)$ containing $\mathfrak{P}$. $\mathfrak{a}$ is regular and closed by Lemma 14. The right identity modulo $\mathfrak{a}$ and the identity modulo $\mathfrak{P}$ is the function $u \in L^{\circ}(\delta)$ such that $U(u) = 1$. If we put

$$\mathfrak{M} = \{f \in L(G) ; \overline{\chi}_{\delta}*g*f*\overline{\chi}_{\delta} \in \mathfrak{a} \quad \text{for all } g \in L(G)\},$$

$\mathfrak{M}$ is a closed regular maximal left ideal in $L(G)$ by Lemma 3, and the right identity modulo $\mathfrak{M}$ is also u. If we denote by

L_x, as before, the operator on $L(G)$ such that $(L_x f)(y) = f(x^{-1}y)$ for all $f \in L(G)$, $\mathfrak{m}$ is invariant under L_x since it is closed. Now, we shall denote by T_x the continuous linear operator induced from L_x on the locally convex topological vector space

$$\mathfrak{H} = L(G)/\mathfrak{m} .$$

Of course the topology in $\mathfrak{H}$ is the quotient topology. Unfortunately, the author does not know whether $\mathfrak{H}$ is complete or not. But we can easily show that the integrals

$$T_f = \int_G T_x f(x)dx \quad (f \in L(G)),$$

$$E(\delta) = \int_K T_k \overline{\chi}_\delta(k)dk$$

converge in $\mathfrak{H}$. We shall denote by $\{f\} \in \mathfrak{H}$ the class which contains $f \in L(G)$. Then we have

$$T_f\{g\} = \{f*g\}, \quad E(\delta)\{f\} = \{\overline{\chi}_\delta * f\}.$$

Since the left ideal $\mathfrak{m}$ is maximal, the representation $\{\mathfrak{H}, T_f\}$ of $L(G)$ is algebraically irreducible.

By Lemma 3, we know that $\{f*\overline{\chi}_\delta\} = \{f\}$ for all $f \in L(G)$. Thus $E(\delta)\{f\} = \{\overline{\chi}_\delta * f * \overline{\chi}_\delta\}$, and therefore we obtain

$$\mathfrak{H}(\delta) = E(\delta)\mathfrak{H} = \{\{f\} ; f \in L(\delta)\}.$$

Moreover we have $\mathfrak{a} = \mathfrak{m} \cap L(\delta)$ by Lemma 3, hence

$$\{\{f\} ; f \in L(\delta)\} \cong L(\delta)/\mathfrak{a} .$$

Since $\mathfrak{p} \subset \mathfrak{a}$ and $\dim(L(\delta)/\mathfrak{p}) < +\infty$, it follows that $\dim \mathfrak{H}(\delta) < +\infty$. Therefore the finite-dimensional irreducible representation $\{\mathfrak{H}(\delta), \tilde{T}_f\}$ of $L(\delta)$ is equivalent to $\{L(\delta)/\mathfrak{a}, L_f = f*\}$.

If $\dim \mathfrak{H}(\delta) = q \cdot d(\delta)$, we obtain a generalized spherical function $\phi(x) = \mathrm{trace}[E(\delta)T_x]$ of type δ of height q. Then there exists a q-dimensional irreducible representation $f \longrightarrow V(f)$ of $L^\circ(\delta)$ such that

$$\tilde{T}_f = V(f) \otimes I_{d(\delta)},$$

and that

$$\phi(f) = \int_G \phi(x) f(x) dx = d(\delta) \cdot \mathrm{trace}[V(f)]$$

for all $f \in L^\circ(\delta)$. By the way, it follows from Lemma 15 that $U(f) = 0$ implies $\tilde{T}_f = 0$ if f is in $L^\circ(\delta)$. Therefore $U(f) = 0$ implies $V(f) = 0$. Thus the mapping $\psi : U(f) \longrightarrow V(f)$ is well-defined and ψ is a homomorphism from the algebra $M(p,\mathbb{C})$ onto $M(q,\mathbb{C})$. Then we have $p = q$ and ψ must be the identity [8]. Consequently $f \longrightarrow U(f)$ is equivalent to $f \longrightarrow V(f)$, and we obtain the equality

$$\phi(f) = d(\delta) \cdot \mathrm{trace}[V(f)] = d(\delta) \cdot \mathrm{trace}[U(f)] = \phi_\tau(f)$$

for all $f \in L^\circ(\delta)$. Thus we have proved that $\phi_\tau \in \Phi_g(\delta)$ and its height is p.

The injectiveness of the mapping $\tau \longrightarrow \phi_\tau$ is obvious. Thus the assertion (i) in Theorem 8 is completely proved.

B. Proof of (ii) in Theorem 8

If $\phi_\tau \in \Phi_g(\delta)$ is positive definite,

$$0 \leqq \phi_\tau(e), \quad |\phi_\tau(x)| \leqq \phi_\tau(e), \quad \phi_\tau(x^{-1}) = \overline{\phi_\tau(x)}.$$

Put $f^*(x) = \overline{f(x^{-1})}$. Then we have clearly $(f^*)^* = f$, $(f*g)^* =$

g^**f^*, $(\overline{X}_\delta * f)^* = f^* * \overline{X}_\delta$, and $\phi_\tau(f^*) = \overline{\phi_\tau(f)}$. We define an "inner product" in L(G) by

$$(f,g) = \phi_\tau(g^**f) = \int_G \phi_\tau(x) g^**f(x)dx,$$

and put $\|f\| = \sqrt{(f,f)}$. Of course $\|f\| = 0$ does not mean $f = 0$ in general.

LEMMA 16. The above inner product is invariant under the operations L_x. And we obtain the inequalities

$$\|f*g\| \leqq \|g\| \cdot \|f\|_{L^1}, \quad \|\overline{X}_\delta * f\| \leqq \sqrt{d(\delta)}\, \|f\|$$

for every $f, g \in L(G)$.

PROOF. We can easily show that the function

$$\psi(z) = \int_{G\times G} \phi_\tau(xzy) g^*(x) g(y)dxdy$$

is positive definite for every $g \in L(G)$. Therefore the inequality $|\psi(z)| \leqq \psi(e)$ implies

$$\left| \int_{G\times G} \phi_\tau(xzy) g^*(x) g(y)dxdy \right| \leqq \psi(e) = \|g\|^2.$$

Hence we obtain

$$\begin{aligned} \|f*g\|^2 &= \int_{G\times G\times G} \phi_\tau(xzy) g^*(x) f^**f(z) g(y)dxdydz \\ &\leqq \int_G |f^**f(z)| \left| \int_{G\times G} \phi_\tau(xzy) g^*(x) g(y)dxdy \right| dz \\ &\leqq \|g\|^2 \cdot \|f^**f\|_{L^1} \leqq \|g\|^2 \cdot \|f\|_{L^1}^2. \end{aligned}$$

Thus the first inequality has been proved. The other properties are easily shown. q.e.d.

Put $N = \{ f \in L(G) ; \|f\| = 0\}$, then N is invariant under the operations L_x, $f*$, $\overline{X}_\delta *$, and $*\overline{X}_\delta$. Thus we can consider L_x, $f*$, $\overline{X}_\delta *$, $*\overline{X}_\delta$, $(,)$, and $\|\cdot\|$ on the quotient space

$$H'(G) = {}^{L(G)}\!/_{N} .$$

Let $H(G)$ be the completion of $H'(G)$ with respect to the norm $\|\cdot\|$, and extend the operations L_x, $f*$, $\overline{X}_\delta *$, and $*\overline{X}_\delta$ on $H(G)$ by continuity. The linear operators L_x are unitary on the Hilbert space $H(G)$. Since $L(\delta) \cap N = \overline{X}_\delta * N * \overline{X}_\delta$, the quotient space

$$H'(\delta) = {}^{L(\delta)}\!/_{L(\delta) \cap N}$$

is identified with $\overline{X}_\delta * H'(G) * \overline{X}_\delta$, and the completion $H(\delta)$ of $H'(\delta)$ is equal to $\overline{X}_\delta * H(G) * \overline{X}_\delta$.

LEMMA 17. Let $\mathfrak{P}$ be the two-sided ideal in $L(\delta)$ defined in Lemma 14. Then we have $\mathfrak{P} = L(\delta) \cap N$.

PROOF. Let u be a function in $L^0(\delta)$ such that $U(u) = 1$. If a function f is in $L(\delta) \cap N$,

$$|f' * \phi_\tau(x)| = |\phi_\tau * f'(x)| = |\phi_\tau(L_x f)| = |\phi_\tau(u * L_x f)|$$
$$= |(L_x f, u^*)| \leq \|L_x f\| \, \|u^*\| = \|f\| \, \|u^*\| = 0.$$

Therefore f belongs to $\mathfrak{P}$. Conversely for every $f \in \mathfrak{P}$,

$$\|f\|^2 = \phi_\tau(f^* * f) = \phi_\tau * f' * (f^*)'(e) = f' * \phi_\tau * (f^*)'(e) = 0.$$

q.e.d.

By Lemma 17, the space $H'(\delta)$ is equal to ${}^{L(\delta)}\!/_{\mathfrak{P}}$ and is

finite-dimensional by Lemma 14, hence $H'(\delta)$ coincides with $H(\delta)$. Let $\mathfrak{V}_0$ be a maximal $L(\delta)$-invariant proper subspace of $H(\delta)$. Of course $\mathfrak{V}_0$ may be equal to $\{0\}$. If we use the same notation to represent a function in $L(G)$ and the class in $H'(G)$ which contains it, u does not belong to $\mathfrak{V}_0$.

LEMMA 18. We define a subspace $\mathfrak{N}_0$ of $H(G)$ by

$$\mathfrak{N}_0 = \{f \in H(G) ; \overline{\chi}_\delta * g * f * \overline{\chi}_\delta \in \mathfrak{V}_0 \text{ for all } g \in L(G)\}.$$

Then we have

(i) $u \notin \mathfrak{N}_0$, $\mathfrak{V}_0 = \mathfrak{N}_0 \cap H(\delta)$, and $f*u - f \in \mathfrak{N}_0$ for all $f \in L(G)$,

(ii) $\mathfrak{N}_0$ is a closed subspace of $H(G)$ and it is maximal in the set of all closed $L(G)$-invariant subspaces of $H(G)$.

PROOF. (i) If $u \in \mathfrak{N}_0$, we have

$$\overline{\chi}_\delta * g * \overline{\chi}_\delta * u = \overline{\chi}_\delta * g * u * \overline{\chi}_\delta \in \mathfrak{V}_0$$

for every $g \in L(G)$. This means $L(\delta)*u \subset \mathfrak{V}_0$. Then $L(\delta)/\mathfrak{P} = H(\delta) \subset \mathfrak{V}_0$ since u is a right and left identity modulo $\mathfrak{P}$. This contradicts the definition of $\mathfrak{V}_0$, hence $u \notin \mathfrak{N}_0$. Next, let f be a function in $L(G)$, then

$$\overline{\chi}_\delta * g * (f*u - f) * \overline{\chi}_\delta = (\overline{\chi}_\delta * g * f * \overline{\chi}_\delta) * u - \overline{\chi}_\delta * g * f * \overline{\chi}_\delta \in \mathfrak{P}$$

for every $g \in L(G)$. Therefore

$$\overline{\chi}_\delta * g * (f*u - f) * \overline{\chi}_\delta = 0 \in \mathfrak{V}_0,$$

i.e. $f*u - f \in \mathfrak{N}_0$. The equality $\mathfrak{V}_0 = \mathfrak{N}_0 \cap H(\delta)$ is a consequence of the maximality of $\mathfrak{V}_0$.

(ii) Let $\mathfrak{N}$ be a closed L(G)-invariant proper subspace of H(G) containing $\mathfrak{N}_0$. Since $H'(G) \not\subset \mathfrak{N}$, u must not be contained in $\mathfrak{N}$. Therefore $\mathfrak{v}_0 = \mathfrak{N} \cap H(\delta)$ by the maximality of $\mathfrak{v}_0$. For an arbitrary $f \in \mathfrak{N}$ and $g \in L(G)$,

$$\bar{\chi}_\delta * g * f * \bar{\chi}_\delta - \bar{\chi}_\delta * g * f = (\bar{\chi}_\delta * g) * (f * \bar{\chi}_\delta - f) \in \mathfrak{N}_0 \subset \mathfrak{N},$$

since $f * \chi_\delta - f \in \mathfrak{N}_0$. As $\bar{\chi}_\delta * g * f \in \mathfrak{N}$, we have $\bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{N} \cap H(\delta) = \mathfrak{v}_0$. Hence $f \in \mathfrak{N}_0$. This proves that $\mathfrak{N} = \mathfrak{N}_0$.

q.e.d.

As in §2, we associate a non negative continuous function e_U with every open subset U of G, which satisfies $\|e_U\|_{L^1} = 1$ and $\mathrm{supp}[e_U] \subset U$.

LEMMA 19. When an open neighborhood U of e converges to e, $e_{xU} * f$ converges to $L_x f$ in H(G) for every $f \in H(G)$. Hence $\mathfrak{N}_0$ is invariant under the operators L_x.

PROOF. If f is in L(G),

$$\begin{aligned} \|e_{xU} * f - L_x f\|^2 &= \phi_\tau(f^* * e_U^* * e_U^* * f) - \phi_\tau(f^* * e_U * f) \\ &\quad - \phi_\tau(f^* * e_U * f) + \phi_\tau(f^* * f). \end{aligned}$$

The right hand side converges to zero if $U \to e$. Therefore for every $f \in H'(G)$, $\|e_{xU} * f - L_x f\| \to 0$ when $U \to e$. For any f in H(G), we can take an element $g \in H'(G)$ such that $\|f - g\| < \varepsilon/3$, where $\varepsilon > 0$ is an arbitrarily given number, and choose a neighborhood U of e for which we have $\|e_U * g - g\| < \varepsilon/3$. Then we obtain

$$\begin{aligned} \|e_{xU}*f - L_x f\| &= \|e_U*f - f\| \\ &= \|e_U*f - e_U*g\| + \|e_U*g - g\| + \|g - f\| \\ &\leqq 2\,\|f - g\| + \|e_U*g - g\| < \varepsilon. \qquad \text{q.e.d.} \end{aligned}$$

To see that $\{H(G), L_x\}$ is a representation of G, we have only to show that $x \longrightarrow L_x f$ is continuous for every $f \in H(G)$, since H(G) is a Hilbert space. For an arbitrary $\varepsilon > 0$, we choose an element $g \in H'(G)$ such that $\|f - g\| < \varepsilon/3$. And we can find an open neighborhood U of e such that $\|g - L_x g\| < \varepsilon/3$ for $x \in U$. Therefore we have

$$\begin{aligned} \|f - L_x f\| &\leqq \|f - g\| + \|g - L_x g\| + \|L_x g - L_x f\| \\ &= 2\|f - g\| + \|g - L_x g\| < \varepsilon \end{aligned}$$

for all $x \in U$. Thus we have shown that $\{H(G), L_x\}$ is a representation of G.

The representation $\{H(G), L_x\}$ is not irreducible in general, so we consider the Hilbert space

$$\mathfrak{H} = \mathfrak{N}_0^{\perp}$$

which denotes the orthogonal complement of $\mathfrak{N}_0$ in H(G). Then, denoting by T_x the restriction of L_x on $\mathfrak{H}$, $\{\mathfrak{H}, T_x\}$ is an irreducible unitary representation of G (the irreducibility follows from Lemma 18, (ii)). As before, we can easily see that $E(\delta)f = \overline{\chi}_\delta * f * \overline{\chi}_\delta$ for $f \in \mathfrak{H}$, and that

$$\mathfrak{H}(\delta) = H(\delta)/\mathfrak{v}_0 .$$

Therefore we obtain $\dim \mathfrak{H}(\delta) < +\infty$. By the same way as in A, we know that ϕ_τ is just the spherical function of type δ

defined by $\{\mathfrak{Q}, T_x\}$.

C. Proof of (iii) in Theorem 8

Let G be σ-compact. We can find denombrable compact subsets F_n such that $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$ and that $G = \bigcup_{n=1}^{\infty} F_n$. We may assume that every compact subset of G is contained in some F_n. (For, if necassary, we consider the compact subsets $F_n \cdot \overline{U}$ instead of F_n where U is an open neighborhood of the unit with compact closure $\overline{U}$. Then $\{F_n \cdot U\,;\, n = 1, 2, \cdots\}$ is an open covering of G.) As was proved in A, ϕ_τ is a function on G. Then

$$\|f\|_n = \sup_{x \in F_n} \int_G |\phi_\tau(xy) f(y)|\,dy \qquad (n = 1, 2, \cdots)$$

are semi-norms on L(G), and $\|f\|_n = 0$ for all n if and only if $f = 0$. We shall denote by $\mathcal{L}(G)$ the Fréchet space which is the completion of L(G) by these semi-norms.

LEMMA 20. The linear operators L_x, $f*$, $\overline{X}_\delta *$, $*\overline{X}_\delta$ defined on L(G) are continuous with respect to the topology in $\mathcal{L}(G)$. Moreover $\{L_x\,;\, x \in F\}$ is equi-continuous for every compact subset F of G.

PROOF. By a simple calculation, we have

$$\|L_x f\|_n \leqq \|f\|_m \qquad (x \in F),$$

where m is an arbitrary integer such that $F_n F \subset F_m$. For $f*$,

we have the inequality

$$\|f*g\|_n \leqq \|f\|_{L^1} \|g\|_m \quad \text{for } g \in L(G),$$

where m is an arbitrary integer such that $F_n \cdot \mathrm{supp}[f] \subset F_m$. The same is true for $\overline{X}_\delta *$ and $*\overline{X}_\delta$. q.e.d.

By this lemma, we can extend the operators L_x, $f*$, $\overline{X}_\delta *$, and $*\overline{X}_\delta$ on the whole of $\mathcal{L}(G)$ by continuity. Let's denote them by the same notations respectively. If we denote by $\mathcal{L}(\delta)$ the completion of $L(\delta)$ in $\mathcal{L}(G)$, we have $\overline{X}_\delta * \mathcal{L}(G) * \overline{X}_\delta = \mathcal{L}(\delta)$.

LEMMA 21. The linear functional $\phi_\tau(f)$ on $L(G)$ is continuous with respect to the topology in $\mathcal{L}(G)$.

PROOF. The linear functional

$$\phi_\tau(f) = d(\delta) \cdot \mathrm{trace}[U(\overline{X}_\delta * f^\circ)]$$

on $L(G)$ is represented in an integral form

$$\phi_\tau(f) = \int_G \phi_\tau(x) f(x) dx.$$

Therefore this lemma is clear. q.e.d.

LEMMA 22. Let $\hat{\mathfrak{P}}$ be the completion of $\mathfrak{P}$ in $\mathcal{L}(G)$, where $\mathfrak{P}$ is the two-sided ideal in $L(\delta)$ defined in Lemma 14. Then we have

(i) $\hat{\mathfrak{P}} \subsetneqq \mathcal{L}(\delta)$ and $\hat{\mathfrak{P}}$ is $L(\delta)$-invariant,

(ii) $u \notin \hat{\mathfrak{P}}$, where u is a function in $L^\circ(\delta)$ such that $U(u) = 1$.

PROOF. (i) For all $f \in \mathfrak{P}$, we have $\phi_\tau(f) = f' * \phi_\tau(e) = 0$. Thus $\phi_\tau \neq 0$ implies $\hat{\mathfrak{P}} \subsetneqq \mathcal{L}(\delta)$. The inclusion $L(\delta) * \hat{\mathfrak{P}} \subset \hat{\mathfrak{P}}$ is clear.

(ii) Since $f*u - f \in \mathfrak{P} \subset \hat{\mathfrak{P}}$ for every $f \in L(\delta)$, we have $f \in \hat{\mathfrak{P}}$ if $u \in \hat{\mathfrak{P}}$. This means $\hat{\mathfrak{P}} = \mathcal{L}(\delta)$, and hence contradicts (i). q.e.d.

As is easily seen, $\hat{\mathfrak{P}} \cap L(\delta) = \mathfrak{P}$. Therefore it can be considered that $L(\delta)/\mathfrak{P}$ is densely contained in $\mathcal{L}(\delta)/\hat{\mathfrak{P}}$. Thus, by Lemma 14, we have

$$\dim\left(\mathcal{L}(\delta)/\hat{\mathfrak{P}}\right) < +\infty .$$

Therefore we can find a closed maximal $L(\delta)$-invariant subspace $\mathfrak{N}_0$ of $\mathcal{L}(\delta)$ containing $\hat{\mathfrak{P}}$. $\mathfrak{N}_0$ does not contain u.

LEMMA 23. We define a subspace $\mathfrak{M}_0$ of $\mathcal{L}(G)$ by

$$\mathfrak{M}_0 = \{f \in \mathcal{L}(G) ; \overline{\chi}_\delta * g * f * \overline{\chi}_\delta \in \mathfrak{N}_0 \text{ for all } g \in L(G)\}.$$

Then we have

(i) $u \notin \mathfrak{M}_0$, $\mathfrak{N}_0 = \mathfrak{M}_0 \cap \mathcal{L}(\delta)$, and $f*u - f \in \mathfrak{M}_0$ for all $f \in L(G)$,

(ii) $\mathfrak{M}_0$ is a closed subspace of $\mathcal{L}(G)$ and it is maximal in the set of all closed $L(G)$-invariant subspaces of $\mathcal{L}(G)$.

The proof of this lemma is formally the same as that of Lemma 18.

LEMMA 24. When an open neighborhood U of e converges to e, $e_{xU}*f$ converges to $L_x f$ in $\mathcal{L}(G)$ for every $f \in \mathcal{L}(G)$. Hence $\mathfrak{m}_o$ is invariant under the operators L_x.

PROOF. We take $f \in \mathcal{L}(G)$ arbitrarily and fix an element $x \in G$. We may assume that xU is always contained in a fixed compact subset F of G. There exists a sequence $f_i \in L(G)$ such that $f_i \longrightarrow f$ in $\mathcal{L}(G)$. Then,

$$\begin{aligned} \|e_{xU}*f - L_x f\|_n &\leq \|e_{xU}*(f - f_i)\|_n + \|e_{xU}*f_i - L_x f_i\|_n \\ &\quad + \|L_x f_i - L_x f\|_n \\ &\leq \|f - f_i\|_m + \|e_{xU}*f_i - L_x f_i\|_n + \|f_i - f\|_m \\ &= 2\|f - f_i\|_m + \|e_{xU}*f_i - L_x f_i\|_n , \end{aligned}$$

where m is a positive integer such that $F_n F \subset F_m$. For an arbitrary real number $\varepsilon > 0$, we fix i for which we have $\|f - f_i\|_m < \varepsilon/3$. Since f_i is a function in L(G), $e_{xU}*f_i$ converges uniformly to $L_x f_i$ on G when $U \longrightarrow e$. Therefore if we take U small enough, the inequality $\|e_{xU}*f_i - L_x f_i\|_n < \varepsilon/3$ is valid. Thus we obtain

$$\|e_{xU}*f - L_x f\|_n < \varepsilon$$

if U is small enough. q.e.d.

To see $\{\mathcal{L}(G), L_x\}$ is a representation of G, it rest only to show that $x \longrightarrow L_x f$ is continuous for every $f \in \mathcal{L}(G)$, since the space $\mathcal{L}(G)$ is a Fréchet space. Take a sequence

f_i in $L(G)$ which converges to f in $\mathcal{L}(G)$. We have

$$\|L_x f - f\|_n \leq \|L_x(f - f_i)\|_n + \|L_x f_i - f_i\|_n + \|f_i - f\|_n ,$$

here we may assume that x belongs to a fixed compact neighborhood F of e in G. If m is an integer such that $F \subset F_m$, we clearly obtain

$$\|L_x f - f\|_n \leq \|f - f_i\|_m + \|f_i - f\|_n + \|L_x f_i - f_i\|_n .$$

Take i large enough at first, and let $x \longrightarrow e$, then we see $\|L_x f - f\|_n \longrightarrow 0$. Therefore $\{\mathcal{L}(G), L_x\}$ is a representation of G.

The representation $\{\mathcal{L}(G), L_x\}$ of G is not irreducible in general. Now we put

$$\mathfrak{H} = \mathcal{L}(G)/\mathfrak{m}_o$$

and denote by T_x the operators on $\mathfrak{H}$ which are naturally induced from L_x. Then $\mathfrak{H}$ is a Fréchet space [2], and the representation $\{\mathfrak{H}, T_x\}$ of G is topologically irreducible by Lemma 23, (ii). If we denote by $\{f\}$ the class of $f \in \mathcal{L}(G)$ in $\mathfrak{H}$, we obtain

$$\mathfrak{H}(\delta) = \{\{f\} ; f \in \mathcal{L}(\delta)\} \cong \mathcal{L}(\delta)/\mathfrak{o}_o$$

since $\mathfrak{o}_o = \mathfrak{m}_o \cap \mathcal{L}(\delta)$. The condition $\dim \mathfrak{H}(\delta) < +\infty$ is of course satisfied. As before, we can show that ϕ_τ is just the spherical function of type δ of height p defined by $\{\mathfrak{H}, T_x\}$. This completes the proof of (iii) in Theorem 8.

§5. Correspondence between representations and spherical functions

It is well-known that the given two irreducible unitary representations are unitary equivalent if and only if the corresponding spherical functions of the same type coincide with each other [5]. But in general case, such a rigid correspondence does not exist.

Let G be a locally compact unimodular group, and K a compact subgroup of G. Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible representation of G, and ϕ a spherical function of type $\delta \in \Omega_K$ defined by $\{\mathfrak{H}, T_x\}$. By Lemma 14,

$$\mathfrak{P} = \{f \in L(\delta) ; f' * \phi = 0\}$$

is a closed regular maximal two-sided ideal in $L(\delta)$. In fact, we obtain the following

LEMMA 25. $\mathfrak{P}$ equals to $\{f \in L(\delta) ; \tilde{T}_f = 0\}$, where $\tilde{T}_f$ are the restrictions of T_f on $\mathfrak{H}(\delta)$.

This is essentially proved by R.Godement in [5].

We took a maximal left ideal $\mathfrak{M}$ in $L(\delta)$ containing $\mathfrak{P}$ in the preceding section. In this case, this maximal left ideal $\mathfrak{M}$ is characterized as follows.

LEMMA 26. For every non zero vector $v \in \mathfrak{H}(\delta)$, $\mathfrak{M} = \{f \in L(\delta) ; \tilde{T}_f v = 0\}$ is a maximal left ideal in $L(\delta)$ containing $\mathfrak{P}$.

Conversely, for every maximal left ideal $\mathfrak{M}$ in $L(\delta)$ containing $\mathfrak{P}$, there exists a unique non zero vector $v \in \mathfrak{H}(\delta)$ up to scalar multiples, such that $\mathfrak{M} = \{f \in L(\delta) ; \tilde{T}_f v = 0\}$.

PROOF. The first half of this lemma is clear. Let's prove the latter half. Suppose, for every non zero vector $v \in \mathfrak{H}(\delta)$, we can find an element $f \in \mathfrak{M}$ such that $\tilde{T}_f v \neq 0$, then the correspondence $f \to T_f$ is an irreducible representation of the algebra $\mathfrak{M}$ on $\mathfrak{H}(\delta)$. Then, by the Burnside's theorem, there exists an element $f_0 \in \mathfrak{M}$ such that $\tilde{T}_{f_0} = 1$. This implies $f*f_0 - f \in \mathfrak{P}$ for all $f \in L(\delta)$ and hence we have $L(\delta) \subset \mathfrak{M}$, but this is impossible. Thus there exists some non zero vector $v \in \mathfrak{H}(\delta)$ such that $\{f \in L(\delta) ; \tilde{T}_f v = 0\} \supset \mathfrak{M}$. This means $\{f \in L(\delta) ; \tilde{T}_f v = 0\} = \mathfrak{M}$ by the maximality of $\mathfrak{M}$. There exists a function $f \in L(\delta)$ such that $\tilde{T}_f v = 0$ and $\tilde{T}_f w \neq 0$ provided that v and w are linearly independent. Therefore the uniqueness of such v is proved. q.e.d.

Let $\mathfrak{M} = \{f \in L(\delta) ; \tilde{T}_f v = 0\}$ be a maximal left ideal in $L(\delta)$ containing $\mathfrak{P}$, then we have

$$\mathfrak{m} = \{f \in L(G) ; \bar{\chi}_\delta * g * f * \bar{\chi}_\delta \in \mathfrak{M} \quad \text{for all } g \in L(G)\}$$
$$= \{f \in L(G) ; T_f v = 0\}.$$

Therefore the linear mapping $\varphi: T_f v \to \{f\}$ of $\mathfrak{H}_0 = \mathfrak{H}_0[K, \delta, v]$ onto $L(G)/\mathfrak{m}$ is bijective, and clearly we obtain

$$\varphi(T_x T_f v) = \{L_x f\}, \quad \varphi(E(\delta) T_f v) = \{\bar{\chi}_\delta * f\} = \{\bar{\chi}_\delta * f * \bar{\chi}_\delta\}.$$

For a topologically irreducible nice representation $\{\mathfrak{H}, T_x\}$ of G, we can consider spherical functions of various types. Namely we can choose not only a class $\delta \in \Omega_K$ arbitrarily, but also a compact subgroup K of G. We shall denote by $\Phi(\{\mathfrak{H}, T_x\})$ the set of all spherical functions defined by $\{\mathfrak{H}, T_x\}$.

THEOREM 9. Let $\{\mathfrak{H}^1, T_x^1\}$ and $\{\mathfrak{H}^2, T_x^2\}$ be two topologically irreducible nice representations of G. Then the following three statements are equivalent.

(i) $\Phi(\{\mathfrak{H}^1, T_x^1\}) \cap \Phi(\{\mathfrak{H}^2, T_x^2\}) \neq \phi$.

(ii) $\Phi(\{\mathfrak{H}^1, T_x^1\}) = \Phi(\{\mathfrak{H}^2, T_x^2\})$.

(iii) There exists a linear bijective mapping $\varphi: \mathfrak{H}_0^1 \to \mathfrak{H}_0^2$ such that $\varphi T_x^1 = T_x^2 \varphi$ for all $x \in G$ and that $\varphi E^1(\delta) = E^2(\delta)\varphi$ for every $\delta \in \Omega_K$ where K is an arbitrary compact subgroup of G.

PROOF. At first we assume (i). Then there exists a compact subgroup K of G and $\delta \in \Omega_K$ such that the spherical function ϕ of type δ defined by $\{\mathfrak{H}^1, T_x^1\}$ is identically equal to that of the same type defined by $\{\mathfrak{H}^2, T_x^2\}$. We take a maximal left ideal $\mathfrak{m}$ of $L(\delta)$ containing $\mathfrak{P} = \{f \in L(\delta) ; f' * \phi = 0\}$. By Lemma 26, we can find non zero vectors $v_1 \in \mathfrak{H}^1(\delta)$ and $v_2 \in \mathfrak{H}^2(\delta)$ for which we have

$$\mathfrak{m} = \{f \in L(\delta) ; \tilde{T}_f^1 v_1 = 0\} = \{f \in L(\delta) ; \tilde{T}_f^2 v_2 = 0\}.$$

Now we put

$$\varphi(T_f^1 v_1) = T_f^2 v_2 ,$$

then φ is a linear mapping of $\mathfrak{H}_o^1$ to $\mathfrak{H}_o^2$ and satisfies the conditions in (iii).

Next we assume (iii). Let K be a compact subgroup of G and δ a class in Ω_K, then $\mathfrak{H}^1(\delta) \subset \mathfrak{H}_o^1$ if $\dim \mathfrak{H}^1(\delta) < +\infty$. Therefore (ii) is clear. q.e.d.

Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible nice representation of G, and ϕ a spherical function of type $\delta \in \Omega_K$ defined by $\{\mathfrak{H}, T_x\}$ where K is a compact subgroup of G. For this function ϕ, we can define a closed regular maximal left ideal $\mathfrak{M}$ in L(G) as we did in the proof of (i) in Theorem 8. Then ϕ is just the generalized spherical function of type δ defined by $\{L(G)/\mathfrak{M}, L_x\}$.

On the other hand, ϕ can be seen as the generalized spherical function of type δ defined by $\{\mathfrak{H}_o, T_x\}$. In this point of view, the topology in $\mathfrak{H}_o$ need not be the relative topology from $\mathfrak{H}$. For instance, we may identify $\mathfrak{H}_o$ with $L(G)/\mathfrak{M}$.

In the case of σ-compact G, we denoted by $\hat{\mathfrak{P}}$ the completion of $\mathfrak{P}$ in $\mathcal{L}(\delta)$, and took a maximal $L(\delta)$-invariant subspace $\mathfrak{N}_o$ of $\mathcal{L}(\delta)$ containing $\hat{\mathfrak{P}}$. If we put $\mathfrak{N} = \mathfrak{N}_o \cap L(\delta)$, $\mathfrak{N}$ is a maximal left ideal in $L(\delta)$ and $\mathfrak{N}_o$ is just the completion of $\mathfrak{N}$ in $\mathcal{L}(\delta)$. Then,

$$\mathfrak{m} = \{f \in L(G) \; ; \; \bar{X}_\delta * g * f * \bar{X}_\delta \in \mathfrak{o} \quad \text{for all } g \in L(G)\}$$

and

$$\mathfrak{m}_o = \{f \in \mathcal{L}(G) \; ; \; \bar{X}_\delta * g * f * \bar{X}_\delta \in \mathfrak{o}_o \quad \text{for all } g \in L(G)\}$$

are combined by the relation $\mathfrak{m}_o \cap L(G) = \mathfrak{m}$. Thus we know that $\mathcal{L}(G)/\mathfrak{m}_o$ is the completion of $L(G)/\mathfrak{m}$ by a "suitable" topology. In other words, we completed $\mathfrak{H}_o$ by a "suitable" topology. Similarly, in the case of positive-definite ϕ, we can see that $H(G)/\mathfrak{n}_o$ is the completion of $L(G)/\mathfrak{m}$ or $\mathfrak{H}_o$.

§6. Irreducible representations of the algebra $L^{\circ}(\delta)$ and $L(\delta)$

Let τ be an equivalence class of p-dimensional irreducible representations of $L^{\circ}(\delta)$, i.e., $\tau \in \mathcal{T}(\delta)$. We choose an irreducible representation $f \to U(f)$ of $L^{\circ}(\delta)$ belonging to τ, then the linear functional

$$\phi_{\tau}(f) = d(\delta) \cdot \mathrm{trace}[U(\overline{\chi}_{\delta} * f^{\circ})]$$

is actually a function on G. In the proof of (i) in Theorem 8, we took a maximal left ideal $\mathfrak{M}$ in $L(\delta)$ containing $\mathfrak{P} = \{f \in L(\delta) \ ; \ f' * \phi_{\tau} = 0\}$. Then the representation $L_f = f*$ of $L(\delta)$ on $L(\delta)/\mathfrak{M}$ is irreducible and $p \cdot d(\delta)$-dimensional. We clearly have

$$\mathfrak{P} = \{f \in L(\delta) \ ; \ L_f = 0 \text{ on } L(\delta)/\mathfrak{M}\}.$$

LEMMA 27. The equivalence class of irreducible representations of $L(\delta)$ which contains $\{L(\delta)/\mathfrak{M}, L_f\}$ is independent of the choice of a maximal left ideal $\mathfrak{M}$ containing $\mathfrak{P}$.

PROOF. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two maximal left ideals in $L(\delta)$ containing $\mathfrak{P}$. We shall denote by $\{f\}_i$ the class of f in $L(\delta)/\mathfrak{M}_i$ $(i = 1, 2)$. Suppose, for every non zero element $\{g\}_2 \in L(\delta)/\mathfrak{M}_2$, we can find an element $f \in \mathfrak{M}_1$ such that $\{f*g\}_2 \neq 0$. Then $\mathfrak{M}_1 * g + \mathfrak{M}_2$ is a left ideal containing $\mathfrak{M}_2$ as a proper subspace. Therefore we obtain $\mathfrak{M}_1 * g + \mathfrak{M}_2 = L(\delta)$,

i.e., $L_f = f*$ is an irreducible representation of $\mathfrak{A}_1$ on $L(\delta)/\mathfrak{A}_2$. Then, by the Burnside's theorem, there exists a function f_o in $\mathfrak{A}_1$ such that $\{f_o*g\}_2 = \{g\}_2$ for all $g \in L(\delta)$, but this implies $f*f_o - f \in \mathfrak{P}$ for all $f \in L(\delta)$. Therefore $L(\delta)$ must be contained in $\mathfrak{A}_1$. This is a contradiction. Thus there exists an element $\{g_o\}_2 \in L(\delta)/\mathfrak{A}_2$ $(g_o \notin \mathfrak{A}_2)$ such that $\{f*g_o\}_2 = \{0\}_2$ for all $f \in \mathfrak{A}_1$. This implies that $\mathfrak{A}_1*g_o \subset \mathfrak{A}_2$. Then the linear mapping $\eta : L(\delta)/\mathfrak{A}_1 \longrightarrow L(\delta)/\mathfrak{A}_2$, defined as

$$\eta(\{f\}_1) = \{f*g_o\}_2,$$

is well-defined. On the other hand, $\{f \in L(\delta) ; f*g_o \in \mathfrak{A}_2\}$ is a proper left ideal in $L(\delta)$ containing $\mathfrak{A}_1$, hence we obtain $\mathfrak{A}_1 = \{f \in L(\delta) ; f*g_o \in \mathfrak{A}_2\}$. Therefore the linear mapping η is injective, and $\eta(L(\delta)/\mathfrak{A}_1)$ is a L_f-invariant subspace of $L(\delta)/\mathfrak{A}_2$. Thus η must be surjective at the same time. This bijective linear mapping η gives an equivalence of the representations $\{L(\delta)/\mathfrak{A}_1, L_f\}$ and $\{L(\delta)/\mathfrak{A}_2, L_f\}$. q.e.d.

By this lemma, we can associate an equivalence class of irreducible representations of $L(\delta)$ with every class $\tau \in \mathcal{T}(\delta)$.

LEMMA 28. Let $\{\tau^\lambda ; \lambda \in \Lambda\}$ be a subset of $\mathcal{T}(\delta)$ consisting of sufficiently many classes. If we denote by σ^λ the associated equivalence class of irreducible representations

of $L(\delta)$ with τ^λ, then the family $\{\sigma^\lambda ; \lambda \in \Lambda\}$ also consists of sufficiently many equivalence classes.

PROOF. Denote by $\mathfrak{q}^\lambda$ the kernel of τ^λ. Then $\mathfrak{q}^\lambda$ are maximal two-sided ideals in $L^\circ(\delta)$ and $\bigcap_\lambda \mathfrak{q}^\lambda = \{0\}$ by the assumption. On the other hand, if we denote by $\mathfrak{P}^\lambda$ the kernel of σ^λ, $\mathfrak{P}^\lambda$ are maximal two-sided ideals in $L(\delta)$ and we have

$$\mathfrak{q}^\lambda = L^\circ(\delta) \cap \mathfrak{P}^\lambda$$

by Lemma 15. Since $\mathfrak{P}^\lambda = \{f \in L(\delta) ; f' * \phi_{\tau^\lambda} = 0\}$, it is clear that $f^\circ \in \mathfrak{P}^\lambda$ if $f \in \mathfrak{P}^\lambda$. Thus the two-sided ideal

$$\mathfrak{P} = \bigcap_\lambda \mathfrak{P}^\lambda$$

in $L(\delta)$ satisfies $\mathfrak{P}^\circ \subset \mathfrak{P}$. Moreover we obtain

$$\mathfrak{P}^\circ = (\bigcap_\lambda \mathfrak{P}^\lambda) \cap L^\circ(\delta) = \bigcap_\lambda (\mathfrak{P}^\lambda \cap L^\circ(\delta)) = \bigcap_\lambda \mathfrak{q}^\lambda = \{0\},$$

but such a two-sided ideal $\mathfrak{P}$ reduces to $\{0\}$ by Lemma 29. Therefore this lemma has been proved. q.e.d.

LEMMA 29. Let $\mathfrak{P}$ be a two-sided ideal in $L(\delta)$ such that $\mathfrak{P}^\circ \subset \mathfrak{P}$, then $L^\circ(\delta) \cap \mathfrak{P} = \mathfrak{P}^\circ \neq \{0\}$ if $\mathfrak{P} \neq \{0\}$.

PROOF. The first equality $L^\circ(\delta) \cap \mathfrak{P} = \mathfrak{P}^\circ$ is clear. We take a non zero function $f \in \mathfrak{P}$. For every $g \in L(G)$, we have

$$f*g(e) = \int_G f(x^{-1})g(x)dx = \int_G \overline{X}_\delta * f * \overline{X}_\delta(x^{-1})g(x)dx$$

$$= \int_G f(x^{-1})\overline{X}_\delta * g * \overline{X}_\delta(x)dx = f*(\overline{X}_\delta * g * \overline{X}_\delta)(e).$$

Therefore, if $f*h(e) = 0$ for all $h \in L(\delta)$, we obtain $f = 0$.

Since $f \neq 0$, we can find a function $h \in L(\delta)$ such that $f*h(e) \neq 0$ and $f*h$ is contained in $\mathfrak{P}$. Thus we have shown that there exists a function $f \in \mathfrak{P}$ such that $f(e) \neq 0$. Then we have $f^\circ(e) = f(e) \neq 0$, hence f° is a non zero function in $\mathfrak{P}^\circ$.

q.e.d.

THEOREM 10. If the algebra $L^\circ(\delta)$ has sufficiently many irreducible representations whose dimensions are $\leq p$, then δ is contained at most p times in every completely irreducible representation and in every topologically irreducible nice representation of G.

PROOF. By Lemma 28, $L(\delta)$ has sufficiently many irreducible representations whose dimensions are $\leq p \cdot d(\delta)$. Thus, by Lemma 5, every completely irreducible representation of $L(\delta)$ is at most $p \cdot d(\delta)$-dimensional. Now, let $\{\mathfrak{H}, T_x\}$ be a completely irreducible or a topologically irreducible nice representation of G, then $\{\mathfrak{H}(\delta), \tilde{T}_f\}$ is a completely irreducible representation of $L(\delta)$. Therefore $\dim \mathfrak{H}(\delta) \leq p \cdot d(\delta)$.

q.e.d.

§7. Spherical matrix functions

Let G be a locally compact unimodular group. Let $M(p,\mathbb{C})$ be the set of all p×p-complex matrices as in §4. A matrix function of degree p means a $M(p,\mathbb{C})$-valued function on G. A matrix function is called continuous if its matrix elements are continuous, and, in the case of Lie groups, it is called analytic if its matrix elements are analytic. We shall say that a matrix function $U = U(x)$ is irreducible if $\{U(x)\ ;\ x \in G\}$ is an irreducible family of matrices. Two matrix functions $U = U(x)$ and $V = V(x)$ are called equivalent if they are of the same degree and if there exists a regular matrix R such that $U(x) = R^{-1}V(x)R$ for all $x \in G$.

Let K be a compact subgroup of G. For every class $\delta \in \Omega_K$, a continuous matrix function $U = U(x)$ is called a spherical matrix function of type δ, if it satisfies the following three conditions:

(1) U is irreducible,

(2) $\chi_\delta * U = U$,

(3) $\int_K U(kxk^{-1}y)dk = U(x)U(y)$ for all $x, y \in G$.

Let $\{\mathfrak{H}, T_x\}$ be a representation of G on a complete locally convex topological vector space $\mathfrak{H}$. If $\dim \mathfrak{H}(\delta) = p \cdot d(\delta)$, the restriction $\tilde{T}_k$ of T_k on $\mathfrak{H}(\delta)$ is a $p \cdot d(\delta)$-dimensional representation of K on $\mathfrak{H}(\delta)$. Moreover, we can take a base $v_1, \cdots, v_{pd(\delta)}$ in $\mathfrak{H}(\delta)$ such that $\tilde{T}_k$ is repre-

sented in the following form

$$\tilde{T}_k = \begin{pmatrix} D(k) & & 0 \\ & \ddots & \\ 0 & & D(k) \end{pmatrix} = I_p \otimes D(k)$$

with respect to this base, where $k \longrightarrow D(k)$ is an irreducible unitary representation of K belonging to δ. Then there exists a p-dimensional representation $f \longrightarrow U(f) = (u_{ij}(f))$ $(1 \leqq i,j \leqq p)$ of $L^\circ(\delta)$ such that the restriction $\tilde{T}_f$ of T_f on $\mathfrak{H}(\delta)$ is written in the tensor product of matrices $U(f)$ and $I_{d(\delta)}$:

$$\tilde{T}_f = U(f) \otimes I_{d(\delta)}.$$

Now let $v_1', \cdots, v_{pd(\delta)}'$ be continuous linear functionals on $\mathfrak{H}$ satisfying

$$(v_i , v_j') = \delta_{ij} \qquad (1 \leqq i,j \leqq pd(\delta)),$$

and put

$$t_{ij}(x) = (E(\delta)T_x v_j , v_i').$$

Then, for every function $f \in L^\circ(\delta)$, we obtain

$$\tilde{T}_f = \int_G (t_{ij}(x))f(x)dx.$$

Therefore, if we put

$$u_{ij}(x) = d(\delta)^{-1} \sum_{\mu=1}^{d(\delta)} t_{(i-1)d(\delta)+\mu,(j-1)d(\delta)+\mu}(x),$$

we have

$$u_{ij}(f) = \int_G u_{ij}(x)f(x)dx \qquad (1 \leqq i,j \leqq p).$$

This shows that the p-dimensional representation $f \longrightarrow U(f)$ of $L^\circ(\delta)$ is given by

$$U(f) = \int_G U(x)f(x)dx$$

where $U(x) = (u_{ij}(x))$. Clearly the matrix function $U = U(x)$ depends on the choice of a base $v_1, \cdots, v_{pd(\delta)}$, but it is uniquely determined up to equivalence. We shall call this matrix function a matrix function of type δ defined by $\{\mathfrak{H}, T_x\}$. If the representation $\{\mathfrak{H}, T_x\}$ is topologically irreducible, the matrix function $U = U(x)$ is obviously irreducible, and the function

$$\phi(x) = d(\delta)\cdot\text{trace}[U(x)]$$

is a spherical function of type δ of height p defined by $\{\mathfrak{H}, T_x\}$.

Even if the representation space $\mathfrak{H}$ is not complete, the same arguments and definitions can be made provided that the projection $E(\delta)$ and the operators T_f $(f \in L(G))$ can be defined on $\mathfrak{H}$. In this case, $\phi(x) = d(\delta)\cdot\text{trace}[U(x)]$ is, of course, a generalized spherical function of type δ of height p.

LEMMA 30. Let $\{\mathfrak{H}, T_x\}$ be a representation of G on a locally convex topological vector space $\mathfrak{H}$ such that the operators $E(\delta)$ and T_f $(f \in L(G))$ can be defined and that the representation $\{\mathfrak{H}, T_f\}$ of L(G) is topologically irreducible. If $\dim \mathfrak{H}(\delta) < +\infty$, a matrix function of type δ defined by $\{\mathfrak{H}, T_x\}$ is a spherical matrix function of type δ.

PROOF. Let $U = U(x)$ be a matrix function of type δ defined by $\{\mathfrak{H}, T_x\}$. Then we have

$$\chi_\delta * u_{ij}(x) = \int_K u_{ij}(k^{-1}x)\chi_\delta(k)dk$$

$$= d(\delta)^{-1}\sum_{\mu=1}^{d(\delta)}\int_K t_{(i-1)d(\delta)+\mu,(j-1)d(\delta)+\mu}(k^{-1}x)\chi_\delta(k)dk$$

$$= d(\delta)^{-1}\sum_{\mu=1}^{d(\delta)}\int_K (E(\delta)T_k T_x v_{(j-1)d(\delta)+\mu}, v'_{(i-1)d(\delta)+\mu})\overline{\chi}_\delta(k)dk$$

$$= d(\delta)^{-1}\sum_{\mu=1}^{d(\delta)} (E(\delta)^2 T_x v_{(j-1)d(\delta)+\mu}, v'_{(i-1)d(\delta)+\mu})$$

$$= u_{ij}(x).$$

This shows that the condition (2) in the definition of a spherical matrix function of type δ is satisfied by U.

Next, for $f, g \in L(G)$, we obtain

$$U(f^\circ * g) = U((f^\circ * g)^\circ) = U(\overline{\chi}_\delta * f^\circ * g^\circ * \overline{\chi}_\delta) = U(\overline{\chi}_\delta * f^\circ)U(\overline{\chi}_\delta * g^\circ)$$

$$= U(f)U(g).$$

From this the condition (3) easily follows. q.e.d.

THEOREM 11. Let $U = U(x)$ be a spherical matrix function of type δ of degree p. Then

$$f \longrightarrow U(f) = \int_G U(x)f(x)dx$$

is a p-dimensional irreducible representation of $L^\circ(\delta)$. Conversely every p-dimensional irreducible representation of $L^\circ(\delta)$ is given in this way from a spherical matrix function of type δ of degree p.

PROOF. Let f, g be two functions in $L^\circ(\delta)$. Then

$$U(f*g) = \int_G U(x)f*g(x)dx = \int_{G\times G} U(x)f(xy^{-1})g(y)dxdy$$

$$= \int_{G\times G} U(xy)f(x)g(y)dxdy = \int_{G\times G\times K} U(kxk^{-1}y)f(x)g(y)dxdy$$

$$= U(f)U(g).$$

Therefore $f \to U(f)$ is a p-dimensional representation of $L^\circ(\delta)$.

Let's prove the irreducibility of the representation $f \to U(f)$. From the condition (3), we obtain

$$U(x) = U(e)U(x)$$

for every $x \in G$, especially $U(e)^2 = U(e)$. Hence there exists a regular matrix R such that

$$R^{-1}U(e)R = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where r is the rank of U(e) and I_r is the unit matrix of degree r. Therefore the last (p - r) rows of the matrix $R^{-1}U(x)R$ are zero for all $x \in G$. This contradicts the irreducibility of $U = U(x)$. Thus U(e) must be the unit matrix of degree p. Now, from the condition (3), we obtain $U = U^\circ$. Using this fact and the equality $\chi_\delta * U = U$, the irreducibility of the representation $f \to U(f)$ is easily proved.

Conversely, let $f \to U(f)$ be a p-dimensional irreducible representation of $L^\circ(\delta)$. Then there exists a generalized spherical function ϕ of type δ of height p such that

$$\phi(f) = \int_G \phi(x)f(x)dx = d(\delta)\cdot\mathrm{trace}[U(f)]$$

for all $f \in L^0(\delta)$ (Theorem 8). Now we take a representation $\{\mathfrak{H}, T_x\}$ of G which gives ϕ, then one of spherical matrix functions $U = U(x)$ of type δ defined by $\{\mathfrak{H}, T_x\}$ gives the representation $f \to U(f)$. q.e.d.

We shall denote by $\mathfrak{U}(\delta)$ the set of all equivalence classes of spherical matrix functions of type δ. Then, by Theorems 8 and 11, we have obtained one-to-one correspondences between the sets $\Phi_g(\delta)$, $\mathfrak{T}(\delta)$, and $\mathfrak{U}(\delta)$. Namely, for every spherical matrix function $U = U(x)$ of type δ of degree p, the function $d(\delta) \cdot \mathrm{trace}[U(x)]$ is in $\Phi_g(\delta)$ and $f \to U(f) = \int_G U(x)f(x)dx$ is a p-dimensional irreducible representation of $L^0(\delta)$. Therefore we may consider spherical matrix functions instead of (generalized) spherical functions. In this point of view, the equality (3) in the definition of spherical matrix functions is a generalization of the equation satisfied by spherical functions of height one.

Let $\{\mathfrak{H}, T_x\}$ be a representation of G on a complete locally convex topological vector space $\mathfrak{H}$, and $U = U(x)$ a matrix function of type δ defined by $\{\mathfrak{H}, T_x\}$. If $U = U(x)$ is not irreducible, there exists a regular matrix R such that

$$R^{-1}U(x)R = \begin{pmatrix} U^1(x) & & * \\ & \ddots & \\ 0 & & U^r(x) \end{pmatrix},$$

where $U^i(x)$ $(1 \leqq i \leqq r)$ are spherical matrix functions of type

δ. Then $f \to U^i(f)$ $(1 \leqq i \leqq r)$ are irreducible representations of $L^\circ(\delta)$. Now we have

$$(R \otimes I_{d(\delta)})^{-1} \tilde{T}_k (R \otimes I_{d(\delta)}) = \tilde{T}_k \quad (k \in K),$$

$$(R \otimes I_{d(\delta)})^{-1} \tilde{T}_f (R \otimes I_{d(\delta)}) = R^{-1} U(f) R \otimes I_{d(\delta)} \quad (f \in L^\circ(\delta)).$$

Therefore we can choose a base of $\mathfrak{H}(\delta)$ with respect to which the matrix function $U = U(x)$ of type δ has the form

$$U(x) = \begin{pmatrix} U^1(x) & & * \\ & \ddots & \\ 0 & & U^r(x) \end{pmatrix}.$$

We shall call the spherical matrix functions $U^i(x)$ $(1 \leqq i \leqq r)$ the irreducible components of $U(x)$. Then we obtain

$$\tilde{T}_f = \begin{pmatrix} U^1(f) \otimes I_{d(\delta)} & & * \\ & \ddots & \\ 0 & & U^r(f) \otimes I_{d(\delta)} \end{pmatrix}$$

for all $f \in L^\circ(\delta)$. We can find a sequence of T_f-invariant subspaces $\{0\} = V_{r+1} \subset V_r \subset \cdots \subset V_2 \subset V_1 = \mathfrak{H}(\delta)$ such that the representations of $L^\circ(\delta)$ naturally induced from $\tilde{T}_f$ on V_i / V_{i+1} $(1 \leqq i \leqq r)$ are equivalent to $U^i(f) \otimes I_{d(\delta)}$. Since the subspaces V_i are invariant under $\tilde{T}_k$, they are also invariant under $\tilde{T}_f$ for $f \in L(\delta)$ by Lemma 10, and the representations of $L(\delta)$ induced from $\tilde{T}_f$ on V_i / V_{i+1} are irreducible.

Let v_i be a vector in V_i such that $v_i \notin V_{i+1}$. We put $\mathfrak{H}_{r+1} = \{0\}$ and denote by $\mathfrak{H}_i$ $(1 \leqq i \leqq r)$ the closure of the

subspace $\{T_f v_i ; f \in L(G)\} + \mathfrak{H}_{i+1}$. Then it is clear that $E(\delta)\mathfrak{H}_i = V_i$ for $1 \leqq i \leqq r$. On the other hand, we denote by $\mathcal{K}_i$ $(1 \leqq i \leqq r)$ the largest closed invariant subspace of $\mathfrak{H}$ which satisfies $\mathcal{K}_i \subset \mathfrak{H}_i$ and $E(\delta)\mathcal{K}_i = V_{i+1}$. Since both $\mathfrak{H}_i$ and $\mathcal{K}_i$ are closed invariant subspace of $\mathfrak{H}$, we can naturally define a representation T_x^i of G from T_x on $\mathfrak{H}_i/\mathcal{K}_i$. Even if the representation space $\mathfrak{H}_i/\mathcal{K}_i$ is not complete, a matrix function of type δ defined by $\{\mathfrak{H}_i/\mathcal{K}_i, T_x^i\}$ is equivalent to $U^i(x)$. If $\mathfrak{H}$ is a Fréchet space, we know that $\mathfrak{H}_i/\mathcal{K}_i$ is also a Fréchet space and hence complete.

Conversely, let $\mathfrak{H}'$ and $\mathcal{K}'$ be closed invariant subspaces of $\mathfrak{H}$ such that $\mathcal{K}' \subset \mathfrak{H}'$ and that the representation T_x' of G naturally defined from T_x on the space $\mathfrak{H}'/\mathcal{K}'$ is topologically irreducible. If $\{\mathfrak{H}'/\mathcal{K}', T_x'\}$ contains δ, a spherical matrix function of type δ defined by $\{\mathfrak{H}'/\mathcal{K}', T_x'\}$ is equivalent to one of $U^i(x)$ $(1 \leqq i \leqq r)$.

§8. Spherical matrix functions on connected Lie groups

Let G be a connected unimodular Lie group, and K a compact analytic subgroup of G. Let U(G), U(K) be the algebras of all distributions on G, K respectively whose carriers reduce to the identity. Then the algebras U(G), U(K) are isomorphic to the universal enveloping algebras of the complexifications of the Lie algebras of G, K respectively.

Let $\{\mathfrak{H}, T_x\}$ be a representation of G on a complete locally convex topological vector space $\mathfrak{H}$. We shall denote by $C_o^\infty(G)$ the set of all infinitely differentiable functions on G with compact supports, then the space

$$\mathfrak{H}_G = \{ \sum T_{f_i} v_i \text{ (finite sum)} ; f_i \in C_o^\infty(G),\ v_i \in \mathfrak{H} \}$$

is called the Gårding subspace of $\mathfrak{H}$. On the Gårding subspace, we can define the so-called Gårding representation T_α of U(G) by

$$(T_\alpha v , v') = \int (T_x v , v') d\alpha(x) \quad \text{for } v \in \mathfrak{H}_G,\ v' \in \mathfrak{H}';$$

where $\mathfrak{H}'$ denotes the space of all continuous linear functionals on $\mathfrak{H}$. On the other hand, a vector $v \in \mathfrak{H}$, for which the $\mathfrak{H}$-valued function $x \to T_x v$ on G is infinitely differentiable, is called a differentiable vector in $\mathfrak{H}$ [3]. We shall denote by $\mathfrak{H}^\infty$ the space of all differentiable vectors in $\mathfrak{H}$. A representation π of U(G) is defined on $\mathfrak{H}^\infty$ by

$$\pi(\alpha)v = \int T_x v d\alpha(x) \qquad (v \in \mathfrak{H}^\infty,\ \alpha \in U(G)).$$

The subspace $\mathfrak{H}^\infty$ is invariant under T_x for $x \in G$, and we can introduce a topology τ in $\mathfrak{H}^\infty$ so that (i) $\mathfrak{H}^\infty$ becomes a complete locally convex topological vector space, (ii) $\{\mathfrak{H}^\infty, T_x\}$ is a topologically irreducible representation of G, and (iii) the operators $\pi(\alpha)$ are continuous on $\mathfrak{H}^\infty$ for all $\alpha \in U(G)$ [3]. If $\{\mathfrak{H}, T_x\}$ is K-finite, we put

$$\mathfrak{H}_K = \sum_{\delta \in \Omega_K} \mathfrak{H}(\delta).$$

Let $\{\mathfrak{H}, T_x\}$ be a K-finite topologically irreducible representation of G on a complete locally convex topological vector space $\mathfrak{H}$. Then we have four important subspaces $\mathfrak{H}_K$, $\mathfrak{H}_o$, $\mathfrak{H}_G$, and $\mathfrak{H}^\infty$ of $\mathfrak{H}$. We know that $\mathfrak{H}_K \subset \mathfrak{H}_o$ and $\mathfrak{H}_K \subset \mathfrak{H}_G \subset \mathfrak{H}^\infty$ and that $\mathfrak{H}_K$ is dense in $\mathfrak{H}$. It is easily shown that the restriction of π on $\mathfrak{H}_G$ is just the Gårding representation. The subspace $\mathfrak{H}_K$ is invariant under all T_α ($\alpha \in U(G)$) and the representation $\{\mathfrak{H}_K, \pi_K(\alpha) = T_\alpha|\mathfrak{H}_K\}$ of U(G) is algebraically irreducible [5]. Of course the subspace $\mathfrak{H}(\delta)$ is the space of all vectors in $\mathfrak{H}_K$ transformed according to δ under $\pi_K(U(K))$. Now we shall denote by Z(G) the center of U(G), then we obtain the following

LEMMA 31. Let $\{\mathfrak{H}, T_x\}$ be a K-finite topologically irreducible representation of G on a complete locally convex topological vector space $\mathfrak{H}$. Then there exists a homomorphism χ of Z(G) into the field of complex numbers $\mathbf{C}$ such

that

$$\pi(\zeta) = \chi(\zeta)\cdot 1$$

for all $\zeta \in Z(G)$, where 1 is the identity operator on $\mathfrak{H}^\infty$. (This homomorphism χ is called the infinitesimal character of $\{\mathfrak{H}, T_x\}$.)

PROOF. It is proved, as in [5], that $\pi(\zeta)$ is a scalar multiple of the identity operator on $\mathfrak{H}_K$. On the other hand, we introduce the topology τ in $\mathfrak{H}^\infty$, then $\mathfrak{H}_K = (\mathfrak{H}^\infty)_K$ is dense in $\mathfrak{H}^\infty$. Therefore $\pi(\zeta)$ is also a scalar multiple of the identity operator on $\mathfrak{H}^\infty$. q.e.d.

Here, let's prove a theorem which gives a characterization of quasi-simple irreducible representations in the sense of Harish-Chandra.

THEOREM 12. Let G be a connected semi-simple Lie group with finite center. If a topologically irreducible representation $\{\mathfrak{H}, T_x\}$ of G on a complete locally convex topological vector space $\mathfrak{H}$ is nice, then

(i) T_ζ is a scalar multiple of the identity operator on $\mathfrak{H}_G$ for all $\zeta \in Z(G)$,

(ii) T_z is a scalar multiple of the identity operator on $\mathfrak{H}$ for all z in the center of G.

PROOF. Let K be a maximal compact subgroup of G.

Since the representation $\{\mathfrak{H}, T_x\}$ is K-finite (Theorem 4), (i) is clear by Lemma 31. For every element z in the center of G, the operator T_z is a scalar multiple of the identity operator on each $\mathfrak{H}(\delta)$ $(\delta \in \Omega_K)$. Therefore it is also a scalar multiple of the identity operator on $\mathfrak{H}_o$. Since $\mathfrak{H}_o$ is dense in $\mathfrak{H}$, (ii) is clear. q.e.d.

Let $\{\mathfrak{H}^1, T^1_x\}$ and $\{\mathfrak{H}^2, T^2_x\}$ be two K-finite topologically irreducible representations of G on complete locally convex topological vector spaces $\mathfrak{H}^1$ and $\mathfrak{H}^2$ respectively. Let ϕ^1, ϕ^2 be the corresponding spherical functions of type δ and $E^1(\delta)$, $E^2(\delta)$ the usual projections onto $\mathfrak{H}^1(\delta)$, $\mathfrak{H}^2(\delta)$ respectively. We assume that $\phi^1 = \phi^2 \neq 0$. Then, by Theorem 9, there exists a linear mapping φ: $\mathfrak{H}^1_o \longrightarrow \mathfrak{H}^2_o$ such that

$$\varphi E^1(\delta') = E^2(\delta')\varphi \quad \text{for all } \delta' \in \Omega_K,$$

$$\varphi T^1_x = T^2_x \varphi \quad \text{for all } x \in G.$$

Then the restriction ψ of φ on the subspace $\mathfrak{H}^1_K$ is a bijective linear mapping from $\mathfrak{H}^1_K$ to $\mathfrak{H}^2_K$. If we denote by π^1_K, π^2_K the algebraically irreducible representations of U(G) on $\mathfrak{H}^1_K$, $\mathfrak{H}^2_K$ respectively, we have the following

LEMMA 32. ψ gives an equivalence between π^1_K and π^2_K.

PROOF. Let $v \in \mathfrak{H}^1_K$, then $\psi(v) \in \mathfrak{H}^2_K$. For $\alpha \in U(G)$, we

take $\delta_1, \cdots, \delta_n \in \Omega_K$ such that

$$T^1_\alpha v \in \mathfrak{V}^1 = \mathfrak{H}^1(\delta_1) + \cdots + \mathfrak{H}^1(\delta_n)$$

$$T^2_\alpha \psi(v) \in \mathfrak{V}^2 = \mathfrak{H}^2(\delta_1) + \cdots + \mathfrak{H}^2(\delta_n),$$

then $\psi | \mathfrak{V}^1$ is an isomorphism from $\mathfrak{V}^1$ onto $\mathfrak{V}^2$. Put $E^i = E^i(\delta_1) + \cdots + E^i(\delta_n)$ $(i = 1, 2)$ and let v' be an arbitrary vector in the dual of $\mathfrak{V}^2$, then

$$\begin{aligned}(\psi T^1_\alpha v , v') &= (\psi E^1 T^1_\alpha v , v') = (E^1 T^1_\alpha v , \psi^*(v')) \\ &= \int (E^1 T^1_x v , \psi^*(v')) d\alpha(x) = \int (\psi E^1 T^1_x v , v') d\alpha(x) \\ &= \int (E^2 T^2_x \psi(v) , v') d\alpha(x) = (T^2_\alpha \psi(v) , v').\end{aligned}$$

This implies $\psi T^1_\alpha = T^2_\alpha \psi$, i.e., $\psi \pi^1_K(\alpha) = \pi^2_K(\alpha)\psi$. q.e.d.

Conversely, we assume that two representations $\{\mathfrak{H}^1_K, \pi^1_K\}$ and $\{\mathfrak{H}^2_K, \pi^2_K\}$ of $U(G)$ are equivalent. Let $\psi : \mathfrak{H}^1_K \to \mathfrak{H}^2_K$ be a linear mapping which gives an equivalence between the two, then $\psi \pi^1_K(\alpha) = \pi^2_K(\alpha)\psi$ for all $\alpha \in U(K)$ and hence we have

$$\psi E^1(\delta) = E^2(\delta)\psi$$

for all $\delta \in \Omega_K$. Let ϕ^1, ϕ^2 the corresponding spherical functions of type δ. Since the analyticity of spherical functions is shown by the same way as in [5], we have

$$\begin{aligned}\alpha(\phi^1) &= \text{trace } [E^1(\delta)T^1_\alpha] = \text{trace}[\psi^{-1}E^2(\delta)\psi\pi^1_K(\alpha)] \\ &= \text{trace}[\psi^{-1}E^2(\delta)\pi^2_K(\alpha)\psi] = \text{trace}[E^2(\delta)T^2_\alpha] = \alpha(\phi^2)\end{aligned}$$

for all $\alpha \in U(G)$. This implies $\phi^1 = \phi^2$. Therefore we have proved the following

THEOREM 13. Let G be a connected unimodular Lie group, and K a compact analytic subgroup of G. Let $\{ \mathfrak{H}^1, T^1_x \}$, $\{ \mathfrak{H}^2, T^2_x \}$ be two K-finite topologically irreducible representations of G on complete locally convex topological vector spaces $\mathfrak{H}^1$, $\mathfrak{H}^2$ respectively. Then the following three statements are equivalent.

(i) There exists a class $\delta \in \Omega_K$ such that the corresponding spherical functions ($\neq 0$) coincide with each other.

(ii) For every $\delta \in \Omega_K$, the corresponding spherical functions coincide with each other.

(iii) $\{ \mathfrak{H}^1, T^1_x \}$ and $\{ \mathfrak{H}^2, T^2_x \}$ are infinitesimally equivalent, i.e., the corresponding algebraically irreducible representations $\{ \mathfrak{H}^1_K, \pi^1_K \}$ and $\{ \mathfrak{H}^2_K, \pi^2_K \}$ of U(G) are equivalent.

The analogous theorem is also valid for spherical matrix functions.

Next, let's prove two theorems on spherical matrix functions. To do this, we introduce some notations. For $\alpha \in U(G)$, we define a distribution α° by $\alpha^\circ(f) = \alpha(f^\circ)$, and denote by $U^\circ(G)$ the algebra of all distributions $\alpha \in U(G)$ satisfying $\alpha = \alpha^\circ$. And also we define a distribution α' by $\alpha'(f) = \alpha(f')$ where $f'(x) = f(x^{-1})$. We often use the notation $f(\alpha)$ for $\alpha(f)$.

THEOREM 14. Let G be a connected unimodular Lie group, and K a compact analytic subgroup of G. Let $U = U(x)$ be an analytic matrix function on G satisfying $U = U^{\circ}$. Then $f \longrightarrow U(f) = \int_G U(x)f(x)dx$ is a p-dimensional irreducible representation of $L^{\circ}(\delta)$ for some class $\delta \in \Omega_K$ if and only if $\alpha \longrightarrow U(\alpha)$ is a p-dimensional irreducible representation of $U^{\circ}(G)$.

PROOF. We assume that $f \longrightarrow U(f)$ is a p-dimensional irreducible representation of $L^{\circ}(\delta)$. Then the function $\phi(x) = d(\delta)\cdot\text{trace}[U(x)]$ is a generalized spherical function of type δ. Let $\{\mathfrak{H}, T_x\}$ be a representation of G by which ϕ is defined. Then $\mathfrak{H}$ is a locally convex topological vector space, on which the projection $E(\delta)$ and the operators T_f ($f \in L(G)$) can be defined, and $\{\mathfrak{H}, T_f\}$ is a topologically irreducible representation of L(G). We can choose a base in $\mathfrak{H}(\delta)$ such that the spherical matrix function of type δ with respect to this base is just the matrix function $U = U(x)$.

On the other hand, we extend the operators T_x by continuity on the completion $\hat{\mathfrak{H}}$ of $\mathfrak{H}$. Then $\{\hat{\mathfrak{H}}, T_x\}$ is a representation of G, but it does not remain topologically irreducible in general. Even if it is not topologically irreducible, the space of all vectors in $\hat{\mathfrak{H}}$ transformed according to δ under $k \longrightarrow T_k$ is $\mathfrak{H}(\delta)$, and it is contained in the Gårding subspace of $\hat{\mathfrak{H}}$. Since $\{\tilde{T}_\alpha = T_\alpha \mid \mathfrak{H}(\delta) ; \alpha \in U^{\circ}(G)\}$ is the set of all linear operators on $\mathfrak{H}(\delta)$ which commute with

all $\tilde{T}_k$ $(k \in K)$, $\tilde{T}_\alpha$ $(\alpha \in U^\circ(G))$ is represented in the following form

$$\tilde{T}_\alpha = U(\alpha) \otimes I_{d(\delta)},$$

and $\alpha \longrightarrow U(\alpha)$ is a p-dimensional irreducible representation of $U^\circ(G)$.

Conversely we assume that $\alpha \longrightarrow U(\alpha)$ is a p-dimensional irreducible representation of $U^\circ(G)$. Let α, β be arbitrary distributions on G, then we have

$$U(\alpha^\circ * \beta) = U(\alpha^\circ * \beta^\circ) = U(\alpha)U(\beta)$$

by Lemma 15 in [5]. From this we easily obtain

$$\int_K U(kxk^{-1}y)dk = U(x)U(y)$$

for $x, y \in G$. The irreducibility of U obviously follows from the assumptions. Finally there exists a class $\delta \in \Omega_K$ such that $\chi_\delta * U \neq 0$ by the Peter-Weyl's theorem. Since

$$U(\alpha)U(\bar{\chi}_\delta) = U(\alpha * \bar{\chi}_\delta) = U(\bar{\chi}_\delta * \alpha) = U(\bar{\chi}_\delta)U(\alpha)$$

for all $\alpha \in U^\circ(G)$, there exists a complex number c such that $U(\bar{\chi}_\delta) = c \cdot I_p$. Therefore

$$\chi_\delta * U(\alpha) = U(\bar{\chi}_\delta * \alpha) = cU(\alpha)$$

for all $\alpha \in U^\circ(G)$, and hence we obtain $c = 1$ because of the equality $\chi_\delta * \chi_\delta = \chi_\delta$. Thus we have $\chi_\delta * U = U$. Now this theorem is completely proved using Theorem 11. q.e.d.

THEOREM 15. Let G be a connected unimodular Lie group, and K a compact analytic subgroup of G. Let $U = U(x)$ be an

irreducible continuous matrix function on G. Then we have

$$\int_K U(kxk^{-1}y)dk = U(x)U(y)$$

for all $x, y \in G$ if and only if (U is analytic and) the equation

$$\alpha' * U = U(\alpha)U$$

is satisfied by every $\alpha \in U^\circ(G)$.

PROOF. At first we assume $\int_K U(kxk^{-1}y)dk = U(x)U(y)$. The equality $U = U^\circ$ is proved as in the proof of Theorem 11. For an arbitrary measure α on G and $\delta \in \Omega_K$, we obtain

$$U(\alpha)U(\overline{\chi}_\delta) = U * \chi_\delta(\alpha) = \chi_\delta * U(\alpha) = U(\overline{\chi}_\delta)U(\alpha).$$

This implies $U(x)U(\overline{\chi}_\delta) = U(\overline{\chi}_\delta)U(x)$ for all $x \in G$. Therefore $U(\overline{\chi}_\delta)$ is a scalar multiple of I_p. If we take $\delta \in \Omega_K$ such that $\chi_\delta * U \neq 0$, then $\chi_\delta * U = U$ and the representation $f \longrightarrow U(f)$ of $L^\circ(\delta)$ is proved to be irreducible as in the proof of Theorem 11. Thus $U = U(x)$ is a spherical matrix function of type δ, and hence it is analytic. Therefore $\alpha \longrightarrow U(\alpha)$ is a p-dimensional irreducible representation of $U^\circ(G)$ by Theorem 14, and for every $\alpha \in U^\circ(G)$ we have

$$\alpha' * U(x) = \int_G U(y^{-1}x)d\alpha'(y) = \int_{G\times K} U(kyk^{-1}x)d\alpha(y)$$

$$= \int_G U(y)U(x)d\alpha(y) = U(\alpha)U(x).$$

Conversely we assume that U is analytic and that $\alpha' * U = U(\alpha)U$ for all $\alpha \in U^\circ(G)$. Putting

$$V_y(x) = \int_K U(kxk^{-1}y)dk,$$

we obtain

$$\alpha' * V_y(x) = \int_{G\times K} U(z^{-1}xkyk^{-1})dkd\alpha'(z) = \int_K \alpha' * U(xkyk^{-1})dk$$

$$= U(\alpha)\int_K U(xkyk^{-1})dk = U(\alpha)V_y(x)$$

for all $\alpha \in U^\circ(G)$. If we put $x = e$, we have $V_y(\alpha) = U(\alpha)V_y(e) = U(\alpha)U(y)$ or $\alpha(V_y(x) - U(x)U(y)) = 0$ for all $\alpha \in U^\circ(G)$. This implies $V_y(x) = U(x)U(y)$ and this theorem is proved.

q.e.d.

§9. A construction of matrix functions on a group of type G = KS

Let G be a locally compact unimodular group, and K a compact subgroup of G. We assume that there exists a closed subgroup S of G such that

$$G = KS, \quad K \cap S = \{e\},$$

and that the decomposition $x = ks$ $(k \in K,\ s \in S)$ is continuous. We shall denote by M a closed subgroup of K which normalizes S.

We fix a class $\delta \in \Omega_K$ and put $d = d(\delta)$ for simplicity. Let $\sigma_1, \cdots, \sigma_r$ be all equivalence classes in Ω_M which occur in δ. We shall denote by p_i $(1 \leqq i \leqq r)$ the multiplicity of σ_i in δ and by q_i the degree of σ_i. We also use the notation σ_i to denote an irreducible unitary matrix representation of M belonging to the equivalence class σ_i. Then we can chose an irreducible unitary matrix representation $k \longrightarrow D(k) = (d_{pq}(k))$ $(1 \leqq p,q \leqq d)$ of K belonging to δ such that

$$D(m) = \begin{pmatrix} \sigma_1(m) \overset{p_1}{\searrow} & & 0 \\ & \ddots & \\ & & \sigma_r(m) \overset{p_r}{\searrow} \\ 0 & & & \ddots \\ & & & & \sigma_r(m) \end{pmatrix}$$

for all $m \in M$. We define integers $t_0, t_1, \cdots, t_r$ by

$$t_0 = 0, \quad t_i = p_1 q_1 + \cdots + p_i q_i \quad (1 \leqq i \leqq r),$$

and put

$$P^{\sigma_i}(k) = \left(d_{pq}(k)\right)_{t_{i-1}+1 \leqq p,q \leqq t_i} \qquad (1 \leqq i \leqq r).$$

We write dm for the normalized Haar measure on M, and put

$$D_M(k) = \int_M D(mkm^{-1})dm, \qquad P_M^{\sigma_i}(k) = \int_M P^{\sigma_i}(mkm^{-1})dm \quad (1 \leqq i \leqq r).$$

Let $s \longrightarrow \Lambda(s) = (\lambda_{\xi\eta})_{1 \leqq \xi,\eta \leqq n}$ be a n-dimensional irreducible matrix representation of S satisfying $\Lambda(msm^{-1}) = \Lambda(s)$ for all $m \in M$. Then, for $x = ks$ $(k \in K, s \in S)$, we put

$$V^{\Lambda}(x) = \overline{D(k)} \otimes \Lambda(s^{-1}) = \begin{pmatrix} \overline{d_{11}}(k)\Lambda(s^{-1}) & \cdots\cdots & \overline{d_{1d}}(k)\Lambda(s^{-1}) \\ \vdots & & \vdots \\ \overline{d_{d1}}(k)\Lambda(s^{-1}) & \cdots\cdots & \overline{d_{dd}}(k)\Lambda(s^{-1}) \end{pmatrix},$$

$$V_M^{\Lambda}(x) = \overline{D_M}(k) \otimes \Lambda(s^{-1}),$$

$$U^{\Lambda}(x) = (V^{\Lambda})^{\circ}(x^{-1}) = \int_K V^{\Lambda}(kx^{-1}k^{-1})dk,$$

$$U_M^{\Lambda}(x) = (V_M^{\Lambda})^{\circ}(x^{-1}).$$

And also we put

$$\tilde{V}^{\sigma_i,\Lambda}(x) = \overline{P^{\sigma_i}}(k) \otimes \Lambda(s^{-1}), \qquad \tilde{V}_M^{\sigma_i,\Lambda}(x) = \overline{P_M^{\sigma_i}}(k) \otimes \Lambda(s^{-1}),$$

$$\tilde{U}^{\sigma_i,\Lambda}(x) = (\tilde{V}^{\sigma_i,\Lambda})^{\circ}(x^{-1}), \qquad \tilde{U}_M^{\sigma_i,\Lambda}(x) = (\tilde{V}_M^{\sigma_i,\Lambda})^{\circ}(x^{-1}),$$

for $1 \leqq i \leqq r$.

For $k \in K$ and $x \in G$, we shall denote by $\overline{x^{-1}k}$ the K-component of $x^{-1}k$ and by $s(x^{-1},k)$ the S-component of $x^{-1}k$, i.e.

$$x^{-1}k = \overline{x^{-1}k} \cdot s(x^{-1},k).$$

LEMMA 33. We have $U_M^{\Lambda} = U^{\Lambda}$ and $\tilde{U}_M^{\sigma_i,\Lambda} = \tilde{U}^{\sigma_i,\Lambda}$ for $1 \leqq i \leqq r$.

PROOF. For $1 \leqq p,q \leqq d$, we obtain

$$\int_K \int_M \overline{d_{pq}}(mk(\overline{x^{-1}}k^{-1})m^{-1})\Lambda(s(x^{-1},k^{-1})^{-1})dmdk$$

$$= \int_K \int_M \overline{d_{pq}}(mk\cdot\overline{x^{-1}}(k^{-1}m^{-1}))\Lambda(s(x^{-1},k^{-1})^{-1})dmdk$$

$$= \int_M \int_K \overline{d_{pq}}(k\cdot\overline{x^{-1}}k^{-1})\Lambda(s(x^{-1},k^{-1}m)^{-1})dkdm$$

$$= \int_K \int_M \overline{d_{pq}}(k\cdot\overline{x^{-1}}k^{-1})\Lambda(m^{-1}s(x^{-1},k^{-1})^{-1}m)dmdk$$

$$= \int_K \overline{d_{pq}}(k\cdot\overline{x^{-1}}k^{-1})\Lambda(s(x^{-1},k^{-1})^{-1})dk.$$

This proves the lemma. q.e.d.

The matrix $P^{\sigma_i}(k)$ is of degree $p_i q_i$, and $P^{\sigma_i}(m)$ is the p_i-times direct sum of $\sigma_i(m)$ for $m \in M$. Now we define $q_i \times q_i$-matrices $P_{\alpha\beta}^{\sigma_i}(k)$ $(1 \leqq \alpha,\beta \leqq p_i)$ by the equality

$$P^{\sigma_i}(k) = \begin{pmatrix} P_{11}^{\sigma_i}(k) & \cdots & P_{1p_i}^{\sigma_i}(k) \\ \vdots & & \vdots \\ P_{p_i 1}^{\sigma_i}(k) & \cdots & P_{p_i p_i}^{\sigma_i}(k) \end{pmatrix},$$

and put

$$A^{\sigma_i}(k) = \left(a_{\alpha\beta}^{\sigma_i}(k)\right)_{1 \leqq \alpha,\beta \leqq p_i}$$

where

$$a^{\sigma_i}_{\alpha\beta}(k) = \frac{1}{q_i} \operatorname{trace} P^{\sigma_i}_{\alpha\beta}(k)$$

$$= \frac{1}{q_i} \sum_{t=1}^{q_i} d_{t_{i-1}+(\alpha-1)q_i+t\,,\,t_{i-1}+(\beta-1)q_i+t}(k).$$

Then, for $p = t_{i-1}+(\alpha-1)q_i+\xi$ and $q = t_{j-1}+(\beta-1)q_j+\eta$ $(1 \leqq i, j \leqq r,$ $1 \leqq \alpha \leqq p_i$, $1 \leqq \beta \leqq p_j$, $1 \leqq \xi \leqq q_i$, $1 \leqq \eta \leqq q_j$), we have

$$\int_M d_{pq}(mkm^{-1})dm = \sum_{\mu,\nu=1}^{d} d_{\mu\nu}(k) \int_M d_{p\mu}(m)\overline{d_{q\nu}}(m)dm$$

$$= \delta_{ij}\ \delta_{\xi\eta}\ \frac{1}{q_i} \sum_{\mu=1}^{q_i} d_{t_{i-1}+(\alpha-1)q_i+\mu\,,\,t_{i-1}+(\beta-1)q_i+\mu}(k)$$

$$= \delta_{ij}\ \delta_{\xi\eta}\ a^{\sigma_i}_{\alpha\beta}(k).$$

This shows that

$$P^{\sigma_i}_M(k) = A^{\sigma_i}(k) \otimes I_{q_i} \quad \text{and} \quad D_M(k) = \begin{pmatrix} P^{\sigma_1}_M(k) & & 0 \\ & \ddots & \\ 0 & & P^{\sigma_r}_M(k) \end{pmatrix}.$$

Therefore, by Lemma 33, the matrix function U^{Λ} is decomposed into the direct sum of $\tilde{U}^{\sigma_i,\Lambda}$ $(1 \leqq i \leqq r)$;

$$U^{\Lambda}(x) = \begin{pmatrix} \tilde{U}^{\sigma_1,\Lambda}(x) & & 0 \\ & \ddots & \\ 0 & & \tilde{U}^{\sigma_r,\Lambda}(x) \end{pmatrix}.$$

Let's decompose $\tilde{U}^{\sigma_i,\Lambda}$ into the q_i-times direct sum of a matrix function. Let R_i be a regular matrix of degree $p_i q_i$ such that

$$R_i^{-1}\overline{P^{\sigma_i}_M}(k)R_i = R_i^{-1}(\overline{A^{\sigma_i}}(k)\otimes I_{q_i})R_i = I_{q_i} \otimes \overline{A^{\sigma_i}}(k),$$

then

$$(R_i \otimes I_n)^{-1} \tilde{V}_M^{\sigma_i, \Lambda}(x)(R_i \otimes I_n) = (R_i \otimes I_n)^{-1} (\overline{P_M^{\sigma_i}}(k) \otimes \Lambda(s^{-1}))(R_i \otimes I_n)$$

$$= (I_{q_i} \otimes \overline{A^{\sigma_i}}(k)) \otimes \Lambda(s^{-1}) = I_{q_i} \otimes (\overline{A^{\sigma_i}}(k) \times \Lambda(s^{-1})).$$

If we put

$$V^{\sigma_i, \Lambda}(x) = \overline{A^{\sigma_i}}(k) \otimes \Lambda(s^{-1}), \quad U^{\sigma_i, \Lambda}(x) = (V^{\sigma_i, \Lambda}) \circ (x^{-1})$$

for $1 \leqq i \leqq r$, the above equality shows that

$$(R_i \otimes I_n)^{-1} \tilde{U}^{\sigma_i, \Lambda}(x) \ (R_i \otimes I_n) = \begin{pmatrix} U^{\sigma_i, \Lambda}(x) & & 0 \\ & \ddots & \\ 0 & & U^{\sigma_i, \Lambda}(x) \end{pmatrix},$$

i.e., $\tilde{U}^{\sigma_i, \Lambda}$ is equivalent to the q_i-times direct sum of $U^{\sigma_i, \Lambda}$. This decomposition of $\tilde{U}^{\sigma_i, \Lambda}$ shows that

$$R^{-1} U^{\Lambda}(x) R = \begin{pmatrix} U^{\sigma_1, \Lambda}(x) \ (q_1) & & & 0 \\ & \ddots & & \\ 0 & & U^{\sigma_r, \Lambda}(x) \ (q_r) & \end{pmatrix}$$

where

$$R = \begin{pmatrix} R_1 \otimes I_n & & 0 \\ & \ddots & \\ 0 & & R_r \otimes I_n \end{pmatrix}.$$

LEMMA 34. The matrix function U^{Λ} satisfies the following functional equations

(i) $\chi_{\delta} * U^{\Lambda} = U^{\Lambda}$,

(ii) $\int_K U^{\Lambda}(kxk^{-1}y)dk = U^{\Lambda}(x)U^{\Lambda}(y)$ for all $x, y \in G$.

Therefore this is true for $U^{\sigma_i, \Lambda}$ $(1 \leq i \leq r)$, too.

PROOF. The proof of this lemma is given by direct calculations :

$$\chi_{\delta} * U^{\Lambda}(x) = \int_K U^{\Lambda}(u^{-1}x)\chi_{\delta}(u)du$$

$$= \int_K \int_K V^{\Lambda}(vx^{-1}uv^{-1})\chi_{\delta}(u)dudv$$

$$= \int_K \int_K V^{\Lambda}(vx^{-1}u)\chi_{\delta}(uv)dudv$$

$$= \sum_{p,q=1}^{d} \int_K \int_K (\overline{D}(v) \otimes I_n)V^{\Lambda}(x^{-1}u)d_{pq}(u)d_{qp}(v)dudv$$

$$= \int_K (\overline{D}(u^{-1}) \otimes I_n)V^{\Lambda}(x^{-1}u)du$$

$$= \int_K V^{\Lambda}(u^{-1}x^{-1}u)du = U^{\Lambda}(x),$$

$$\int_K U^{\Lambda}(kxk^{-1}y)dk = \int_K \int_K V^{\Lambda}(uy^{-1}kx^{-1}k^{-1}u^{-1})dudk$$

$$= \int_K \int_K V^{\Lambda}(uk^{-1}y^{-1}kx^{-1}u^{-1})dudk$$

$$= \int_K \int_K (\overline{D}(uk^{-1}) \otimes I_n)V^{\Lambda}(y^{-1}kk_1s_1)dudk \qquad (x^{-1}u^{-1} = k_1s_1)$$

$$= \int_K \int_K (\overline{D}(uk_1k^{-1}) \otimes I_n)\, V^{\Lambda}(y^{-1}ks_1)\,dudk$$

$$= \int_K \int_K (\overline{D}(uk_1k^{-1}) \otimes I_n)(\overline{D}(k_2) \otimes I_n) V^{\Lambda}(s_2s_1)\,dudk$$

$$(y^{-1}k = k_2s_2)$$

$$= \int_K \int_K (\overline{D}(uk_1) \otimes I_n)(\overline{D}(k^{-1}k_2) \otimes I_n) \times$$

$$\times (I_d \otimes \Lambda(s_1^{-1}))(I_d \otimes \Lambda(s_2^{-1}))\,dudk$$

$$= \int_K \int_K (\overline{D}(uk_1) \otimes \Lambda(s_1^{-1}))(\overline{D}(k^{-1}k_2) \otimes \Lambda(s_2^{-1}))\,dudk$$

$$= \int_K \int_K V^{\Lambda}(ux^{-1}u^{-1}) V^{\Lambda}(k^{-1}y^{-1}k)\,dudk$$

$$= U^{\Lambda}(x)U^{\Lambda}(y). \qquad \text{q.e.d.}$$

COROLLARY. Let σ be an equivalence class in Ω_M which occurs only once in δ. Then, if σ is of degree one or Λ is one-dimensional, $U^{\sigma,\Lambda}$ is a spherical matrix function of type δ, namely it is a spherical function of type δ of height one.

Hereafter, we consider only for $i = 1$, but the same arguments can be made for $i = 2, \cdots, r$ with some trivial modifications.

Let $(\ ,\ )$ be the usual inner product in $\mathbb{C}^{q_1 n}$, and $\mathfrak{H}^{\sigma_1,\Lambda}$ the space of all $\mathbb{C}^{q_1 n}$-valued measurable functions φ on K such that

$$\int_K (\varphi(k), \varphi(k))\,dk < +\infty,$$

$$\varphi(km) = \{\sigma_1(m^{-1}) \otimes I_n\}\, \varphi(k) \qquad \text{for all } m \in M.$$

Then $\mathfrak{H}^{\sigma_1,\Lambda}$ is a Hilbert space with the inner product

$$\langle \varphi , \psi \rangle = \int_K (\varphi(k), \psi(k))dk.$$

For every element $x \in G$, we define a continuous linear operator $T_x^{\sigma_1,\Lambda}$ on $\mathfrak{H}^{\sigma_1,\Lambda}$ by

$$(T_x^{\sigma_1,\Lambda} \varphi)(k) = \{I_{q_1} \otimes \Lambda(s(x^{-1},k)^{-1})\} \varphi(\overline{x^{-1}}k).$$

Then $\{\mathfrak{H}^{\sigma_1,\Lambda}, T_x^{\sigma_1,\Lambda}\}$ is a representation of G and the subspace $\mathfrak{H}^{\sigma_1,\Lambda}(\delta)$, which is the set of all vectors in $\mathfrak{H}^{\sigma_1,\Lambda}$ transformed according to δ under $k \to T_k^{\sigma_1,\Lambda}$, is exactly equal to $\overline{\chi}_\delta * \mathfrak{H}^{\sigma_1,\Lambda}$.

We shall denote by e_j $(1 \leq j \leq q_1 n)$ the column vector in $\mathbb{C}^{q_1 n}$ of which the jth component is equal to 1 and all of the others are equal to 0. Then, for $1 \leq i \leq q_1$ and $1 \leq \xi \leq n$, we have

$$\{\sigma_1(m) \otimes \Lambda(s)\} e_{(i-1)n+\xi} = \sum_{j=1}^{q_1} \sum_{\eta=1}^{n} \sigma^1_{ji}(m) \lambda_{\eta\xi}(s) e_{(j-1)n+\eta}$$

where $\sigma^1_{ji}(m)$ denotes the (j,i)-component of the matrix $\sigma_1(m)$.

LEMMA 35. The space $\mathfrak{H}^{\sigma_1,\Lambda}(\delta)$ is $dp_1 n$-dimensional, and we may choose the set of functions

$$\varphi_{\mu,\alpha,\xi}(k) = \sqrt{\frac{d}{q_1}} \sum_{i=1}^{q_1} \overline{d_{\mu,(\alpha-1)q_1+i}(k)}\, e_{(i-1)n+\xi}$$

for $1 \leq \mu \leq d$, $1 \leq \alpha \leq p_1$, and $1 \leq \xi \leq n$ as an orthonormal base in $\mathfrak{H}^{\sigma_1,\Lambda}(\delta)$.

PROOF. The space $\mathfrak{H}^{\sigma_1,\Lambda}(\delta) = \overline{\chi}_\delta * \mathfrak{H}^{\sigma_1,\Lambda}$ is clearly a subspace of the linear space generated by the d^2q_1n functions

$$\overline{d_{\mu,\nu}}(k)e_{(i-1)n+\xi} \quad (1 \leqq \mu,\nu \leqq d,\ 1 \leqq i \leqq q_1,\ 1 \leqq \xi \leqq n).$$

Therefore a function $\varphi \in \mathfrak{H}^{\sigma_1,\Lambda}(\delta)$ can be written in the form

$$\varphi(k) = \sum_{\mu,\nu=1}^{d} \sum_{i=1}^{q_1} \sum_{\xi=1}^{n} a(i,\xi,\mu,\nu)\overline{d_{\mu\nu}}(k)e_{(i-1)n+\xi}$$

where $a(i,\xi,\mu,\nu) \in \mathbb{C}$. Then, for every $m \in M$,

$$\begin{aligned}
\varphi(km) &= \sum_{\mu,\nu=1}^{d} \sum_{i=1}^{q_1} \sum_{\xi=1}^{n} a(i,\xi,\mu,\nu)\overline{d_{\mu\nu}}(km)e_{(i-1)n+\xi} \\
&= \sum_{\mu,\nu=1}^{d} \sum_{i=1}^{q_1} \sum_{\xi=1}^{n} \sum_{t=1}^{d} a(i,\xi,\mu,\nu)\overline{d_{\mu t}}(k)\overline{d_{t\nu}}(m)e_{(i-1)n+\xi} \\
&= \sum_{\mu,t=1}^{d} \sum_{i=1}^{q_1} \sum_{\xi=1}^{n} \left\{ \sum_{\nu=1}^{d} a(i,\xi,\mu,\nu)\overline{d_{t\nu}}(m) \right\} \overline{d_{\mu t}}(k)e_{(i-1)n+\xi}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi(km) &= \{\sigma_1(m^{-1}) \otimes I_n\}\ \varphi(k) \\
&= \sum_{\mu,\nu=1}^{d} \sum_{i=1}^{q_1} \sum_{\xi=1}^{n} a(i,\xi,\mu,\nu)\overline{d_{\mu\nu}}(k) \left\{ \sum_{j=1}^{q_1} \sigma'_{ji}(m^{-1})e_{(j-1)n+\xi} \right\} \\
&= \sum_{\mu,\nu=1}^{d} \sum_{j=1}^{q_1} \sum_{\xi=1}^{n} \left\{ \sum_{i=1}^{q_1} a(i,\xi,\mu,\nu)\sigma'_{ji}(m^{-1}) \right\} \overline{d_{\mu\nu}}(k)e_{(j-1)n+\xi}.
\end{aligned}$$

Therefore we obtain

$$\sum_{j=1}^{q_1} a(j,\xi,\mu,\nu)\overline{\sigma'_{ji}}(m) = \sum_{t=1}^{d} a(i,\xi,\mu,t)\overline{d_{\nu t}}(m)$$

for all $m \in M$, where $1 \leqq i \leqq q_1$, $1 \leqq \xi \leqq n$, and $1 \leqq \mu,\nu \leqq d$. For $\nu = (\alpha-1)q_1 + b$ $(1 \leqq \alpha \leqq p_1,\ 1 \leqq b \leqq q_1)$,

$$\overline{d_{\nu t}}(m) = \begin{cases} \overline{\sigma^1_{bc}}(m) & \text{if } t = (\alpha-1)q_1 + c \quad (1 \leqq c \leqq q_1), \\ 0 & \text{otherwise,} \end{cases}$$

and for $\nu \geqq p_1 q_1 + 1$, $\overline{d_{\nu t}}(m)$ and $\overline{\sigma^1_{ji}}(m)$ are linearly independent. Thus the above relation is equivalent to

$$\sum_{j=1}^{q_1} a(j,\xi,\mu,(\alpha-1)q_1+b)\overline{\sigma^1_{ji}}(m) = \sum_{c=1}^{q_1} a(i,\xi,\mu,(\alpha-1)q_1+c)\overline{\sigma^1_{bc}}(m)$$

$$a(i,\xi,\mu,\nu) = 0 \qquad \text{if } \nu \geqq p_1 q_1 + 1$$

for $1 \leqq i,b \leqq q_1$, $1 \leqq \xi \leqq n$, $1 \leqq \alpha \leqq p_1$, and $1 \leqq \mu \leqq d$. This implies

$$a(i,\xi,\mu,(\alpha-1)q_1+i) = a(b,\xi,\mu,(\alpha-1)q_1+b),$$

$$a(i,\xi,\mu,(\alpha-1)q_1+c) = 0 \quad \text{if } c \neq i,$$

$$a(i,\xi,\mu,\nu) = 0 \quad \text{if } \nu \geqq p_1 q_1 + 1,$$

and the lemma is clear from these relations. q.e.d.

Now we represent integers $i = 1, 2, \cdots, dp_1 n$ in the form

$$i = (\xi-1)p_1 d + (\alpha-1)d + \mu \qquad (1 \leqq \xi \leqq n,\ 1 \leqq \alpha \leqq p_1,\ 1 \leqq \mu \leqq d),$$

and put

$$v_i = \varphi_{\mu,\alpha,\xi}.$$

Then, for every $u \in K$, we have

$$(T_u^{\sigma_1,\Lambda} v_i)(k) = (T_u^{\sigma_1,\Lambda} v_{(\xi-1)p_1 d + (\alpha-1)d + \mu})(k)$$

$$= (T_u^{\sigma_1,\Lambda} \varphi_{\mu,\alpha,\xi})(k)$$

$$= \sqrt{\frac{d}{q_1}} \sum_{i=1}^{q_1} \overline{d_{\mu,(\alpha-1)q_1+i}}(u^{-1}k) e_{(i-1)n+\xi}$$

$$= \sqrt{\frac{d}{q_1}} \sum_{i=1}^{q_1} \sum_{t=1}^{d} d_{t\mu}(u)\overline{d_{t,(\alpha-1)q_1+i}}(k)\, e_{(i-1)n+\xi}$$

$$= \sum_{t=1}^{d} d_{t\mu}(u)\, \varphi_{t,\alpha,\xi}(k)$$

$$= \sum_{t=1}^{d} d_{t\mu}(u) v_{(\xi-1)p_1 d+(\alpha-1)d+t}(k).$$

Therefore the restriction $\tilde{T}_k^{\sigma_1,\Lambda}$ of $T_k^{\sigma_1,\Lambda}$ on $\mathfrak{H}^{\sigma_1,\Lambda}(\delta)$ is represented in the form

$$\tilde{T}_k^{\sigma_1,\Lambda} = I_{p_1 n} \otimes D(k)$$

with respect to this base $v_1, v_2, \cdots, v_{dp_1 n}$. The matrix function of type δ defined by $\{\mathfrak{H}^{\sigma_1,\Lambda}, T_x^{\sigma_1,\Lambda}\}$ is of degree $p_1 n$, and its matrix elements $u_{ij}^{\sigma_1,\Lambda}$ ($i = (\xi-1)p_1 + \alpha$, $j = (\eta-1)p_1 + \beta$, $1 \leqq \alpha,\beta \leqq p_1$, $1 \leqq \xi,\eta \leqq n$) are given by

$$u_{ij}^{\sigma_1,\Lambda}(x) = \frac{1}{d} \sum_{\mu=1}^{d} \left\langle E(\delta) T_x^{\sigma_1,\Lambda} v_{(j-1)d+\mu}, v_{(i-1)d+\mu} \right\rangle$$

$$= \frac{1}{d} \sum_{\mu=1}^{d} \int_K \left\langle T_{ux}^{\sigma_1,\Lambda} v_{(j-1)d+\mu}, v_{(i-1)d+\mu} \right\rangle \overline{\chi}_\delta(u)du$$

$$= \frac{1}{d} \sum_{\mu=1}^{d} \int_K \int_K \Big(\{I_{q_1} \otimes \Lambda(s(x^{-1},u^{-1}k)^{-1})\}\, \varphi_{\mu,\beta,\eta}(\overline{x^{-1}}(u^{-1}k)),$$

$$\varphi_{\mu,\alpha,\xi}(k) \Big) \overline{\chi}_\delta(u)dudk$$

$$= \frac{1}{q_1} \sum_{\mu=1}^{d} \sum_{s,t=1}^{q_1} \int_K \int_K \Big(\{I_{q_1} \otimes \Lambda(s(x^{-1},u^{-1}k)^{-1})\}$$

$$\overline{d_{\mu,(\beta-1)q_1+s}}(\overline{x^{-1}}(u^{-1}k))\, e_{(s-1)n+\eta},$$

$$\overline{d_{\mu,(\alpha-1)q_1+t}}(k)\, e_{(t-1)n+\xi} \Big) \overline{\chi}_\delta(u)dudk$$

$$= \frac{1}{q_1}\sum_{\mu=1}^{d}\sum_{s,t=1}^{q_1}\sum_{c=1}^{n}\int_K\int_K \Big(\overline{d_{\mu,(\beta-1)q_1+s}}(\overline{x^{-1}}(u^{-1}k))\lambda_{c\eta}(s(x^{-1},u^{-1}k)^{-1})$$

$$\times e_{(s-1)n+c}, \overline{d_{\mu,(\alpha-1)q_1+t}}(k)e_{(t-1)n+\xi}\Big)\overline{\chi}_\delta(u)dudk$$

$$= \frac{1}{q_1}\sum_{\mu=1}^{d}\sum_{t=1}^{q_1}\int_K\int_K d_{\mu,(\alpha-1)q_1+t}(k)\overline{d_{\mu,(\beta-1)q_1+t}}(\overline{x^{-1}}(u^{-1}k))\times$$

$$\times\lambda_{\xi\eta}(s(x^{-1},u^{-1}k)^{-1})\overline{\chi}_\delta(u)dudk$$

$$= \frac{1}{q_1}\sum_{\mu=1}^{d}\sum_{t=1}^{q_1}\sum_{a,b=1}^{d}\Big\{d\int_K d_{\mu,(\alpha-1)q_1+t}(k)\overline{d_{ab}}(k)dk\Big\}\times$$

$$\times\Big\{\int_K \overline{d_{\mu,(\beta-1)q_1+t}}(\overline{x^{-1}}u^{-1})\lambda_{\xi\eta}(s(x^{-1},u^{-1})^{-1})\overline{d_{ba}}(u)du\Big\}$$

$$= \frac{1}{q_1}\sum_{\mu=1}^{d}\sum_{t=1}^{q_1}\int_K \overline{d_{(\alpha-1)q_1+t,\mu}}(u)\overline{d_{\mu,(\beta-1)q_1+t}}(\overline{x^{-1}}u^{-1})\times$$

$$\times\lambda_{\xi\eta}(s(x^{-1},u^{-1})^{-1})du$$

$$= \frac{1}{q_1}\sum_{t=1}^{q_1}\int_K \overline{d_{(\alpha-1)q_1+t,(\beta-1)q_1+t}}(u\cdot\overline{x^{-1}}u^{-1})\lambda_{\xi\eta}(s(x^{-1},u^{-1})^{-1})du$$

$$= \int_K \overline{a^{\sigma_1}_{\alpha\beta}}(u\cdot\overline{x^{-1}}u^{-1})\lambda_{\xi\eta}(s(x^{-1},u^{-1})^{-1})du.$$

This shows that the matrix function of type δ defined by $\{\mathfrak{Q}^{\sigma_1,\Lambda}, T_x^{\sigma_1,\Lambda}\}$ is equal to $U^{\sigma_1,\Lambda}$.

Especially, we may consider the case $M = \{e\}$. In this case, we shall denote by T_x^{Λ} (where Λ is a finite-dimensional irreducible representation of S) the operator on $L^2(K)$ such that

$$(T_x^{\Lambda}\varphi)(k) = \Lambda(s(x^{-1},k)^{-1})\ \varphi(\overline{x^{-1}}k).$$

The above argument can be applied to this case, and we can

show that U^{Λ} is a matrix function of type δ defined by $\{L^2(K), T_x^{\Lambda}\}$.

Now we have proved the following

THEOREM 16. Let G be a locally compact unimodular group, K a compact subgroup of G, and δ an equivalence class in Ω_K. We assume that there exists a closed subgroup S of G such that $G = KS$, $K \cap S = \{e\}$, and that the decomposition $x = ks$ ($k \in K$, $s \in S$) is continuous. Let M be a closed subgroup of K which normalizes S, and Λ a finite-dimensional irreducible representation of S satisfying $\Lambda(msm^{-1}) = \Lambda(s)$ for all $m \in M$. Then

(i) U^{Λ} is equivalent to a matrix function of type δ defined by the representation $\{L^2(K), T_x^{\Lambda}\}$ of G.

(ii) Let σ_i ($1 \leqq i \leqq r$) be all equivalence classes in Ω_M which occur p_i-times in δ. If we write q_i for the degree of σ_i, then the matrix functions $U^{\sigma_i, \Lambda}$ are of degree p_i, and U^{Λ} is equivalent to

$$\begin{pmatrix} U^{\sigma_1,\Lambda} & & & & 0 \\ & \ddots & & & \\ & & U^{\sigma_1,\Lambda} & & \\ & & & \ddots & \\ & & & & U^{\sigma_r,\Lambda} \\ & & & & \ddots \\ 0 & & & & U^{\sigma_r,\Lambda} \end{pmatrix}.$$

(iii) $U^{\sigma_i,\Lambda}$ is equivalent to a matrix function of type δ defined by the representation $\{\mathfrak{H}^{\sigma_i,\Lambda}, T_x^{\sigma_i,\Lambda}\}$ of G.

The irreducible components of U^{Λ} or $U^{\sigma_i,\Lambda}$ $(1 \leqq i \leqq r)$ are spherical matrix functions of type δ. The author does not know how many spherical matrix functions are obtained in this way. But, if G is a motion group or a connected semi-simple Lie group with finite center, we denote by K a maximal compact subgroup of G, then we obtain all of the spherical matrix functions of type $\delta \in \Omega_K$ in this way.

1. For a motion group G (i.e., G has a compact subgroup K and a closed abelian normal subgroup A such that $G = KA$, $K \cap A = \{e\}$), R.Godement studied the form of an irreducible representation of $L^0(\delta)$ in [5], and the above result follows immediately from his study.

2. Next, let's consider the case that G is a connected semi-simple Lie group with finite center. Let $\mathfrak{G}$ be the Lie algebra of G, and $\mathfrak{G} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of $\mathfrak{G}$, where, as usual, $\mathfrak{k}$ denotes a maximal compact subalgebra. Let $\mathfrak{h}^-$ be a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{G} = \mathfrak{k} + \mathfrak{h}^- + \mathfrak{n}$ an Iwasawa decomposition of $\mathfrak{G}$, and $G = KAN$ the corresponding Iwasawa decomposition of G. We shall denote by M the centralizer of A in K.

Since every finite-dimensional irreducible representation of $S = AN$ is one-dimensional, it is identified with a

one-dimensional representation of A. And every one-dimensional representation λ of A satisfies $\lambda(mam^{-1}) = \lambda(a)$ for all $m \in M$. Therefore, for a fixed class $\delta \in \Omega_K$, we can define the matrix functions U^{λ}, $U^{\sigma_i,\lambda}$ $(1 \leqq i \leqq r)$, and the representations $\{L^2(K), T_x^{\lambda}\}$, $\{\mathfrak{H}^{\sigma_i,\lambda}, T_x^{\sigma_i,\lambda}\}$ of G as above, where $\sigma_1, \cdots, \sigma_r$ are all of the classes in Ω_M which occur in δ.

Let $\{\mathfrak{H}, T_x\}$ be a topologically irreducible nice representation of G on a complete locally convex topological vector space $\mathfrak{H}$. Then it is K-finite by Theorem 4, and has an infinitesimal character. Therefore we know that Theorem 5.5.1.5 in [9] remains true for our representation $\{\mathfrak{H}, T_x\}$. Namely, there exists a one-dimensional representation λ of A and T_x-invariant closed subspaces H_1, H_2 of $L^2(K)$ such that (i) $H_1 \subset H_2$, (ii) the representation γ of G induced from T_x on the Hilbert space $H = H_1/H_2$ is topologically irreducible, (iii) $\dim H(\delta) < +\infty$ for all $\delta \in \Omega_K$, where $H(\delta)$ is the set of all vectors in H transformed according to δ under $\gamma(K)$, and (iv) $\{\mathfrak{H}, T_x\}$ is infinitesimally equivalent to γ. Therefore, by Theorem 13, we obtain the following theorem which gives a generalization of the well known formula on the construction of spherical functions of type 1.

THEOREM 17. Let G be a connected semi-simple Lie group with finite center, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G, where $\mathfrak{k}$ denotes a maximal compact

subalgebra. Let $\mathfrak{h}^-$ be a maximal abelian subalgebra of $\mathfrak{p}$, $\mathfrak{g} = \mathfrak{k} + \mathfrak{h}^- + \mathfrak{n}$ an Iwasawa decomposition of $\mathfrak{g}$, and $G = KAN$ the corresponding Iwasawa decomposition of G. Let M be the centralizer of A in K, and $\sigma_1, \cdots, \sigma_r$ the all classes in Ω_M which occur in $\delta \in \Omega_K$. Then, for every spherical matrix function U of type δ, there exists a one-dimensional representation λ of A such that U is equivalent to an irreducible component of $U^{\sigma_i, \lambda}$ for some i.

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