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Department of Mathematics
KYOTO UNIVERSITY

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AUTOMORPHIC FORMS AND ALGEBRAIC EXTENSIONS OF NUMBER FIELDS

BY

HIROSHI SAITO

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Preface

These notes contains the contents of my doctorial thesis. In these notes, I give a result on an arithmetical relation between Hilbert cusp forms over totally real algebraic number fields and cusp forms of one variable.

H. Saito

February 3, 1975

Contents

Automorphic forms and algebraic extensions of number fields

by

Hiroshi SAITO

§ 0. Introduction

0.0. The purpose of this paper is to study an arithmetical relation between Hilbert cusp forms over a totally real algebraic number field and cusp forms of one variable by using the theory of Hecke operators.

Let F be a totally real algebraic number field, and $\mathfrak{o}$ be its maximal order. For an even positive integer κ, let $S_\kappa(\Gamma)$ denote the space of Hilbert cusp forms of weight κ with respect to the subgroup $\Gamma = GL_2(\mathfrak{o})_+$ consisting of all elements with totally positive determinants in $GL_2(\mathfrak{o})$. For a place (archimedean or non-archimedean) v of F, let F_v be the completion of F at v. For a non-archimedean place $v\ (= \mathfrak{p})$, let $\mathfrak{o}_\mathfrak{p}$ be the ring of $\mathfrak{p}$-adic integers of F_v. Let F_A be the adele ring of F, and consider the adele group $GL_2(F_A)$. Let $\mathfrak{U}_F$ be the open subgroup $\prod_{\mathfrak{p}:\text{non-archimedean}} GL_2(\mathfrak{o}_\mathfrak{p}) \times \prod_{v:\text{archimedean}} GL_2(F_v)$ of $GL_2(F_A)$.

Then we can consider the Hecke ring $R(\mathcal{U}_F, GL_2(F_A))$ and its action T on $S_\kappa(\Gamma)$ as in G. Shimura [18]. For some technical reasons, we shall work with a certain subring $R^0(\mathcal{U}_F, GL_2(F_A))$ of $R(\mathcal{U}_F, GL_2(F_A))$. It's precise definition will be given in §5.1, but roughly speaking $R^0(\mathcal{U}_F, GL_2(F_A))$ is the subring consisting of all elements of $R(\mathcal{U}_F, GL_2(F_A))$ which are relatively prime to the discriminant of the extension F/Q.

For the ordinary modular group $SL_2(Z)$ ($= GL_2(Z)_+$), we also consider its adelization $\mathcal{U}_Q = \prod GL_2(Z_p) \times GL_2(R)$ and the Hecke ring $R(\mathcal{U}_Q, GL_2(Q_A))$. The latter is acting on the space $S_\kappa(SL_2(Z))$ of cusp forms.

<u>0.1.</u> <u>The space $SS_\kappa(\Gamma)$.</u> Suppose F is a cyclic extension of Q of degree ℓ. Fixing a generator σ of the Galois group $Gal(F/Q)$, we define an operator T_σ on $S_\kappa(\Gamma)$ by the permutation of variables, namely $T_\sigma f(z_1, \dots, z_\ell) = f(z_2, \dots, z_\ell, z_1)$. Using this T_σ, we define a new subspace $SS_\kappa(\Gamma)$ of $S_\kappa(\Gamma)$, to be called "the space of symmetric Hilbert cusp forms", as follows:

$$SS_\kappa(\Gamma) = \{ f \in S_\kappa(\Gamma) \mid T(e)T_\sigma f = T_\sigma T(e)f \text{ for any } e \in R(\mathcal{U}_F, GL_2(F_A$$

Obviously $SS_\kappa(\Gamma)$ is stable under the action of $R(\mathcal{U}_F, GL_2(F_A))$, and we get a new representation T_S of the Hecke ring.

$R(\mathfrak{M}_F,\ GL_2(F_A))$ (or $R^0(\mathfrak{M}_F,\ GL_2(F_A))$) on the space $\mathfrak{S}_\kappa(\Gamma)$.

Now we assume

0) The weight $\kappa \geqq 4$.

1) The degree $\ell = [F : Q]$ is a prime.

2) The class number of F is one.

3) $\mathfrak{O}$ has a unit of any signature distribution.

4) F is tamely ramified over Q.

As a consequence of 2) and 4), the conductor of F/Q is a prime number q.

The purpose of this paper is to show that the representation T_S of $R^0(\mathfrak{M}_F,\ GL_2(F_A))$ on $\mathfrak{S}_\kappa(\Gamma)$ can be obtained from the spaces of cusp forms $S_\kappa(SL_2(Z))$ and $S_\kappa(\Gamma_0(q),\ \chi)$ for various characters χ of $(Z/qZ)^\times$ of order ℓ , where of course

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \;\middle|\; c \equiv 0 \mod. q \right\}.$$

To give a meaningful description for the above, we shall define a "natural" homomorphism $\lambda :\ R^0(\mathfrak{M}_F,\ GL_2(F_A)) \longrightarrow R^0(\mathfrak{M}_Q,\ GL_2(Q_A))$ in the next section 0.2. Here $R^0(\mathfrak{M}_Q,\ GL_2(Q_A))$ is defined as a subring consisting of all elements of $R(\mathfrak{M}_Q,\ GL_2(Q_A))$ which are relatively prime to the conductor q of F/Q. Then $R^0(\mathfrak{M}_Q,\ GL_2(Q_A))$ is acting not only on $S_\kappa(SL_2(Z))$ but also on

$S_\kappa(\Gamma_0(q), \chi)$ via natural injection

$R^0(\mathfrak{U}_Q, GL_2(Q_A)) \hookrightarrow R(\Gamma_0(q), GL_2(Q))$, hence it has representations T_1 on $S_\kappa(SL_2(Z))$ and T_χ on $S_\kappa(\Gamma_0(q), \chi)$. Thus $S_\kappa(SL_2(Z))$ (resp. $S_\kappa(\Gamma_0(q), \chi)$) can be viewed as a $R^0(\mathfrak{U}_F, GL_2(F_A))$-module by the action $T_1 \circ \lambda$ (resp. $T_\chi \circ \lambda$). Now our main result (Th. 3) claims that there exists a subspace S of $\bigoplus_\chi S_\kappa(\Gamma_0(q), \chi)$ such that

$$\mathfrak{S}_\kappa(\Gamma) \simeq S_\kappa(SL_2(Z)) \oplus S$$

$$(\text{and} \quad \bigoplus_\chi S_\kappa(\Gamma_0(q), \chi) \simeq S \oplus S \ )$$

as $R^0(\mathfrak{U}_F, GL_2(F_A))$-modules, where in $\bigoplus_\chi$, χ runs through all the characters of order ℓ of $(Z/qZ)^\times$. The above result will be derived by standard arguments from the following equality of the traces of the operators:

<u>Theorem</u>

$$(*) \quad \mathrm{tr}\ T_{\mathfrak{S}}(e) = \mathrm{tr}\ T_1(\lambda(e)) + \frac{1}{2} \sum_\chi \mathrm{tr}\ T_\chi(\lambda(e))$$

for any $e \in R^0(\mathfrak{U}_F, GL_2(F_A))$.

The proof of this last equality will occupy the most part of this paper.

0.2. The homomorphism $\lambda : R^0(\mathfrak{U}_F, GL_2(F_A)) \rightarrow R^0(\mathfrak{U}_Q, GL_2(Q_A))$.

Let $\mathfrak{a}$ (resp. n) be an integral ideal of F (resp. a positive integer), and $T(\mathfrak{a})$ (resp. $T(n)$) be the sum of all integral element in $R(\mathfrak{U}_F, GL_2(F_A))$ (resp. $R(\mathfrak{U}_Q, GL_2(Q_A))$) of norm $\mathfrak{a}$ (resp. n). For a prime ideal $\mathfrak{p}$ of F (resp. a prime p), let $T(\mathfrak{p}, \mathfrak{p})$ (resp. $T(p, p)$) denote the double coset $\mathfrak{U}_F \alpha \mathfrak{U}_F$ (resp. $\mathfrak{U}_Q \alpha \mathfrak{U}_Q$), where the $\mathfrak{p}$-component (resp. p-component) of α is $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ (resp. $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$) with a prime element π of $\mathcal{O}_{\mathfrak{p}}$, and the other component of α is the identity. We define elements $U(\mathfrak{p}^m)$ (resp. $U(p^m)$) of $R(\mathfrak{U}_F, GL_2(F_A))$ (resp. $R(\mathfrak{U}_Q, GL_2(Q_A))$) for a prime ideal $\mathfrak{p}$ of F (resp. a prime p) and a non-negative integer m by

$$U(\mathcal{O}) = 2\,T(\mathcal{O})$$

$$(\text{resp. } U(1) = 2\,T(1)\ )$$

$$U(\mathfrak{p}^m) = \begin{cases} T(\mathfrak{p}) & , m = 1 \\ T(\mathfrak{p}^m) - N\mathfrak{p}\, T(\mathfrak{p}, \mathfrak{p})\, T(\mathfrak{p}^{m-2}) & , m \geqq 2 \end{cases}$$

$$\left(\text{resp. } U(p^m) = \begin{cases} T(p) & , m = 1 \\ T(p^m) - p\, T(p, p)\, T(p^{m-2}) & , m \geqq 2 \end{cases}\right)$$

, where $N\mathfrak{q}$ is the cardinality of $\mathcal{O}/\mathfrak{q}$. Then the correspondence $U(\mathfrak{z}^m) \longrightarrow U(N\mathfrak{z}^m)$ can be extended to a homomorphism λ from $R(\mathfrak{N}_F, GL_2(F_A))$ to $R(\mathfrak{N}_Q, GL_2(Q_A))$.

0.3. We give an outline of each section. In § 1, we define the space $\mathfrak{S}_{\kappa}(\Gamma)$ and make some preliminary consideration on the representation T_S . In § 2, by using Selberg's trace formula, we shows that $\mathrm{tr}\ T_S(T(\sigma\iota))$ can be expressed as a sum extended over twisted conjugacy classes (c.f. (2.12.1)). In § 3, we study the twisted conjugacy classes and in particular determine the numbers $c_\sigma(f, r, \Lambda)$ and $c_\sigma(\alpha, r, \Lambda)$ explicitly (c.f. § 3.6 , § 3.12). In § 4, by making use of the results of § 2 and § 3, we give an explicit formula for $\mathrm{tr}\ T_S(T(\sigma\iota))$. In § 5, from the explicit formula for $\mathrm{tr}\ T_S(T(\sigma\iota))$, $\mathrm{tr}\ T_1(T(n))$ and $\mathrm{tr}\ T_\chi(T(n))$, we deduce our main result.

0.4. Applications. Our result is related to the recent works of the following authors.

(I) In their joint work [2], K. Doi and H. Naganuma studied a relation between cusp forms with respect to $SL_2(Z)$ and Hilbert cusp forms over real quadratic fields. More precisely, let $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, $a_1 = 1$, be the Dirichlet series associated with a cusp form of weight κ with respect to $SL_2(Z)$ which is a common eigen-function for all Hecke operators, and let χ be the real character corresponding to a real quadratic field $F = Q(\sqrt{D})$ in the sense of the class field theory. If we put $\varphi(s, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}$, then $\varphi(s)\varphi(s,\chi)$ can be expressed in the following form with suitable coefficients $C_{\mathfrak{a}}$ which are defined for every integral ideal $\mathfrak{a}$ in F :

$$\varphi(s)\varphi(s,\chi) = \sum_{\mathfrak{a}} C_{\mathfrak{a}} N\mathfrak{a}^{-s} .$$

For a Grössen-character ξ of F, we set

$$D(s, \varphi, \chi, \xi) = \sum_{\mathfrak{a}} \xi(\mathfrak{a}) C_{\mathfrak{a}} N\mathfrak{a}^{-s} .$$

In [2] , K. Doi and H. Naganuma tried to prove a functional equation of $D(s, \varphi, \chi, \xi)$, and proved it for the case where the conductor of ξ is one, and showed that if the maximal order of F is an Euclidean domain, the Dirichlet series $\varphi(s)\varphi(s,\chi)$ is actually associated with a Hilbert cusp form over F and the function

$$h(z_1, z_2) = \sum_{\substack{\mathfrak{a}=(\mu) \\ \frac{\mu}{\sqrt{q}} \gg 0}} C_{\mathfrak{a}} \sum_{\varepsilon \in E_+} \exp\left(2\pi\sqrt{-1}\left(\frac{\mu\varepsilon}{\sqrt{q}} z_1 + \left(\frac{\mu\varepsilon}{\sqrt{q}}\right)^{\sigma} z_2\right)\right)$$

on the product $H \times H$ of the complex upper half planes is a Hilbert cusp form over F . Moreover in [14] H. Naganuma showed that a similar result holds also for cusp forms of "Neben" type (in Hecke's sense) with a prime level. Now, from our present result for $\ell = 2$, it can be proved that $\varphi(s)\,\varphi(s,\chi)$ is the Dirichlet series associated with a Hilbert cusp form over a real quadratic field F , and that Doi-Naganuma's construction is "injective" (see text Th.3, Cor2) under the condition for F in this paper. In fact, an effort to show this injectivity is the main motivation of our study.

(II) In [12] , H. Jacquet studied the similar theme as Doi-Naganuma's, in a more general (adelic and representation-theoretic) point of view, hence this result should have a close connection to ours.

(III) F. Hirzebruch [9] [10] and R. Busam [1] gave a dimension formula for the subspace $S_k(\hat{\Gamma})$ of $S_k(\Gamma)$ consisting of elements f such that $T_\sigma f = (-1)^{k/2} f$. Since there is an obvious relation

$$\dim S_k(\hat{\Gamma}) = \frac{1}{2}\left(\dim S_k(\Gamma) + (-1)^{k/2} \dim S_k^{\sigma}(\Gamma) \right),$$

our result can be viewed as a generalization of their formula.

C.5. Notation. As usual, Z, Q, R and C denote respectively the ring of rational integers, the rational number field, the real number field, and the complex number field. For a rational prime p, Z_p and Q_p denote the ring of p-adic integers and the field of p-adic numbers, respectively. For every element $z \in C$, we denote by z and $\mathrm{Im}(z)$ the complex conjugate and the imaginary part of z, respectively. We denote by ϕ the empty set, and for a set S by $|S|$ the cardinality of S (however if $z \in C$, $|z|$ denotes the ordinary absolute value of z). For a ring S with the unity 1, we denote by $S^{\times}$ the multiplicative group of the invertible elements of S, and by $M_2(S)$ the ring of 2 by 2 matrices over S, and we put $GL_2(S) = M_2(S)^{\times}$. If S is commutative, we denote by $\det s$ (resp. $\mathrm{tr}\, s$) the determinant (resp. trace) of s for $s \in M_2(S)$ and identify the center of $M_2(S)$ with S. For subsets S_{ij} $(1 \leq i, j \leq 2)$ of S, $(S_{ij}) \subset M_2(S)$ denotes the set $\{ s = (s_{ij}) \in M_2(S) \mid s_{ij} \in S_{ij} \}$.

The author would like to express his hearty thanks to Prof. K. Doi and Prof. H. Hijikata for their valuable

suggestions and encouragement.

§1. Definition of the space $\mathfrak{S}_k(\mathrm{T}_\sigma)$.

1.1. Let F be a totally real algebraic number field, which is a cyclic extension of the rational number field Q of a prime degree ℓ. And let $\mathcal{O}$ be the maximal order of F, E the group of units of F and E_+ its subgroup of totally positive elements of E. We assume that the class number of F is equal to one and that the index $[E : E_+]$ is equal to 2^ℓ. Moreover we shall assume that F is a tamely ramified extension of Q later. We denote by $\mathcal{G}$ the Galois group of the extension F/Q. We fix a generator of $\mathcal{G}$, and denote it by σ. If we fix an embedding of F into R and identify F with its image, then all the distinct embedding of F into R is given by $a \longmapsto {}^{\sigma_i}a$ for $\sigma_i = \sigma^{i-1}$. Let H be the complex upper half plane and H^ℓ be the product of ℓ copies of H. Let $GL_2(R)_+$ be the subgroup of $GL_2(R)$ consisting of all elements g such that $\det g > 0$, and let $GL_2(R)_+^\ell$ be the product of ℓ copies of $GL_2(R)_+$. Then $GL_2(R)_+^\ell$ acts on H^ℓ by

$$gz = (g^{(1)}z^{(1)}, \ldots\ldots, g^{(\ell)}z^{(\ell)}),$$

$$g^{(i)}z^{(i)} = \frac{a^{(i)}z^{(i)} + b^{(i)}}{c^{(i)}z^{(i)} + d^{(i)}}, \qquad g^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix},$$

for $g = (g^{(1)}, \ldots, g^{(\ell)}) \in GL_2(R)_+^\ell$ and $z = (z^{(1)}, \ldots, z^{(\ell)}) \in H^\ell$ as an analytic transformation group. Let $GL_2(F)_+$ be the subgroup of $GL_2(F)$ consisting of all element g such that $\det g$ is totally positive. With σ_i's , we can embed $GL_2(F)_+$ into $GL_2(R)_+^{\ell}$ by

$$g \longmapsto ({}^{\sigma_1}g, \ldots, {}^{\sigma_\ell}g)$$

where ${}^{\sigma_i}g = \begin{pmatrix} {}^{\sigma_i}a & {}^{\sigma_i}b \\ {}^{\sigma_i}c & {}^{\sigma_i}d \end{pmatrix}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F)_+$. With this embedding, we concider $GL_2(F)_+$ as a transformation group on H^{ℓ}. Let κ be an even positive integer. Put

$$j(g, z) = \prod_{i=1}^{\ell} (c^{(i)}z^{(i)} + d^{(i)})^{-\kappa} |\det g^{(i)}|^{\kappa/2}$$

for $g = (g^{(i)}) \in GL_2(R)_+^{\ell}$, $z \in H^{\ell}$. Let Γ be the subgroup $GL_2(\mathcal{O})_+ = GL_2(\mathcal{O}) \cap GL_2(F)_+$ of $GL_2(\mathcal{O})$. We denote by $S_\kappa(\Gamma)$ the space of all functions $f(z)$ on H^{ℓ} satisfying the following conditions.

(S1) $f(z)$ is holomorphic on H^{ℓ}.

(S2) $f(\gamma z) = j(\gamma, z)^{-1} f(z)$ for $\gamma \in \Gamma$.

(S3) $f(z)$ is regular at every parabolic point x of Γ and the constant term in the Fourier expansion of $f(z)$ at x vanishes.

Let $\mathfrak{F}_r$ be a fundamental domain of Γ in H^{ℓ}. In the space $S_\kappa(\Gamma)$, we have an inner product given by

$$(1.1.1) \qquad \langle f, g \rangle = \int_{\mathfrak{F}} f(z)\overline{g(z)} \left(\prod y^{(i)}\right)^{\kappa} dz \qquad \text{for } f, g \in S_\kappa(\Gamma)$$

where $z^{(i)} = x^{(i)} + \sqrt{-1}y^{(i)}$, $dz = \prod \frac{dx^{(i)}dy^{(i)}}{y^{(i)2}}$, and $\overline{g(z)}$ denotes the complex conjugate of $g(z)$.

<u>1.2.</u> For a place v of F, we denote by F_v the completion of F at v. We shall use $\mathfrak{p}$ to denote finite places. Let $\mathcal{O}_{\mathfrak{p}}$ be the ring of $\mathfrak{p}$-adic integers in $F_{\mathfrak{p}}$. And let F_A and $F_A^{\times}$ be the adele ring and idele group of F respectively. We denote the subgroup $\prod_{\mathfrak{p}} GL_2(\mathcal{O}_{\mathfrak{p}}) \times \prod_{v} GL_2(F_v)$ of $GL_2(F_A)$ by $\mathcal{U}_F$. Then for any element α of $GL_2(F_A)$, $\mathcal{U}_F$ and $\alpha\mathcal{U}_F\alpha^{-1}$ are commensurable with each other, hence we can define the Hecke ring $R(\mathcal{U}_F, GL_2(F_A))$ as in G. Shimura [18]. Namely $R(\mathcal{U}_F, GL_2(F_A))$ is a free Z-module generated by all $\mathcal{U}_F\alpha\mathcal{U}_F$ ($\alpha \in GL_2(F_A)$) with a structure of ring as well. And in our case $R(\mathcal{U}_F, GL_2(F_A))$ is a commutative ring. Now we define a representation of $R(\mathcal{U}_F, GL_2(F_A))$ in the vector space $S_k(\Gamma)$. For a $\mathcal{U}_F$-double $\mathcal{U}_F\alpha\mathcal{U}_F$, by the assumption on F, $\mathcal{U}_F\alpha\mathcal{U}_F \cap GL_2(F)_+$ is a Γ-double coset. Let $\mathcal{U}_F\alpha\mathcal{U}_F \cap GL_2(F)_+ = \bigcup_{\nu=1}^{d} g_\nu\Gamma$ be a disjoint union. For f of $S_k(\Gamma)$, put

$$(T(\mathcal{U}_F\alpha\mathcal{U}_F)f)(z) = \sum_{\nu=1}^{d} j(g_\nu^{-1}, z)f(g_\nu^{-1}, z) \quad ,$$

then $T(\mathcal{U}_F\alpha\mathcal{U}_F)f$ is also contained in $S_k(\Gamma)$, and T can be extended to a linear mapping of $R(\mathcal{U}_F, GL_2(F_A))$ into the ring of endomorphisms of $S_k(\Gamma)$. It is actually a ring homomorphism, and gives a representation of $R(\mathcal{U}_F, GL_2(F_A))$ in $S_k(\Gamma)$. $T(\mathcal{U}_F\alpha\mathcal{U}_F)$ is a normal operator with respect to the inner product given by (1.1.1) and $R(\mathcal{U}_F, GL_2(F_A))$ is a commutative ring, hence there exists a basis of $S_k(\Gamma)$ consisting of common eigen-functions for all $T(\mathcal{U}_F\alpha\mathcal{U}_F)$ ([8]).

Now we define another linear operator T_σ in $S_k(\Gamma)$. Let

T_σ be an automorphism of H^ℓ given by

$$T_\sigma(z^{(1)}, \ldots, z^{(\ell)}) = (z^{(2)}, \ldots, z^{(\ell)}, z^{(1)})$$

for $z = (z^{(1)}, \ldots, z^{(\ell)}) \in H^\ell$. Then as elements of the transformation group of H^ℓ, T_σ and $g \in GL_2(F)_+$ satisfy the following relation,

$$(1.2.1) \qquad T_\sigma g = {}^\sigma g T_\sigma \qquad ,$$

where ${}^\sigma g = \begin{pmatrix} {}^\sigma a & {}^\sigma b \\ {}^\sigma c & {}^\sigma d \end{pmatrix}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(F)_+$. For $f \in S_k(\Gamma)$, we see easily $f(T_\sigma z)$ is also contained in $S_k(\Gamma)$. In fact, the condition (S1) in the definition of $S_k(\Gamma)$ is obvious. As to (S2), for $\gamma \in \Gamma$ we have

$$\begin{aligned} f(T_\sigma \gamma z) &= f(T_\sigma({}^{\sigma_1}\gamma z^{(1)}, \ldots, {}^{\sigma_\ell}\gamma z^{(\ell)})) \\ &= f({}^{\sigma_2}\gamma z^{(2)}, \ldots, {}^{\sigma_1}\gamma z^{(1)}) \\ &= \prod (c^{(i)} z^{(i)} + d^{(i)}) f(z^{(2)}, z^{(3)}, \ldots, z^{(1)}) \\ &= j(\gamma, z)^{-1} f(T_\sigma z) \quad . \end{aligned}$$

Hence $f(T_\sigma z)$ also satisfies (S2). The condition (S3) is easy to see in our case. Actually, by the assumption on F there exists only one cusp up to Γ-equivalence and we may take $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ as a representative. Let $\mathfrak{d}$ be the different of the extension F/Q. Then f(z) has the Fourier expansion of the form

$$f(z) = \sum_{\mathfrak{a}} C(\mathfrak{a}) \sum_{\substack{\mathfrak{a}/\mathfrak{d} = (\mu) \\ \mu \gg 0}} \exp(2\pi\sqrt{-1}({}^{\sigma_1}\mu z^{(1)} + \ldots + {}^{\sigma_\ell}\mu z^{(\ell)}))$$

, where in the summation, $\mathfrak{a}$ runs through all the integral ideals of F and μ runs through all totally positive elements of F such that $\mathfrak{a}/\mathfrak{d} = (\mu)$. And $C(\mathfrak{a})$'s are the Fourier coefficients. Then the Fourier expansion of $f(T_\sigma z)$ is of the form

$$f(T_\sigma z) = \sum_{\mathfrak{a}} C(\mathfrak{a}) \sum_{\substack{\mathfrak{a}/\mathfrak{d}=(\mu) \\ \mu \gg 0}} \exp(2\pi\sqrt{-1}({}^{\sigma_1}\mu z^{(2)} + \ldots + {}^{\sigma_\ell}\mu z^{(1)})) \quad .$$

Hence the condition (S3) is obvious. For $f \in S_\kappa(\Gamma)$, put

$$(T_\sigma f)(z) = f(T_\sigma z)$$

, then T_σ defines a C-linear operator in $S_\kappa(\Gamma)$, and T_σ obviously induces an automorphism in $S_\kappa(\Gamma)$.

Making use of this operator T_σ and the operators $T(\mathfrak{n}_F a \mathfrak{n}_F)$, we define the subspace $\mathbb{S}_\kappa(\Gamma)$ of $S_\kappa(\Gamma)$ as follows. We denote by $\mathbb{S}_\kappa(\Gamma)$ the set of all elements of $S_\kappa(\Gamma)$ which satisfy

$$(1.2.2) \qquad T_\sigma T(e)f = T(e)T_\sigma f \quad ,$$

for all $e \in R(\mathfrak{n}_F, GL_2(F_A))$. Then it is easy to see $\mathbb{S}_\kappa(\Gamma)$ is a subspace of $S_\kappa(\Gamma)$. We show that $\mathbb{S}_\kappa(\Gamma)$ is stable under the action of $R(\mathfrak{n}_F, GL_2(F_A))$. Extend the automorphism σ to F_A, $GL_2(F_A)$ and $R(\mathfrak{n}_F, GL_2(F_A))$ naturally, and denote it also by σ. Then by (1.2.1) we see easily

$$(1.2.3) \qquad T_\sigma T(e)f = T({}^{\sigma^{-1}}e)T_\sigma f$$

for $f \in S_\kappa(\Gamma)$ and $e \in R(\mathfrak{n}_F, GL_2(F_A))$. If f is contained in $\mathbb{S}_\kappa(\Gamma)$, then for any e and $e' \in R(\mathfrak{n}_F, GL_2(F_A))$,

$$
\begin{aligned}
T_\sigma T(e')(T(e)f) &= T_\sigma T(e)T(e')f \\
&= T({}^{\sigma^{-1}}e)T_\sigma T(e')f \\
&= T({}^{\sigma^{-1}}e)T(e')T_\sigma f \\
&= T(e')T_\sigma(T(e)f)
\end{aligned}
$$

Hence $T(e)f$ is also contained in $\mathfrak{S}_\kappa(\Gamma)$, and we obtain a representation of $R(\mathfrak{U}_F, GL_2(F_A))$ in the space $\mathfrak{S}_\kappa(\Gamma)$. We denote this representation by T_S.

In the rest of this section, we give some preliminary consideration on this representation.

<u>1.3.</u> First we make some remarks on the representation T. It is known that the Hecke ring $R(\mathfrak{U}_F, GL_2(F_A))$ is isomorphic to the tensor product of the Hecke rings $R(GL_2(\mathfrak{o}_\mathfrak{p}), GL_2(F_\mathfrak{p}))$ with respect to $GL_2(\mathfrak{o}_\mathfrak{p})$ and $GL_2(F_\mathfrak{p})$ by the correspondence,

$$\mathfrak{U}_F\, \alpha\, \mathfrak{U}_F \longleftrightarrow \otimes\, GL_2(\mathfrak{o}_\mathfrak{p})\alpha_\mathfrak{p} GL_2(\mathfrak{o}_\mathfrak{p}) \quad ,$$

where $\alpha_\mathfrak{p}$ denotes the $\mathfrak{p}$-component of $\alpha \in GL_2(F_A)$. And it is also known that $R(GL_2(\mathfrak{o}_\mathfrak{p}), GL_2(F_\mathfrak{p}))$ is generated as a ring by the $GL_2(\mathfrak{o}_\mathfrak{p})$-double cosets $GL_2(\mathfrak{o}_\mathfrak{p})\begin{pmatrix} 1 & \\ & \pi \end{pmatrix}GL_2(\mathfrak{o}_\mathfrak{p})$, $GL_2(\mathfrak{o}_\mathfrak{p})\begin{pmatrix} \pi & \\ & \pi \end{pmatrix}GL_2(\mathfrak{o}_\mathfrak{p})$ and $GL_2(\mathfrak{o}_\mathfrak{p})\begin{pmatrix} \pi^{-1} & \\ & \pi^{-1} \end{pmatrix}GL_2(\mathfrak{o}_\mathfrak{p})$, where π is a prime element of $\mathfrak{o}_\mathfrak{p}$. We see easily that the double coset of the form $\mathfrak{U}_F\alpha\mathfrak{U}_F$ with $\alpha \in F_A^\times$ acts on $\mathfrak{S}_\kappa(\Gamma)$ as an identity, hence the representation T is determined by the restriction of it to the subring $R_I(\mathfrak{U}_F, GL_2(F_A))$ of $R(\mathfrak{U}_F, GL_2(F_A))$, which generated by the double cosets $\mathfrak{U}_F \alpha \mathfrak{U}_F$ such that the right $M_2(\mathcal{O}_F)$-ideal $\bigcap \alpha_{\mathfrak{p}} M_2(\mathcal{O}_{\mathfrak{p}})$ is integral. For an integral ideal $\mathfrak{a}$ of F, we denote by $T(\mathfrak{a})$ the sum of all the double cosets $\mathfrak{U}_F \alpha \mathfrak{U}_F$ such that the right $M_2(\mathcal{O})$-ideal $\bigcap \alpha_{\mathfrak{p}} M_2(\mathcal{O}_{\mathfrak{p}})$ is integral and of the norm $\mathfrak{a}$. Then by the well-known formula for the Hecke ring $R(\mathfrak{U}_F, GL_2(F_A))$, T is determined by the action of $T(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$.

Now we will describe the space $\mathbb{S}_\kappa(\Gamma)$ using $T(\mathfrak{a})$. We note that $T_S(T(\mathfrak{a}))$ and $T_S(T({}^{\sigma}\mathfrak{a}))$ are equal to each other as operators on $\mathbb{S}_\kappa(\Gamma)$ for any integral ideal $\mathfrak{a}$. This is easily seen by (1.2.3) and the definition (1.2.2) of $\mathbb{S}_\kappa(\Gamma)$. As remarked before, there exists a basis of $S_\kappa(\Gamma)$ consisting of common eigen-functions for all $T(e)$, and for a common eigen-function f for all $T(e)$, we have the following.

<u>Lemma 1.1.</u> Let f be a common eigen-function for all operators $T(e)$. Then f belongs to $\mathbb{S}_\kappa(\Gamma)$ if and only if the eigen-value $a(\mathfrak{a})$ of f for $T(T(\mathfrak{a}))$ is equal to that $a({}^{\sigma}\mathfrak{a})$ for $T(T({}^{\sigma}\mathfrak{a}))$ for all integral ideals $\mathfrak{a}$.

<u>Proof.</u> If f belongs to $\mathbb{S}_\kappa(\Gamma)$, then by the above remark $a(\mathfrak{a})$ is equal to $a({}^{\sigma}\mathfrak{a})$. On the other hand, if $a(\mathfrak{a})$ is equal to $a({}^{\sigma}\mathfrak{a})$, then by (1.2.3) f satisfies the condition (1.2.2) for $T(T(\mathfrak{a}))$, hence also for all $T(e)$. And f belongs to $\mathbb{S}_\kappa(\Gamma)$.

<u>Corollary 1.2.</u> $\mathbb{S}_\kappa(\Gamma)$ is the subspace of $S_\kappa(\Gamma)$ generated by the common eigen-functions for all $T(e)$ such that the

eigen-values for $T(T(\mathfrak{a}))$ and for $T(T({}^{\sigma}\mathfrak{a}))$ are equal to each other for all integral ideals $\mathfrak{a}$.

<u>Proof.</u> We note that there exists a basis of $\mathfrak{S}_{\kappa}(\Gamma)$ consisting of common eigen-functions for all $T_S(e)$ in the same way as in the case of $S_{\kappa}(\Gamma)$. Hence our corollary easily follows from Lemma 1.1. .

Let f be a common eigen-function for all $T(e)$. Then we see easily that f satisfies the condition $a(\mathfrak{a}) = a({}^{\sigma}\mathfrak{a})$ for all the integral ideals $\mathfrak{a}$ if and only if f satisfies the condition $a(\mathfrak{p}) = a({}^{\sigma}\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of F such that $\mathfrak{p} \neq {}^{\sigma}\mathfrak{p}$. For a prime number p which decomposes by the extension F/Q, let $p = \mathfrak{p}_1 \ldots \mathfrak{p}_{\ell}$ be the prime ideal decomposition in F. Then the above condition is equivalent to that f satisfies $a(\mathfrak{p}_1) = \ldots\ldots = a(\mathfrak{p}_{\ell})$ for all p which decompose by the extension F/Q. This fact shows that the definition of $\mathfrak{S}_{\kappa}(\Gamma)$ does not depend on the choice of a generator σ of $\mathfrak{G}$.

The operators $T_S(e)$ in $\mathfrak{S}_{\kappa}(\Gamma)$ are normal with respect to the inner product (1.1.1), hence they generate a commutative semi-simple algebra of operators over C. Hence the representation T_S is determined by all traces $tr(T_S(e))$ of $T_S(e)$, and also by all the traces $tr\, T_S(T(\mathfrak{a}))$ of $T_S(T(\mathfrak{a}))$. On $tr\, T_S(T(\mathfrak{a}))$, we can prove the following.

<u>Proposition 1.3.</u> For an integral ideal $\mathfrak{a}$ of F, we have

$$(1.3.1) \qquad tr(T_S(T(\mathfrak{a}))) = tr(T_{\sigma}T(T(\mathfrak{a}))) = tr(T(T(\mathfrak{a}))T_{\sigma}) \quad .$$

<u>Proof.</u> First we note that in the space $S_{\kappa}(\Gamma)$ a common

eigen-function f for all $T(T(\mathfrak{a}))$ is determined by its eigen-values $\{a(\mathfrak{a})\}$ up to a constant. This follows from a general result Th. 2 of [13], and can be proved easily in our case in the following way. Let

$$f(z) = \sum_{\mathfrak{a}} C(\mathfrak{a}) \sum_{\substack{(\mu)=\mathfrak{a}/\mathfrak{d} \\ \mu \gg 0}} \exp(2\pi\sqrt{-1}(\sum \mu^{\sigma_i} z^{(i)}))$$

be the Fourier expansion of f at the cusp $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$. For an integral ideal $\mathfrak{b}$, let $\{C'(\mathfrak{a})\}$ be the Fourier coefficients of $T(T(\mathfrak{b}))f$. Let $\Xi(\mathfrak{b})_A$ be the union of all double cosets in $T(\mathfrak{b})$. We see that $\Xi(\mathfrak{b})_A \cap GL_2(F)_+ = \bigcup_{\substack{(a)(d)=\mathfrak{b} \\ a\gg 0,\ d\gg 0}} \bigcup_{b \bmod d} \begin{pmatrix} d & b \\ 0 & a \end{pmatrix} \Gamma$

is a disjoint union. Here a and b are totally positive elements of $\mathfrak{O}$ such that $(a)(d) = \mathfrak{b}$ and run through a complete system of representatives of the equivalence classes with respect to the relation $x \sim x' \Longleftrightarrow x = x'e$ for some $e \in E$. And b is an element of $\mathfrak{O}$ and runs through a complete system of representatives of $\mathfrak{O}$ mod.d. Since $\sum_{b \bmod d} \exp(-2\pi\sqrt{-1}\sum (\frac{\mu b}{d})^{\sigma_i})$ is equal to $N(d)$ or 0 according as $(\mu/d)\mathfrak{d}$ is integral or not, we obtain

$$C'(\mathfrak{a}) = \sum_{\mathfrak{r} | (\mathfrak{a},\mathfrak{b})} N(\frac{\mathfrak{r}^2}{\mathfrak{b}})^{k/2} N(\frac{\mathfrak{b}}{\mathfrak{r}}) C(\frac{\mathfrak{a}\mathfrak{b}}{\mathfrak{r}^2}) \quad .$$

In this formula taking $\mathfrak{a}$ to be (1), we obtain

$$C'(1) = N(\mathfrak{b})^{1-\frac{k}{2}} C(\mathfrak{b}) \quad .$$

If f is an eigen-function for $T(T(\mathfrak{b}))$ with the eigen-value $a(\mathfrak{b})$, then this value is equal to $a(\mathfrak{b})C(1)$ and we obtain

$$(1.3.2) \qquad C(\mathfrak{b}) = N(\mathfrak{b})^{\kappa/2-1}a(\mathfrak{b})C(1)$$

Hence if f is a common eigen-function for all $T(T(\mathfrak{a}))$, all the Fourier coefficients are determined by the eigen-values $a(\mathfrak{b})$ up to a constant, hence f is determined up to a constant. Now if $f \in S_\kappa(\Gamma)$ is a common eigen-function for all $T(T(\mathfrak{a}))$ with eigen-values $a(\mathfrak{a})$, then by (1.2.3) $T_\sigma f$ is also a common eigen-function for all $T(T(\mathfrak{a}))$ with eigen-values $a({}^\sigma\mathfrak{a})$. Hence by Lemma 1.1. f belongs to $\mathfrak{S}_\kappa(\Gamma)$ if and only if f and $T_\sigma f$ have the same eigen-values for all $T(T(\mathfrak{a}))$, and then by the above remark f and $T_\sigma f$ differ only up to a constant. From this it follows that f is an eigen-function also for T_σ, and that T_σ transforms $\mathfrak{S}_\kappa(\Gamma)$ into itself. On the other hand, if a common eigen-function for all $T(T(\mathfrak{a}))$ does not belong to $\mathfrak{S}_\kappa(\Gamma)$, then $T_\sigma f$ belongs to a eigen-space different from that of f with respect to the representation T, and it is obvious that $T_\sigma f$ also does not belong to $\mathfrak{S}_\kappa(\Gamma)$. Hence the traces of the restrictions of the operators $T(T(\mathfrak{a}))T_\sigma$ and $T_\sigma T(T(\mathfrak{a}))$ on the orthogonal complement of $\mathfrak{S}_\kappa(\Gamma)$ in $S_\kappa(\Gamma)$ with respect to the inner product (1.1.1) are both equal to zero. Since a common eigen-function $f \in \mathfrak{S}_\kappa(\Gamma)$ for all $T(T(\mathfrak{a}))$ is also an eigen-function for T_σ and T_σ^ℓ is an identity operator in $S_\kappa(\Gamma)$, there exists a ℓ-th root of unity ξ which satisfies

$$T_\sigma f = \xi f \qquad .$$

If it is shown that ξ is equal to 1, then for a common eigen-function f for all $T(T(\mathfrak{a}))$ with the eigen-values $a(\mathfrak{a})$ of $\mathfrak{S}_k(\Gamma)$, we have

$$T_S(T(\mathfrak{a}))T_\sigma f = T_\sigma T_S(T(\mathfrak{a}))f = a(\mathfrak{a})f$$

and our proposition will be proved. Hence we show $\xi = 1$. Actually let

$$f(z) = \sum_{\mathfrak{a}} C(\mathfrak{a}) \sum_{\substack{(\mu)=\mathfrak{a}/\vartheta \\ \mu \gg 0}} \exp(2\pi\sqrt{-1}(\sum {}^{\sigma_i}\mu z^{(i)}))$$

be the Fourier expansion, then

$$T_\sigma f(z) = \sum_{\mathfrak{a}} C(\mathfrak{a}) \sum_{\substack{(\mu)=\mathfrak{a}/\vartheta \\ \mu \gg 0}} \exp(2\pi\sqrt{-1}({}^{\sigma_1}\mu z^{(2)} + {}^{\sigma_2}\mu z^{(3)} + \dots + {}^{\sigma_l}\mu z^{(1)}))$$

$$= \sum_{\mathfrak{a}} C({}^{\sigma}\mathfrak{a}) \sum_{\substack{(\mu)=\mathfrak{a}/\vartheta \\ \mu \gg 0}} \exp(2\pi\sqrt{-1}(\sum {}^{\sigma_i}\mu z^{(i)})) \quad .$$

If f is an eigen-function for T_σ with the eigen-value ξ, then it holds the following

$$C(\mathfrak{a}) = \xi C(\mathfrak{a})$$

for all ideals such that ${}^{\sigma}\mathfrak{a} = \mathfrak{a}$. Taking $\mathfrak{a}$ to be (1), we obtain $C(1) = \xi C(1)$. Now if f is a common eigen-function for all $T_S(T(\mathfrak{a}))$, we see by (1.3.2) that C(1) is not equal to zero. Hence ξ is equal to 1 and our proposition is proved.

As a corollary of the proof of Proposition 1.3., we obtain the following

Corollary 1.4. If $f \in \mathfrak{S}_k(\Gamma)$ is a common eigen-function for all $T_S(T(\mathfrak{n}))$, then f is an eigen-function also for T_σ with the eigen-value 1. And we have

$$(1.3.1)' \qquad \mathrm{tr}\, T_S(T(\mathfrak{n})) = \mathrm{tr}(T_\sigma^{-1} T(T(\mathfrak{n}))) = \mathrm{tr}(T(T(\mathfrak{n}))T_\sigma^{-1}) .$$

Thus the calculation of the trace of the operator $T_S(T(\mathfrak{n}))$ in the space $\mathfrak{S}_k(\Gamma)$ is reduced to that of the operator $T(T(\mathfrak{n}))T_\sigma^{-1}$ or $T_\sigma^{-1}T(T(\mathfrak{n}))$ in the space $S_k(\Gamma)$. In the following three sections, we shall compute the trace of $T_S(T(\mathfrak{n}))$ in $\mathfrak{S}_k(\Gamma)$.

§ 2. Selberg's trace formula

2.1. As to the detail of what is stated in 2.1. and 2.2. we refer to [16] § 3, 4 , [17] § 2 , and [6] Exposé 8, 10 .

For $z, z' \in H^{\ell}$, put

$$k(z, z') = \prod \left(\frac{z^{(i)} - \bar{z}'^{(i)}}{2\sqrt{-1}} \right)^{-\kappa} ,$$

then we have

$$k(z, z') = \overline{k(z', z)}$$

$$k(gz, gz)j(g, z)\overline{j(g, z')} = k(z, z') \quad .$$

We denote by $H_{\kappa}^{2}(\Gamma)$ (resp. $H_{\kappa}^{\infty}(\Gamma)$) the space of all functions on H^{ℓ} satisfying the conditions (S1) and (S2) in the definition of $S_{\kappa}(\Gamma)$ and

$$\|f\|_2 = \left[\int_{\mathcal{F}} |k(z, z)|^{-1} |f(z)|^2 dz \right]^{1/2} < \infty$$

$$(\text{resp. } \|f\|_{\infty} = \sup_{z \in \mathcal{F}} |k(z, z)|^{-1/2} |f(z)| < \infty) \quad .$$

Then $H_{\kappa}^{2}(\Gamma)$ (resp. $H_{\kappa}^{\infty}(\Gamma)$) forms a Banach space with respect to the norm $\| \ \|_2$ (resp. $\| \ \|_{\infty}$). The space $H_{\kappa}^{\infty}(\Gamma)$ is a closed subspace of $H_{\kappa}^{2}(\Gamma)$ and coincides with $S_{\kappa}(\Gamma)$. For $z, z' \in H^{\ell}$, put

$$K(z, z') = \sum_{\gamma \in \Gamma,\ \gamma \bmod. E} k(z, \gamma z')\overline{j(\gamma, z')} \quad .$$

Then $K(z, z')$ converges absolutely and uniformly on any compact set in $H^{\ell} \times H^{\ell}$. We have

$$K(z, z') = \overline{K(z', z)}$$

$$K(\gamma z, \gamma' z')\, j(\gamma, z)\, \overline{j(\gamma', z')} = K(z, z')$$

for $\gamma, \gamma' \in \Gamma$. And

$$\frac{K(z, z')}{|k(z, z)|^{1/2}\, |k(z', z')|^{1/2}}$$

is bounded on $H^{\ell} \times H^{\ell}$, and it holds

$$f(z) = \left(\frac{\kappa-1}{4\pi}\right)^{\ell} \int_{\mathfrak{F}} \frac{K(z, z')\, f(z')}{k(z', z')}\, dz'$$

for every $f \in H_{\kappa}^{\infty}(\Gamma)$. Let $\Xi(\sigma)_A$ be the union of all the double cosets which appears in $T(\sigma)$, and let $\Xi(\sigma)_A \cap GL_2(F)_+ = \bigcup_{\nu=1}^{d} g_{\nu}\Gamma$ be a disjoint union. Then we have

$$\begin{aligned}
tr(T(T(\sigma))T_{\sigma}^{-1}) &= \left(\frac{\kappa-1}{4\pi}\right)^{\ell} \int_{\mathfrak{F}} \sum_{\nu=1}^{d} \frac{K(T_{\sigma}^{-1} g_{\nu}^{-1} z, z)\, j(g_{\nu}^{-1}, z)}{k(z, z)}\, dz \\
&= \left(\frac{\kappa-1}{4\pi}\right)^{\ell} \int_{\mathfrak{F}} \sum_{\nu=1}^{d} \sum_{\gamma \in \Gamma,\ \mathrm{mod.}E} \frac{k(T_{\sigma}^{-1} g_{\nu}^{-1} z, \gamma z)\, \overline{j(\gamma, z)}\, j(g_{\nu}^{-1}, z)}{k(z, z)}\, dz \\
&= \left(\frac{\kappa-1}{4\pi}\right)^{\ell} \int_{\mathfrak{F}} \sum_{\nu=1}^{d} \sum_{\gamma \in \Gamma,\ \mathrm{mod.}E} \frac{k(z, g_{\nu}\gamma T_{\sigma} z)\, \overline{j(g_{\nu}\gamma, T_{\sigma} z)}}{k(z, z)}\, dz
\end{aligned}$$

Consequently we have by (1.3.3)'

(2.1.1)

$$tr(T_S(T(\sigma))) = tr(T(T(\sigma))T_{\sigma}^{-1}) = \left(\frac{\kappa-1}{4\pi}\right)^{\ell} \int_{\mathfrak{F}} \sum_{g \in \Xi(\sigma)_A \cap GL_2(F)_+\ \mathrm{mod.}E} \frac{k(z, g T_{\sigma} z)\, \overline{j(g, T_{\sigma} z)}}{k(z, z)}\, dz$$

We will calculate this integral explicitly following the method of H. Shimizu ([16], [17]). In the case where $\mathfrak{n} = \mathcal{O}$, the calculation of the above integral has been treated in R. Busam [1]. But there, the explicit calculation of them was carried out only in the case where $\ell = 2$.

2.2. As we noted before, all the parabolic points of Γ are Γ-equivalent to the infinite point $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$. We take it as a representative of the Γ-equivalence class of parabolic points. Let $\Gamma_\infty^{(1)}$ be the group of all $\gamma \in \Gamma$ leaving $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ fixed and Γ_∞ be the group consisting of all parabolic transformations in $\Gamma_\infty^{(1)}$. Put $U_\infty = \left\{ z \in H^\ell \mid \operatorname{Im} z^{(i)} > d \right\}$, where d is a suitable positive number. Let V_∞ be a fundamental domain of $\Gamma_\infty^{(1)}$ in U_∞. Then we may assume that $\mathcal{F}$ is of the form

$$\mathcal{F} = \mathcal{F}_c \cup V_\infty$$

where $\mathcal{F}_c$ is a relatively compact set in H^ℓ ([16]). We note that $|\log(y^{(i)}/y^{(j)})|$ is bounded in V_∞. This is easily seen by the section 9 in [16].

In the following we write $B = \Xi(\mathfrak{n})_A \cap GL_2(F)_+$ for the sake of simplicity. We denote by $\langle GL_2(F)_+, T_\sigma \rangle$ the group generated by $GL_2(F)_+$ and T_σ with the relation (1.?.1). Then we may consider the group $\langle GL_2(F)_+, T_\sigma \rangle$ acts on H^ℓ. Let $B_\infty^{(1)}$ (resp. $(BT_\sigma)_\infty^{(1)}$) the set of all elements in B (resp. BT_σ) leaving $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ fixed, where we consider BT_σ as a subset of $\langle GL_2(F)_+, T_\sigma \rangle$. Then we have $(BT_\sigma)_\infty^{(1)} = B_\infty^{(1)} T_\sigma$.

In fact, if an element gT_σ of BT_σ leaves $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ fixed, then since T_σ leaves $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ fixed, g also does. On the other hand if g leaves $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ fixed, then gT_σ also does.

To exchange the integral and the summation in (2.1.1), we proceed as in [16] and [17]. From [17], we quote the following lemmas. For $g \in GL_2(F)$, put ${}^{\sigma_i}g = g^{(i)} = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix}$. Then we have

Lemma 2.1. (Shimizu)

1) For $\varepsilon > 0$, we have

$$\sum_{g} \prod_{i=1}^{\ell} \frac{1}{(c^{(i)}/\det g^{(i)})} \left(\frac{1}{c^{(i)2}/\det g^{(i)} + 1} \right)^{\varepsilon} < \infty$$

g running over all the representatives of $\Gamma_\infty \backslash B - B_\infty^{(1)} / \Gamma_\infty$.

2) For $\varepsilon > 0$, we have

$$\sum_{g} \prod_{i=1}^{\ell} \left(\frac{|\det g^{(i)}|^{1/2}}{|a^{(i)} + d^{(i)}|} \right)^{\varepsilon} < \infty$$

g running over all the representatives of $B_\infty^{(1)} / \Gamma_\infty$.

Using the above lemma, we can prove the following lemmas which are analogues of Lemma 13 and 14 of [17].

Lemma 2.2. If $\kappa \geq 4$, the integral

$$\int_{V_\infty} \sum_{\substack{g \in B - B_\infty^{(1)} \\ g \bmod . E}} \frac{k(z, gT_\infty z)\overline{j(g, T_\infty z)}}{k(z, z)} dz$$

is termwise integrable.

Lemma 2.3. For $z \in H^{\ell}$, put $I(z) = \prod \operatorname{Im} z^{(i)}$. Then for $\kappa \geq 4$, we have

$$\int_{V_\infty} \sum_{g \in B_\infty^{(1)}} \frac{k(z, gT_\sigma z)\overline{j(g, T_\sigma z)}}{k(z, z)} dz$$

$$= \lim_{s \to 0} \sum_{\substack{g \in B_\infty^{(1)} \\ g \bmod. E}} \int_{V_\infty} \frac{k(z, gT_\sigma z)\overline{j(g, T_\sigma z)}}{I(z)^s k(z, z)} dz \quad .$$

The proof of the above two lemmas proceed in a quite similar way as that of Lemma 13 and 14 in [17], if it is noted that $|\log y^{(i)}/y^{(j)}|$ is bounded in V_∞ and that $(BT_\sigma)_\infty^{(1)} = B_\infty^{(1)}T_\sigma$. And we omit the proof.

On account of Lemma 2.2. and 2.3. we obtain

$$(2.2.1) \quad \left(\frac{4\pi}{\kappa-1}\right)^{\ell} \operatorname{tr} T_S(T(\sigma)) = \left(\frac{4\pi}{\kappa-1}\right)^{\ell} \operatorname{tr}(T(T(\sigma))T_\sigma^{-1})$$

$$= \sum_{\substack{g \bmod. E \\ g \in B - B_\infty^{(1)}}} \int_{\mathcal{F}} \frac{k(z, gT_\sigma z)\overline{j(g, T_\sigma z)}}{k(z, z)} dz$$

$$+ \lim_{s \to 0} \sum_{\substack{g \bmod. E \\ g \in B_\infty^{(1)}}} \left[\int_{\mathcal{F}_c} \frac{k(z, gT_\sigma z)\overline{j(g, T_\sigma z)}}{k(z, z)} dz \right.$$

$$\left. + \int_{V_\infty} \frac{k(z, gT_\sigma z)\overline{j(g, T_\sigma z)}}{I(z)^s k(z, z)} dz \right] \quad .$$

2.3. Before going further, we study some properties of the transformations of H^{ℓ} of the form gT_{σ}. For $g \in GL_2(F)_+$, put

$$Ng = {}^{\sigma_1}g\, {}^{\sigma_2}g \ldots {}^{\sigma_\ell}g \qquad . \tag{2.3.1}$$

Then by (1.2.1) $(gT_{\sigma})^{\ell}$ is equal to Ng as elements of the transformation group of H^{ℓ}. The element Ng is of one of the following types; i) $Ng \in F^{\times}$, ii) Ng is elliptic, iii) Ng is hyperbolic and no fixed point of Ng is a cusp, iv) Ng is hyperbolic and one of the f xed points of Ng is a cusp, v) Ng is parabolic, vi) Ng is mixed. Let $\overline{H}^{\ell}$ be the set of all ℓ-tuples $z = (z^{(1)}, \ldots, z^{(\ell)})$ with $z^{(i)} \in C$, $\mathrm{Im}\, z^{(i)} \geqslant 0$ or $z^{(i)} = \sqrt{-1}\infty$. The set $\overline{H}^{\ell} - H^{\ell}$ is called the boundary of H^{ℓ}. If an element $z = (z^{(i)})$ of $\overline{H}^{\ell}$ be a fixed point of gT_{σ}, i.e. $({}^{\sigma_1}gz^{(2)}, {}^{\sigma_2}gz^{(3)}, \ldots, {}^{\sigma_\ell}gz^{(1)}) = (z^{(1)}, \ldots, z^{(\ell)})$, then we obtain

$$z^{(1)} = Ngz^{(1)}, \quad z^{(2)} = {}^{\sigma_2}g \ldots {}^{\sigma_\ell}gz^{(1)}, \quad \ldots, \quad z^{(n)} = {}^{\sigma_\ell}gz^{(1)} \qquad . \tag{2.3.2}$$

Conversely we consider $Ng \in GL_2(F)_+$ as an element of $GL_2(R)_+$ by the embedding σ_1 of F into R and assume Ng has a fixed point $z^{(1)}$ in $\overline{H}$ as such. Define $z^{(i)}$ for $i \geqslant 2$ by (2.3.2), then the element $z = (z^{(i)})$ of $\overline{H}^{\ell}$ is a fixed point of gT_{σ}. Hence the set of fixed points of gT_{σ} in $\overline{H}^{\ell}$ is in one to one correspondence with the set of fixed points of Ng as an element of $GL_2(R)_+$ in $\overline{H}$. And we see easily that the set of the fixed points of gT_{σ} in $\overline{H}^{\ell}$ is contained in that of Ng in $\overline{H}^{\ell}$. If Ng is of type i), then the set of the fixed points of gT_{σ} in

H^{ℓ} consists of all the points of the form $(z, {}^{\sigma_2}g \dots {}^{\sigma_\ell}gz, \dots , {}^{\sigma_\ell}gz)$ for some $z \in H$ and is holomorphically isomorphic to H. If Ng is of type ii), the set of the fixed point of Ng consists of a unique inner point of $\overline{H}^{\ell}$. Hence gT_{σ} also has only one fixed point in $\overline{H}^{\ell}$ which is the same point as that of Ng. If Ng is of type iii), the set of the fixed points of Ng consists of 2^{ℓ} points contained in the boundary of $\overline{H}^{\ell}$, and they are not cusps. The fixed points of gT_{σ} in $\overline{H}^{\ell}$ are two points of them, and are both not cusps. If Ng is of type iv), the set of the fixed points of Ng consists of 2^{ℓ} points contained in the boundary of $\overline{H}^{\ell}$, and two of then are cusps of Γ. And if $z = (z^{(i)})$ is one of its cusp, then the fixed point $z' = (z^{(i)'})$ of Ng with $z^{(i)'} \neq z^{(i)}$ for all i $(1 \leq i \leq \ell)$ is also a cusp. The fixed points of gT_{σ} are two points of 2^{ℓ} fixed points of Ng. If one of the fixed point of gT_{σ} is a cusp, then the other is also a cusp. In fact, let $z_j = (z_j^{(i)})$ $= (z_j^{(1)}, g^{(2)} \dots g^{(\ell)} z_j^{(1)}, \dots , g^{(\ell)} z_j^{(1)})$, $z_j^{(1)} \in \overline{H} - H$, $j = 1, 2$, be the fixed points of gT_{σ}, then it holds that $z_1^{(i)} \neq z_2^{(i)}$ for all i. Hence if z_1 is a cusp of Γ, then z_2 also a cusp. We show gT_{σ} fixes two cusps of Γ if $\ell \neq 2$. Actually, let z_1 and z_2 be the cusps of the fixed points of Ng. Then there exists an element h of $GL_2(F)_+$ such that $h(0, \dots , 0) = z_1$ and $h(\sqrt{-1}\infty, \dots , \sqrt{-1}\infty) = z_2$. Since $h^{-1}Ngh$ leaves $(0, \dots, 0)$ and $(\sqrt{-1}\infty, \dots, \sqrt{-1}\infty)$ fixed, it is a diagonal matrix. The set of the fixed points of $h^{-1}g^{\sigma}hT_{\sigma}$ is contained in that of $h^{-1}Ngh$, hence it hold one of the followings;

i) $h^{-1}g^{\sigma}h(\sqrt{-1}\infty) = (\sqrt{-1}\infty)$, $h^{-1}g^{\sigma}h(0) = (0)$,
ii) $h^{-1}g^{\sigma}h(\sqrt{-1}\infty) = (0)$, $h^{-1}g^{\sigma}h(0) = (\sqrt{-1}\infty)$. According to i) or ii), $h^{-1}g^{\sigma}h$ is of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ or $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Since $N(h^{-1}g^{\sigma}h) = h^{-1}Ngh$, in the case where $\ell \neq 2$, $h^{-1}g^{\sigma}h$ must be of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Hence the fixed points of $h^{-1}g^{\sigma}hT_{\sigma}$ are $(0,\ldots,0)$ and $(\sqrt{-1}\infty,\ldots,\sqrt{-1}\infty)$, and the fixed points of gT_{σ} are two cusps of Γ. In the case where $\ell = 2$, it can occur that $h^{-1}g^{\sigma}h$ is of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ and in this case, the fixed point of $h^{-1}g^{\sigma}hT_{\sigma}$ are $(0,\sqrt{-1}\infty)$ and $(\sqrt{-1}\infty, 0)$. Hence neither of the fixed points of gT_{σ} are cusps of Γ. If Ng is of type v), the set of the fixed points of Ng consists of a unique cusp of Γ. Hence gT_{σ} also has a unique fixed point which is the same as that of Ng. We show that the case vi) does not occur. In fact, we see by the definition of Ng

$$^{\sigma_1}Ng = Ng \ , \quad ^{\sigma_2}Ng = g^{-1}(Ng)g \ , \quad \ldots$$

$$^{\sigma_\ell}Ng = {}^{\sigma_{\ell-1}}g^{-1}\ {}^{\sigma_{\ell-2}}g^{-1} \ldots {}^{\sigma_1}g^{-1}(Ng)\ {}^{\sigma_1}g \ldots {}^{\sigma_{\ell-2}}g\ {}^{\sigma_{\ell-1}}g \ .$$

This show that Ng is not of type vi). Summing up the above results, we obtain

Proposition 2.4. An element gT_{σ} of $GL_2(F)_+T_{\sigma}$ is of one of the following types.

i) $Ng \in F^{\times}$ and the set of the fixed points of gT_{σ} in H^{ℓ} is holomorphically isomorphic to H.

ii) Ng is elliptic and the set of the fixed points of gT_{σ} in

$\overline{H}^{\ell}$ consists of a unique inner point of $\overline{H}^{\ell}$.

iii) Ng is hyperbolic and none of fixed points is a cusp of Γ. The set of the fixed points of gT_{σ} in $\overline{H}^{\ell}$ consists of two boundary points, which are not cusps of Γ.

iva) Ng is hyperbolic and one of its fixed points is a cusp. The set of the fixed point of gT_{σ} in $\overline{H}^{\ell}$ consists of two cusps of Γ.

ivb) Ng is hyperbolic and one of its fixed points is a cusp of Γ. The set of the fixed points of gT_{σ} in $\overline{H}^{\ell}$ consists of two boundary points, which are not cusps of Γ.

v) Ng is parabolic and the set of the fixed points of gT_{σ} in $\overline{H}^{\ell}$ consists of a unique cusp of Γ.

The type ivb) can occur only in the case where $\ell = 2$.

We will call an element $g \in GL_2(F)_+$ is of type v, e, h, h_a, h_b or p accordings as gT_{σ} is of type i), ii), iii), iva), ivb) or v) in the above proposition.

Now we define two equivalence relations $\underset{(\Gamma, E)}{\approx}$ and $\underset{\Gamma}{\approx}$ in $GL_2(F)$ by

$$(2.3.3) \qquad g \underset{(\Gamma,E)}{\approx} g' \iff g = \varepsilon \gamma^{-1} g^{\sigma} \gamma, \quad \text{for } \gamma \in \Gamma, \ \varepsilon \in E$$

$$(2.3.4) \qquad g \underset{\Gamma}{\approx} g' \iff g = \gamma^{-1} g^{\sigma} \gamma, \quad \text{for } \gamma \in \Gamma .$$

The condition (2.3.3) (resp. (2.3.4)) is equivalent to that $gT_{\sigma} = \varepsilon \gamma^{-1} gT_{\sigma} \gamma$ for $\gamma \in \Gamma$, $\varepsilon \in E$ (resp. $\varepsilon = 1$) in $\langle GL_2(F)_+, T_{\sigma} \rangle$. Let $\Gamma(gT_{\sigma})$ (resp. $\widetilde{\Gamma}(gT_{\sigma})$) be the group of all $\gamma \in \Gamma$ which satisfy $\varepsilon\gamma^{-1}g^{\sigma}\gamma = g$ for $\varepsilon \in E$ (resp. $\varepsilon = 1$). Then we see easily $\tilde{\Gamma}(gT_{\sigma})E$ is a subgroup of $\Gamma(gT_{\sigma})$ of finite index, since the 1st Galois cohomology group $H^1(\mathcal{G}, E)$ of E is a finite group. For $g \in GL_2(F)$, put

$$(2.3.5) \qquad Z_{\sigma}(g) = \left\{ x \in M_2(F) \;\middle|\; g^{\sigma}x = xg \right\}$$

then $Z_{\sigma}(g)$ is a $\mathbb{Q}$-algebra and $\tilde{\Gamma}(gT) = Z_{\sigma}(g) \cap \Gamma$. We will study in § 3 the equivalence relation $\underset{\Gamma}{\approx}$ and the $\mathbb{Q}$-algebra $Z_{\sigma}(g)$. Here we give a direct consequence of Prop. 3.2., which is needed for the later calculation.

Proposition 2.5. The notation being as above, let g be an element of $GL_2(F)_+$.

i) If g is of type v, $\tilde{\Gamma}(gT_{\sigma})/\tilde{\Gamma}(gT_{\sigma}) \cap E$ is a Fuchsian group of the 1st kind as a subgroup of $GL_2(\mathbb{R})_+$.

ii) If g is of type e, $\tilde{\Gamma}(gT_{\sigma})/\tilde{\Gamma}(gT_{\sigma}) \cap E$ is a finite cyclic group.

iii) If g is of type h, h_b, or p, $\tilde{\Gamma}(gT_{\sigma})/\tilde{\Gamma}(gT_{\sigma}) \cap E$ is a free abelian group of rank one.

iv) If g is of type h_a, $\tilde{\Gamma}(gT_{\sigma})/\tilde{\Gamma}(gT_{\sigma}) \cap E = \{1\}$.

Before the computation of the integral (2.2.1), we prove the following.

Lemma 2.6. Let B_v be the set of all elements of type v in B. Then the integral

$$\int_{V_{\infty}} \sum_{\substack{g \in B_v \cap B_{\infty}^{(1)} \\ \text{mod. } E}} \frac{k(z,\, gT_{\sigma}z)\overline{j(g,\, T_{\sigma}z)}}{k(z,\, z)}\, dz$$

is termwise integrable.

<u>Proof</u>. First we show that the set $B_V \cap B_\infty^{(1)}$ divides into a finite number of classes with respect to the equivalence relation $\underset{\Gamma_\infty^{(1)}}{\approx}$ given by $g \underset{\Gamma_\infty^{(1)}}{\approx} g' \iff g = \gamma^{-1} g' {}^\sigma\gamma$, for $\gamma \in \Gamma_\infty^{(1)}$. An element $g \in B_V \cap B_\infty^{(1)}$ is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, b, d \in \mathcal{O}$ and the ideal (ad) is fixed. Hence by considering the element $\gamma^{-1} g T_\sigma \gamma$ for a suitable $\gamma = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix}$ $(\in \Gamma_\infty^{(1)})$ with $\varepsilon, \varepsilon' \in E$, we may assume that a and d are contained in a finite set. For fixed a and d, $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is contained in $B_V \cap B_\infty^{(1)}$ if and only if $N_{F/Q}\left(\frac{a}{d}\right) = 1$ and $\varphi_{a/d}\left(\frac{b}{d}\right) = 0$, where

$$\varphi_{a/d}\left(\frac{b}{d}\right) = \sigma_1\left(\frac{b}{d}\right) + \sigma_1\left(\frac{a}{d}\right)\sigma_2\left(\frac{b}{d}\right) + \dots + \sigma_1\left(\frac{a}{d}\right)\sigma_2\left(\frac{a}{d}\right)\dots\sigma_{\ell-1}\left(\frac{a}{d}\right)\sigma_\ell\left(\frac{b}{d}\right).$$

Hence we may assume $N_{F/Q}\, a = N_{F/Q}\, d$, and for such a and d we see easily that the elements b of $\mathcal{O}$ which satisfy $\varphi_{a/d}\left(\frac{b}{d}\right) = 0$ form a free Z-module M of rank $\ell - 1$. Now for $b' \in \mathcal{O}$, we have

$$\begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix}^{-1} g \, {}^\sigma\!\begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + a\,{}^\sigma b' - b'd \\ 0 & d \end{pmatrix},$$

we see that the set $\{ a\,{}^\sigma b' - b'd \mid b' \in \mathcal{O} \}$ is a Z-submodule of M of rank $\ell-1$, hence it is a submodule of M of finite index. From this it follows that $B_V \cap B_\infty^{(1)}$ divides into a finite number of $\underset{\Gamma_\infty^{(1)}}{\approx}$-equivalence classes. Hence to prove our assertion it is enough to prove that

$$S_g(z) = \sum_{\substack{g' = \gamma^{-1} g^{\sigma}\gamma \\ \gamma \in \Gamma_\infty^{(1)} \\ \mathrm{mod.}\, E}} \left| \frac{k(z,\ gT_\sigma z)\overline{j(g,\ T_\sigma z)}}{k(z,\ z)} \right|$$

is integrable, where the sum runs over all the elements g' of the form $\gamma^{-1}g^{\sigma}\gamma$ for some $\gamma \in \Gamma_\infty^{(1)}$ modulo E. For $g \in B_v \cap B_\infty^{(1)}$, put

$$\Gamma_\infty(gT_\sigma) = \left\{ \gamma \in \Gamma_\infty^{(1)} \ \middle|\ \gamma^{-1}g^{\sigma}\gamma = g \right\}$$

, then we see that

$$\Gamma_\infty(gT_\sigma) = \left\{ \begin{pmatrix} \varepsilon & b' \\ 0 & \varepsilon \end{pmatrix} \ \middle|\ \varepsilon = \pm 1,\ b' \in \mathcal{O},\ a^{\sigma}b' = b'd \right\}$$

and that all b''s of $\mathcal{O}$ which satisfy $a^{\sigma}b' = b'd$ form a free $\mathbb{Z}$-module of rank one. And it is enough to show that the function

$$\left| \frac{k(z,\ gT_\sigma z)\overline{j(g,\ T_\sigma z)}}{k(z,\ z)} \right|$$

is integrable on a fundamental domain $U_\infty / \Gamma_\infty(gT_\sigma)$ of $\Gamma_\infty(gT_\sigma)$ in U_∞. This can be verified by explicit calculation.

<u>2.4.</u> We classify all the elements in B with respect to the equivalence relation $\underset{(\Gamma,E)}{\approx}$ ((2.3.3)). We denote the class containing g by $[g]$. Let $\Gamma(gT_\sigma)$ be as in 2.3.. Let $\{g_0\}$ be a complete system of representatives of the above equivalence classes in B, and for each g_0, $\{\delta\}$ be a system of representatives of $\Gamma(g_0T_\sigma)\backslash\Gamma$. We set

$$\mathcal{F}_{g_0} = \bigcup_{\delta} \delta \mathcal{F}$$

, then $\mathcal{F}_{g_0}$ is a fundamental domain of $\Gamma(g_0T_\sigma)$ in H^ℓ. And we set

$$\mathcal{F}_{g_0}^{\;*} = \mathcal{F}_{g_0} - \bigcup_{\delta^{-1}g_0\sigma\delta \in B_\infty^{(1)}} \delta V_\infty \quad .$$

In notice of the fact that $I(z) = I(z')|j(g, z')|^s$ for $z = gz'$, by (2.2.1) and Lemma 2.6. we obtain

$$(2.4.1) \qquad \left(\frac{4\pi}{\kappa-1}\right)^\ell \operatorname{tr} T_S(T(\alpha)) = \left(\frac{4\pi}{\kappa-1}\right)^\ell \operatorname{tr}(T(T(\alpha))T_\sigma^{-1})$$

$$= \sum_{g_0 \,:\, [g_0] \cap (B_\infty^{(1)} - B_v) = \phi} \int_{\mathcal{F}_{g_0}} \frac{k(z,\, g_0T_\sigma z)\overline{j(g_0,\, T_\sigma z)}}{k(z,\, z)}\, dz$$

$$+ \lim_{s\to 0} \sum_{g_0 : [g_0] \cap (B_\infty^{(1)} - B_v) \neq \phi} \left[\int_{\mathcal{F}_{g_0}^{\;*}} \frac{k(z,\, g_0T_\sigma z)\overline{j(g_0,\, T_\sigma z)}}{k(z,\, z)}\, dz \right.$$

$$\left. + \sum_{\delta^{-1}g_0\sigma\delta \in B_\infty^{(1)}} \int_{\delta V_\infty} \frac{k(z,\, g_0T_\sigma z)\overline{j(g_0,\, T_\sigma z)}}{I(z)^s |j(\delta^{-1},\, z)|^s k(z,\, z)}\, dz \right]$$

In the following, we calculate the integrals in (2.4.1).

__2.5.__ g_0 is of type v. Let $\sigma_1, \ldots, \sigma_\ell$ be as in 1.1., and for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(F)_+$, put $g^{(i)} = {}^{\sigma_i}g = \begin{pmatrix} {}^{\sigma_i}a & {}^{\sigma_i}b \\ {}^{\sigma_i}c & {}^{\sigma_i}d \end{pmatrix}$. By σ_1, we may consider $\Gamma(g_0T_\sigma)$ as a subgroup of $GL_2(R)_+$, and then by Prop. 2.5. $\Gamma(g_0T_\sigma)$ is a Fuchsian group of the 1st kind. Let $\mathcal{F}_0$ be a fundamental domain of $\Gamma(g_0T_\sigma)$ in H, then as a fundamental domain of $\Gamma(g_0T_\sigma)$ in H^ℓ, we can take the set $\mathcal{F}_0 \times H \times \ldots \times H$. Put

$$I(g_0) = \int_{\mathcal{F}g_0} \frac{k(z, g_0T_\sigma z)\overline{j(g_0, T_\sigma z)}}{k(z, z)} dz$$

$$= \int_{\mathcal{F}_0 \times H \times \ldots \times H} \frac{k(z, g_0T_\sigma z)\overline{j(g_0, T_\sigma z)}}{k(z, z)} dz$$

We set for $z, z' \in H$ $k_0(z, z') = \left(\frac{z - \bar{z}'}{2\sqrt{-1}}\right)^{-\kappa}$ and for

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})_+$, $z \in H$, $j_0(g, z) = (cz + d)^{-\kappa}|\det g|^{\kappa/2}$.

And we consider the following integral I_2.

$$I_2 = \int_H \frac{k_0(z^{(1)}, g_0^{(1)}z^{(2)})k_0(z^{(2)}, g_0^{(2)}z^{(3)})\overline{j_0(g_0^{(1)}, z^{(2)})}\,\overline{j_0(g_0^{(2)}, z^{(3)})}}{k_0(z^{(2)}, z^{(2)})} \frac{dx^{(2)}dy^{(2)}}{y^{(2)^2}}$$

, where $z^{(2)} = x^{(2)} + \sqrt{-1}y^{(2)}$. We see

$$I_2 = \int_H \frac{k_0(g_0^{(1)^{-1}}z^{(1)}, z^{(2)})k_0(z^{(2)}, g_0^{(2)}z^{(3)})j_0(g_0^{(1)^{-1}}, z^{(1)})\overline{j_0(g_0^{(2)}, z^{(3)})}}{k_0(z^{(2)}, z^{(2)})} \frac{dx^{(2)}dy^{(2)}}{y^{(2)^2}}$$

As a function of $z^{(2)}$, $f(z^{(2)}) = k_0(z^{(2)}, g_0^{(2)}z^{(3)})$ satisfies the condition

$$\|f\|_2 = \left[\int_H |k_0(z, z)^{-1/2}f(z)|^2 dz\right]^{1/2} < \infty .$$

Hence by Th. 3, Exposé 10, [6], we have

$$I_2 = \frac{4\pi}{\kappa-1} k_o(g_o^{(1)}z^{(1)}, g_o^{(2)}z^{(3)}) j_o(g_o^{(1)^{-1}}, z^{(1)}) \overline{j_o(g_o^{(2)}, z^{(3)})}$$

$$= \frac{4\pi}{\kappa-1} k_o(z^{(1)}, g_o^{(1)}g_o^{(2)}z^{(3)}) \overline{j(g_o^{(1)}g_o^{(2)}, z^{(3)})}$$

By the same calculation for $i \geqslant 3$, we obtain

$$I = \left(\frac{4\pi}{\kappa-1}\right)^{\ell-1} \int_{\mathcal{F}_o} \frac{k_o(z^{(1)}, Ng_o z^{(1)}) j(Ng_o, z^{(1)})}{k_o(z^{(1)}, z^{(1)})} \frac{dx^{(1)}dy^{(1)}}{y^{(1)2}}$$

Since $Ng \in F^{\times}$, we see

$$I = \left(\frac{4\pi}{\kappa-1}\right)^{\ell-1} \int_{\mathcal{F}_o} \frac{dxdy}{y^2}$$

$$= \left(\frac{4\pi}{\kappa-1}\right)^{\ell-1} v(H/\Gamma(gT_\sigma)) \qquad ,$$

where $v(H/\Gamma(gT_\sigma))$ is the volume of a fundamental domain of $\Gamma(gT_\sigma)$ in H with respect to the invariant measure $\frac{dxdy}{y^2}$

2.6. g_o is of type e. In this case, by Prop. 2.5. $\Gamma(gT_\sigma)/E$ is a finite group. We consider $\Gamma(g_oT_\sigma)$ as a subgroup of $GL_2(\mathbb{R})_+$ by σ_1, and let $\mathcal{F}_o$ be a fundamental domain of $\Gamma(g_oT_\sigma)$ in H. Then by the same calculation as in 2.5., we obtain

$$I = \int_{\mathcal{F}_{g_o}} \frac{k(z, g_oT_\sigma z)\overline{j(g_o, T_\sigma z)}}{k(z, z)} dz$$

$$= \left(\frac{4\pi}{\kappa-1}\right)^{\ell-1} \int_{\mathcal{F}_0} \frac{k_0(z^{(1)}, Ng_0 z^{(1)}) j(Ng_0, z^{(1)})}{k_0(z^{(1)}, z^{(1)})} \frac{dx^{(1)}dy^{(1)}}{y^{(1)2}}$$

, where $z^{(1)} = x^{(1)} + \sqrt{-1}y^{(1)}$. Since Ng is an elliptic element, there exists a unique fixed point z_0 in H. Let η, ζ be the eigen-values of Ng and suppose that we have for $z \in H$

$$(2.6.1) \qquad \frac{Ng_0 z - z_0}{Ng_0 z - \overline{z_0}} = \eta\zeta^{-1} \frac{z - z_0}{z - \overline{z_0}} .$$

Then we have

$$I = \left(\frac{4\pi}{\kappa-1}\right)^{\ell} \frac{1}{[\Gamma(g_0 T_\sigma) : E]} \frac{\zeta^{\kappa-1}}{\eta-\zeta} (\det Ng_0)^{1-\frac{\kappa}{2}} .$$

2.7. g_0 is of type h. Let $\mathcal{F}_0$ be a fundamental domain of $\widetilde{\Gamma}(gT_\sigma)$ in H. Then by the same calculation as in 2.5., we obtain

$$I = \int_{\mathcal{F}_{g_0}} \frac{k(z, g_0 T_\sigma z)\overline{j(g_0, T_\sigma z)}}{k(z, z)} dz$$

$$= \left(\frac{4\pi}{\kappa-1}\right)^{\ell-1} \frac{1}{[\Gamma(gT_\sigma) : \widetilde{\Gamma}(g_0 T_\sigma)E]} \int_{\mathcal{F}_0} \frac{k_0(z^{(1)}, Ng_0 z^{(1)})\overline{j(Ng_0, z^{(1)})}}{k_0(z^{(1)}, z^{(1)})} \frac{dx^{(1)}dy^{(1)}}{y^{(1)2}}$$

, where $z^{(1)} = x^{(1)} + \sqrt{-1}y^{(1)}$. Ther exists an element $h \in GL_2(\mathbb{R})_+$ such that $h^{-1}Ng_0 h$ is of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a, d \in \mathbb{R}^\times$, and then obviously $a \neq d$. By Prop. 2.5., $\widetilde{\Gamma}(g_0 T_\sigma) / \widetilde{\Gamma}(g_0 T_\sigma) \cap E$ is a free abelian group of rank one, hence we may assume

$\mathcal{F}_0 = h \ \mathcal{F}_0'$, where $\mathcal{F}_0' = \left\{ z = x + \sqrt{-1}y \in H \mid -\infty < x < \infty ,\ 1 < y < A \right\}$

with a positive number A. Hence we see

$$I = \left(\frac{4\pi}{\kappa-1}\right)^{\ell-1} \frac{1}{[\Gamma(g_0T_\sigma) : \tilde{\Gamma}(g_0T_\sigma)E]} \iint_{\substack{-\infty < x < \infty \\ 1 < y < A}} \left(\frac{a}{d}\right)^{\kappa/2} \frac{y^{\kappa-2}}{(z - \frac{a}{d}\bar{z})^{\kappa}} dxdy$$

$$= 0 \quad .$$

<u>2.8.</u> g_0 is of type h_a. By Prop. 2.5., $\tilde{\Gamma}(g_0T_\sigma)/\tilde{\Gamma}(g_0T_\sigma)\cap E = \{1\}$ and H^ℓ is a fundamental domain of $\tilde{\Gamma}(g_0T_\sigma)$ in H^ℓ. We may assume that g_0T_σ is of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}T_\sigma$, since every cusp of Γ is Γ-equivalent to $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ and then the fixed points of g_0T_σ are $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ and $({}^{\sigma_1}\alpha, {}^{\sigma_2}\alpha, \ldots, {}^{\sigma_\ell}\alpha)$ with $\alpha \in F^\times$. Let γ_0 be an element of Γ such that $\gamma_0(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty) = ({}^{\sigma_1}\alpha, {}^{\sigma_2}\alpha, \ldots, {}^{\sigma_\ell}\alpha)$. Then the set of all $\gamma \in \Gamma$ which satisfy $\gamma^{-1}g_0{}^{\sigma}\gamma \in B_\infty^{(1)}$ is the union $\gamma_0\Gamma_\infty^{(1)} \cup \Gamma_\infty^{(1)}$. We denote by h the matrix $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in GL_2(F)_+$, then $h(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty) = (\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$ and $h(0, \ldots, 0) = ({}^{\sigma_1}\alpha, \ldots, {}^{\sigma_\ell}\alpha)$, and put $g_0' = h^{-1}g_0{}^{\sigma}h$ and $\gamma_0' = h^{-1}\gamma_0 h$. Then g_0' is a diagonal matrix and $\gamma_0'(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty) = (0, \ldots, 0)$. And the contribution I of the conjugacy class $[g_0]$ to the integral (2.4.1) is

$$I = \lim_{s \to 0} \left(\int_{h^{-1}\mathcal{F}_{g_0}^*} \frac{k(z,\ g_0' T_\sigma z)\overline{j(g_0',\ T_\sigma z)}}{k_0(z,\ z)}\, dz \right.$$

$$\left. + \sum_{\delta \in \Gamma(g_0 T_\sigma) \backslash \Gamma_\infty^{(1)} \cup \gamma_0 \Gamma_\infty^{(1)}} \int_{h^{-1}\delta V_\infty} \frac{k(z,\ g_0' T_\sigma z)\overline{j(g_0',\ T_\sigma z)}}{I(z)^s\, |j(\delta^{-1}h,\ z)|^s k_0(z,z)}\, dz \right)$$

We see that $\bigcup_{g \in h^{-1}\Gamma_\infty^{(1)} h} g(h^{-1}V_\infty) = h^{-1}U_\infty = U(d_1)$ and

$\bigcup_{g \in \gamma_0' h^{-1}\Gamma_\infty^{(1)} h} g(h^{-1}V_\infty) = \gamma_0' h^{-1} U(d) = U'(d_2)$ for some positive numbers d_1 and d_2, where $U(d_1) = \left\{ z \in H^\ell \mid \prod \mathrm{Im}(z_i) > d_1 \right\}$ and $U'(d_2) = \left\{ z \in H^\ell \mid \prod \mathrm{Im}(z_i)/|z_i|^2 < d_2 \right\}$. We note $j(\gamma h,\ z) = 1$ and $j(\gamma \gamma_0^{-1} h,\ z) = j(\gamma_0'^{-1},\ z)$ for $\gamma \in \Gamma_\infty^{(1)}$.

$$(2.8.1) \qquad I = \lim_{s \to 0} \frac{1}{[\Gamma(gT_\sigma) : \tilde{\Gamma}(gT_\sigma)E]} (I_1 + I_2 + I_3)$$

where

$$I_1 = \int_{U(d_1)} \frac{k(z,\ g_0' T_\sigma z)\overline{j(g_0',\ T_\sigma z)}}{I(z)^s k_0(z,\ z)}\, dz$$

$$I_2 = \int_{U'(d_2)} \frac{k(z,\ g_0' T_\sigma z)\overline{j(g_0',\ T_\sigma z)}}{I(z)^s |j(\gamma_0'^{-1},\ z)|^s}\, dz \qquad ,$$

and

$$I_3 = \int_{H^\ell - U(d_1) - U'(d_2)} \frac{k(z,\ g_0' T_\sigma z)\overline{j(g_0',\ T_\sigma z)}}{k(z,\ z)}\, dz$$

We show the integrals I_1 and I_2 vanish. For $g_0' = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, put $\lambda_i = {}^{\sigma_i}\!\left(\frac{a}{d}\right)$. Then we have

$$I_1 = (2\sqrt{-1})^{\kappa\ell} \prod \lambda_i \int_{U(d_1)} \frac{{y^{(1)}}^{\kappa-s} \cdots {y^{(\ell)}}^{\kappa-s}}{(z^{(1)}-\lambda_1\bar{z}^{(2)})^{\kappa}(z^{(2)}-\lambda_2\bar{z}^{(3)})^{\kappa}\cdots(z^{(\ell)}-\lambda_\ell\bar{z}^{(1)})^{\kappa}}\, dz$$

, where $z^{(i)} = x^{(i)} + \sqrt{-1}y^{(i)}$. Now we consider the integral I_1',

$$I_1' = \int_{\mathbb{R}^\ell} \frac{dx^{(1)} \cdots dx^{(\ell)}}{(z^{(1)}-\lambda_1\bar{z}^{(2)})^{\kappa}(z^{(2)}-\lambda_2\bar{z}^{(3)})^{\kappa}\cdots(z^{(\ell)}-\lambda_\ell\bar{z}^{(1)})^{\kappa}}$$

We note $\prod \lambda_i \neq 1$ and $\lambda_i > 0$, since g_o is of type h_a,

and put $u_1 = x^{(1)}-\lambda_1 x^{(2)}$, $u_2 = x^{(2)}-\lambda_2 x^{(3)}$, ... , $u_\ell = x^{(\ell)}-\lambda_\ell x^{(1)}$,

then

$$I_1' = \frac{1}{|1-\prod\lambda_i|}\int_{R^\ell} \frac{du_1 \cdots du_\ell}{(u_1+\sqrt{-1}(y^{(1)}+\lambda_1 y^{(2)}))^{\kappa}(u_2+\sqrt{-1}(y^{(2)}+\lambda_2 y^{(3)}))^{\kappa}\cdots(u_\ell+\sqrt{-1}(y^{(\ell)}+\lambda_\ell y^{(1)}))^{\kappa}}$$

$$= 0$$

hence I_1 vanishes. Put $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $g_o'' = \gamma^{-1} g_o'^{\sigma}\gamma$, then

g_o'' is also a diagonal matrix, and we have

$$I_2 = \int_{\gamma^{-1}U(d_2)'} \frac{k(z, g_o'' T_\sigma z)\overline{j(g_o'', T_\sigma z)}}{I(z)^s j(\gamma_o^{-1}\gamma\ , z)k(z, z)}\, dz$$

Since $\gamma^{-1}U(d_2)' = U(d_2)$ and $\gamma_o^{-1}\gamma$ is of the form $\begin{pmatrix} * & * \\ c & * \end{pmatrix}$, we see $I_2 = 0$ by the same calculation as above. Put
$W = H^\ell - U(d_1) - U(d_2)'$, then

$$I_3 = (2\sqrt{-1})^{\kappa\ell} (\prod \lambda_i)^{\kappa/2} \int_W \frac{(\prod y^{(i)})^{\kappa}}{(z^{(1)}-\lambda_1 z^{(2)})^{\kappa} (z^{(2)}-\lambda_2 z^{(3)})^{\kappa} \cdots (z^{(\ell)}-\lambda_\ell z^{(1)})^{\kappa}} dz \quad .$$

In the rest of 2.8, we write $e[\theta] = \exp(\sqrt{-1}\,\theta)$ for the sake of simplicity. Then

$$I_3 = (2\sqrt{-1})^{\kappa\ell} (\prod \lambda_i)^{\kappa/2} \int_{W_1} \frac{(\prod \rho_i)^{\kappa-1} (\prod \sin \theta_i)^{\kappa-2}}{(\rho_1 e[\theta_1]-\lambda_1 \rho_2 e[-\theta_2])^{\kappa} (\rho_2 e[\theta_2]-\lambda_2 \rho_3 e[-\theta_3])^{\kappa} \cdots}$$

$$\cdots \times \frac{1}{(\rho_\ell\, e[\theta_\ell]-\lambda_\ell \rho_\ell e[-\theta_1])^{\kappa}} d\rho_1 \cdots d\rho_\ell\, d\theta_1 \cdots d\theta_\ell$$

, where $z^{(i)} = \rho_i e[\theta_i]$, and $W_1 = \{ 0 < \theta_i < \pi ,\ \rho_i > 0,$ $\prod \sin \theta_i / d_2 < \prod \rho_i < d_1 / \prod \sin \theta_i ,\ 1 \leqslant i \leqslant \ell \}$. Put $\gamma_1 = \rho_2/\rho_1$, $\gamma_2 = \rho_3/\rho_2$, ..., $\gamma_{\ell-1} = \rho_\ell/\rho_{\ell-1}$, and $\gamma_\ell = \rho_1 \cdots \rho_\ell$, then

$$I_3 = \frac{(2\sqrt{-1})^{\kappa\ell} (\prod \lambda_i)^{\kappa/2}}{\ell} \int_{W_2} \frac{(\prod \gamma_i)^{-1} (\prod \sin \theta_i)^{\kappa-2}}{(e[\theta_1]-\lambda_1 \gamma_1 e[-\theta_2])^{\kappa} \cdots (e[\theta_{\ell-1}]-\lambda_{\ell-1}\gamma_{\ell-1} e[-\theta_\ell])^{\kappa}}$$

$$\times \frac{1}{(e[\theta_\ell]-\lambda_\ell (\gamma_1 \cdots \gamma_{\ell-1})^{-1} e[-\theta_1])^{\kappa}} d\gamma_1 \cdots d\gamma_\ell\, d\theta_1 \cdots d\theta_\ell$$

, where $W_2 = \{ 0 < \theta_i < \pi ,\ 1 \leqslant i \leqslant \ell ,\ \gamma_j > 0,\ 1 \leqslant j \leqslant \ell-1,$ $\prod \sin \theta_i / d_2 < \gamma_\ell < d_1 / \prod \sin \theta_i \}$. Put $W_3 = \{ 0 < \theta_i < \pi ,\ 1 \leqslant i \leqslant \ell,$ $\gamma_j > 0,\ 1 \leqslant j \leqslant \ell-1 \}$, then we see

$$(2.8.2) \qquad I_3 = \frac{(2\sqrt{-1})^{\kappa\ell} (\prod \lambda_i)^{\kappa/2}}{\ell} \left\{ (\log d_1 d_2) I_3' + I_3'' \right\}$$

, where

$$I_3' = \int_{W_3} \frac{(\gamma_1 \dots \gamma_{\ell-1})^{-1} (\prod \sin \theta_i)^{\kappa-2}}{(e[\theta_1] - \lambda_1\gamma_1\, e[-\theta_2])^{\kappa} \dots (e[\theta_{\ell-1}] - \lambda_{\ell-1}\gamma_{\ell-1}\, e[-\theta_\ell])^{\kappa}}$$

$$\times \frac{1}{(e[\theta_\ell] - \lambda_\ell(\gamma_1 \dots \gamma_{\ell-1})^{-1}\, e[-\theta_1])^{\kappa}} d\gamma_1 \dots d\gamma_{\ell-1}\, d\theta_1 \dots d\theta_{\ell-1}$$

and

$$I_3'' = \int_{W_3} \frac{(\gamma_1 \dots \gamma_{\ell-1})^{-1} (\prod \sin \theta_i)^{\kappa-2} \log(\prod \sin \theta_i)^{-2}}{(e[\theta_1] - \lambda_1\gamma_1\, e[-\theta_2])^{\kappa} \dots (e[\theta_{\ell-1}] - \lambda_{\ell-1}\gamma_{\ell-1}\, e[-\theta_\ell])^{\kappa}}$$

$$\times \frac{1}{(e[\theta_\ell] - \lambda_\ell(\gamma_1 \dots \gamma_{\ell-1})^{-1}\, e[-\theta_1])^{\kappa}} d\gamma_1 \dots d\gamma_{\ell-1}\, d\theta_1 \dots d\theta_{\ell-1}$$

To compute the integral I_3', we consider the following integral J.

$$J = \int_0^{\infty}\int_0^{\pi} \frac{\gamma_{\ell-1}^{-1}(\sin\theta_\ell)^{\kappa-2}}{(e[\theta_{\ell-1}] - \lambda_{\ell-1}\gamma_{\ell-1}\, e[-\theta_\ell])^{\kappa} (e[\theta_\ell] - \lambda_\ell(\gamma_1 \dots \gamma_{\ell-1})^{-1}\, e[-\theta_\ell])^{\kappa}} d\theta_\ell\, d\gamma_{\ell-1} .$$

Put $z = x + \sqrt{-1}y = \gamma_{\ell-1}\, e[\theta_\ell]$, then

$$J = \int_0^{\infty}\int_{-\infty}^{\infty} \frac{y^{\kappa-2}}{(e[\theta_{\ell-1}] - \lambda_{\ell-1}\bar{z})^{\kappa} (z - \lambda_\ell(\gamma_1 \dots \gamma_{\ell-2})^{-1} e[-\theta_1])^{\kappa}} dx dy$$

$$= \frac{2\pi\sqrt{-1}(2\kappa-2)!(-1)^{\kappa-1}}{((\kappa-1)!)^2(\lambda_{\ell-1})^{\kappa}} \int_0^{\infty} \frac{y^{\kappa-2}}{(2\sqrt{-1}y + e[\theta_{\ell-1}]\lambda_{\ell-1}^{-1} - \lambda_\ell(\gamma_1 \dots \gamma_{\ell-2})^{-1} e[-\theta_1])^{2\kappa-1}} dy$$

$$= \frac{4\pi}{(2\sqrt{-1})^{\kappa}(\kappa-1)} \frac{1}{(e[\theta_{\ell-1}] - \lambda_{\ell-1}\lambda_{\ell}(\gamma_1 \ldots \gamma_{\ell-2})^{-1} e[-\theta_1])^{\kappa}}$$

By the same calculation for (θ_i, γ_{i-1}), $2 \leq i \leq \ell-1$, we obtain

$$I_3' = \frac{(4\pi)^{\ell-1}}{(2\sqrt{-1})^{\kappa(\ell-1)} (\kappa-1)^{\ell-1}} \int_0^{\pi} \frac{(\sin\theta_1)^{\kappa-2}}{(e[\theta_1] - \lambda_1 \cdots \lambda_\ell e[-\theta_1])} d\theta_1$$

Since $\prod \lambda_i \neq 1$, we see

$$I_3' = 0 \quad .$$

Put

$$J_i = \int_{W_3} \frac{(\gamma_1 \ldots \gamma_{\ell-1})^{-1} (\prod \sin\theta_j)^{\kappa-2} \log(\sin\theta_i)^{-2}}{(e[\theta_1] - \lambda_1\gamma_1 e[-\theta_2])^{\kappa} \ldots (e[\theta_{\ell-1}] - \lambda_\ell\gamma_{\ell-1} e[-\theta_\ell])^{\kappa}}$$

$$\times \frac{1}{(e[\theta_\ell] - \lambda_\ell(\gamma_1 \ldots \gamma_{\ell-1})^{-1} e[-\theta_1])^{\kappa}} d\gamma_1 \ldots d\gamma_{\ell-1} d\theta_1 \ldots d\theta_\ell \quad .$$

Then we have

$$(2.8.3) \qquad I_3'' = \sum_{i=1}^{\ell} J_i$$

For J_i, $i \geq 2$, we set $\tilde{\gamma}_j = \gamma_{i+j-1}$, $1 \leq j \leq \ell-i$,

$\tilde{\gamma}_{\ell-i+1} = (\gamma_1 \ldots \gamma_{\ell-1})^{-1}$, $\tilde{\gamma}_j = \gamma_{j-\ell+i-1}$, $\ell-i+2 \leq j \leq \ell-1$,

$\tilde{\theta}_j = \theta_{i+j-1}$ (resp. $\tilde{\lambda}_j = \lambda_{i+j-1}$), $1 \leq j \leq \ell-i+1$,

$\tilde{\theta}_j = \theta_{j-\ell+i-1}$ (resp. $\tilde{\lambda}_j = \lambda_{j-\ell+i-1}$), $\ell-i+2 \leq j \leq \ell$.

Then, $\prod \lambda_j = \prod \tilde{\lambda}_j$, $(\tilde{\gamma}_1 \dots \tilde{\gamma}_{\ell-1})^{-1} = \gamma_{i-1}$ and we see

$$J_i = \int_{W_3} \frac{(\tilde{\gamma}_1 \dots \tilde{\gamma}_{\ell-1})^{-1} (\prod \sin \tilde{\theta}_j)^{\kappa-2} \log(\sin \tilde{\theta}_1)^{-2}}{(e[\tilde{\theta}_1] - \tilde{\lambda}_1\tilde{\gamma}_1 e[-\tilde{\theta}_2])^{\kappa} \dots (e[\tilde{\theta}_{\ell-1}] - \tilde{\lambda}_{\ell-1}\tilde{\gamma}_{\ell-1} e[-\tilde{\theta}_\ell])^{\kappa}}$$

$$\times \frac{1}{(e[\tilde{\theta}_\ell] - \tilde{\lambda}_\ell(\tilde{\gamma}_1 \dots \tilde{\gamma}_{\ell-1})^{-1} e[-\tilde{\theta}_{\ell-1}])^{\kappa}} d\tilde{\gamma}_1 \dots d\tilde{\gamma}_{\ell-1} d\tilde{\theta}_1 \dots d\tilde{\theta}_\ell$$

where $W_3 = \left\{ 0 < \tilde{\theta}_i < \pi , \quad 1 \leqslant i \leqslant \ell, \quad \tilde{\gamma}_i > 0, \quad 1 \leqslant i \leqslant \ell-1 \right\}$.

By the same calculation as in the case of I_3', we see $J_1 = \dots\dots = J_\ell$, and

$$(2.8.4) \qquad J_1 = \frac{(4\pi)^{\ell-1}}{(2\sqrt{-1})^{\kappa(\ell-1)} (\kappa-1)^{\ell-1}} \int_0^{\pi} \frac{(\sin\theta)^{\kappa-2} \log(\sin\theta)^{-2}}{(e[\theta] - \prod\lambda_i e[-\theta])} d\theta .$$

By an explicit calculation, we see

$$\int_0^{\pi} \log(\sin\theta)^{-2} \frac{(\sin\theta)^{\kappa-2}}{(e[\theta] - \prod\lambda_i e[-\theta])^{\kappa}} d\theta$$

$$= \begin{cases} \dfrac{4\pi}{(2\sqrt{-1})^{\kappa}(\kappa-1)(1-\prod\lambda_i)(\prod\lambda_i)^{\kappa-1}} & \text{if } \prod\lambda_i > 1 \\[2ex] \dfrac{4\pi}{(2\sqrt{-1})^{\kappa}(\kappa-1)(\prod\lambda_i - 1)} & \text{if } \prod\lambda_i < 1 \end{cases} .$$

Hence by (2.8.3) and (2.8.4)

(2.8.5)

$$I_3'' = \begin{cases} \dfrac{(4\pi)^{\ell}}{(2\sqrt{-1})^{\kappa\ell}(\kappa-1)^{\ell}} \dfrac{|\prod d_i|^{\kappa-1}}{|\prod d_i| - |\prod a_i|} (\det Ng_0')^{1-\kappa/2} & \text{if } \prod \lambda_i > 1 \\ \dfrac{(4\pi)^{\ell}}{(2\sqrt{-1})^{\kappa\ell}(\kappa-1)^{\ell}} \dfrac{|\prod a_i|^{\kappa-1}}{|\prod a_i| - |\prod d_i|} (\det Ng_0')^{1-\kappa/2} & \text{if } \prod \lambda_i < 1 , \end{cases}$$

where $\sigma_i g_0 = \begin{pmatrix} a_i & 0 \\ 0 & d_i \end{pmatrix}$. In any case, it holds

$$I_3'' = -\left(\frac{4\pi}{\kappa-1}\right)^{\ell} \frac{(\operatorname{Min}(|\prod a_i|, |\prod d_i|))^{\kappa-1}}{|\prod a_i - \prod d_i|} (\det Ng_0')^{1-\kappa/2} .$$

Since $h^{-1}Ng_0 h = Ng_0'$, $\det Ng_0 = \det Ng_0'$ and we see that $\prod a_i$ and $\prod d_i$ are the eigen-value of Ng_0. We denote them by ξ, η, then we obtain by (2.8.1), (2.8.2) and (2.8.4),

$$I = -\frac{1}{[\Gamma(g_0 T_\sigma) : \tilde{\Gamma}(g_0 T_\sigma)E]} \left(\frac{4\pi}{\kappa-1}\right)^{\ell} \frac{(\operatorname{Min}(|\xi|, |\eta|))^{\kappa-1}}{|\xi - \eta|} (\det Ng_0)^{1-\kappa/2} .$$

<u>2.9.</u> g_0 is of type h_b. By Prop. 2.5, $\tilde{\Gamma}(g_0 T_\sigma)/\tilde{\Gamma}(g_0 T_\sigma) \cap E$ is a free abelian group of rank one. We can show the integral

$$I = \int_{\mathcal{F}_{g_0}} \frac{k_0(z, g_0 T_\sigma z)\, j(g_0, T_\sigma z)}{k_0(z, z)} \, dz$$

vanishes by the same calculation as in 2.7. We omit the details.

2.10. g_0 is of type p. We may assume $g_0 \in B_\infty^{(1)}$ since every cusp is Γ-equivalent to $(\sqrt{-1}\infty, \ldots, \sqrt{-1}\infty)$. We note that if $\gamma^{-1} g_0 T_\sigma \gamma \in B_\infty^{(1)}$ for $\gamma \in \Gamma$, then γ is contained in $\Gamma_\infty^{(1)}$. Since $j(\gamma, z) = 1$ for $\gamma \in \Gamma$, we see that the contribution of the conjugacy classes $[g_0]$ of type p to (2.4.1) equals

$$\lim_{s\to 0} \sum_{[g_0]} \left[\int_{\mathcal{F}_{g_0}} \frac{k(z, g_0 T_\sigma z)}{k(z, z)} dz + \int_{U_\infty \cap \mathcal{F}_{g_0}} \frac{k(z, g_0 T_\sigma z)}{I(z)^s k(z, z)} dz \right]$$

$$= \lim_{s\to 0} \sum_{[g_0]} \int_{\mathcal{F}_{g_0}} \frac{k(z, g_0 T_\sigma z)}{I(z)^s k(z, z)} dz \quad .$$

Put $g_0 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and we consider the following integral I,

$$I = \int_{\mathcal{F}_{g_0}} \frac{k(z, g_0 T_\sigma z)}{I(z)^s k(z, z)} dz \quad .$$

Let M be the set of all element m of $\mathcal{O}$ which satisfy $a^\sigma m = md$, then M is a Z-module of rank one. And we see easily that $\tilde{\Gamma}(g_0 T_\sigma) = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \mid m \in M \right\}$. Let m_0 be a generator of M, and $\mathcal{F}_0$ the subset of H^ℓ given by

$$\mathcal{F}_0 = \left\{ (z^{(i)} = x^{(i)} + \sqrt{-1}y^{(i)}) \in H^\ell \;\middle|\; \begin{array}{l} 0 < x^{(1)} < |m_0| , \quad -\infty < x^{(i)} < \infty , \\ 2 \leq i \leq \ell , \quad y^{(i)} > 0, \ 1 \leq i \leq \ell \end{array} \right\}.$$

Then we may take $\mathcal{F}_0$ as a fundamental domain of $\tilde{\Gamma}(g_0 T_\sigma)$ in H^ℓ. Put $^{\sigma_i} g_0 = \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}$ and $\lambda_i = \frac{a_i}{d_i}$, $\mu_i = \frac{b_i}{d_i}$. Then we have

$$I = \frac{(2\sqrt{-1})^{\kappa\ell}}{[\Gamma(g_0 T_\sigma) : \tilde{\Gamma}(g_0 T_\sigma) E]} \int_{\mathcal{F}} \frac{(\prod y^{(i)})^{\kappa-2-s}}{(z^{(1)} - \lambda_1 \bar{z}^{(2)} - \mu_2)^{\kappa} (z^{(2)} - \lambda_2 \bar{z}^{(3)} - \mu_2)^{\kappa} \cdots}$$

$$\times \frac{1}{\cdots (z^{(\ell)} - \lambda_\ell \bar{z}^{(1)} - \mu_\ell)^{\kappa}} dx^{(1)} \cdots dx^{(\ell)} dy^{(1)} \cdots dy^{(\ell)} .$$

Put $Y_1 = y_1$, $Y_2 = \lambda_1 y_2$, $\ldots$, $Y_\ell = \lambda_1 \cdots \lambda_{\ell-1} y_\ell$, and $A = \mu_1 + \lambda_1 \mu_2 + \ldots + \lambda_1 \cdots \lambda_{\ell-1} \mu_\ell$. Then we see

$$I = \frac{(2\pi\sqrt{-1})^{\ell-1} (2\sqrt{-1})^{\kappa\ell} (-1)^{\ell-1} (\ell(\kappa-1))! \, |m_0| \, \lambda_1{}^s (\lambda_1 \lambda_2)^s \cdots (\lambda_1 \cdots \lambda_{\ell-1})^s}{((\kappa-1)!)^\ell [\Gamma(g_0 T_\sigma) : \tilde{\Gamma}(g_0 T_\sigma) E]}$$

$$\int_0^\infty \cdots \int_0^\infty \frac{(\prod Y_i)^{\kappa-2-s}}{(2\sqrt{-1}(Y_1 + \ldots + Y_\ell) - A)^{\ell(\kappa-1)+1}} dY_1 \cdots dY_\ell .$$

By some calculation, we see

$$\int_0^\infty \cdots \int_0^\infty \frac{(\prod Y_i)^{\kappa-2-s}}{(2\sqrt{-1}(Y_1 + \ldots + Y_\ell) - A)^{\ell(\kappa-1)+1}} dY_1 \cdots dY_\ell$$

$$= \frac{\prod_{i=1}^{\ell} B(\kappa-1-s,\ (\ell-i)(\kappa-1)+1+is)}{(2\sqrt{-1})^{\ell(\kappa-1)+1}} \left(\frac{-2\sqrt{-1}}{A}\right)^{1+\ell s}$$

where $B(x, y)$ is the beta-function. We note

$$\lim_{s\to 0} \frac{(\ell(\kappa-1))!}{((\kappa-1)!)^{\ell}} \prod_{i=1}^{\ell} B(\kappa-1-s,\ (\ell-i)(\kappa-1)+1+is) = \frac{1}{(\kappa-1)^{\ell}} .$$

Hence we see that the contribution of the conjugacy classes $[g_0]$ of type p to the integral (2.4.1) equals

$$\lim_{s\to 0} \left(\frac{4\pi}{\kappa-1}\right)^{\ell} \sum_{[g_0]} \frac{1}{2\pi} \frac{1}{[\Gamma(g_0T_\sigma) : \tilde{\Gamma}(g_0T_\sigma)E]} |m_0(g_0)|$$

$$\times\ \lambda_1^{s}(g_0)(\lambda_1(g_0)\lambda_2(g_0))^{s}\cdots(\lambda_1(g_0)\cdots\lambda_{\ell-1}(g_0))^{s} \left(\frac{-\sqrt{-1}}{A(g_0)}\right)^{1+\ell s}$$

where for a representative g_0 of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ of a conjugacy class $[g_0]$, $\lambda_i(g_0) = {}^{\sigma_i}(a/d)$, $\mu_i(g_0) = {}^{\sigma_i}(b/d)$, and

(2.10.1) $\quad A(g_0) = \mu_1(g_0) + \lambda_1(g_0)\mu_2(g_0) + \cdots + \lambda_1(g_0)\cdots\lambda_{\ell-1}(g_0)\mu_\ell(g_0)$.

And $m(g_0)$ denotes an element of $\mathcal{O}$ such that $\begin{pmatrix} 1 & m(g_0) \\ 0 & 1 \end{pmatrix}$ is a generator of $(\tilde{\Gamma}(g_0T_\sigma))/\tilde{\Gamma}(g_0T_\sigma)\cap E$

<u>2.11.</u> For $i = v, e, h, p$, we denote by C_i a complete system of representatives of elements of type v, e, h_a, p in B ($= \Sigma'\mathcal{O}'_A \cap GL_2(F)_+$) with respect to the equivalence

relation $\underset{(\Gamma, E)}{\approx}$ ((2.3.3.)).

For $i = p$, we take the representatives from $B_{\infty}^{(1)}$. Then by 2.5, 2.6, 2.7, 2.8, 2.9, and 2.10, we obtain the following theorem.

Theorem 1. κ is even and $\kappa \geqslant 4$, the trace of $T_S(T(\sigma))$ in $\mathfrak{S}_\kappa(\Gamma)$ is given by the following formula.

(2.11.1)
$$\operatorname{tr} T_S(T(\sigma)) = \frac{\kappa - 1}{4\pi} \sum_{g \in C_v} v(H/\Gamma(gT_\sigma))$$

$$+ \sum_{g \in C_e} \frac{1}{[\Gamma(gT_\sigma) : E]} \frac{\zeta(Ng)^{\kappa-1}}{\eta(Ng) - \zeta(Ng)} (\det Ng)^{1-\frac{\kappa}{2}}$$

$$- \sum_{g \in C_h} \frac{1}{[\Gamma(gT_\sigma) : \tilde{\Gamma}(gT_\sigma)E]} \frac{(\operatorname{Min}(|\eta(Ng)|, |\zeta(Ng)|))^{\kappa-1}}{|\eta(Ng) - \zeta(Ng)|} (\det Ng)^{1-\frac{\kappa}{2}}$$

$$+ \lim_{s \to 0} \sum_{g \in C_p} \frac{1}{2\pi} \frac{|m(g)|\, \lambda_1(g)^s (\lambda_1(g)\lambda_2(g))^s \cdots (\lambda_1(g) \cdots \lambda_{\ell-1}(g))^s}{[\Gamma(gT_\sigma) : \tilde{\Gamma}(gT_\sigma)E]}$$

$$\times \left(\frac{-\sqrt{-1}}{A(g)} \right)^{1+\ell s} .$$

Here $v(H/\Gamma(gT_\sigma))$ denotes the volume of a fundamental domain of $\Gamma(gT_\sigma)$ in H with respect to the invariant measure $\frac{dxdy}{y^2}$.

For an element g of type e $\zeta(Ng)$ and $\eta(Ng)$ denote the eigenvalues of Ng which satisfy (2.6.1). For an element g

of type h_a, $\eta(Ng)$ and $\zeta(Ng)$ denote the eigenvalues of Ng. For an element g of type p in $B_\infty^{(1)}$, $A(g)$ is defined by (2.10.1) and $m(g)$ denotes an element of $\mathcal{O}$ such that $\begin{pmatrix} 1 & m(g) \\ 0 & 1 \end{pmatrix}$ is a generator of $\tilde{\Gamma}(gT_\sigma)/\tilde{\Gamma}(gT_\sigma) \cap E$.

<u>2.12.</u> For the sake of later use, we rewrite the formula (2.11.1) in Th.1 slightly. First we note that if g is an element of type e in B, then $g' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} g^\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is also an element of type e in B. Since $Ng' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} Ng \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$, it holds for some $z_0' \in H$

$$\frac{Ng'z - z_0'}{Ng'z - \overline{z_0'}} = \overline{\eta(Ng)\zeta(Ng)^{-1}}\,\frac{z - z_0'}{z - \overline{z_0'}}$$

where $\eta(Ng)$ and $\zeta(Ng)$ denote the eigenvalues of Ng which satisfy (2.6.1). Hence if we denote by $\eta(Ng')$ $\zeta(Ng')$ the eigenvalues of Ng' which satisfy (2.6.1) for g', then $\eta(Ng') = \zeta(Ng)$ and $\zeta(Ng') = \eta(Ng)$. If C_e is a complete system of representatives of elements of type e in B with respect to $\underset{(\Gamma,E)}{\approx}$, then $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} C_e^\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is also a complete system of them. Hence we see the contribution of elements of type e to $\operatorname{tr} T_S(T(\mathfrak{n}))$ equals

$$(2.12.1)\quad \frac{1}{2}\sum_{g\in C_e}\frac{1}{[(gT_\sigma):E]}\frac{\eta(Ng)^{\kappa-1}}{\eta(Ng)-\zeta(Ng)}(\det Ng)^{1-\frac{\kappa}{2}}$$

$$+\sum_{g\in C_e}\frac{1}{[\Gamma(gT_\sigma):E]}\frac{\zeta(Ng)^{\kappa-1}}{\zeta(Ng)-\eta(Ng)}(\det Ng)^{1-\frac{\kappa}{2}}$$

$$=-\frac{1}{2}\sum_{g\in C_e}\frac{1}{[\Gamma(gT_\sigma):E]}\frac{\eta(Ng)^{\kappa-1}-\zeta(Ng)^{\kappa-1}}{\eta(Ng)-\zeta(Ng)}(\det Ng)^{1-\frac{\kappa}{2}}\ .$$

Next we consider the difference between the equivalence relations $\underset{(\Gamma,E)}{\approx}$ and $\underset{\Gamma}{\approx}$. For an element g of B, the set of the elements in B which is $\underset{(\Gamma,E)}{\approx}$ equivalent to g is equal to $S(g) = \{\varepsilon\gamma^{-1}g^\sigma\gamma \mid \varepsilon\in E,\ \gamma\in\Gamma\}$. We consider the number of $\underset{\Gamma}{\approx}$ equivalence classes in $S(g)$. Let ε_0 be an element of E such as $N_{F/Q}\varepsilon_0 = -1$. For $\varepsilon\in E$ with $N_{F/Q}\varepsilon = 1$, put $a_\sigma = \varepsilon$, then a_σ determines a 1 cocycle $\{a_\tau\}$, $\tau\in\mathfrak{g}$, of $\mathfrak{g}$ in E. Let $\{a_\tau^{(i)}\}$, $1\leq i\leq |H^1(\mathfrak{g}, E)|$, be a complete system of representatives of $H^1(\mathfrak{g}, E)$, and put $a_\sigma^{(i)} = \varepsilon_i$ with $\varepsilon_i\in E$. Then we see that each element in $S(g)$ is $\underset{\Gamma}{\approx}$-equivalent to εg for some element ε of $\overline{E} = \{\varepsilon_i,\ \varepsilon_0\varepsilon_i,\ 1\leq i\leq |H^1(\mathfrak{g}, E)|\}$. For $\overline{\varepsilon}$, $\overline{\varepsilon}'\in\overline{E}$, suppose $\gamma^{-1}\overline{\varepsilon}g^\sigma\gamma = \overline{\varepsilon}'g$ with $\gamma\in\Gamma$, then $\gamma\in\Gamma(gT_\sigma)$. Conversely for $\gamma\in\Gamma(gT_\sigma)$ and $\overline{\varepsilon}\in E$, there exist $\varepsilon\in E$ and $\overline{\varepsilon}'\in\overline{E}$ such that $(\varepsilon\gamma)^{-1}\overline{\varepsilon}g^\sigma(\varepsilon\gamma) = \overline{\varepsilon}'g$. We see $\overline{\varepsilon}'$ is determined uniquely by γ, and $\overline{\varepsilon} = \overline{\varepsilon}'$ if and only if $\gamma\in\widetilde{\Gamma}(gT_\sigma)E$. Hence it follows that $S(g)$ divides into $2H^1(\mathfrak{g}, E)/[\Gamma(gT_\sigma):\widetilde{\Gamma}(gT_\sigma)E]$ equivalence classes with respect to the relation $\underset{\Gamma}{\approx}$. Let $\tilde{C}_i$, $i = v, e, h, p$, be a complete system of representatives of the elements in B $(= \Xi(\mathfrak{O})_A \cap GL_2(F)_+)$ of type i with respect to $\underset{\Gamma}{\approx}$. For $i = p$, we take $\tilde{C}_i$ from $B_\infty^{(1)}$. Then we obtain the following theorem.

<u>Theorem 1'</u>. The assumption and the notations being as in Th.1 , we have

$$(2.12.1) \qquad \mathrm{tr}\ T_S(T(\mathfrak{N})) = \frac{1}{2\,|H^1(\mathfrak{g}, E)|}\left\{ \frac{\kappa-1}{4\pi} \sum_{g\in\tilde{C}_v} v(\ H/\tilde{\Gamma}(gT_\sigma)\ ) \right.$$

$$- \frac{1}{2} \sum_{g\in\tilde{C}_e} \frac{1}{[\tilde{\Gamma}(gT_\sigma)E : E]} \frac{\eta(Ng)^{\kappa-1} - \zeta(Ng)^{\kappa-1}}{\eta(Ng) - \zeta(Ng)} (\det Ng)^{1-\kappa/2}$$

$$- \sum_{g\in\tilde{C}_h} \frac{(\ \mathrm{Min}(|\eta(Ng)|,\ |\zeta(Ng)|)\ )^{\kappa-1}}{|\eta(Ng) - \zeta(Ng)|} (\det Ng)^{1-\kappa/2}$$

$$\left. + \lim_{s\to 0} \sum_{g\in\tilde{C}_p} \frac{|m(g)|\ \lambda_1(g)^s(\lambda_1(g)\lambda_2(g))^s \cdots (\lambda_1(g)\cdots\lambda_{l-1}(g))^s}{2\pi} \left(-\frac{\sqrt{-1}}{A(g)}\right)^{1+ls} \right\}$$

where $v(\ H/\tilde{\Gamma}(gT_\sigma)\ )$ denotes the volume of a fundamental domain of $\tilde{\Gamma}(gT_\sigma)$ in H with respect to the invariant measure $\dfrac{dxdy}{y^2}$.

§3. Twisted conjugacy classes

3.1. Let $\mathfrak{r}$ be a Dedekind domain, and k its quotient field. In this section, we denote by F one of the followings; i) a cyclic extension of k of prime degree ℓ, ii) the direct product of ℓ-copies of k. In the case of ii), we consider k a subring of F by diagonal embedding. We denote by $\mathcal{O}$ the integral closure of r in F. In the case of ii), $\mathcal{O} = r \oplus \dots \oplus r$ (ℓ-copies). In the case of i), we denote by $\mathfrak{G}$ the Galois group of the extension F/k, and we fix a generator σ in the following. In the case of ii), we denote by σ the k-linear automorphism of F given by

$$\sigma : (x_1, x_2, \dots , x_\ell) \longrightarrow (x_2, \dots , x_\ell, x_1)$$

for $(x_1, x_2, \dots , x_\ell) \in F$, and denote by $\mathfrak{G}$ the group of k-linear automorphisms of F generated by σ. We extend the map σ to $M_2(F)$ by component-wise action, and denote it also by σ.

For a subgroup H of $GL_2(F)$, we define an equivalence relation $\underset{H}{\approx}$ in $GL_2(F)$ by

$$(3.1.1) \qquad g \underset{H}{\approx} g' \qquad h^{-1} g\, {}^{\sigma}h = g' \quad \text{for } h \in H .$$

For an element g of $GL_2(F)$, put

$$Ng = g\, {}^{\sigma}g \dots {}^{\sigma^{\ell-1}}g .$$

Since $g\, {}^{\sigma}(Ng) g^{-1} = Ng$, the determinant $\det Ng$ and the trace tr Ng of Ng are contained in k. For $g \in GL_2(F)$, we set

$$Z_\sigma(g) = \{ x \in M_2(F) \mid g^\sigma x = xg \}$$

and

$$Z(Ng) = \{ x \in M_2(F) \mid (Ng)x = xNg \} \quad .$$

Denote by σ_g the map from $M_2(F)$ to itself given by

$$\sigma_g : x \longrightarrow g^\sigma x g^{-1}$$

for $x \in M_2(F)$. Then we see easily the following.

<u>Lemma 3.1</u>. Let the notation be as above.

(i) $Z_\sigma(g)$ is a k-algebra containing Ng.

(ii) $Z(Ng)$ is a F-algebra containing $Z_\sigma(g)$.

(iii) For $x \in GL_2(F)$, it holds $Z_\sigma(x^{-1}g^\sigma x) = x^{-1}Z_\sigma(g)x$ and $Z(x^{-1}(Ng)x) = x^{-1}Z(Ng)x$.

(iv) The restriction $\sigma_g|Z(Ng)$ of σ_g to $Z(Ng)$ induces a k-linear automorphism of $Z(Ng)$ such that the restriction of σ_g to F equals σ. The set of all elements of $Z(Ng)$ fixed by σ_g coincides with $Z_\sigma(g)$.

<u>Remark 3.2</u>. i) If Ng is not contained in $F^\times$, then $Z_\sigma(g) = k + kNg$ and $Z(Ng) = F + FNg$, and in particular, $Z_\sigma(g)$ and $Z(g)$ are commutative. Hence if we denote by $f(X)$ the characteristic polynomial of Ng, then $f(X)$ is contained in $k[X]$, and it holds $Z_\sigma(g) \simeq k[X]/(f(X))$ and $Z(g) \simeq F[X]/(f(X)) \simeq k[X]/(f(X)) \otimes_k F$. The k-algebra $k[X]/(f(X))$ is one of the followings; a) $k \oplus k$, b) an unramified extension

of k of degree 2, c) a ramified extension of k of degree 2, d) $k + k\Delta$ with $\Delta^2 = 0$.

ii) If Ng is contained in $F^\times$, then $Z(Ng) = M_2(F)$. If we put $a_\sigma = g$, then a_σ determines a 1-cocycle $\{a_\tau\}$, $\tau \in \mathfrak{g}$, of $\mathfrak{g}$ in $PGL_2(F)$, and a class of $H^1(\mathfrak{g}, PGL_2(F))$. The k-algebra $Z_\sigma(g)$ is a quaternion algebra over k.

3.2. If we take F as in § 1 and § 2, and $k = Q$, then the definition of Ng and $Z_\sigma(g)$ in this section coincides with that in 2.3 ((2.3.1), (2.3.5)). Here we prove the results on $Z_\sigma(g)$ used in § 2.

Proposition 3.3. Let F be as in 1.1, and $k = Q$. Then $Z_\sigma(g) \cap F$ is equal to Q, and it holds the followings.

i) If g is of type v, $Z_\sigma(g)$ is a quaternion algebra over Q and $Z(Ng) = M_2(F)$.

ii) If g is of type e, $Z_\sigma(g)$ is a imaginary quadratic field, and $Z(Ng)$ is a totally imaginary quadratic extension of F.

iii) If g is of type h, $Z_\sigma(g)$ is a real quadratic field, and $Z(Ng)$ is a totally real quadratic extension of F.

iv_a) If g is of type h_a, $Z_\sigma(g)$ is isomorphic to $Q \oplus Q$, and $Z(Ng)$ is isomorphic to $F \oplus F$.

iv_b) If g is of type h_b, $Z_\sigma(g)$ is isomorphic to F, and $Z(Ng)$ is isomorphic to $F \oplus F$.

v) If g is of type p, $Z_\sigma(g)$ is isomorphic to the Q-algebra $Q \oplus Q\Delta$, where $\Delta^2 = 0$, and $Z(Ng)$ is isomorphic to the F-algebra $F \oplus F\Delta$.

Proof. The first assertion and the assertions i), ii), iii) easily follow from the definition of type v, e, h, and the result of 2.1.

iv_a) In this case, there exists $h \in GL_2(F)_+$ such that $h^{-1}g^{\sigma}h$ and $h^{-1}Ngh$ are diagonal matrices. Hence our assertion easily follows from (iii) of Lemma 3.1.

iv_b) There exists $h \in GL_2(F)_+$ such that $h^{-1}(Ng)h$ is a diagonal matrix and $g' = h^{-1}g^{\sigma}h$ is of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$, hence $Z(h^{-1}Ngh) = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$. Since $\sigma_{g'}$ induces in $Z(h^{-1}Ngh)$ the automorphism $\sigma_{g'} : \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \longmapsto {}^{\sigma}\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$ for $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in Z(h^{-1}Ngh)$. Hence $Z_{\sigma}(g)$ is isomorphic to F.

v) We see there exists an element $h \in GL_2(F)_+$ such that $g' = h^{-1}g^{\sigma}h = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in F$. Hence $Z(h^{-1}Ngh) = F + F\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Since $\sigma_{g'}$ induces in $Z(h^{-1}Ngh)$ the automorphism $\sigma_{g'} : \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \longmapsto \begin{pmatrix} {}^{\sigma}x & {}^{\sigma}y \\ 0 & {}^{\sigma}x \end{pmatrix}$ for $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in Z(h^{-1}Ngh)$, we have $Z_{\sigma}(h^{-1}g^{\sigma}h) = Q + Q\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and our assertion is proved.

3.3. We consider to classify $GL_2(F)$ into $\underset{GL_2(F)}{\approx}$ -equivalence classes. For a subgroup H of $GL_2(F)$, we denote by $\underset{H}{\sim}$ the equivalence relation in $GL_2(F)$ defined by

$$(3.3.1) \qquad g \underset{H}{\sim} g' \iff g = h^{-1}g'h \quad \text{for } h \in H .$$

Then we see $g \underset{H}{\approx} g'$ implies $Ng \underset{H}{\sim} Ng'$. Hence N induces a map from equivalence classes with respect to $\underset{H}{\approx}$ to those with respect to $\underset{H}{\sim}$. For $H = GL_2(F)$, we can prove the following.

Lemma 3.4. The map from $\{g \in GL_2(F) \mid Ng \notin F\} / \underset{GL_2(F)}{\approx}$ to $(N(GL_2(F)) - F^{\times})/ \underset{GL_2(F)}{\sim}$ induced by N is bijective.

Proof. Since the surjectivity is obvious, we prove the map is injective. Assume $Ng_1 \underset{GL_2(F)}{\sim} Ng_2$, for $g_1, g_2 \in GL_2(F)$. As $\det Ng_i$ and $\operatorname{tr} Ng_i$ are contained in k, there exists an element g of $GL_2(k)$ $(\subset GL_2(F))$ which is $\underset{GL_2(F)}{\approx}$-equivalent to Ng_i for $i = 1, 2$. Namely there exist $x_1, x_2 \in GL_2(F)$ such that $x_1^{-1}(Ng_1)x_1 = x_2^{-1}(Ng_2)x_2 = g$. Put $g_i' = x_i^{-1} g_i {}^{\sigma}x_i$, then $g_i' \underset{GL_2(F)}{\approx} g_i$ and $N(g_i') = g$ for $i = 1, 2$. Since ${}^{\sigma}N(g_i') = N(g_i')$ and $N(g_i') = g_i' {}^{\sigma}N(g_i') g_i'^{-1} = g_i' N(g_i') g_i'^{-1}$, g_i' is contained in $Z(g) = Z(Ng_i')$. And by (iv) of Lemma 3.1 we see that the k-linear automorphisms ${}^{\sigma}g_1'$ and ${}^{\sigma}g_2'$ coincide with σ on $Z(g)$. Hence $Z_{\sigma}(g_1') = Z_{\sigma}(g_2')$ and they are contained in $M_2(k)$ $(\subset M_2(F))$. Since $Z(g)$ is commutative and ${}^{\sigma}Z(g) = Z(g)$, $N(g_1') = N(g_2')$ implies $N(g_1'^{-1} g_2') = 1$. Then by Hilbert's theorem 90 for k-algebra $Z_{\sigma}(g_i')$, there exists $x \in Z(g)$ such that $g_1'^{-1} g_2' = x^{-1} {}^{\sigma}x$, hence $g_1' = x g_2' {}^{\sigma}x^{-1}$. This implies $g_1' \underset{GL_2(F)}{\approx} g_2'$, hence $g_1 \underset{GL_2(F)}{\approx} g_2$, and our lemma is proved.

Let B be a commutative finite dimensional k-algebra. Then we can extend σ to $B \otimes_k F$ naturally, we denote it also by σ. For $x \in B \otimes_k F$, put

$$N_{B \otimes F/B}(x) = x^{\sigma}x \dots {}^{\sigma^{l-1}}x ,$$

Then $N_{B \otimes F/B}(x)$ is contained in B. We call $N_{B \otimes F/B}$ the norm from $B \otimes F$ to B. Then we can prove

Lemma 3.5. An element g of $GL_2(F) - F^{\times}$ belongs to $N(GL_2(F)) - F^{\times}$ if and only if the characteristic polynomial $f(X)$ of g belongs to $k[X]$, and it holds

$$(3.3.2) \qquad \widetilde{X} \in N_{K \otimes F/K}((K \otimes F)^{\times})$$

, where $K = k[X]/(f(X))$, and $\widetilde{X}$ is the element of K represented by X.

Proof. As remarked before, the characteristic polynomial of any element of $N(GL_2(F))$ is contained in $k[X]$. Hence we assume the characteristic polynomial of g belongs to $k[X]$. If $N(\bar{g}) = g$ and $g' = x^{-1}gx$ for $\bar{g}, x \in GL_2(F)$, then $N(x^{-1}\bar{g}{}^{\sigma}x) = x^{-1}gx = g'$. Hence if g belongs to $N(GL_2(F))$, any element $g' \in GL_2(F)$ such that $g' \underset{GL_2(F)}{\sim} g$ also belongs to $N(GL_2(F))$. Now any two elements of $GL_2(F) - F^{\times}$ which have the same characteristic polynomials are $\underset{GL_2(F)}{\sim}$ - equivalent to each other. Since the characteristic polynomial of g belongs to $k[X]$, there exists an element of $M_2(k)$ which is $\underset{GL_2(F)}{\sim}$ -equivalent to g. By the above remark, we may assume g

belongs to $M_2(k)$ from the first. Put $Z_\sigma = k + kg$ and $Z = F + Fg$, then $Z_\sigma \simeq K$ and $Z \simeq K \otimes F$ canonically. The map N induces a map from Z to Z_σ, and this coincides with the norm map from $K \otimes F$ to K by the above isomorphism. If there exists an element $\bar{g}$ of $GL_2(F)$ such that $N(\bar{g}) = g$, then we see as in the proof of Lemma 3.4 that $\bar{g} \in Z$ and $Z_\sigma(\bar{g}) = Z_\sigma$ and $Z(N\bar{g}) = Z$. Our assertion easily follows from this.

By the above two lemmas, we can determine $\{g \in GL_2(F) \mid g \in N(GL_2(F)) - F^\times\} / \underset{GL_2(F)}{\approx}$ completely.

Now we consider the elements g of $GL_2(F)$ such that $Ng \in F^\times$. If $Ng \in F^\times$, we see $Ng \in k^\times$, and N defines a map from $\{g \in GL_2(F) \mid Ng \in F^\times\} / \underset{GL_2(F)}{\approx}$ to $k^\times \cap N(GL_2(F))$.
For a subgroup H and a subgroup H' of $F^\times$ we define an equivalence relation in $GL_2(F)$ $\underset{(H,H')}{\approx}$ by

$$g \underset{(H,H')}{\approx} g' \iff g = \varepsilon h^{-1} g' h \quad \text{for } h \in H, \quad \varepsilon \in H' .$$

Then we can prove the following.

Lemma 3.6. The map from $\{g \in GL_2(F) \mid Ng \in F^\times\} / \underset{GL_2(F)}{\approx}$ to $k^\times \cap N(GL_2(F))$ induced by N is bijective.

Proof. For an element g of $GL_2(F)$ such that $Ng \in F^\times$, put $a_\sigma = g$ and $a_{\sigma^i} = a_\sigma {}^\sigma a_{\sigma^{i-1}}$ for i, $1 \leq i \leq \ell - 1$ inductively. Then $\{a_\tau\}$, $\tau \in \mathcal{G}$, determines a 1-cocycle of $\mathcal{G}$ in $PGL_2(F)$, and determines a class of $H^1(\mathcal{G}, PGL_2(F))$. We see this map induces a bijective map from

$\{ g \in GL_2(F) \mid Ng \in F^\times \} \Big/ \underset{(GL_2(F), F^\times)}{\widetilde{\approx}}$ to $H^1(\mathfrak{g}, PGL_2(F))$.

Now the following exact sequence

$$1 \longrightarrow F^\times \longrightarrow GL_2(F) \longrightarrow PGL_2(F) \longrightarrow 1$$

induces a injective map from $H^1(\mathfrak{g}, PGL_2(F))$ to $H^2(\mathfrak{g}, F^\times)$. And we fixed a generator σ of $\mathfrak{g}$, there is an isomorphism from $H^2(\mathfrak{g}, F^\times)$ to $\hat{H}^0(\mathfrak{g}, F^\times)$, where $\hat{H}^0(\mathfrak{g}, F^\times)$ is the modified 0-th cohomology group of $\mathfrak{g}$ in $F^\times$ and is equal to $k^\times/N_{F/k}(F^\times)$ (c.f. Ch VIII, [15]). On the other hand N induces a map from $\{ g \in GL_2(F) \mid Ng \in F^\times \} \Big/ \underset{(GL_2(F), F^\times)}{\widetilde{\approx}}$ to $N(GL_2(F)) \cap k^\times/N_{F/k}(F^\times)$. Then we see the following diagram is commutative.

$$\begin{array}{ccc}
\{ g \in GL_2(F) \mid Ng \in F^\times \} \Big/ \underset{GL_2(F)}{\approx} & \xrightarrow{\ N\ } & N(GL_2(F)) \cap k^\times \\
\downarrow & & \downarrow \\
\{ g \in GL_2(F) \mid Ng \in F^\times \} \Big/ \underset{(GL_2(F), F^\times)}{\widetilde{\approx}} & \longrightarrow & N(GL_2(F)) \cap k^\times/N_{F/k}F^\times \\
\wr\| & & \downarrow \cap \\
H^1(\mathfrak{g}, PGL_2(F)) & \hookrightarrow \ H^2(\mathfrak{g}, F) \simeq \hat{H}^0(\mathfrak{g}, F^\times) & = k^\times/N_{F/k}F^\times
\end{array}$$

For tow element g_1, g_2 of $\{ g \in GL_2(F) \mid Ng \in F^\times \}$, assume $Ng_1 = Ng_2$. Then by the above diagram $g_1 \underset{(GL_2(F), F^\times)}{\widetilde{\approx}} g_2$, hence there exist $a \in F^\times$ and $x \in GL_2(F)$ such that $g_1 = ax^{-1} g_2 {}^\sigma x$. Taking N of the both sides, we see $N_{F/k} a = 1$.

By Hilbert's theorem 90, there exists $a' \in F^{\times}$ such that $a = a'^{-1}\sigma a'$, hence $g_1 = (a'x)^{-1} g_2 {}^{\sigma}(a'x)$, and our assertion is proved.

From the proof of the above lemma we see

Cororally 3.7. The map from $\{ g \in GL_2(F) \mid Ng \in F^{\times}\} / \underset{(GL_2(F),F)}{\approx}$ to $N(GL_2(F)) \cap k^{\times} / N_{F/k}F^{\times}$ induced by N is bijective.

Remark 3.8. i) If F is a field, the cohomology classes $H^1(\mathfrak{g}, PGL_2(F))$ are in one to one correspondence with the isomorphism classes of quaternion algebras D over k such that $D \otimes_k F$ is isomorphic to $M_2(F)$. Unless $[F:k] = 2$, $D \otimes_k F \cong M_2(F)$ implies $D \cong M_2(k)$, and $H^1(\mathfrak{g}, PGL_2(F))$ consists of only one class.

ii) If F is not a field, we see $H^1(\mathfrak{g}, PGL(F))$ consists of only one class, and $N_{F/k}F^{\times} = k^{\times}$. Hence $\{g \in GL_2(F) \mid Ng \in F^{\times}\} / \underset{GL_2(F)}{\approx}$ consists of only one class.

3.4. Let k and $\mathfrak{r}$ be as in 3.1. Let B a finite dimensional algebra over k. A subset Λ of B is called an $\mathfrak{r}$-order if firstly it is a finitely generated $\mathfrak{r}$-module such that $\Lambda \otimes k = B$, and secondly it is a subring of B containing the unity. If F is a field, $\mathcal{O}$-order of a finite dimensional F-algebra is defined in the same way as above. For $g \in GL_2(F)$ with $Ng \notin F^{\times}$, let $C_{\sigma}(g)$ denote the set of all elements of $GL_2(F)$ which are $\underset{GL_2(F)}{\approx}$-equivalent to g, i.e.

(3.4.1) $$C_\sigma(g) = \{ x^{-1} g^\sigma x \mid x \in GL_2(F) \} ,$$

and for an $\underline{r}$-order Λ of $Z_\sigma(g)$, put

(3.4.2) $$C_\sigma(g, \Lambda) = \{ x^{-1} g^\sigma x \mid x \in GL_2(F),\ Z_\sigma(g) \cap x M_2(\mathcal{O}) x^{-1} = \Lambda \} .$$

Then $C_\sigma(g)$ is the disjoint union $\bigcup_\Lambda C_\sigma(g, \Lambda)$, where Λ runs through all $\underline{r}$-orders of $Z_\sigma(g)$. Let U be the subgroup $GL_2(\mathcal{O})$ of $GL_2(F)$, then ${}^\sigma U = U$, and let Ξ be a union of U-double cosets in $GL_2(F)$. For $g \in GL_2(F)$ with $Ng \notin F^\times$, Λ and Ξ, put

(3.4.3) $$\mathfrak{M}_\sigma(g, \Xi) = \{ x \in GL_2(F) \mid x^{-1} g^\sigma x \in \Xi \} .$$

(3.4.4) $$\mathfrak{M}_\sigma(g, \Xi, \Lambda) = \{ g \in GL_2(F) \mid x^{-1} g^\sigma x \in \Xi ,\ Z_\sigma(g) \cap x M_2(\mathcal{O}) x^{-1} = \Lambda \} .$$

Then $\mathfrak{M}_\sigma(g, \Xi)$ is the disjoint union $\bigcup_\Lambda \mathfrak{M}_\sigma(g, \Xi, \Lambda)$. For $g \in GL_2(F)$ with $Ng \in F^\times$, we define $C_\sigma(g)$ and $\mathfrak{M}_\sigma(g, \Xi)$ by (3.4.1) and (3.4.3), and we modify the definition of $C_\sigma(g, \Lambda)$ and $\mathfrak{M}_\sigma(g, \Xi, \Lambda)$ as follows. For a quaternion algebra D over k, we define in the set of all $\underline{r}$-orders of D an equivalence relation by

(3.4.5) $$\Lambda \sim \Lambda' \iff \Lambda = x^{-1} \Lambda' x , \quad \text{for } x \in D^\times$$

for $\mathcal{O}$-orders Λ and Λ'. And for an $\underline{r}$-order Λ of $Z_\tau(g)$, put

(3.4.2)' $$C_\sigma(g, \Lambda) = \{ x^{-1} g^\sigma x \mid x \in GL_2(F),\ Z_\sigma(g) \cap x M_2(\mathcal{O}) x^{-1} \sim \Lambda \}$$

(3.4.4)' $$\mathfrak{M}_\sigma(g, \Xi, \Lambda) = \{ x \in GL_2(F) \mid x^{-1} g^\sigma x \in \Xi ,\ Z_\sigma(g) \cap x M_2(\mathcal{O}) x^{-1} \sim \Lambda \} .$$

Then $C_\sigma(g) = \bigcup_{\Lambda/\sim} C_\sigma(g, \Lambda)$ and $\mathfrak{M}_\sigma(g, \Xi) = \bigcup_{\Lambda/\sim} \mathfrak{M}_\sigma(g, \Xi, \Lambda)$ are disjoint unions. In any case, we see it holds $Z_\sigma(g)^\times \mathfrak{M}_\sigma(g, \Xi)U = \mathfrak{M}_\sigma(g, \Xi)$ and $Z_\sigma(g)^\times \mathfrak{M}_\sigma(g, \Xi, \Lambda)U = \mathfrak{M}_\sigma(g, \Xi, \Lambda)$. Hence $\mathfrak{M}_\sigma(g, \Xi)$ and $\mathfrak{M}_\sigma(g, \Xi, \Lambda)$ divide into double cosets with respect to $Z_\sigma(g)^\times$ and U. We can easily verify the following.

Lemma 3.9. Let the notation be as above.

(i) The map from $C_\sigma(g) \cap \Xi$ to $Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi)$ induced by the correspondence

$$(3.4.5) \qquad x^{-1} g^\sigma x \longmapsto Z_\sigma(g)x$$

is bijective.

(ii) The correspondence (3.4.5) in (i) induces a bijection

$$C_\sigma(g, \Lambda) \cap \Xi \simeq Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi, \Lambda) \quad .$$

(iii) The correspondence (3.4.5) induces a bijection

$$C_\sigma(g, \Lambda) \cap \Xi / \underset{U}{\approx} \; \simeq Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi, \Lambda)/U \quad .$$

(iv) For $x \in GL_2(F)$, $x^{-1}\Lambda x$ is an $\underline{r}$-order of $x^{-1}Z_\sigma(g)x = Z_\sigma(x^{-1}g^\sigma x)$, and it holds

$$\mathfrak{M}_\sigma(x^{-1}g^\sigma x, \Xi, x^{-1}\Lambda x) = x^{-1}\mathfrak{M}_\sigma(g, \Xi, \Lambda) \quad .$$

The correspondence $Z_\sigma(g)^\times \bar{g}U \longrightarrow Z_\sigma(x^{-1}g^\sigma x)^\times (x^{-1}\bar{g})U$ induces a bijective map

$$Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi, \Lambda)/U \simeq Z_\sigma(x^{-1}g^\sigma x) \backslash \mathfrak{M}_\sigma(x^{-1}g^\sigma x, \Xi, x^{-1}\Lambda x)/U.$$

3.5. In the rest of this section, we assume, besides the assumption in 3.1, that k is a p-field of characteristic 0 in the sense of [21] and that $\underline{r}$ is its maximal order. In this section we denote by p the prime element of $\underline{r}$, and in § 4 and 5 we use p to denote a prime. Let v be the discrete valuation of k determined by $v(p) = 1$. Then F is one of the followings; i) the direct product of ℓ-copies of k, ii) the unramified extension of k of degree ℓ, iii) a totally ramified extension of k of degree ℓ. In the case of iii) we assume F is a tamely ramified extension of k. In the case of ii) and iii), let π be a prime element of $\mathcal{O}$, $\mathfrak{P}$ be the maximal ideal of $\mathcal{O}$, and w be the discrete valuation of F determined by $w(\pi) = 1$. For a non-negative integer r, we define the finite union $\Xi(r)$ of U-double cosets as follows. If F is of type i), put

$$\Xi(r) = \{ g \in M_2(\mathcal{O}) \mid \det g \in (p^r \underline{r}^\times) \times \underline{r}^\times \times \cdots \times \underline{r}^\times \}$$

If F is of type ii) or iii), put

$$\Xi(r) = \{ g \in M_2(\mathcal{O}) \mid \det g \in \pi^r \mathcal{O}^\times \} .$$

In the following, we calculate $\left| C_\sigma(g, \Lambda) \cap \Xi / \underset{U}{\approx} \right|$ or the number of double cosets $Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi, \Lambda)/U$ for $\Xi = \Xi(r)$. When F is of type iii), i.e. a totally ramified extension of k, we assume $r = 0$. We note if the set $\mathfrak{M}_\sigma(g, \Xi(r), \Lambda)$ is not empty, $Ng \in \Lambda$, hence Ng is integral, since $\Xi(r) \subset M_2(\mathcal{O})$.

3.6. First we treat the case where $Ng \notin F$. Let $f(X) = X^2 - sX + n$ be a polynomial in $\underline{r}[X]$, g an element of $GL_2(F)$ with the characteristic polynomial $f(X)$, and $K(f)$ the k-algebra $k[X]/(f(X))$. Then there exists a natural isomorphism φ_g from $K(f)$ to the k-algebra $k[g]$ given by $\varphi_g(\tilde{X}) = g$, where we denote by $k[g]$ the k-algebra $k + kg$. Let Λ be an $\underline{r}$-order of $K(f)$. We define a non-negative integer $c_\sigma(f, r, \Lambda)$ for f, r, and Λ as follows. If $g \notin N(GL_2(F))$, we set $c_\sigma(f, r, \Lambda) = 0$. If $g \in N(GL_2(F))$, put $g = N\bar{g}$ with some $\bar{g} \in GL_2(F)$. Since $Z_\sigma(\bar{g}) = k[g]$, φ_g is an isomorphism from $K(f)$ to $Z_\sigma(\bar{g})$. Put

$$c_\sigma(f, r, \Lambda) = \left| Z_\sigma(\bar{g})^{\times} \backslash \mathfrak{M}_\sigma(\bar{g}, \Xi(r), \varphi_g(\Lambda))/U \right| .$$

Then by iv) of Lemma 3.9, this definition of $c_\sigma(f, r, \Lambda)$ is independent of the choice of g and $\bar{g}$. And by iii) of Lemma 3.9, we see $c_\sigma(f, r, \Lambda) = \left| C_\sigma(\bar{g}, \varphi_g(\Lambda)) \cap \Xi(r)/\underset{U}{\approx} \right|$. As noted in 3.5, if $\mathfrak{M}_\sigma(\bar{g}, \Xi(r), \varphi_g(\Lambda)) \neq \phi$, then $\Lambda \ni g = N\bar{g}$. Hence $c_\sigma(f, r, \Lambda) = 0$ for Λ which does not contain g. If there exists $\bar{g} \in \Xi(r)$ such that $N\bar{g}$, $v(n) = r, \ell r$, or r according as F is of type i), ii) or iii) in 3.5. Hence we may compute $c_\sigma(f, r, \Lambda)$ only for such $f(X)$.

Let g and $f(X)$ be as above. Then $g \in N(GL_2(F))$ if and only if the condition (3.3.2) is satisfied for $f(X)$. As to the condition (3.3.2), we give some remarks in the following. If F is not a field, the condition is satisfied for all $f(X)$. Next assume F is a field. If $K(f)$ is of type a) or d) in

i) of Remark 3.2, there exist $\alpha, \beta \in k^{\times}$ such that $f(X) = (X - \alpha)(X - \beta)$. And the condition (3.3.2) is equivalent to that $\alpha, \beta \in N_{F/k}(F^{\times})$. This is obvious if $K(f)$ is of type a). If $K(f)$ is of type d), put $\widetilde{X} = \alpha + \Delta$, then $\Delta^2 = 0$. If $\widetilde{X} \in N_{K(f)\otimes F/K(f)}((K(f)\otimes F)^{\times})$, it is obvious $\alpha \in N_{F/k}(F^{\times})$. If $\alpha \in N_{F/k}(F^{\times})$, put $N_{F/k}\overline{\alpha} = \alpha$, and let $\overline{\beta}$ be an element of F such that $Tr_{F/k}(\overline{\beta}/\overline{\alpha}) = 1$. Then $\overline{\alpha} + \overline{\beta}\Delta \in (K(f)\otimes F)^{\times}$ satisfies $N_{K(f)\otimes F/K(f)}(\overline{\alpha} + \overline{\beta}\Delta) = \alpha + \Delta$. In the case where $K(f)$ is of type b) or c) in i) of Remark 3.2, the condition (3.3.2) is equivalent to that $n \in N_{F/k}(F^{\times})$ if $K(f)\otimes F$ is a field. This is nothing but ([21], Ch XII, Th.4, Cor.3). If $K(f)\otimes F$ is not a field, the condition (3.3.2) is satisfied for all such $f(X)$.

3.7. We quote the following result of H.Hijikata from [8]. Let R be a discrete valuation ring, π a prime element, $P = \pi R$ its maximal ideal, K its quotient field. (Our notation differs from that of [8]) Let g be an integral element of $M_2(K)$, not in the center K, with the characteristic polynomial $f(X) = X^2 - sX + n$. Let Λ be an R-order of $K + Kg$ containing g, ρ a non-negative integer such that $[\Lambda : R + Rg] = [R : P]^{\rho}$. For g, put

$$C(g, \Lambda) = \{ x^{-1}gx \mid (K + Kg)\cap xM_2(R)x^{-1} = \Lambda \} .$$

We denote by $\underset{GL_2(R)}{\sim}$ the equivalence relation in $GL_2(K)$ given by

$$g \underset{\widetilde{GL_2}(R)}{\sim} g' \iff g = x^{-1}g'x \ , \quad \text{for} \quad x \in GL_2(R)$$

Then by Th. 2.2 and Cor. 2.6 of [8], we have the following lemma.

Lemma 3.10. (Hijikata) The notation being as above, then $C(g,\Lambda) \cap M_2(R) / \widetilde{GL_2}(R)$ consists of only one class, and a representative of it given by

$$\begin{pmatrix} \xi & \pi^{\rho} \\ -\pi^{-\rho} f(\xi) & s-\xi \end{pmatrix}$$

, where ξ is an element of R which satisfies $f(\xi) \equiv 0 \bmod . P^{2\rho}$ and $2\xi \equiv s \bmod . P^{\rho}$.

In the following we apply this lemma taking $K = F$ or k .

3.8. Let F be the direct product of ℓ-copies of k. For $f(X) = X^2 - sX + n$ with $s, n \in \underline{r}$, and an $\underline{r}$-order Λ of $K(f)$ containing $\widetilde{X}$, by Lemma 3.12 there exists $g \in M_2(\underline{r})$ with the characteristic polynomial $f(X)$ such that $k[g] \cap M_2(\underline{r}) = \varphi_g(\Lambda)$, where φ_g is the isomorphism from $K(f)$ to $k[g]$ given by $\varphi_g(\widetilde{X}) = g$ as in 3.6. We consider g as an element of $M_2(\mathcal{O})$ by the diagonal embedding, then $k[g] \cap M_2(\mathcal{O}) = \varphi_g(\Lambda)$. For such g, there exists an element $\overline{g}$ of $\Xi(r)$ such that $N\overline{g} = g$ if and only if $n \in p^r \underline{r}^{\times}$. For, only if part is obvious, and if $n \in p^r \underline{r}^{\times}$, put $\overline{g} = (p_1(g), 1, \dots, 1)$, where $p_1(g)$ is the projection of $g \in M_2(\mathcal{O})$ $(= M_2(\underline{r}) \oplus \dots \oplus M_2(\underline{r}))$ to the 1st component. Then it is obvious that $\overline{g} \in \Xi(r)$, $N(\overline{g}) = g$ and $Z_{\sigma}(\overline{g}) = k[g]$. For this $\overline{g}$ and an element $x = (x_1, \dots , x_{\ell})$

$\in GL_2(F)$, $x^{-1}\bar{g}^{\sigma}x \in \Xi(r)$, then $x_2^{-1}x_3, \ldots,$ $x_{\ell-1}^{-1}x_\ell$, $x_\ell^{-1}x_1 \in M_2(\underline{r})^\times$. Hence if $x^{-1}\bar{g}^{\sigma}x \in \Xi(r)$, there exist $x' \in GL_2(k)$ ($\subset GL_2(F)$) and $u \in U$ such that $x = x'u$. Assume $k[g] \cap (x'u)M_2(\mathcal{O})(x'u)^{-1} = \varphi_g(\Lambda)$, hence $k[g] \cap x'M_2(\mathcal{O})x'^{-1} = \varphi_g(\Lambda)$. Projecting this equality to the 1st component, we see $k[g] \cap x'M_2(\underline{r})x'^{-1} = \varphi_g(\Lambda)$. Since $M_2(\underline{r})$ is maximal order, by Lemma 3.10 we see $x'^{-1}gx' \underset{\widetilde{GL_2(\underline{r})}}{\ } g$ and x' is contained in $k[g]^\times M_2(\underline{r})^\times$, hence $\mathcal{M}_\sigma(\bar{g}, \Xi(r), \varphi_g(\Lambda)) \subset Z_\sigma(g)^\times U$. Convesely $Z_\sigma(g)^\times U$ is obviously contained in $\mathcal{M}_\sigma(\bar{g}, \Xi(r), \varphi_g(\Lambda))$, hence we see $c_\sigma(f, r, \Lambda) = 1$. Thus we obtain the following proposition.

<u>Proposition 3.11</u>. Let F be the direct product of ℓ-copies of k, $f(x) = X^2 - sX + n$ a polynomial in $\underline{r}[X]$ with $n \in p^r\underline{r}^\times$. Then we have

$$c_\sigma(f, r, \Lambda) = 1$$

for any $\underline{r}$-order Λ of $K(f)$ containing $\check{X}$.

<u>3.9.</u> Now we consider the case where F is a field. For $\bar{g} \in GL_2(F)$ with $N\bar{g} \notin F^\times$ and an $\underline{r}$-order Λ of $Z_\sigma(\bar{g})$ containing Ng, assume $\mathcal{M}_\sigma(\bar{g}, \Xi(r), \Lambda) \neq \phi$. Then there exists $x \in GL_2(F)$ such that $Z_\sigma(g) \cap xM_2(\mathcal{O})x^{-1} = \Lambda$. For this x, $Z(N\bar{g}) \cap xM_2(\mathcal{O})x^{-1}$ is an $\mathcal{O}$-order of $Z(N\bar{g})$, and if we denote it by $\bar{\Lambda}$, then $\bar{\Lambda}$ satisfies

$$(3.9.1) \qquad \begin{cases} \bar{\Lambda} \cap Z_\sigma(g) = \Lambda \\ \bar{\Lambda} \supset \mathcal{O}[\Lambda] \end{cases}$$

, where $\mathcal{O}[\Lambda]$ is the $\mathcal{O}$-order generated by Λ. Define $C(N\bar{g}, \bar{\Lambda})$ in the same way as in 3.7, namely, put

$$C(N\bar{g}, \bar{\Lambda}) = \{ x^{-1}N(\bar{g})x \mid x \in GL_2(F),\ Z(N\bar{g}) \cap xM_2(\mathcal{O})x^{-1} = \bar{\Lambda} \} .$$

Then N induces a natural map from $C_{\sigma}(\bar{g}, \Lambda) \cap \Xi(r) / \underset{U}{\approx}$ to $\bigcup_{\bar{\Lambda}} C(N\bar{g}, \bar{\Lambda}) \cap M_2(\mathcal{O}) / \underset{U}{\sim}$, where $\bar{\Lambda}$ runs through all $\mathcal{O}$-order of $Z(Ng)$ which satisfy (3.9.1).

In the rest of this section, we denote the k-algebra $k[X]/(f(X))$ by K for the sake of simplicity. Let $f(x) = X^2-sX+n$ be a polynomial in $\underline{r}[X]$ as before and put $L = K \otimes_k F$. We define $\underline{r}$-orders $\Lambda_K(m)$ of K (resp. $\mathcal{O}$-orders $\Lambda_L(m)$ of L) as follows. In the case where K is of type a), b), or c) in i) of Remark 3.2, then for a non-negative integer m, put

(3.9.2) $$\Lambda_K(m) = \underline{r} + p^m\Lambda_K$$

$$(\text{resp.} \quad \Lambda_L(m) = \mathcal{O} + \pi^m\Lambda_L\ )$$

, where Λ_K (resp. Λ_L) is the maximal order of K (resp. L). In the case of d), for any integer m, put

(3.9.3) $$\Lambda_K(m) = \underline{r} + p^m\underline{r}[\tilde{X}]$$

$$(\text{resp.} \quad \Lambda_L(m) = \mathcal{O} + \pi^m\mathcal{O}[\tilde{X}]\ )$$

Then we see that $\Lambda_K(m)$ (resp. $\Lambda_L(m)$) is $\underline{r}$-order (resp. $\mathcal{O}$-order) of K (resp. L) and any $\underline{r}$-order (resp. $\mathcal{O}$-order) of K (resp. L) is $\Lambda_K(m)$ (resp. $\Lambda_L(m)$) for some non-negative integer m in the case of a), b), and c), and for some integer

m in the case of d).

For $f(X)$, let δ_1 be the largest integer such that $\tilde{X}/p^{\delta_1}$ is integral and δ_2 be the integer such that $\mathfrak{r}[\tilde{X}/p^{\delta_1}] = \Lambda_K(\delta_2)$. Then we see $\mathfrak{r}[\tilde{X}] = \Lambda_K(\delta_1 + \delta_2)$, and an $\mathfrak{r}$-order $\Lambda_K(m)$ contains $\tilde{X}$ if and only if $\delta_1 + \delta_2 \geqslant m \geqslant 0$ in the case where K is of type a), b), or c) in i) of Remark 3.2, and $m \leqslant \delta_1 + \delta_2$ in the case where K is of type d) in i) of Remark 3.2.

<u>3.10.</u> Let F be the unramified extension of k of degree ℓ. Let f, K and L be as in 3.9. Then for an $\mathfrak{r}$-order Λ of K, an $\mathfrak{O}$-order $\bar{\Lambda}$ of L satisfying the condition (3.9.1) is uniquely determined by Λ, more precisely we can prove the following.

<u>Lemma 3.12.</u> Let F be as above, and $\Lambda_K(m)$ and $\Lambda_L(m)$ be as in 3.9. Then

i) $\Lambda_L(m) \cap K = \Lambda_K(m)$

ii) $\mathfrak{O}[\Lambda_K(m)] = \Lambda_L(m)$

<u>Proof.</u> First we prove i) under the assumption of ii). Assume $\Lambda_L(m) \cap K = \Lambda_K(m')$ for some integer m'. Then $\Lambda_L(m) \supset \mathfrak{O}[\Lambda_K(m')]$, hence $\Lambda_L(m) \supset \Lambda_L(m')$ by ii), and $m \leqslant m'$. But $\Lambda_K(m') \supset \Lambda_K(m)$, hence $m = m'$, and i) is proved. If ii) is holds for $m = 0$, then it is obvious that it holds also for any integer m. It is enough to prove that $\mathfrak{O}[\Lambda_K(0)] = \Lambda_L(0)$, but this follows easily from the fact that F is an unramified extension of k.

Let g be an element of $GL_2(F)$ with the characteristic polynomial $f(X) = X^2 - sX + n$ with $v(n) = \ell r$. Assume there exists an element $\bar{g}$ of $GL_2(F)$ such that $N\bar{g} = g$. For an $\underline{r}$-order Λ of K containing $\tilde{X}$, denote by $\bar{\Lambda}$ the $\mathcal{O}$-order $\mathcal{O}[\Lambda]$ of L. Then by the above lemma, N induces a map from $C_\sigma(\bar{g}, \Lambda) \cap \Xi(r) / \underset{U}{\approx}$ to $C(N\bar{g}, \bar{\Lambda}) \cap M_2(\mathcal{O}) / \underset{U}{\sim}$. By Lemma 3.10, $C(N\bar{g}, \varphi_g(\bar{\Lambda})) \cap M_2(\mathcal{O}) / \underset{U}{\sim}$ consists of only one class. By Lemma 3.10, there exists g' of $GL_2(k)$ which has the characteristic polynomial $f(X)$ and satisfies the condition $k[g'] \cap M_2(\underline{r}) = \varphi_{g'}(\Lambda)$. Then we see $F[g] \cap M_2(\mathcal{O}) = \varphi_{g'}(\bar{\Lambda})$, where we extend $\varphi_{g'}$ to the isomorphism from L to $F[g]$ naturally, and denote it also by $\varphi_{g'}$. Hence we may take g' as a representative of $C(N\bar{g}, \varphi_g(\bar{\Lambda})) \cap M_2(\mathcal{O}) / \underset{U}{\sim}$. Let $\bar{g}'$ be an element of $GL_2(F)$ such that $N\bar{g}' = g'$, then by Lemma 3.1, we see $\bar{g}' \in Z(g')$ and N coincides with the norm map from $Z(g') = Z_\sigma(\bar{g}') \otimes F$ to $Z_\sigma(\bar{g}')$, since $g' \in M_2(k)$. We show the following relation.

(3.10.1) $$C_\sigma(\bar{g}, \varphi_g(\Lambda)) \cap \Xi(r) \cap \{ \bar{g}'' \in GL_2(F) \mid N\bar{g}'' = g' \}$$
$$= \{ x^{-1} \bar{g}' {}^\sigma x \mid x \in Z(g'),\ x^{-1} \bar{g}' {}^\sigma x \in \varphi_{g'}(\bar{\Lambda}) \}$$

Since $g \underset{GL_2(F)}{\sim} g'$, we see $C_\sigma(\bar{g}, \varphi_g(\Lambda)) = C_\sigma(\bar{g}', \varphi_{g'}(\Lambda))$. If $N\bar{g}'' = g'$ for $\bar{g}'' \in GL_2(F)$, by Lemma 3.4 there exists $x \in GL_2(F)$ such that $\bar{g}'' = x^{-1} \bar{g}' {}^\sigma x$. Since $N\bar{g}' = N\bar{g}''$, it follows $x \in Z(N\bar{g}') = Z(g')$. For $x^{-1} \bar{g}' {}^\sigma x$ with $x \in Z(g')$, we see it holds

$$Z_\sigma(\bar{g}') \cap x M_2(\mathcal{O}) x^{-1} = \varphi_{g'}(\Lambda)$$

Hence

$$C_\sigma(g,\ \varphi_g(\Lambda)) \cap \{ \bar{g}'' \in GL_2(F) \mid N\bar{g}'' = g' \} = \{ x^{-1}\bar{g}'\,{}^\sigma x \mid x \in Z(g')^\times \} .$$

By the way for $x^{-1}\bar{g}'\,{}^\sigma x$ with $x \in Z(g')$, we have

$$x^{-1}\bar{g}'\,{}^\sigma x \in \Xi(r) \iff x^{-1}\bar{g}'\,{}^\sigma x \in M_2(\mathcal{O})$$

$$\iff x^{-1}\bar{g}'\,{}^\sigma x \in M_2(\mathcal{O}) \cap Z(g') = \varphi_{g'}(\bar{\Lambda})$$

, since $v(n) = \ell r$ and ${}^\sigma Z(g') = Z(g')$.

Put

$$C_K(\bar{g}',\ \varphi_{g'}(\bar{\Lambda})) = \{ x^{-1}\bar{g}'\,{}^\sigma x \mid x \in Z(g')^\times,\ \ x^{-1}\bar{g}'\,{}^\sigma x \in \varphi_{g'}(\bar{\Lambda}) \} .$$

, then $C_K(\bar{g}',\ \varphi_{g'}(\Lambda))$ is a subset of $C_\sigma(g,\ \varphi_g(\Lambda)) \cap \Xi(r)$ and the inclusion map induces the following bijective map.

(3.10.2) $$C_K(\bar{g}',\ \varphi_{g'}(\Lambda)) \big/ \widetilde{\varphi_{g'}(\bar{\Lambda})}^\times \xrightarrow{\ \sim\ } C_\sigma(g,\ \varphi_g(\Lambda)) \cap \Xi(r) \big/ \tilde{\tilde{U}}$$

In fact, it is obviously surjective by (3.10.1), and we show it is injective. For two elements $x_1^{-1}\bar{g}'\,{}^\sigma x_1,\ x_2^{-1}\bar{g}'\,{}^\sigma x_2$ of $C_K(\bar{g}',\ \varphi_g(\Lambda'))$, assume there exists an element u of U such that $u^{-1}x_1^{-1}\bar{g}'\,{}^\sigma x_1\,{}^\sigma u = x_2^{-1}\bar{g}'\,{}^\sigma x_2$. Then $u^{-1}N\bar{g}'u = N\bar{g}'$, hence u is contained in $Z(N\bar{g}') \cap U = \varphi_{g'}(\bar{\Lambda})^\times$, and $x_1^{-1}\bar{g}'\,{}^\sigma x_1 \ \widetilde{\varphi_{g'}(\bar{\Lambda})^\times}\ x_2^{-1}\bar{g}'\,{}^\sigma x_2$.

For an element $\tilde{\tilde{X}}$ of L such that $N_{K\otimes F/K}(\tilde{\tilde{X}}) = \tilde{X}$, put

(3.10.3) $$M(\tilde{\tilde{X}},\ r,\ \Lambda) = \{ x \mid x \in L^\times,\ \ x^{-1}\tilde{\tilde{X}}\,{}^\sigma x \in \bar{\Lambda} = \mathcal{O}[\Lambda] \} .$$

Then we see in the same way as in Lemma 3.9 that $C(\bar{g}',\ \varphi_{g'}(\Lambda)) / \widetilde{\varphi_{g'}(\bar{\Lambda})}^\times$ is in one to one correspondence with the double cosets $K^{\times}\backslash M(\overline{\overline{X}}, r, \Lambda)/\overline{\Lambda}^{\times}$ with respect to $K^{\times}$ and $\Lambda^{\times}$. We note if $M(\overline{\overline{X}}, r, \Lambda) \neq \phi$, there exists $\overline{\overline{X}}' \in \overline{\Lambda} = \mathcal{O}[\Lambda]$ such that $\overline{\overline{X}}' = x^{-1}\overline{\overline{X}}^{\sigma}x$ with $x \in L^{\times}$, and for such $\overline{\overline{X}}'$, we have $|K^{\times}\backslash M(\overline{\overline{X}}, r, \Lambda)/\overline{\Lambda}^{\times}| = |K^{\times}\backslash M(\overline{\overline{X}}', r, \Lambda)/\overline{\Lambda}^{\times}|$. Hence by Lemma 3.5 and (3.10.2)

<u>Lemma 3.13</u>. Let the notation be as above. If $\widetilde{X} \notin N_{K\otimes F/K}(\overline{\Lambda})$, then $c_{\sigma}(f, r, \Lambda) = 0$. If there exists an element $\overline{\overline{X}}$ of Λ such that $N_{K\otimes F/K}(\overline{\overline{X}}) = \widetilde{X}$, we have

$$c_{\sigma}(f, r, \Lambda) = |K^{\times}\backslash M(\overline{\overline{X}}, r, \Lambda)/\overline{\Lambda}^{\times}| ,$$

where $M(\overline{\overline{X}}, r, \Lambda)$ is given by (3.10.3).

In the rest of 3.10, we denote $\widetilde{X}$ by g and use the notation $\overline{g}$ to denote an element of L such that $N_{L/K}(\overline{g}) = g$.

We will determine the number of the double cosets $K^{\times}\backslash M(\overline{g}, r, \Lambda)/\overline{\Lambda}^{\times}$. First we prove some results on the unit groups of $\mathcal{O}$-orders of L. For a non-negative integer m, put

$$(3.10.4) \qquad U_F(m) = \begin{cases} \mathcal{O}^{\times} & , m = 0 \\ 1 + \mathfrak{p}^m & , m \geq 1 \end{cases} \qquad U_k(m) = \begin{cases} \underline{r}^{\times} & , m=0 \\ 1 + p^m\underline{r} & , m \geq 1 . \end{cases}$$

Then $U_F(m)$ (resp. $U_k(m)$) is a subgroup of $\mathcal{O}^{\times}$ (resp. $\underline{r}^{\times}$). When K is of type a), b), or c) in i) of Remark 3.2, for a non-negative m put

$$(3.10.5) \qquad U_L(m) = \begin{cases} \Lambda_L(0)^{\times} & , m=0 \\ 1 + \pi^m\Lambda_L(0) & , m \geq 1 \end{cases} \qquad U_K(m) = \begin{cases} \Lambda_K(0)^{\times} & , m=0 \\ 1 + p^m\Lambda_K(0) & , m \geq 1 . \end{cases}$$

If K is of type d), there exist $\alpha \in r$ and $\Delta \in K$ such that $g = \alpha + \Delta$ and $\Delta^2 = 0$, and put for any integer m

$$(3.10.6) \qquad U_L(m) = 1 + \pi^m \theta \Delta \quad , \quad U_K(m) = 1 + p^m \underline{r} \Delta \quad .$$

Then $U_L(m)$ (resp. $U_K(m)$) is a subgroup of $\Lambda_L(m)^\times$ (resp. $\Lambda_K(m)^\times$) and satisfies $\Lambda_L(m)^\times = \theta^\times U_L(m)$ (resp. $\Lambda_K(m)^\times = \theta^\times U_K(m)$). For a $\mathfrak{g}$-module A, put $\hat{H}^0(\mathfrak{g}, A) = A^{\mathfrak{g}}/NA$, where $A^{\mathfrak{g}} = \{ a \in A \mid {}^\sigma a = a \}$ and $NA = \{ a\, {}^\sigma a \ldots {}^{\sigma^{\ell-1}} a \mid a \in A \}$ (c.f. [15], Ch VIII). Then we can prove

Lemma 3.14. Let F, K, L, $\Lambda_K(m)$, and $\Lambda_L(m)$ be as in Lemma 3.12.

i) $\hat{H}^0(\mathfrak{g}, \Lambda_L(m)^\times) = 1$, i.e. $\Lambda_K(m)^\times = N_{L/K}(\Lambda_L(m)^\times)$.

ii) $H^1(\mathfrak{g}, \Lambda_L(m)^\times) = 1$.

, where m is a non-negative integer if K is of type a), b) or c), and an integer if K is of type d) in i) of Remark 3.2.

Proof. First we show the following Sublemma.

Sublemma. Let $U_F(m)$ be as above, then

i) $\hat{H}^0(\mathfrak{g}, U_F(m)) = 1$ and $H^1(\mathfrak{g}, U_F(m)) = 1$

for every non-negative integer m.

ii) $\hat{H}^0(\mathfrak{g}, \mathfrak{P}^m) = 0$, and $H^1(\mathfrak{g}, \mathfrak{P}^m) = 0$

for any integer m.

Proof. i) The assertion $\hat{H}^0(\mathfrak{g}, U_F(m)) = 1$ is nothing but([1?], Ch V, Prop.1). Since $H^1(\mathfrak{g}, F^\times) = 1$ and $\pi \in p\mathcal{O}^\times$, we see $H^1(\mathfrak{g}, U_F(0)) = 1$. For $m \geqslant 1$, we prove $H^1(\mathfrak{g}, U_F(m)) = 1$ by induction on m. Assume $H^1(\mathfrak{g}, U_F(m-1)) = 1$. From the exact sequence

$$1 \longrightarrow U_F(m) \longrightarrow U_F(m-1) \longrightarrow U_F(m-1)/U_F(m) \longrightarrow 1$$

, we obtain the exact sequence

$$\hat{H}^0(\mathfrak{g}, U_F(m-1)/U_F(m)) \longrightarrow H^1(\mathfrak{g}, U_F(m)) \longrightarrow H^1(\mathfrak{g}, U_F(m-1))$$

Since $H^1(\mathfrak{g}, U_F(m-1)) = 1$, it is enough to show that $\hat{H}^0(\mathfrak{g}, U_F(m-1)/U_F(m)) = 1$. By the way $U_F(m-1)/U_F(m) \simeq (\mathcal{O}/\mathfrak{P})^\times$ for $m = 1$, and $U_F(m-1)/U_F(m) \simeq \mathcal{O}/\mathfrak{P}$ for $m \geqslant 2$. Since $\mathcal{O}/\mathfrak{P}$ is a finite field, we have $\hat{H}^0(\mathfrak{g}, U_F(m-1)/U_F(m)) = 1$.

ii) Since $\mathfrak{P}^m \simeq \mathcal{O}$ as $\mathfrak{g}$-modules, we may assume $m = 0$. Then the first assertion follows from([21], Ch VIII, Prop. 4). Since $\mathcal{O}/\mathfrak{P}$ is a cyclic extension of $\underline{r}/p\underline{r}$ of degree ℓ, we can show easily that there exists an element $a \in \mathcal{O}$ such that $\mathcal{O} = {}^{\sigma_1}a\underline{r} + {}^{\sigma_2}a\underline{r} + \dots + {}^{\sigma_\ell}a\underline{r}$, where $\sigma_i = \sigma^{i-1}$. The assertion easily follows from this.

Now we prove our lemma. If K is of type d), we see $\Lambda_L(m)^\times \simeq \mathcal{O}^\times \times U_L(m)$ and $U_L(m) \simeq \mathfrak{P}^m$ as $\mathfrak{g}$-modules, hence the assertion follows directly from the sublemma. If K is of type a), b), or c), we consider the following exact sequence

$$1 \longrightarrow \mathcal{O}^\times \cap U_L(m) \longrightarrow \mathcal{O}^\times \times U_F(m) \longrightarrow \Lambda_L(m)^\times \longrightarrow 1 \quad .$$

We see $\mathcal{O}^\times \cap U_L(m) = U_F(m)$, and we have the following exact sequence.

$$\hat{H}^{0}(\mathfrak{g},\ \mathcal{O}^{\times}\times U_L(m)) \longrightarrow \hat{H}^{0}(\mathfrak{g},\ \Lambda_L(m)^{\times}) \longrightarrow H^{1}(\mathfrak{g},\ U_F(m))$$

$$\longrightarrow H^{1}(\mathfrak{g},\ \mathcal{O}^{\times}\times U_L(m)) \longrightarrow H^{1}(\mathfrak{g},\ \Lambda_L(m)^{\times}) \longrightarrow H^{2}(\mathfrak{g},\ U_F(m))$$

Since $\hat{H}^{0}(\mathfrak{g},\ U_F(m)) \simeq H^{2}(\mathfrak{g},\ U_F(m))$, by the sublemma it is enough to prove $\hat{H}^{0}(\mathfrak{g},\ U_L(m)) = H^{1}(\mathfrak{g},\ U_L(m)) = 1$. We prove this by induction on m. First we prove for $m = 0$. If K is of type a), $\Lambda_L(0)^{\times} \simeq \mathcal{O}^{\times}\times\mathcal{O}^{\times}$, hence our assertion follows from i) of the sublemma. If K is of type b) and $\ell \neq 2$, or K is of type c), L is an unramified extension of K, and our assertion can be proved in the same way as i) of the sublemma. If K is of type b) and $\ell = 2$, $L \simeq F \oplus F$ and we may assume σ acts on $F \oplus F$ by

$$\sigma\ :\ (a,\ b) \longmapsto ({}^{\sigma}b,\ {}^{\sigma}a)$$

for $(a,\ b) \in F \oplus F$, and $\Lambda_L(0)^{\times} = \mathcal{O}\times\mathcal{O}$. Hence our assertion is obvious. For a positive integer m, we consider the exact sequence

$$1 \longrightarrow U_L(m) \longrightarrow U_L(m-1) \longrightarrow U_L(m-1)/U_L(m) \longrightarrow 1 \quad .$$

Assume $\hat{H}^{0}(\mathfrak{g},\ U_L(m-1)) = H^{1}(\mathfrak{g},\ U_L(m-1)) = 1$. Then to prove $\hat{H}^{0}(\mathfrak{g},\ U_L(m)) = H^{1}(\mathfrak{g},\ U_L(m)) = 1$, it is enough to show $\hat{H}^{0}(\mathfrak{g},\ U_L(m-1)/U_L(m)) = H^{1}(\mathfrak{g},\ U_L(m-1)/U_L(m)) = 1$. We show this separately. We see $U_L(m-1)/U_L(m) \simeq (\Lambda_L/\mathfrak{z}\Lambda_L)^{\times}$ for $m = 1$, and $\simeq \Lambda_L/\mathfrak{z}\Lambda_L$ for $m \geqslant 2$. If K is of type a), $\Lambda_L/\mathfrak{z}\Lambda_L \simeq \mathcal{O}/\mathfrak{z} \oplus \mathcal{O}/\mathfrak{z}$, and if K is of type b) and $\ell \neq 2$, $\Lambda_L/\mathfrak{z}\Lambda_L$ is a finite field. Our assetion for this cases is well known. If K is of type b) and $\ell = 2$, $\Lambda_L/\mathfrak{z}\Lambda_L$ is isomorphic to $\mathcal{O}/\mathfrak{z} \oplus \mathcal{O}/\mathfrak{z}$ and σ acts on $\mathcal{O}/\mathfrak{z} \oplus \mathcal{O}/\mathfrak{z}$ by

$$\sigma : (a, b) \longmapsto ({}^{\sigma}b, {}^{\sigma}a)$$

for $(a, b) \in \mathcal{O}/\mathfrak{z} \oplus \mathcal{O}/\mathfrak{z}$. Hence our assertion for this case is obvious. If K is of type c), we denote by $\mathfrak{P}$ the maximal ideal of Λ_L and consider the exact sequences

$$0 \longrightarrow \mathfrak{P}/\mathfrak{z}\Lambda_L \longrightarrow \Lambda_L/\mathfrak{z}\Lambda_L \longrightarrow \Lambda_L/\mathfrak{P} \longrightarrow 0$$

$$1 \longrightarrow 1+\mathfrak{P}/1+\mathfrak{z}\Lambda_L \longrightarrow (\Lambda_L/\mathfrak{z}\Lambda_L)^{\times} \longrightarrow (\Lambda_L/\mathfrak{P})^{\times} \longrightarrow 1$$

Since $\mathfrak{P}/\mathfrak{z}\Lambda_L \simeq \Lambda_L/\mathfrak{P}$, and $1+\mathfrak{P}/1+\mathfrak{z}\Lambda_L \simeq \Lambda_L/\mathfrak{P}$, our assertion easily follows from the fact for finite fields as in the case where K is of type a) or b). Thus our lemma is proved.

As a corollary of the proof, we obtain

Corollary 3.15. The notation being as in Prop. 3.14, we have

$$\hat{H}^0(\mathcal{G}, U_L(m)) = 1, \quad \text{and} \quad H^1(\mathcal{G}, U_L(m)) = 1 \quad .$$

Using this lemma, we can determine $c_{\mathcal{J}}(f, r, \Lambda)$ for $r = 0$.

Proposition 3.16. The notation be as above, let F be the unramified extension of k of degree ℓ, and $f(X) = X^2-sX+n$ be a polynomial in $\underline{r}[X]$ with $v(n) = 0$. Then we have

$$c_\sigma(f, 0, \Lambda) = 1$$

for all $\underline{r}$-order Λ of K containing g.

Proof. Since $r = 0$, g is contained in $\Lambda^\times$. Hence by Lemma 3.14, there exists $\bar{g}$ of $\mathcal{O}[\Lambda]^\times$ such that $N_{L/K}(\bar{g}) = g$. Let's consider the set $M(\bar{g}, 0, \Lambda)$. For $x \in L^\times$, $x \in M(\bar{g}, 0, \Lambda)$ if and only if $x^{-1}\sigma x\bar{g} \in \mathcal{O}[\Lambda]$. Hence we have $x^{-1}\sigma x \in \mathcal{O}[\Lambda]^\times$ and $N_{L/K}(x^{-1}\sigma x) = 1$. By Lemma 3.14, there exists x' of $\mathcal{O}[\Lambda]^\times$ such that $x^{-1}\sigma x = x'^{-1}\sigma x'$. From this, it follows $M(\bar{g}, 0, \Lambda) = K^\times\mathcal{O}[\Lambda]^\times$ and $c_\sigma(f, 0, \Lambda) = 1$.

In the following we treat the case where $r > 0$. Let F be the unramified extension of k with $[F : k] = \ell$ as above, and $f(X) = X^2-sX+n$ be a polynomial in $\underline{r}[X]$ with $v(n) = \ell r$. We denote the k-algebra $k[X]/(f(X))$ be K as before. Let δ_1 and δ_2 be as in 3.9. Then if K is of type a), we have the following.

Proposition 3.17. Let the notation be as above, assume K is of type a), i.e. $K \simeq k \oplus k$, and let α and β be two elements of $\underline{r}$ such that $f(X) = (X-\alpha)(X-\beta)$. Then,

i) $c_\sigma(f, r, \Lambda_K(m)) \neq 0$ only if $\ell \mid v(\alpha), v(\beta)$.

ii) If $\ell \mid v(\alpha), v(\beta)$, then

$$c_\sigma(f, r, \Lambda_K(m)) = \begin{cases} 1 & , m = 0 \\ \dfrac{N\mathfrak{z}^m(1-\frac{1}{N\mathfrak{z}})}{Np^m(1-\frac{1}{Np})} & , 0 < m \leq \delta_1/\ell \\ \dfrac{N\mathfrak{z}^{\delta_1/\ell}}{Np^{\delta_1/\ell}} & , \delta_1/\ell < m \leq \delta_1/\ell + \delta_2 \\ 0 & , \delta_1/\ell + \delta_2 < m \end{cases} .$$

, where $N\mathfrak{z}$ and Np denote $|\mathcal{O}/\mathfrak{z}|$ and $|\underline{r}/p\underline{r}|$ respectively.

Proof. By assumption, K is isomorphic to $F \oplus F$, and by this isomorphism, we may identify g with (α, β) of $F \oplus F$. If we put $(\alpha, \beta) = p^{\delta_1}(u, v)$ with $u, v \in \mathcal{O}$, then one of u and v is a unit of $\mathcal{O}$ and $v(u-v) = \delta_2$. We see $(\alpha, \beta) \in N_{L/K}(L^\times)$ if and only if $\ell \mid v(\alpha)$, $\ell \mid v(\beta)$, and i) is proved. Hence we assume $\ell \mid v(\alpha)$ and $\ell \mid v(\beta)$. Then we see $\delta_1 = \mathrm{Min}(v(\alpha), v(\beta))$ and $\ell \mid \delta_1$. Let $(\bar\alpha, \bar\beta)$ be an element of $\Lambda_L(m)$ such that $N_{L/K}((\bar\alpha, \bar\beta)) = (\alpha, \beta)$. Then we see that $(\bar\alpha, \bar\beta)$ is of the form $p^{\delta_1/\ell}(\bar u, \bar v)$ with $\bar u, \bar v \in \mathcal{O}$, and that

$$m \leq \delta_1/\ell + w(\bar u - \bar v) \leq \delta_1/\ell + v(u-v) = \delta_1/\ell + \delta_2 .$$

Hence by Lemma 3.13, $c_\sigma(f, r, \Lambda_K(m)) = 0$ for m, $m > \delta_1/\ell + \delta_2$. If $m \leq \delta_1/\ell + \delta_2$, we see there exists $(\bar u, \bar v) \in \Lambda_L(\delta_2)$ such that $N_{L/K}(\bar u, \bar v) = (u, v)$. For, if $(u, v) \in \Lambda_K(\delta_2)^\times$, i.e. $v(\alpha) = v(\beta)$, this follows from Lemma 3.14, and otherwise $\delta_2 = 0$, and the assertion is obvious. Put $(\bar{\alpha}, \bar{\beta}) = p^{\delta_1/\ell}(\bar{u}, \bar{v})$, then $N_{L/K}(\bar{\alpha}, \bar{\beta}) = (\alpha, \beta)$ and $(\bar{\alpha}, \bar{\beta}) \in \Lambda_L(\delta_1/\ell + \delta_2)$. Let's consider the set $M((\bar{\alpha}, \bar{\beta}), r, \Lambda_K(m))$. An element (x, y) of $L^\times = (F \oplus F)^\times$ is contained in $M((\bar{\alpha}, \bar{\beta}), r, \Lambda_K(m))$ if and only if $(x, y)^{-1}(\bar{\alpha}, \bar{\beta})^\sigma(x, y) \in \Lambda_L(m)$. Since one of $\bar{u}$ and $\bar{v}$ is a unit of $\mathfrak{o}$ and $w(\bar{u}-\bar{v}) = \delta_2$, we see

$$(x, y)^{-1}(\bar{\alpha}, \bar{\beta})^\sigma(x, y) \in \Lambda_L(m) \iff w(x^{-1}\sigma x\bar{u} - y^{-1}\sigma y\bar{v}) \geqslant m - \delta_1/\ell$$

$$\iff w(x^{-1}\sigma x - y^{-1}\sigma y) \geqslant m - \delta_1/\ell \quad .$$

Hence for m , $0 \leqslant m \leqslant \delta_1/\ell$, $M((\bar{\alpha}, \bar{\beta}), r, \Lambda_K(m)) = K^\times\Lambda_L(0)^\times$.

For m, $\delta_1/\ell < m \leqslant \delta_1/\ell + \delta_2$, we see

$$w(x^{-1}\sigma x - y^{-1}\sigma y) \geqslant m - \delta_1/\ell \iff (x, y)^{-1\,\sigma}(x, y) \in \Lambda_L(m - \delta_1/\ell) .$$

By Lemma 3.14, there exists $(\bar{u}', \bar{v}') \in \Lambda_L(m - \delta_1/\ell)^\times$ such that $(x, y)^{-1\,\sigma}(x, y) = (\bar{u}', \bar{v}')^{-1\,\sigma}(\bar{u}', \bar{v}')$. From this, we see easily $M((\bar{\alpha}, \bar{\beta}), r, \Lambda_K(m)) = K^\times\Lambda_L(m - \delta_1/\ell)^\times$ for m, $\delta_1/\ell < m \leqslant \delta_1/\ell + \delta_2$.

Hence by Lemma 3.13 we have $c_\sigma(f, r, \Lambda_K(m)) = 1$ for $m = 1$, and for $0 < m \leqslant \delta_1/\ell$,

$$\begin{aligned} c_\sigma(f, r, \Lambda_K(m)) &= |\, K^\times\Lambda_L(0)^\times \;/\; K^\times\Lambda_L(m)^\times \,| \\ &= |\Lambda_L(0)^\times/\Lambda_L(m)^\times| \;/\; |\Lambda_K(0)^\times/\Lambda_K(m)^\times| \\ &= N\mathfrak{P}^m(1 - 1/N\mathfrak{P}) \;/\; Np^m(1 - 1/Np) \end{aligned}$$

For m, $\delta_1/\ell < m \leq \delta_1/\ell + \delta_2$, we have

$$c_\sigma(f, r, \Lambda_K(m)) = |K^\times \Lambda_L(m - \delta_1/\ell)^\times / K^\times \Lambda_L(m)^\times|$$

$$= |\Lambda_L(m - \delta_1/\ell)^\times / \Lambda_L(m)^\times| \,/\, |\Lambda_K(m - \delta_1/\ell)^\times / \Lambda_K(m)^\times|$$

$$= N\mathfrak{P}^{\delta_1/\ell} / Np^{\delta_1/\ell}$$

If K is of type b), we can prove the following.

Proposition 3.18. The notation being as in Prop. 3.17, assume K is of type b), i.e. the unramified **extension** of k with $[K:k] = 2$.

i) If $\ell \neq 2$, we have

$$c_\sigma(f, r, \Lambda_K(m)) = \begin{cases} 1 & , \quad m = 0 \\ \dfrac{N\mathfrak{P}^m(1 + 1/N\mathfrak{P})}{Np^m(1 + 1/Np)} & , \quad 0 < m \leq \delta_1/\ell \\ \dfrac{N\mathfrak{P}^{\delta_1/\ell}}{Np^{\delta_1/\ell}} & , \quad \delta_1/\ell < m \leq \delta_1/\ell + \delta_2 \\ 0 & , \quad \delta_1/\ell + \delta_2 < m \end{cases}$$

ii) Assume $\ell = 2$. If $r = \delta_1$ is odd, then we have

$$c_\sigma(f, r, \Lambda_K(m)) = \begin{cases} \delta_1 + 1 & , \quad m = 0 \\ (\delta_1 - 2m + 1)\dfrac{N\mathfrak{P}^m(1 - 1/N\mathfrak{P})}{Np^m(1 + 1/Np)} & , \quad 0 < m \leq (\delta_1 - 1)/2 \\ 0 & , \quad (\delta_1 - 1)/2 < m \end{cases}$$

If $r = \delta_1$ is even, we have

$$c_\sigma(f, r, \Lambda_K(m)) = \begin{cases} \delta_1 + 1 & , \quad m = 0 \\ (\delta_1 - 2m+1)\dfrac{N\mathfrak{z}^m(1-1/N\mathfrak{z})}{Np^m(1+1/Np)} & , \quad 0 < m \leq \delta_1/2 \\ \dfrac{N\mathfrak{z}^{\delta_1/2}}{Np^{\delta_1/2}} & , \quad \delta_1/2 < m \leq \delta_1/2 + \delta_2 \\ 0 & , \quad \delta_1/2 + \delta_2 < m \end{cases}$$

Proof. First assume $\ell \neq 2$, then L is the unramified extension of K with $[L : K] = \ell$. By the assumption $v(n) = \ell r$, there exists an element $\bar{g}$ of $\Lambda_L(0)$ such that $N_{L/K}(\bar{g}) = g$. By this note and Lemma 3.14, we can prove our result for $\ell \neq 2$ in the same way as Prop. 3.17, and we omit the details. Next assume $\ell = 2$, then L is isomorphic to $F \oplus F$, and we may assume F is diagonally embedded in $F \oplus F$ and σ acts on $F \oplus F$ by

$$\sigma : (x, y) \longmapsto ({}^\sigma y, {}^\sigma x)$$

for $(x, y) \in F \oplus F$. Hence for $(x, y) \in F \oplus F$, $N_{L/K}(x, y) = (x{}^\sigma y, {}^\sigma xy)$ and $K = \{(x, {}^\sigma x) \mid x \in F\}$. For g, there exists u of $\mathcal{O}^\times$ such that $g = p^{\delta_1}(u, {}^\sigma u)$, and $\delta_1 = r$. Assume δ_1 is odd. If $\bar{g} = (x, y) \in \Lambda_L(m)$ satisfies $N_{L/K}(\bar{g}) = g$, then $w(x) + w(y) = \delta_1$. Since δ_1 is odd,

$\mathrm{Min}(w(x), w(y)) \leqslant (\delta_1 - 1)/2$. From this it follows $c_\sigma(f, r, \Lambda_K(m)) = 0$ for m, $m > (\delta_1 - 1)/2$. If $0 \leqslant m \leqslant (\delta_1 - 1)/2$, put $\bar{g} = (p^{\delta_1 - m}u, p^m)$, then $N_{L/K}(\bar{g}) = g$ and $\bar{g} \in \Lambda_L(m)$. Let's consider the set $M(\bar{g}, r, \Lambda_K(m))$. For (x, y) of L, we see easily that $(x, y)^{-1}\bar{g}\,{}^{\sigma}(x, y) \in \Lambda_L(m)$ if and only if $0 \leqslant w(x) - w(y) \leqslant \delta_1 - 2m$. Hence $M(\bar{g}, r, \Lambda_L(m)) = \bigcup_{i=0}^{\delta_1 - 2m} K^\times(p^i, 1)\Lambda_L(0)^\times$, where the union is disjoint. Since $|K^\times \backslash K^\times(p^i, 1)\Lambda_L(0)^\times / \Lambda_L(m)^\times|$ $= |\Lambda_L(0)^\times / \Lambda_L(m)^\times| \;/\; |\Lambda_K(0)^\times / \Lambda_K(m)^\times|$, our assetion for δ_1 odd is proved. Now assume δ_1 is even. If $\bar{g} = (x, y)$ of $\Lambda_L(m)$ satisfies $N_{L/K}(\bar{g}) = g$, then $w(x) + w(y) = \delta_1$. If $w(x) \neq w(y)$, then $\mathrm{Min}(w(x), w(y)) < \delta_1/2$. Hence $m \leq \delta_1/2 = \delta_1/2 + \delta_2$. If $w(x) = w(y)$, put $\bar{g} = p^{\delta_1/2}(u_1, u_2)$ with u_1, u_2 of $\mathcal{O}^\times$. Since $w(u_1 - u_2) \leqslant w(u_1/u_2 - {}^{\sigma}(u_1/u_2))$ $= v(u - {}^{\sigma}u) = \delta_2$, $m \leqslant \delta_1/2 + \delta_2$. Hence it follows that $c_\sigma(f, r, \Lambda_K(m)) = 0$ for m, $m > \delta_1/2 + \delta_2$. On the other hand, assume $m \leqslant \delta_1/2 + \delta_2$, then there exists (u_1, u_2) of $\Lambda_L(\delta_2)^\times$ such that $N_{L/K}(u_1, u_2) = (u, {}^{\sigma}u)$ by Lemma 3.14, and put $\bar{g} = p^{\delta_1/2}(u_1, u_2)$. Then $N_{L/K}(\bar{g}) = g$ and $\bar{g} \in \Lambda_L(\delta_1/2 + \delta_2)$ $\subset \Lambda_L(m)$. Let's consider the set $M(\bar{g}, r, \Lambda_K(m))$. Let (x, y) be an element of L such that $w(x) \neq w(y)$. Then (x, y) is contained in $M(\bar{g}, r, \Lambda_K(m))$ if and only if $m - \delta_1/2 \leqslant w(x) - w(y) \leqslant \delta_1/2 - m$. Let (x, y) be an element of L such that $w(x) = w(y)$. Then (x, y) is contained in $M(\bar{g}, r, \Lambda_K(m))$ if and only if $w(x^{-1}u_1{}^{\sigma}y - y^{-1}u_2{}^{\sigma}x) \geqslant m - \delta_1/2$. Hence for m, $0 \leqslant m \leqslant \delta_1/2$, we have

$$M(\bar{g}, r, \Lambda_K(m)) = \Big(\bigcup_{i=-(\delta_1/2-m)}^{\delta_1/2-m} K^{\times}(p^i, 1)\Lambda_L(0)^{\times} \Big)$$

, where the union is disjoint. And for m, $\delta_1/2 < m \leqslant \delta_1/2 + \delta_2$, we have

$$M(\bar{g}, r, \Lambda_K(m)) = K^{\times}\Lambda_L(m - \delta_1/2)^{\times} \quad .$$

Our result for δ_1 even follows easily from this, and our proposition is proved.

If K is of type c), we have the following.

<u>Proposition 3.19</u>. The notation being as in Prop. 3.17 , assume K is of type c), i.e. a ramified extension of k with $[K : k] = 2$. If ℓr is odd, we have

$$c_{\sigma}(f, r, \Lambda_K(m)) = \begin{cases} \dfrac{N\mathfrak{z}^m}{Np^m} & , \quad 0 \leqslant m \leqslant (2\delta_1+1-\ell)/2\ell \\ 0 & , \quad (2\delta_1+1-\ell)/2\ell < m \end{cases}$$

And if ℓr is even, we have

$$c_{\sigma}(f, r, \Lambda_K(m)) = \begin{cases} \dfrac{N\mathfrak{z}^m}{Np^m} & , \quad 0 \leqslant m \leqslant \delta_1/\ell \\ & \end{cases}$$

$$\begin{cases} \dfrac{N\mathfrak{P}^{\delta_1/\ell}}{Np^{\delta_1/\ell}} , & \delta_1/\ell < m \leqslant \delta_1/\ell + \delta_2 \\ 0 , & \delta_1/\ell + \delta_2 < m \end{cases}$$

<u>Proof.</u> Since K is of type c), L is an unramified extension of K. Assume ℓr is odd, then $2\delta_1 + 1 = \ell r$ and $(2\delta_1+1-\ell)/2\ell$ is an integer. By the assumption $v(n) = \ell r$, there exists $\bar{g}$ of Λ_L such that $N_{L/K}(\bar{g}) = g$. Then we see that $p^{-\delta_1}g$ and $p^{-(2\delta_1+1-\ell)/2\ell}\bar{g}$ are prime elements of K and L, respectively. And $\bar{g} \in \Lambda_L(m)$ if and only if $0 \leqslant m \leqslant (2\delta_1+1-\ell)/2\ell$, hence $c_\sigma(f, r, \Lambda_K(m)) = 0$ for m, $m > (2\delta_1+1-\ell)/2\ell$. Let m be an integer such that $0 \leqslant m \leqslant (2\delta_1+1-\ell)/2\ell$ and $\bar{g}$ be an element of $\Lambda_L(m)$ such that $N_{L/K}(\bar{g}) = g$. Let's consider the set $M(\bar{g}, r, \Lambda_K(m))$. We see that for any element x of $L^\times$, $x^{-1}\bar{g}\,{}^\sigma x$ is contained in $\Lambda_L(m)$. Hence $M(\bar{g}, r, \Lambda_K(m)) = K^\times\Lambda_L(0)^\times$. From this we see

$$c_\sigma(f, r, \Lambda_K(m)) = |\Lambda_L(0)^\times/\Lambda_L(m)^\times| \;/\; |\Lambda_K(0)^\times/\Lambda_K(m)^\times| = N\mathfrak{P}^m/Np^m .$$

Assume ℓr is even. Then there exists $u \in \Lambda_K(0)^\times$ such that $g = p^{\delta_1}u$. An element $\bar{g}$ of L such that $N_{L/K}(\bar{g}) = g$ is of the form $\bar{g} = p^{\delta_1/\ell}\bar{u}$ with $\bar{u} \in \Lambda_L(0)^\times$. By the definition of δ_2, $u \in \Lambda_K(\delta_2)^\times$ but $\notin \Lambda_K(\delta_2+1)^\times$. Hence if $\bar{g} \in \Lambda_L(m)$, then $m \leqslant \delta_1/\ell + \delta_2$, and $c_\sigma(f, r, \Lambda_K(m)) = 0$ for $m > \delta_1/\ell + \delta_2$. By Lemma 3.14, there exists $\bar{u}$ of $\Lambda_L(\delta_2)^\times$ such that $N_{L/K}(\bar{u}) = u$. Put $\bar{g} = p^{\delta_1/\ell}\bar{u}$, then $\bar{g} \in \Lambda_L(m)$ if $0 \leqslant m \leqslant \delta_1/\ell + \delta_2$.

Let's consider the set $M(\bar{g}, r, \Lambda_K(m))$. For m, $0 \leqslant m \leqslant \delta_1/\ell$, we see $M(\bar{g}, r, \Lambda_K(m)) = K^\times \Lambda_L(0)^\times$. Hence if $0 \leqslant m \leqslant \delta_1/\ell$, $c_\sigma(f, r, \Lambda_K(m)) = |\Lambda_L(0)^\times/\Lambda_L(m)^\times| \,/\, |\Lambda_K(0)^\times/\Lambda_K(m)^\times| = N\mathfrak{P}^m/Np^m$.
For m, $\delta_1/\ell < m$, we see

$$x \in M(\bar{g}, r, \Lambda_K(m)) \iff x^{-1}\sigma x p^{\delta_1/\ell}\bar{u} \in \Lambda_L(m) \iff x^{-1}\sigma x \in \Lambda_L(m - \delta_1/\ell) .$$

Hence by Lemma 3.14, we see $M(\bar{g}, r, \Lambda_K(m)) = K^\times \Lambda_L(m - \delta_1/\ell)$, and $c_\sigma(f, r, \Lambda_K(m)) = N\mathfrak{P}^{\delta_1/\ell}/Np^{\delta_1/\ell}$.

If K is of type d), we have the following.

<u>Proposition 3.20.</u> The notation being as in Prop. 3.17, assume K is of type d), i.e. $K \simeq k + k\Delta$ with $\Delta^2 = 0$. Let α be an element of $\underline{r}$ such that $f(X) = (X-\alpha)^2$.

(i) $c_\sigma(f, r, \Lambda_K(m)) \neq 0$ only if $\ell \mid v(\alpha)$.

(ii) If $\ell \mid v(\alpha)$, then we have

$$c_\sigma(f, r, \Lambda_K(m)) = \begin{cases} N\mathfrak{P}^{\delta_1/\ell}/Np^{\delta_1/\ell}, & m \leqslant -(\ell-1)\delta_1/\ell \\ 0, & -(\ell-1)\delta_1/\ell < m \end{cases}$$

<u>Proof.</u> Put $g = \alpha + \Delta$, then $\Delta^2 = 0$ and $K = k + k\Delta$. For any integer m, $\Lambda_L(m) = \mathcal{O} + \mathcal{O}\pi^m\Delta$ and $\Lambda_K(m) = \underline{r} + \underline{r}p^m\Delta$. As noted in 3.6, $g \in N_{L/K}(L^\times) \iff \alpha \in N_{F/k}(F^\times)$, and $\alpha \in N_{F/k}(F^\times) \iff \ell \mid v(\alpha)$ in this case, hence i) is proved. Assume $\ell \mid v(\alpha)$ and $N_{L/K}(\bar{g}) = g$ with $\bar{g} \in L^\times$, and put

$\bar{g} = x + y\Delta$, where $x, y \in \mathcal{O}$. Then we see $N_{F/k}(x) = \alpha$ and $\alpha Tr_{F/k}(y/x) = 1$. Since $v(\alpha) = \delta_1$, it follows that $w(x) = \delta_1/\ell$ and $w(y/x) \leqslant -\delta_1$, hence $w(y) \leqslant -(\ell-1)\delta_1/\ell$. This implies $c_\sigma(f, r, \Lambda_K(m)) = 0$, for $m, -(\ell-1)\delta_1/\ell < m$. Let x be an element of $\mathcal{O}$ such that $N_{F/k}(x) = \alpha$, then there exists $y \in \mathfrak{P}^{-(\ell-1)\delta_1/\ell}$ such that $Tr_{F/k}(y/x) = 1/\alpha$. Put $\bar{g} = x + y\Delta$, then $N_{L/K}(\bar{g}) = g$ and $\bar{g} \in \Lambda_L(-(\ell-1)\delta_1/\ell)$. Let's consider the set $M(\bar{g}, r, \Lambda_K(m))$ for $m, m \leqslant -(\ell-1)\delta_1/\ell$. An element $x'+ y'\Delta$ of $L^\times$ belongs to $M(\bar{g}, r, \Lambda_K(m))$ if and only if $(x'+ y'\Delta)^{-1\sigma}(x'+ y'\Delta)(x + y\Delta) \in \Lambda_L(m)$ by definition. We see

$$(x'+ y'\Delta)^{-1\sigma}(x'+ y'\Delta)(x + y\Delta) \in \Lambda_L(m)$$

$$\Longleftrightarrow (x'+ y'\Delta)^{-1\sigma}(x'+ y'\Delta) \in \Lambda_L(m - \delta_1/\ell)^\times .$$

By Lemma 3.14, we obtain $M(\bar{g}, r, \Lambda_K(m)) = K^\times\Lambda_L(m - \delta_1/\ell)^\times$, and $c_\sigma(f, r, \Lambda_K(m)) = N\mathfrak{P}^{\delta_1/\ell}/N\mathfrak{p}^{\delta_1/\ell}$.

3.11. Let F be a tamely ramified extension of k of degree ℓ . In this case we assume $r = 0$. Moreover if $n \notin N_{F/k}(F^\times)$, it is obvious $c_\sigma(f, r, \Lambda_K(m)) = 0$, where $f(X) = X^2-sX+n$. Hence assume $v(n) = 0$ and $n \in N_{F/k}(F^\times)$. First we prove the result corresponding to Lemma 3.12.

Lemma 3.21. Let F be a tamely ramified extension of k of degree ℓ , and K, L, $\Lambda_K(m)$, $\Lambda_L(m)$ be as in 3.9.

i) If K is of type a), or b), then

$$\Lambda_L(m) \cap K = \begin{cases} \Lambda_K(0) & \text{if } m = 0 \\ \Lambda_K(n) & \text{if } \ell n - (\ell-1) \leqslant m \leqslant \ell n \end{cases}$$

and

$$\theta[\Lambda_K(n)] = \Lambda_L(\ell n)$$

ii) If K is of type c) and $\ell \neq 2$, then

$$\Lambda_L(m) \cap K = \begin{cases} \Lambda_K(0) & \text{if } 0 \leqslant m \leqslant (\ell-1)/2 \\ \Lambda_K(n) & \text{if } \ell n + \frac{\ell-1}{2} - (\ell-1) \leqslant m \leqslant \ell n + \frac{\ell-1}{2} \end{cases}$$

and

$$\theta[\Lambda_K(n)] = \Lambda_L(\ell n + \frac{\ell-1}{2})$$

If K is of type c) and $\ell = 2$, then

$$\Lambda_L(m) \cap K = \begin{cases} \Lambda_K(0) & \text{if } 0 \leqslant m \leqslant 1 \\ \Lambda_K(n) & \text{if } 2n \leqslant m \leqslant 2n+1 \end{cases}$$

and

$$\theta[\Lambda_K(n)] = \Lambda_L(2n+1)$$

iii) If K is of type d), then

$$\Lambda_L(m) \cap K = \Lambda_K(n) \qquad \text{if } \ell n - (\ell-1) \leqslant m \leqslant \ell n$$

and

$$\theta[\Lambda_K(n)] = \Lambda_L(\ell n)$$

Proof. First we prove the second assertions. It is enough to prove them for $n = 0$ as in the proof of Lemma 3.12. We can easily verify them separately, and omit the details. As to the first assertions we can prove in the same way as Lemma 3.12. For example, assume K is of type a), and for a non-negative integer m, put $\Lambda_L(m) \cap K = \Lambda_K(n)$ with a non-negative integer n. Then $\mathcal{O}[\Lambda_L(m) \cap K] = \Lambda_L(\ell n)$, hence $m \leq \ell n$. If $m \leq \ell(n-1)$, then $\Lambda_L(m) \supset \Lambda_K(n-1)$, and $\Lambda_K(n) \supset \Lambda_K(n-1)$ by the assumption on n. This is a contradiction. In the other cases, we can prove the first assertions in the same way and omit the details.

We define $U_F(m)$, $U_k(m)$, $U_L(m)$, $U_K(m)$ as in 3.10 by (3.10.4), (3.10.5) and (3.10.6). Then $U_F(m)$ (resp. $U_k(m)$) is a subgroup of $\mathcal{O}^\times$ (resp. $\underline{r}^\times$). And $U_L(m)$ (resp. $U_K(m)$) is a subgroup of $\Lambda_L(m)^\times$ (resp. $\Lambda_K(m)^\times$) and satisfies $\Lambda_L(m)^\times = \mathcal{O}^\times U_L(m)$ (resp. $\Lambda_K(m)^\times = \mathcal{O}^\times U_K(m)$).

Lemma 3.22. Let F, K, L, $\Lambda_K(m)$, $\Lambda_L(m)$, $U_F(m)$, and $U_L(m)$ as above.

i) $H^1(\mathfrak{g}, \Lambda_L(0)^\times) \simeq \begin{cases} Z_\ell \times Z_\ell & \text{if } K \text{ is of type a)} \\ Z_\ell & \text{if } K \text{ is of type b), d), or type c) and } \ell \neq 2 \\ 1 & \text{if } K \text{ is of type c) and } \ell = 2 \end{cases}$

, where we denote by Z_ℓ the cyclic group of order ℓ.

ii) $H^1(\mathfrak{g}, \Lambda_L(m)^\times) \simeq Z_\ell$, for $m \geqslant 1$ if K is of type a), b) or c) and for any integer m if K is of type d).

Proof. First we prove the following.

Sublemma. Let $U_F(m)$ be as above.

i) $$\hat{H}^0(\mathfrak{g}, U_F(m)) \simeq \begin{cases} Z_\ell & , m = 0 \\ 1 & , m \geqslant 1 \end{cases}$$

$$H^1(\mathfrak{g}, U_F(m)) \simeq \begin{cases} Z_\ell & , m = 0 \\ 1 & , m \geqslant 1 \end{cases}$$

ii) $\hat{H}^0(\mathfrak{g}, \mathfrak{P}^m) = H^1(\mathfrak{g}, \mathfrak{P}^m) = 0$ for any integer m.

Proof. i) Put $a_\sigma = \pi^{-1}\,{}^\sigma\pi$, then a_σ determines a 1-cocycle $\{a_\tau\}$, $\tau \in \mathfrak{g}$, of $\mathfrak{g}$ in $\Lambda_L(0)^\times$. It is easily to see $\{a_\tau\}$ gives a generator of $H^1(\mathfrak{g}, U_F(0))$ and is of order ℓ . Hence $H^1(\mathfrak{g}, U_F(0)) \simeq Z_\ell$. The assertion for $\hat{H}^0(\mathfrak{g}, U_F(0))$ easily follows from the local class field theory. We prove $H^1(\mathfrak{g}, U_F(m)) = 1$, $m \geqslant 1$, by induction on m. First we show $H^1(\mathfrak{g}, U_F(1)) = 1$. Assume $N_{F/k}(x) = 1$ for $x \in U_F(1)$, then there exists $y \in F^\times$ such that $x = y^{-1}\,{}^\sigma y$, since $H^1(\mathfrak{g}, F^\times) = 1$ Put $y = \pi^i u$ with some integer i and $u \in \mathcal{O}^\times$, then $x \equiv ({}^\sigma\pi/\pi)^i \equiv 1 \mod. \mathfrak{P}$. Since the extension F/k is tamely ramified, it follows that ℓ divides i. Since $\pi^\ell \in p\mathcal{O}^\times$, we may assume $y = u$ with $u \in \mathcal{O}^\times$. For u, there exists $u' \in \underline{r}^\times$ such that $u \equiv u' \mod. \mathfrak{P}$. Put $y' = uu'^{-1}$, then $y' \in U_F(1)$

and $x = y'^{-1}\,{}^{\sigma}y'$. Hence $H^1(\mathcal{G}, U_F(1)) = 1$. We assume $H^1(\mathcal{G}, U_F(m)) = 1$ for $m \geqslant 1$, and prove $H^1(\mathcal{G}, U_F(m+1)) = 1$. By the exact sequence

$$1 \longrightarrow U_F(m+1) \longrightarrow U_F(m) \longrightarrow U_F(m)/U_F(m+1) \longrightarrow 1$$

we obtain the exact sequence

$$\hat{H}^0(\mathcal{G}, U_F(m)/U_F(m+1)) \longrightarrow H^1(\mathcal{G}, U_F(m+1)) \longrightarrow H^1(\mathcal{G}, U_F(m))$$

Hence it is enough to prove $\hat{H}^0(\mathcal{G}, U_F(m)/U_F(m+1)) = 1$. But this follows easily from the fact $(\ell, |U_F(m)/U_F(m+1)|) = 1$. Now we see $U_F(m)^{\mathcal{G}} = U_k(m/\ell)$ if $\ell \mid m$ and $U_F(m)^{\mathcal{G}} = U_k([m/\ell] + 1)$ if $\ell \nmid m$. And the assertion $\hat{H}^0(\mathcal{G}, U_F(m)) = 1$ for $m \geqslant 1$ is an easy consequence of ([illegible], Ch V, § 3, Cor. 3 of Prop. 5). ii) The assertion $\hat{H}^0(\mathcal{G}, \mathfrak{p}^m) = 0$ easily follows from ([illegible], Ch VIII, § 1, Prop. 4). We prove $H^1(\mathcal{G}, \mathfrak{p}^m) = 0$. Put $a_\tau = ({}^{\tau}\pi/\pi)^m$, then a_τ determines a 1-cocycle $\{a_\tau\}$, $\tau \in \mathcal{G}$, of $\mathcal{G}$ in $\mathcal{O}^\times$. We consider $\mathcal{O}$ as a $\mathcal{G}$-module in the following way. If we make $\mathcal{G}$ act on $\mathcal{O}$ by $\tau(x) = a_\tau\,{}^{\tau}x$, then we obtain another $\mathcal{G}$-module, and we denote it by $\tilde{\mathcal{O}}$. Then $\mathfrak{p}^m$ is isomorphic to $\tilde{\mathcal{O}}$ as $\mathcal{G}$-modules by the map $x \longmapsto \pi^{-m}x$ for $x \in \mathfrak{p}^m$. If $\sum_{i=0}^{\ell-1} \sigma^i(x) = 0$, for $x \in \tilde{\mathcal{O}}$,

put $x_1 = x$, $x_2 = x + \sigma(x)$, ... , $x_{\ell-1} = x + \sigma(x) + \ldots + \sigma^{\ell-2}(x)$,

and $y = \sum_{i=1}^{\ell-1} x_i$. Then we see $\sigma(y) = y - \ell x$, hence $x = (y/\ell) - \sigma(y/\ell)$. Since $\ell \in \mathcal{O}^\times$, it is proved that $H^1(\mathcal{G}, \mathfrak{p}^m) = 0$.

Now we prove our lemma. i) We see $\Lambda_L(0)^\times \simeq \mathcal{O}^\times \times \mathcal{O}^\times$ if K is of type a), and $\simeq \mathcal{O}^\times \times \Lambda_L(0)^\times$ if K is of type d). If K is of type d), $\Lambda_L(0) \simeq \mathcal{O}$ as $\mathfrak{G}$-modules. Hence the assertions for such K follows from the sublemma. If K is of type b), or of type c) and $\ell \neq 2$, we can prove the assertion i) in the same way as the sublemma, since L is a tamely ramified extension of K of degree ℓ. If K is of type c) and $\ell = 2$, two cases can occur, i.e. 1) $L = K \otimes F$ is a field, or 2) $L \simeq F \oplus F$. In the case of 1), L is an unramified extension of K, and the assetion can be proved in the same way as Lemma 3.14. In the case of 2), the assertion is obvious.
ii) For $m \geqslant 1$, $\Lambda_L(m)^\times = \mathcal{O}^\times U_L(m)$. If K is of type d), $\Lambda_L(m)^\times \simeq \mathcal{O}^\times \times U_L(m)$, and $U_L(m) \simeq \mathfrak{p}^m$. Hence our assertion easily follows from the sublemma. We assume K is of type a), b), or c). We consider the following exact sequence

$$1 \longrightarrow \mathcal{O}^\times \cap U_L(m) \ (= U_F(m)) \longrightarrow \mathcal{O}^\times \times U_L(m) \longrightarrow \Lambda_L(m)^\times \longrightarrow 1$$

In the same way as in Lemma 3.14, it is enough to prove $H^1(\mathfrak{G}, U_L(m)) = 1$ for $m \geqslant 1$, since $\hat{H}^0(\mathfrak{G}, U_F(m)) = H^1(\mathfrak{G}, U_F(m)) = 1$ by the sublemma. Consider the exact sequence

$$1 \longrightarrow U_L(m+1) \longrightarrow U_L(m) \longrightarrow U_L(m)/U_L(m+1) \longrightarrow 1$$

Since $(|U_L(m)/U_L(m+1)|, \ell) = 1$, we have $\hat{H}^0(\mathfrak{G}, U_L(m)/U_L(m+1)) = 1$. Hence it follows from $H^1(\mathfrak{G}, U_L(m)) = 1$ that $H^1(\mathfrak{G}, U_L(m+1)) = 1$, and it is enough to prove it for $m = 1$. We see

$U_L(1) \simeq U_F(1) \times U_F(1)$ if K is of type a). Our assertion for such K follows from the sublemma. If K is of type b) or c), we can prove it in a similar way as i), and we omit the details.

Corollary 3.23. The notation being as in Lemma 3.22, then we have $H^0(\mathfrak{g}, U_L(m)) = H^1(\mathfrak{g}, U_L(m)) = 1$ for $m \geqslant 1$ if K is of type a), b), or c) and for any integer m if K is of type d).

The assertion $H^1(\mathfrak{g}, U_L(m)) = 1$ is shown in the proof of the above lemma, and the assertion $\hat{H}^0(\mathfrak{g}, U_L(m)) = 1$ can be proved in the similar say as in Lemma 3.14 by using the above sublemma. We omit the details.

Remark 3.24. If K is of type a) and $m = 0$, a complete system of the representatives of $H^1(\mathfrak{g}, \Lambda_L(m)^{\times})$ is given by the 1-cocycles determined by $a_\sigma = (\pi^i, \pi^j)^{\sigma}(\pi^i, \pi^j)^{-1}$, $0 \leqslant i, j \leqslant \ell-1$. In the other cases, that is given by the 1-cocycles determined by $a_\sigma = \pi^{i\sigma}\pi^{-i}$, $0 \leqslant i \leqslant \ell - 1$.

Now we determine $c_0(f, 0, \Lambda_K(m))$ according to the type of K. Let F be a tamely ramified extension of k with $[F:k] = \ell$, and $f(X) = X^2 - sX + n$ be a polynomial in $r[X]$ such that $v(n) = 0$ and $n \in N_{F/k}(F^{\times})$. We denote by K the k-algebra $k[X]/(f(X))$, and by $\tilde{X}$ the class represented by X as before. Let δ_1 and δ_2 be as in 3.9, then $\delta_1 = 0$, since we assume $v(n) = 0$, and $r[\tilde{X}] = \Lambda_K(\delta_2)$. We denote by χ_i, $1 \leqslant i \leqslant \ell$, the characters of $k^\times$ corresponding to the extension F in the sense of the local class field theory. Since F is a tamely ramified extension of k, they induce the characters of $(\underline{r}/p\underline{r})^\times$. We denote them also by χ_i, $1 \leqslant i \leqslant \ell$, and assume χ_1 is the identity character. Then for $x \in \underline{r}^\times$, we have

$$x \in N_{F/k}(F^\times) \iff \sum_{i=1}^{\ell} \chi_i(\bar{x}) = \ell$$

, where $\bar{x}$ is the class of $\underline{r}/p\underline{r}$ represented by x. If K is of type a), we have the following.

<u>Proposition 3.25</u>. Notation being as above, assume K is of type a), i.e. $K \simeq k \oplus k$. Let α and β be elements of $\underline{r}$ such that $f(X) = (X-\alpha)(X-\beta)$. Then, we have

$$c_\sigma(f, 0, \Lambda_K(m)) = \begin{cases} \ell \sum_{i=1}^{\ell} \chi_i(\alpha) = \ell \sum_{i=1}^{\ell} \chi_i(\beta) & , 0 \leqslant m \leqslant \delta_2 \\ 0 & , \delta_2 < m . \end{cases}$$

<u>Proof</u>. It is obvious that there exists $\tilde{\tilde{X}}$ of L such that $N_{L/K}(\tilde{\tilde{X}}) = \tilde{X}$ if and only if $\sum_{i=1}^{\ell} \chi_i(\alpha) = \sum_{i=1}^{\ell} \chi_i(\beta) = \ell$, and if there does not exist such $\tilde{\tilde{X}}$, then $\sum_{i=1}^{\ell} \chi_i(\alpha) = \sum_{i=1}^{\ell} \chi_i(\beta) = 0$. On the other hand if $\sum_{i=1}^{\ell} \chi_i(\alpha) = \ell$, there exists $\tilde{\tilde{X}} \in \Lambda_L(\ell\delta_2)^\times$ such that $N_{L/K}(\tilde{\tilde{X}}) = \tilde{X}$, where $L = K \otimes F$. If $\delta_2 = 0$, this is obvious. If $\delta_2 > 0$, put $\tilde{X} = xu$ with $x \in \underline{r}^\times$

and $u \in U_K(\delta_2)$, then $x \equiv \alpha \bmod \mathfrak{p}\underline{r}$. Hence there exists $\bar{x} \in \mathcal{O}^\times$ such that $N_{F/k}(\bar{x}) = x$. By Cor. 3.23 there exists $\bar{u} \in U_L(\ell\delta_2)$ such that $N_{L/K}(\bar{u}) = u$, since $U_L(\ell\delta_2)^{\mathcal{J}} = U_K(\delta_2)$. Put $\bar{\bar{X}} = \bar{x}\bar{u}$, then $\bar{\bar{X}}$ satisfies the above conditions. Let ξ be an element of $\underline{r}$ such that $f(\xi) \equiv 0 \bmod. (p\underline{r})^{2\delta_2}$ and $2\xi \equiv s \bmod. (p\underline{r})^{\delta_2}$. As ξ we may take α. If $\delta_2 < m$, $\Lambda_K(m) \not\ni \tilde{X}$, hence $c_\sigma(f, 0, \Lambda_K(m)) = 0$. For m, $0 \leqslant m \leqslant \delta_2$, by Lemma 3.10 ,

$g = \begin{pmatrix} \xi & p^{\delta_2 - m} \\ -p^{\delta_2 - m} f(\xi) & s - \xi \end{pmatrix}$ is an element of $M_2(\underline{r})$ such that $k[g] \cap M_2(\underline{r}) = \varphi_g(\Lambda_K(m))$ and $F[g] \cap M_2(\mathcal{O}) = \varphi_g(\Lambda_L(\ell m))$, where φ_g is the isomorphism from L to $F[g]$ given by $\varphi_g(\tilde{X}) = g$. Put $\bar{g} = \varphi_g(\bar{\bar{X}})$, and let's consider the set $\mathfrak{M}_\sigma(\bar{g}, 0, \varphi_g(\Lambda_K(m)))$. Now N induces a map from $C_\sigma(\bar{g}, \Lambda) \cap \Xi(0) / \tilde{\tilde{U}}$ to $\bigcup_{\bar{\Lambda}} C(N\bar{g}, \bar{\Lambda}) \cap M_2(\mathcal{O}) / \tilde{U}$ (see 3.9) , where $\bar{\Lambda}$ are the $\mathcal{O}$-orders of $Z(N\bar{g})$ which satisfy (3.9.1). If $m = 0$, then the $\mathcal{O}$-order of $Z(N\bar{g})$ which satisfies (3.9.1) for $\Lambda = \varphi_g(\Lambda_K(0))$ is $\varphi_g(\Lambda_L(0))$ by Lemma 3.21. Since $C(N\bar{g}, \varphi_g(\Lambda_L(0))) \cap M_2(\mathcal{O}) / \tilde{U}$ consists of only one class, hence for $x \in \mathfrak{M}_\sigma(\bar{g}, 0, \varphi_g(\Lambda_K(0)))$, there exists $u \in U$ such that $N(x^{-1}\bar{g}\,{}^\sigma x) = u^{-1}N\bar{g}u$. It follows that $\mathfrak{M}_\sigma(\bar{g}, 0, \varphi_g(\Lambda_K(0)))$ is contained in $Z(N\bar{g})^\times U = \varphi_g(L^\times)U$. Since $g \in M_2(k)$, φ_g is a $\mathcal{J}$-isomorphism from L to $Z(N\bar{g})$. For xu with $x \in Z(N\bar{g})^\times$ and $u \in U$, we see

$$xu \in \mathfrak{M}_\sigma(\bar{g}, 0, \varphi_g(\Lambda_K(0))) \iff x^{-1}\bar{g}\,{}^\sigma x \in \varphi_g(\Lambda_L(0)^\times)$$

$$\iff x^{-1}\,{}^\sigma x \in \varphi_g(\Lambda_L(0)^\times) \quad .$$

By Lemma 3.22 and Remark 3.24, there exist two integers i, j, $0 \leqslant i, j \leqslant \ell-1$, and $u \in \Lambda_I(0)^{\times}$ such that $x^{-1\sigma}x = \varphi_g((\pi^i, \pi^j)^{-1\,\sigma}(\pi^i, \pi^j)u^{-1\sigma}u)$. We see $x \in \varphi_g(K^{\times}(\pi^i, \pi^j))U$, hence $\mathcal{M}_\sigma(\bar{g}, 0, \varphi_g(\Lambda_K(0))) = \bigcup_{i,j=0}^{\ell-1} \varphi_g(K^{\times}(\pi^i, \pi^j))U$. From this it follows

$$\begin{aligned} c_\sigma(f, 0, \Lambda_K(0)) &= \left| \varphi_g(K^{\times}) \backslash \bigcup_{i,j} \varphi_g(K^{\times}(\pi^i, \pi^j))U / U \right| \\ &= \left| K^{\times} \backslash K^{\times} \bigcup_{i,j} (\pi^i, \pi^j)\Lambda_K(0)^{\times} / \Lambda_K(0)^{\times} \right| \\ &= \ell^2 . \end{aligned}$$

In the case where $m \geqslant 1$, for i, $0 \leqslant i \leqslant \ell-1$, put $h_i = \begin{pmatrix} 1 & 0 \\ 0 & \pi^i \end{pmatrix}$. Since

$$h_i^{-1} g h_i = \begin{pmatrix} \xi & p^{\delta_2 - m}\pi^i \\ -p^{-(\delta_2 - m)}\pi^{-i} f(\xi) & s - \xi \end{pmatrix},$$

by Lemma 3.10, $h_i^{-1} g h_i$ is an element of $C(g, \Lambda_L(\ell m - i)) \cap M_2(\theta)$. By Lemma 3.10, $C(g, \Lambda_L(\ell m-i)) \cap M_2(\theta) / \tilde{U}$ consists of only one class, hence in the same way as above we see

$$\mathcal{M}_\sigma(\bar{g}, 0, \varphi_g(\Lambda_K(m))) \subset \bigcup_{i=0}^{\ell-1} \varphi_g(L^{\times})h_i U .$$

For $xh_i u \in \varphi_g(L^{\times})h_i U$, where $x \in \varphi_g(L^{\times})$, $u \in U$, we see

$$\begin{aligned} xh_i u \in \mathcal{M}_\sigma(g, 0, \varphi_g(\Lambda_K(m))) &\iff h_i^{-1}x^{-1}\bar{g}\,{}^{\sigma}x\,{}^{\sigma}h_i \in M_2(\theta)^{\times} \\ &\iff h_i^{-1}x^{-1}\bar{g}\,{}^{\sigma}x h_i (h_i^{-1\sigma}h_i) \in M_2(\theta)^{\times} \end{aligned}$$

$$\Longleftrightarrow \quad h_i^{-1}x^{-1}\bar{g}\,\sigma x h_i \in M_2(\mathcal{O})^{\times}$$

$$\Longleftrightarrow \quad x^{-1}\bar{g}\,\sigma x \in \varphi_g(\Lambda_L(\ell m - i)) \ .$$

By Lemma 3.22 and Remark 3.24, there exists an integer j, $0 \leq j \leq \ell-1$, such that $x \in \varphi_g(K^{\times}\pi^{j})U$.
Hence we see $\mathfrak{M}_{\sigma}(\bar{g},\ 0,\ \varphi_g(\Lambda_K(m))) = \bigcup_{i,j} \varphi_g(K^{\times}\pi^{i})h_j U$ and $c_{\sigma}(f,\ 0,\ \Lambda_K(m)) = \ell^2$.

If K is of type b), we can prove the following.

Proposition 3.26. Let notation be as in Prop. 3.25. Assume K is of type b), i.e. the unramified extension of k with $[K:k] = 2$. Then for $m = 0$, we have

$$c_{\sigma}(f,\ 0,\ \Lambda_K(0)) = \ell \quad .$$

In the case where $m \geq 1$, let α be an element of r such that $f(\alpha) \equiv 0 \bmod. pr$, then

$$c_{\sigma}(f,\ 0,\ \Lambda_K(m)) = \begin{cases} \ell\left(\sum \chi_i(\alpha)\right) & ,\ 1 \leq m \leq \delta_2 \\ 0 & ,\ \delta_2 < m \end{cases} \quad .$$

Proof. By the assumption $n \in N_{F/k}(\mathcal{O}^{\times})$, there exists $\bar{\bar{X}}$ of $\Lambda_L(0)^{\times}$ such that $N_{L/K}(\tilde{\bar{X}}) = \tilde{X}$. In the case where $\delta_2 \geq 1$, there exists α of $\underline{r}$ such that $f(\alpha) \equiv 0 \bmod. p\underline{r}$. Put $\tilde{X} = xu$, where $x \in \underline{r}^{\times}$ and $u \in U_K(\delta_2)$, then $x \equiv \alpha \bmod. p\underline{r}$. If there exists $\tilde{\tilde{X}}$ of $\Lambda_L(m)$ with $m \geq 1$ such that $N_{L/K}(\tilde{\tilde{X}}) = \tilde{X}$, then we see $\sum \chi_i(\alpha) = \ell$. If $\sum \chi_i(\alpha) = \ell$, then there exists $\bar{x}$ of $\mathcal{O}^\times$ such that $N_{F/k}(\bar{x}) = x$. And by Cor. 3.23, there exists $\bar{u}$ of $U_L(\ell\delta_2)$ such that $N_{L/K}(\bar{u}) = u$, since $U_L(\ell\delta_2) = U_K(\delta_2)$. Put $\bar{\tilde{X}} = \bar{x}\bar{u}$, then $N_{L/K}(\bar{\tilde{X}}) = \tilde{X}$ and $\bar{\tilde{X}} \in \Lambda_L(\ell\delta_2)$. With these facts we can prove our proposition in the similar way as Prop. 3.25 and we omit the details.

<u>Proposition 3.27</u>. Let the notation be as in Prop. 3.25. Assume K is of type c), i.e. a ramified extension of k with $[K:k] = 2$. Let α be an element of $\underline{r}$ such that $f(\alpha) \equiv 0$ mod.pr . Then we have

$$c_\sigma(f, G, \Lambda_K(m)) = \begin{cases} \ell + \frac{\ell}{2}\sum_{i\neq 1} \chi_i(\alpha) & , m = 0 \\ \ell\,(\sum \chi_i(\alpha)) & , 1 \leq m \leq \delta_2 \\ 0 & , \delta_2 < m . \end{cases}$$

<u>Proof</u>. First assume $\ell \neq 2$. By the assumption $n \in N_{F/k}(\mathcal{O}^\times)$, there exists $\bar{\tilde{X}} \in \Lambda_L(0)^\times$ such that $N_{L/K}(\bar{\tilde{X}}) = \tilde{X}$, and we see $\sum \chi_i(\alpha) = \ell$. We show that if $\sum \chi_i(\alpha) = \ell$, there exists $\bar{\tilde{X}} \in \Lambda_L(\ell\delta_2 + (\ell-1)/2)$ such that $N_{L/K}(\bar{\tilde{X}}) = \tilde{X}$. Let π_L and π_K be prime elements of L and K respectively. For a non-negative integer m, put

$$\bar{U}_K(m) = \begin{cases} \Lambda_K(0)^\times & , m=0 \\ 1 + \pi_K^m \Lambda_K(0) & , m \geq 1 \end{cases} \qquad \bar{U}_L(m) = \begin{cases} \Lambda_L(0)^\times & , m=0 \\ 1 + \pi_L^m \Lambda_L(0) & , m \geq 1 . \end{cases}$$

Then $\bar{U}_K(m)$ (resp. $\bar{U}_L(m)$) is a subgroup of $\Lambda_K(0)^\times$ (resp. $\Lambda_L(0)^\times$) and $U_K(m) = \bar{U}_K(2m)$ (resp. $U_L(m) = \bar{U}_L(2m)$). We see $\Lambda_K(m)^\times = \underline{r}^\times U_K(m) = \underline{r}^\times \bar{U}_K(2m+1)$ (resp. $\Lambda_L(m)^\times = \mathcal{O}^\times U_L(m) = \mathcal{O}^\times \bar{U}_L(2m+1)$), since each element of $\bar{U}_K(2m)/\bar{U}_K(2m+1)$ (resp. $\bar{U}_L(2m)/\bar{U}_L(2m+1)$) is represented by an element of $\underline{r}^\times$ (resp. $\mathcal{O}^\times$). Hence $\Lambda_K(\delta_2)^\times = \underline{r}^\times \bar{U}_K(2\delta_2+1)$ and $\Lambda_L(\ell\delta_2+(\ell-1)/2)^\times = \mathcal{O}^\times \bar{U}_L(2\ell\delta_2+\ell)$. By ([15], Ch V, § 3, Cor.3 of Prop.5), $N_{L/K}(\bar{U}_L(2\ell\delta_2+\ell)) = \bar{U}_K(2\delta_2+1)$, and our assertion follows from this in the same way as in the proof of Prop.3.26. We note there exists $\xi \in \underline{r}$ such that $f(\xi) \equiv 0 \bmod.(p\underline{r})^{2\delta_2+1}$ and $2\xi \equiv s \bmod.(p\underline{r})^{\delta_2+1}$, since K is a ramified extension of k. Put $g = \begin{pmatrix} \xi & p^{\delta_2} \\ -p^{-\delta_2}f(\xi) & s-\xi \end{pmatrix}$, then g is an element of $M_2(k)$ which satisfies $k[g]\cap M_2(\underline{r}) = \varphi_g(\Lambda_K(0))$ and $F[g]\cap M_2(\mathcal{O}) = \varphi_g(\Lambda_L((\ell-1)/2)$.

For i, $0 \leqslant i \leqslant \frac{\ell-1}{2}$, put $h_i = \begin{pmatrix} 1 & 0 \\ 0 & \pi^i \end{pmatrix}$, then $h_i^{-1}gh_i$ is an element of $C(g, \Lambda_L((\ell-1)/2 - i))\cap M_2(\mathcal{O})$. Put $\bar{g} = \varphi_g(\bar{\bar{X}})$, $\bar{\bar{X}} \in \Lambda_L(\ell\delta_2 + (\ell-1)/2)$, then we see as in the proof of Prop.3.25 that $\mathfrak{M}_\sigma(g, 0, \Lambda_K(0)) = \bigcup_{i=0}^{\ell-1} \bigcup_{j=0}^{(\ell-1)/2} K^\times \pi^i h_j U$ and we have

$c_\sigma(f, 0, \Lambda_K(0)) = \frac{1}{2}\ell(\ell+1) = \ell + \frac{\ell}{2}\sum_{i\neq 1} \chi_i(\alpha)$, since $\sum \chi_i(\alpha) = \ell$.

For $m \geqslant 1$, we can deduce our result in the same way as above. Next assume $\ell = 2$. By the assumption $n \in N_{F/k}(\mathcal{O}^\times)$, there exists $\tilde{\tilde{X}} \in \Lambda_L(0)^\times$ such that $N_{L/K}(\tilde{\tilde{X}}) = \tilde{X}$. For $m \geq 1$, if there exists $\tilde{\tilde{X}} \in \Lambda_L(m)^\times$ such that $N_{L/K}(\tilde{\tilde{X}}) = \tilde{X}$, then $\sum \chi_i(\alpha) = 2$. Let $\overline{U}_K(m)$ be as above, then $\Lambda_K(\delta_2)^\times = r^\times \overline{U}_K(2\delta_2+1)$ and $\Lambda_L(2\delta_2+1)^\times = \theta^\times U_L(2\delta_2+1)$. We see $U_L(2\delta_2+1)^{\mathfrak{J}} = \overline{U}_K(2\delta_2+1)$, hence by Cor. 3.23 $N_{L/K}(U_L(2\delta_2+1)) = \overline{U}_K(2\delta_2+1)$. If $\sum \chi_i(\alpha) = 2$, by the above fact we can show there exists $\tilde{\tilde{X}} \in \Lambda_L(\delta_2+1)^\times$ such that $N_{L/K}(\tilde{\tilde{X}}) = \tilde{X}$ in the same way as in the proof of Prop. 3.25. Using these facts, we easily obtain our result and omit the details.

If K is of type d), we can easily prove the following in the same way as above and omit the proof.

<u>Proposition 3.28</u>. Let the notation be as in Prop. 3.25. Assume that K is of type d), i.e. $K \simeq k + k\Delta$ with $\Delta^2 = 0$. Let α be an element of r such that $f(\alpha) \equiv 0 \mod . \mathfrak{p}r$. Then we have,

$$c_\sigma(f, 0, \Lambda_K(m)) = \ell(\sum \chi_i(\alpha))$$

for any non-negative integer m.

<u>3.12</u>. In the following 3.12 $\sim$ 3.15, we treat the case where $Ng \in F^\times$. In this case the k-algebra $Z_\sigma(g)$ is isomorphic to a quaternion algebra over k. For a quaternion algebra D over k, let $\sim$ be the equivalence relation (3.4.5) in all r-orders of D as in 3.4. Let α be a non-zero element of k. Assume $\alpha \in N(GL_2(F))$, and let $\bar{g}$ be an element of $GL_2(F)$ such that $N\bar{g} = \alpha$. Then $Z_\sigma(\bar{g})$ is determined by α up to isomorphisms

over k, and is independent of the choice of $\bar{g}$ by Lemma 3.6. For $\alpha \in N(GL_2(F)) \cap k^{\times}$, we denote by $D(\alpha)$ the quaternion algebra over k determined by α in the above way. Let α be an element of $N(GL_2(F)) \cap \underline{r}$, r an non-negative integer, and Λ an $\underline{r}$-order of $D(\alpha)$. For a triple (α, r, Λ) we define a non-negative integer $c_{\sigma}(\alpha, r, \Lambda)$ as in 3.6. Let $\bar{g}$ be an element of $GL_2(F)$ such that $N\bar{g} = \alpha$. The k-algebra $Z_{\sigma}(\bar{g})$ is isomorphic to $D(\alpha)$, and let φ be an isomorphism from $D(\alpha)$ to $Z_{\sigma}(\bar{g})$. For $\bar{g}$, r and Λ, let $\mathfrak{M}_{\sigma}(\bar{g}, \Xi(r), \Lambda)$ and $C_{\sigma}(\bar{g}, \Lambda)$ be as (3.4.2)' and (3.4.4)' for $\Xi = \Xi(r)$, namely

$$\mathfrak{M}_{\sigma}(\bar{g}, r, \Lambda) = \left\{ x \in GL_2(F) \;\middle|\; x^{-1}\bar{g}\,{}^{\sigma}x \in \Xi(r),\ Z_{\sigma}(\bar{g}) \cap xM_2(\mathcal{O})x^{-1} \sim \varphi(\Lambda) \right\}$$

$$C_{\sigma}(\bar{g}, \Lambda) = \left\{ x^{-1}\bar{g}\,{}^{\sigma}x \;\middle|\; x \in GL_2(F),\ Z_{\sigma}(\bar{g}) \cap xM_2(\mathcal{O})x^{-1} \sim \varphi(\Lambda) \right\} .$$

We note $\mathfrak{M}_{\sigma}(\bar{g}, r, \Lambda)$ and $C_{\sigma}(\bar{g}, \Lambda)$ are independent of the choice of φ. Put

$$c_{\sigma}(\alpha, r, \Lambda) = \left| Z_{\sigma}(\bar{g})^{\times} \backslash \mathfrak{M}_{\sigma}(\bar{g}, r, \Lambda) / U \right|$$

, then we see $c_{\sigma}(\alpha, r, \Lambda)$ is independent of the choice of $\bar{g}$. By Lemma 3.9, the double cosets $Z_{\sigma}(\bar{g})^{\times} \backslash \mathfrak{M}_{\sigma}(\bar{g}, r, \Lambda) / U$ is in one to one correspondence with $C_{\sigma}(\bar{g}, \Lambda) \cap \Xi(r) / \underset{U}{\approx}$. In the following we will determine $c_{\sigma}(\alpha, r, \Lambda)$ according to the type of F. When F is a ramified extension, we assume $r = 0$ as before.

3.13. Let F be the direct product of ℓ-copies of k. If $\alpha \in N(GL_2(F))$ and $c_{\sigma}(\alpha, r, \Lambda) \neq 0$, we see $v(\alpha) = r/2$. Hence we may assume r is even and $v(\alpha) = r/2$. Then we can prove the following.

Proposition 3.29. Let F be the direct product of ℓ-copies of k. Assume r is even, and let α be an element of $\underline{r}$ with $v(\alpha) = r/2$. Then,

(i) There exists $\bar{g} \in GL_2(F)$ such that $N\bar{g} = \alpha$, and $D(\alpha)$ is isomorphic to $M_2(k)$.

(ii) We have

$$c_\sigma(\alpha, r, \Lambda) = \begin{cases} 1 & , \Lambda \sim M_2(\underline{r}) \\ 0 & , \text{otherwise} \end{cases} .$$

Proof. By the assumption, $M_2(F)$ is isomorphic to $M_2(k) \oplus \dots\dots \oplus M_2(k)$ (ℓ-copies). Put $\bar{g} = (\alpha, 1, \dots, 1)$, then $N\bar{g} = \alpha$, and it is easy to see that $D(\alpha)$ is isomorphic to $M_2(k)$. Let $x = (x_i)$ be an element of $GL_2(F)$ such that $x^{-1}\bar{g}\,{}^\sigma x \in \Xi(r)$, then we see $x_1^{-1}x_2, \dots, x_{\ell-1}^{-1}x_\ell \in M_2(\underline{r})^\times$. Hence there exists $u \in U$ and $x' \in GL_2(k)$ such that $x = x'u$. From this, we see $\mathfrak{M}_\sigma(\bar{g}, r, \Lambda) \neq \phi$ only if $\Lambda \sim M_2(\underline{r})$ and $c_\sigma(\alpha, r, \Lambda) = 1$ for $\Lambda = M_2(\underline{r})$.

3.14. Let F be the unramified extension of k with $[F:k] = \ell$. If $\alpha \in N(GL_2(F))$ and $c_\sigma(\alpha, r, \Lambda) \neq 0$, we see ℓr is even and $v(\alpha) = \ell r/2$. Hence we assume ℓr is even and $v(\alpha) = \ell r/2$. First we assume $\ell = 2$. Let D be a quaternion algebra over k. We define r-orders $\Lambda(m)$ of D for non-negative integers m as follows. Let R be $M_2(\underline{r})$ if $D = M_2(k)$, and let R be the maximal order if D is the division quaternion algebra. Let $\bar{F}$ be a k-subalgebra of D such that $\bar{F}$ is isomorphic to F and $\bar{F} \cap R$ is the maximal order of $\bar{F}$. For a non-negative integer m, put

$$\Lambda(m) = \bar{F} \cap R + p^m R \quad .$$

Then $\Lambda(m)$ is an r-order of D and the equivalence class with respect to (3.4.5) containing $\Lambda(m)$ is independent of the choice of $\bar{F}$. By considering the indeces as additive groups of $\Lambda(m)$ in a maximal order of D which contains $\Lambda(m)$, we see easily $\Lambda(m) \nsim \Lambda(m')$ if $m \neq m'$. Then we can prove the following.

Proposition 3.30. Let F be the unramified extension with $[F : k] = 2$, α be an element of $\underline{r}$ with $v(\alpha) = r$, and $\Lambda(m)$ be as above.

(i) There exists $\bar{g} \in GL_2(F)$ such that $N\bar{g} = \alpha$, and $D(\alpha)$ is isomorphic to $M_2(k)$ or the division quaternion algebra over k according as r is even or odd.

(ii) We have

$$c_\sigma(\alpha, r, \Lambda) = \begin{cases} 1 & , \Lambda \sim \Lambda(m),\ 0 \leqslant m \leqslant [r/2] \\ 0 & , \text{otherwise} \end{cases}$$

Proof. (i) If $v(\alpha) = r$ is even, there exists $\bar{\alpha} \in F^\times$ such that $N_{F/k}(\bar{\alpha}) = \alpha$, and we may take $\bar{\alpha}$ as $\bar{g}$. If r is odd, $v(\alpha p^{-1})$ is even, hence there exists $\bar{\alpha} \in F^\times$ such that $N_{F/k}(\bar{\alpha}) = \alpha p^{-1}$. Put $\bar{g} = \bar{\alpha}\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$, then $N\bar{g} = \alpha$.

(ii) First we prove the following lemma.

Lemma 3.31. The notation being as in Prop. 3.30, let u_0 be an element of $\mathcal{O}$ such that $\mathcal{O} = \underline{r} + \underline{r}u_0$, $\bar{g}$ be as in the proof of Prop. 3.30, (i), and φ be as in 3.12.

(i) If r is even, then the union

$$GL_2(F) = \bigcup_m Z_\sigma(\bar{g})^\times h_m U$$

is disjoint, where $h_m = \begin{pmatrix} 1 & 0 \\ u_0 & p^m \end{pmatrix}$, and m runs through all non-negative integers. And we have

$$Z_\sigma(g) \cap h_m M_2(\mathcal{O}) h_m^{-1} \sim \varphi(\Lambda(m)) \quad .$$

(ii) If r is odd, the union

$$GL_2(F) = \bigcup_m Z_\sigma(\bar{g})^\times h_m U$$

is disjoint, where $h_m = \begin{pmatrix} 1 & 0 \\ 0 & p^{m+1} \end{pmatrix}$, and m runs through all non-negative integers. And we have

$$Z_\sigma(\bar{g}) \cap h_m M_2(\mathcal{O}) h_m^{-1} \sim \varphi(\Lambda(m)) \quad .$$

Proof. (i) Since $g \in F^\times$, $Z_\sigma(\bar{g}) = M_2(k)$. In this proof, we denote by $\sim$ the equivalence relation in $GL_2(F)$ given by

$$g \sim g' \iff g' \in Z_\sigma(\bar{g}) g U$$

Then we see for an element $g \in GL_2(F)$ there exist $u \in \mathcal{O}^\times$ and a non-negative integer i such that $g \sim \begin{pmatrix} 1 & 0 \\ u & p^i \end{pmatrix}$. Put

$u = a + bu_o$ with $a, b \in \underline{r}$. Then $g \sim \begin{pmatrix} 1 & 0 \\ bu_o & p^i \end{pmatrix}$, and we see $g \sim \begin{pmatrix} 1 & 0 \\ 0 & p^i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ u_o & 1 \end{pmatrix}$ if $v(b) \geqslant i$, and $g \sim \begin{pmatrix} 1 & 0 \\ u_o & p^j \end{pmatrix}$ if $v(b) < i$, where $j = v(b) - i$, hence the equality holds.

Note that if $h \sim h'$, then $Z_\sigma(\bar{g}) \cap hM_2(\mathcal{O})h^{-1} \sim Z_\sigma(\bar{g}) \cap h'M_2(\mathcal{O})h'^{-1}$.

Hence to prove the union is disjoint it is enough to show that $\Lambda = Z_\sigma(\bar{g}) \cap h_m M_2(\mathcal{O}) h_m^{-1} \sim \varphi(\Lambda(m))$. Let $f_o(X) = X^2 - s_o X + n_o$ be the minimal polynomial of u_o over k, and put

$$\bar{F} = \left\{ \begin{pmatrix} a & b \\ -bn_o & a+bs_o \end{pmatrix} \middle| \ a, b \in k \right\}, \quad \bar{\mathcal{O}} = \left\{ \begin{pmatrix} a & b \\ -bn_o & a+bs_o \end{pmatrix} \middle| \ a, b \in \underline{r} \right\}.$$

Then $\bar{F}$ is isomorphic to F and $\bar{F}$ contains $\bar{\mathcal{O}}$ as its maximal order. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z_\sigma(\bar{g}) = M_2(k)$, we see by an explicit calculation,

$$g \in \Lambda \iff a + bu_o, \ -bu_o + d, \ \frac{1}{p^m}(-bu_o^2 + (d-a)u_o + c) \in \mathcal{O}$$

$$\iff a, b, d \in \underline{r}, \quad a + bs_o - d, \ bn_o + c \in p^n\underline{r}$$

Hence we see $\Lambda = \bar{\mathcal{O}} + p^m M_2(\underline{r})$, and $\Lambda \sim \varphi(\Lambda(m))$.

(ii) By the definition of $\bar{g}$, we have

$$Z_\sigma(\bar{g}) = \left\{ \begin{pmatrix} a & b \\ p^\sigma b & {}^\sigma a \end{pmatrix} \middle| \ a, b \in F \right\}.$$

For $g \in GL_2(F)$ we see that $g \sim \begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix}$ or $\begin{pmatrix} u & 0 \\ p^i & p^j \end{pmatrix}$ with $u \in \mathcal{O}^\times$ and positive integers i, j. Since

$$\begin{pmatrix} {}^{\sigma}u & -p^{i-1} \\ -p^{i} & u \end{pmatrix} \begin{pmatrix} u & 0 \\ p^{i} & p^{j} \end{pmatrix} = \begin{pmatrix} {}^{\sigma}uu - p^{2i-1} & -p^{i+j-1} \\ 0 & p^{j}u \end{pmatrix}$$

, we see $\begin{pmatrix} u & 0 \\ p^{i} & p^{j} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & p^{j} \end{pmatrix}$, hence the equality holds.
To prove that the union is disjoint, it is enough to show $\Lambda = Z_{\sigma}(\bar{g}) \cap h_m M_2(\mathcal{O}) h_m^{-1} \sim \varphi(\Lambda(m))$ as above. By some calculation, we see $\Lambda = \bar{\mathcal{O}} + p^m M_2(\underline{r})$, where $\bar{\mathcal{O}} = \left\{ \begin{pmatrix} a & 0 \\ 0 & {}^{\sigma}a \end{pmatrix} \middle| \ a \in \mathcal{O} \right\}$, hence $\Lambda \sim \varphi(\Lambda(m))$, and our assertion is proved.

Now we prove the assertion (ii) of our proposition. First assume r is even. Then by (i) of the above lemma, for any element $\bar{g}' \in C_{\sigma}(\bar{g})$, there exists a non-negative integer m such that $\bar{g}' \underset{U}{\approx} h_m^{-1} \bar{g}\, {}^{\sigma}h_m$. Then again by (i) of the above lemma, we see $\mathcal{M}_{\sigma}(\bar{g}, r, \Lambda) \neq \phi$ only if $\Lambda \sim \Lambda(m)$ for some non-negative integer m. Since $h_m^{-1}\, {}^{\sigma}h_m = \begin{pmatrix} 1 & 0 \\ p^{-m}({}^{\sigma}u_0 - u_0) & 1 \end{pmatrix}$, we see

$$h_m^{-1} \bar{g}\, {}^{\sigma}h_m \in \Xi(r) \iff m \leq r/2 \quad .$$

Our assertion for an even integer r easily follows from this and the above lemma. Next assume r is odd. Then by (ii) of the above lemma, we see $\mathcal{M}_{\sigma}(\bar{g}, r, \Lambda) \neq \phi$ only if $\Lambda \sim \Lambda(m)$ for a non-negative integer m. Since
$h_m^{-1} \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} {}^{\sigma}h_m = \begin{pmatrix} 0 & p^{m+1} \\ p^{-m} & 0 \end{pmatrix}$, we see

$$h_m^{-1} \bar{g}\, {}^{\sigma}h_m \in \Xi(r) \iff m \leq (r-1)/2 \quad .$$

Our assertion for an odd integer r easily follows from this and the above lemma.

Next assume ℓ is an odd prime. For a non-negative integer m, put

$$\Lambda(m) = \underline{r} + p^m M_2(\underline{r}) .$$

Then we see $\Lambda(m)$ is an $\underline{r}$-order and $\Lambda(m) \not\sim \Lambda(m')$ if $m \neq m'$. As noted before, we may assume ℓr is even, hence r is even.

<u>Proposition 3.32</u>. Let F be the unramified extension of k with $[F : k] = \ell$, where $\ell \neq 2$. Assume r is even. Let α be an element of r with $v(\alpha) = \ell r/2$ and $\Lambda(m)$ be as above.

(i) There exists $\bar{g} \in GL_2(F)$ such that $N\bar{g} = \alpha$, and $D(\alpha)$ is isomorphic to $M_2(k)$.

(ii) We have

$$c_\sigma(\alpha, r, \Lambda) = \begin{cases} 1 & , \Lambda \sim \Lambda(0) \\ Np^{\ell m-(\ell-1)}(Np^{(\ell-1)} - 1) \,/\, PGL_2(\underline{r}/(p\underline{r})^m) & , \Lambda \sim \Lambda(m),\ 1 \leqslant m \leqslant r/2 \\ 0 & , \text{otherwise} \end{cases}$$

<u>Proof</u>. The assertion (i) is obvious. And we may take $\bar{g}$ from $F^\times$. We fix such $\bar{g}$ in the following. Let S be the set of all elements $x \in \mathcal{O}$ which satisfy the condition $x \not\equiv {}^\sigma x \mod. \mathfrak{P}$.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\underline{r})$ and $x \in S$, put $\gamma x = \dfrac{ax+b}{cx+d}$, then we see γx is also contained in S, hence $GL_2(\underline{r})$ acts on

S. For any non-negative integer m, $GL_2(r/p^m\underline{r})$ acts on $\{S \bmod. \mathfrak{Z}^i\}$ in a similar way . Then we can prove

Lemma 3.33. The notation being as above, then

$$GL_2(F) = \bigcup_{m=0}^{\infty} \bigcup_{x \in \{S \bmod. \mathfrak{Z}^m\}/GL_2(\underline{r}/p^m\underline{r})} Z_\sigma(\bar{g})^\times h_m(x) U$$

is a disjoint union, where $h_m(x) = \begin{pmatrix} 1 & 0 \\ x & p^m \end{pmatrix}$. And we have

$$Z_\sigma(\bar{g}) \cap h_m(x) M_2(\mathcal{O}) h_m(x)^{-1} \sim \varphi(\Lambda(m)) .$$

Proof. Since $\bar{g} \in F^\times$, $Z_\sigma(\bar{g}) = M_2(k)$. In this proof, we denote by $\sim$ the equivalence relation in $GL_2(F)$ given by

$$g \sim g' \iff Z_\sigma(\bar{g}) g U \ni g' .$$

Then for $g \in GL_2(F)$, we see $g \sim \begin{pmatrix} 1 & 0 \\ x & p^i \end{pmatrix}$ for some $x \in S$ and a non-negative integer i, hence the equality holds. If $i \neq j$, $\begin{pmatrix} 1 & 0 \\ x & p^i \end{pmatrix} \not\sim \begin{pmatrix} 1 & 0 \\ x' & p^j \end{pmatrix}$ for any $x, x' \in S$. To prove this, it is enough to show

$$\Lambda = Z_\sigma(\bar{g}) \cap h_i(x) M_2(\mathcal{O}) h_i(x)^{-1} \sim \varphi(\Lambda(i)) .$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z_\sigma(\bar{g}) = M_2(k)$, we see

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Lambda \iff a+bx,\ -bx+d,\ p^{-i}(-bx^2+(d-a)x+c) \in \mathcal{O} .$$

Since $x \in S$, it follows $b \equiv c \equiv a-d \equiv 0 \bmod. p^i\underline{r}$. Hence we see $\Lambda \sim \varphi(\Lambda(i))$. To prove the union is disjoint, it is enough to show that for $x, x' \in S$

$$\begin{pmatrix} 1 & 0 \\ x & p^i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ x' & p^i \end{pmatrix} \Longleftrightarrow x \equiv \gamma x' \bmod. \mathfrak{P}^i \quad \text{for} \quad \gamma \in GL_2(\underline{r}).$$

($\Longleftarrow$) Since $GL_2(\underline{r})$ is generated by the elements of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we may verify our assertion for such elements. This can be done by explicit calculation.

($\Longrightarrow$) Assume $\begin{pmatrix} 1 & 0 \\ x & p^i \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ x' & p^i \end{pmatrix}$ for $x, x' \in S$. Then there exists an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ such that

$$\begin{pmatrix} 1 & 0 \\ x & p^i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x' & p^i \end{pmatrix}^{-1} = \frac{1}{p^i}\begin{pmatrix} ap^i - bx' & b \\ cp^{2i} + (ax-dx')p^i - xx'b & dp^i + bx \end{pmatrix}$$

is contained in $GL_2(k)$. From this, we see $b \in p^i\underline{r}$ and $ad \in \mathcal{O}^\times$. Put $b' = p^{-i}b$, $a' = a - b'x'$, $d' = d + b'x$, and $c' = cp^i + (ax-dx') - b'xx'$, then $a', b', c', d' \in \underline{r}$. We see $x(a'+b'x') \equiv c'+d'x' \bmod. \mathfrak{P}^i$. Since $a'+b'x = a$, $a'+b'x \in \underline{r}^\times$, and we see $a'd'-b'c' \equiv ad \bmod. p^i\underline{r}$. Hence if we put $\gamma = \begin{pmatrix} d' & c' \\ b' & a' \end{pmatrix}$, $\gamma \in GL_2(\underline{r})$ and $x \equiv \gamma x' \bmod. \mathfrak{P}^i$ and our assertion is proved.

Now we return to the proof of our proposition. We see $|\{S \bmod. \mathfrak{P}^i\}| = 1$ if $i = 0$, and $= Np^{\ell i-(i-1)}(Np^{(i-1)} - 1)$ if $i \geqslant 1$. For $x \in S$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\underline{r})$, $\gamma x \equiv x \bmod. \mathfrak{P}^i$

if and only if $a-d \equiv b \equiv c \equiv 0 \mod . p^i \underline{r}$. By these facts and Lemma 3.33, we can prove our proposition in the same way as Prop. 3.30.

3.15. Let F be a tamely ramified extension of k with $[F:k] = \ell$. We assume $r = 0$. First we treat the case where $\ell \neq 2$. For a non-negative integer m, we set

$$\Lambda(m) = \begin{pmatrix} \underline{r} & \underline{r} \\ p^m \underline{r} & \underline{r} \end{pmatrix}$$

Then $\Lambda(m)$ is an r-order of $M_2(k)$. If $\alpha \in N(GL_2(F))$ and $c_\sigma(\alpha, 0, \Lambda) \neq 0$, then $v(\alpha) = 0$, and we assume $r = 0$. Then we can prove the following.

Proposition 3.34. Let F be a tamely ramified extension of k with $[F:k] = \ell$, where $\ell \neq 2$, α be an element of $\underline{r}^\times$, and $\Lambda(m)$ be as above.

(i) There exists $\bar{g} \in GL_2(F)$ such that $N\bar{g} = \alpha$ if and only if $\alpha \in N_{F/k}(\mathcal{O}^\times)$. If $\alpha \in N_{F/k}(\mathcal{O}^\times)$, we have

$$D(\alpha) \simeq M_2(k)$$

(ii) Assume $\alpha \in N_{F/k}(\mathcal{O}^\times)$. Then we have

$$c_\sigma(\alpha, 0, \Lambda) = \begin{cases} \ell & , \Lambda \sim \Lambda(0) \\ \dfrac{\ell(\ell-1)}{2} & , \Lambda \sim \Lambda(1) \\ 0 & , \text{otherwise} \end{cases}$$

Proof. If there exists $\tilde{g}$ such that $N\tilde{g} = \alpha$, then $\alpha^2 \in N_{F/k}(F^\times)$. Since $[F : k] = \ell$ is odd, $\alpha \in N_{F/k}(\mathcal{O}^\times)$. The converse is obvious, and we may take $\tilde{g}$ from $\mathcal{O}^\times$. In the following, we assume $\tilde{g} \in \mathcal{O}^\times$, hence $Z_\sigma(\tilde{g}) = M_2(k)$. If $x^{-1}\tilde{g}\,{}^\sigma x \in \Xi(0) = U$ for $x \in GL_2(F)$, then $x^{-1}\,{}^\sigma x \in U$. If we put $a_\sigma = x^{-1}\,{}^\sigma x$, then a_σ determines a 1-cocycle $\{a_\tau\}$, $\tau \in \mathfrak{g}$, of $\mathfrak{g}$ in U. And we see the correspondence $x^{-1}\tilde{g}\,{}^\sigma x \longmapsto \{a_\tau\}$ gives a bijective map

$$C_\sigma(\tilde{g}) \cap U \,/\, \widetilde{\widetilde{U}} \longrightarrow H^1(\mathfrak{g}, U)$$

For a pair (i, j) of integers such that $0 \leqslant i \leqslant j \leqslant \ell$, put $x_{ij} = \begin{pmatrix} \pi^i & 0 \\ 0 & \pi^j \end{pmatrix}$, and $a_\sigma(i, j) = x_{ij}^{-1}\,{}^\sigma x_{ij}$. Then $a_\sigma(i, j)$ determines a 1-cocycle $a_\tau(i, j)$, $\tau \in \mathfrak{g}$, of $\mathfrak{g}$ in U. By the assumption that F is a tamely ramified extension of k, we see the set $\{\, \{a_\tau(i, j)\} \mid 0 \leq i < j \leq \ell \,\}$ gives a complete system of representatives of $H^1(\mathfrak{g}, U)$. For such i, j we see

$$Z_\sigma(\tilde{g}) \cap x_{ij} M_2(\mathcal{O}) x_{ij}^{-1} \sim \begin{cases} \varphi(\Lambda(0)) & \text{, if } i = j \\ \varphi(\Lambda(1)) & \text{, if } i < j \end{cases}$$

Our assertions easily follow from this.

Next we treat the case where $\ell = 2$. As in the case where $\ell \neq 2$, we may assume $v(\alpha) = 0$. For the quaternion $M_2(k)$ and a non-negative integer m, we put

$$\Lambda(m) = \begin{pmatrix} \underline{r} & \underline{r} \\ p^m\underline{r} & \underline{r} \end{pmatrix}$$

as in the case where $\ell \neq 2$. For the division quaternion algebra we denote by $\Lambda(0)$ its maximal order.

Proposition 3.35. Let F be a tamely ramified extension of k with $[F:k]=2$, α be an element of $\underline{r}^{\times}$, and $\Lambda(m)$ be as above.

(i) There exists $\bar{g} \in GL_2(F)$ such that $N\bar{g} = \alpha$. If $\alpha \in N_{F/k}(\mathcal{O}^{\times})$ $D(\alpha)$ is isomorphic to $M_2(k)$, and if $\alpha \notin N_{F/k}(\mathcal{O}^{\times})$, $D(\alpha)$ is isomorphic to the division quaternion algebra over k.

(ii) If $\alpha \in N_{F/k}(\mathcal{O}^{\times})$, we have

$$c_{\sigma}(\alpha, 0, \Lambda) = \begin{cases} 2 & , \Lambda \sim \Lambda(0) \\ 1 & , \Lambda \sim \Lambda(1) \\ 0 & , \text{otherwise} \end{cases}.$$

If $\alpha \notin N_{F/k}(\mathcal{O}^{\times})$, we have

$$c_{\sigma}(\alpha, 0, \Lambda) = \begin{cases} 1 & , \Lambda \sim \Lambda(0) \\ 0 & , \text{otherwise} \end{cases}.$$

Proof. (i) If $\alpha = N_{F/k}(\bar{\alpha})$ with $\bar{\alpha} \in \mathcal{O}^{\times}$, put $\bar{g} = \bar{\alpha}$, then $N\bar{g} = \alpha$ and $Z_{\sigma}(\bar{g}) = M_2(k)$. If $\alpha \notin N_{F/k}(\mathcal{O}^{\times})$, put $\bar{g} = \begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$, then $N\bar{g} = \alpha$. We see

$$Z_{\sigma}(\bar{g}) = \left\{ \begin{pmatrix} a & b \\ \alpha^{\sigma} b & a \end{pmatrix} \;\middle|\; a, b \in F \right\}$$

and $Z_{\sigma}(\bar{g})$ is the division quaternion algebra over k.

(ii) The assertion for the case where $\alpha \in N_{F/k}(\mathcal{O}^\times)$ can be proved in the same way as Prop. 3.34. Assume $\alpha \notin N_{F/k}(\mathcal{O}^\times)$, then for $x \in GL_2(F)$, we see

$$x^{-1}\bar{g}\,{}^{\sigma}x \in \Xi(\mathcal{O}) = U \iff x \in Z_\sigma(\bar{g})^\times U \quad .$$

Our assertion easily follows from this.

§ 4. Explicit formula for tr $T_S(T(\mathfrak{o}))$

4.1. We will lead the formula (2.12.1) of Th.1' into a more explicit form. Let F be as in § 1, i.e. a totally real algebraic number field which satisfies the following conditions;

(1) F is a cyclic extension of Q of prime degree ℓ.

(2) The class number of F is equal to one.

(3) The index $[E : E_+]$ is equal to 2^{ℓ},

,where E is the group of all units of F, and E_+ is its subgroup consisting of all totally positive elements of E. Hence the conductor q of F/Q is a prime. Moreover in the following we assume

(4) F/Q is a tamely ramified extension

, and the conductor q and the degree ℓ are prime to each other. We denote by $\mathcal{O}$ the maximal order of F, and let F_v, $F_{\mathfrak{p}}$ and $\mathcal{O}_{\mathfrak{p}}$ be as in § 1, where we denote by v (resp. $\mathfrak{p}$) archimedean (resp. non-archimedean) places of F.

For a prime p, put $F_p = F \otimes_Q Q_p$, $\mathcal{O}_p = \mathcal{O} \otimes_Z Z_p$ and $F_\infty = F \otimes_Q R$.

Then F_p (resp. F_∞) is one of i), ii) and iii) of 3.5. Let σ be the generator of the Galois group $\mathfrak{g}$ of the extension F/Q fixed in § 1, then σ can be extended to F_p (resp. F_∞) as Q_p(resp. R)-linear automorphism of F_p (resp. F_∞). We denote it also by σ. In such a situation, we can apply the results of § 3.

Let F_A (resp. Q_A) be the adele ring of F (resp. Q). Then σ can be extended to F_A, we denote it also by σ. Let

$\mathfrak{U}_F$ be the subgroup $\prod_{\mathfrak{p}} GL_2(\mathcal{O}_{\mathfrak{p}}) \times \prod_v GL_2(F_v)$ of $GL_2(F_A)$ as in § 1, and $\mathfrak{U}_Q$ be the subgroup $\prod_p GL_2(Z_p) \times GL_2(R)$ of $GL_2(Q_A)$. For an integral ideal $\mathfrak{n}$ of F, let $\Xi(\mathfrak{n})_A$ be the union of all $\mathfrak{U}_F$-double cosets in $T(\mathfrak{n})$, where $T(\mathfrak{n})$ is the element of $R(\mathfrak{U}_F, GL_2(F_A))$ given in 1.3. In the following, we assume that $\mathfrak{n}$ is prime to the conductor q, and that $\mathfrak{n}$ is divided by at most one prime factor of p if p decomposes in F. Then $\Xi(\mathfrak{n})_A$ is of the form $\prod_p \Xi(\mathfrak{n})_p \times \prod_v GL_2(F_v)$, where $\Xi(\mathfrak{n})_p$ is a union of $GL_2(\mathcal{O}_p)$-double cosets, and we may assume $\Xi(\mathfrak{n})_p$ is of the form $\Xi_p(r)$ for some non-negative integer r, where $\Xi_p(r)$ is the union of $GL_2(\mathcal{O}_p)$-double cosets $\Xi(r)$ defined in 3.5. Put $\Xi(\mathfrak{n}) = \Xi(\mathfrak{n})_A \cap GL_2(F)$ (resp. $\Xi(\mathfrak{n})_+ = \Xi(\mathfrak{n})_A \cap GL_2(F)_+$), then $\Xi(\mathfrak{n})$ (resp. $\Xi(\mathfrak{n})_+$) is a union of $GL_2(\mathcal{O})$ (resp. Γ)-double cosets. Let g be an element of $GL_2(F)_+$, and let $Z_\sigma(g)$, $C_\sigma(g)$ and $\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}))$ be as in 3.1 and 3.4. Put $\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+) = \{ x \in GL_2(F) \mid x^{-1}g^\sigma x \in \Xi(\mathfrak{n})_+ \}$. For a Z-order Λ of $Z_\sigma(g)$, let $C_\sigma(g, \Lambda)$ and $\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda)$ be as in 3.4 and put

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+, \Lambda) = \{ x \in \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda) \mid x^{-1}g^\sigma x \in \Xi(\mathfrak{n})_+ \}$$

Then $C_\sigma(g) = \bigcup_\Lambda C_\sigma(g, \Lambda)$ (resp. $C_\sigma(g) \cap \Xi(\mathfrak{n})_+ = \bigcup_\Lambda C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+$, where Λ runs through all the Z-order of $Z_\sigma(g)$. Here we note $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})$ (resp. $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+$) $\neq \phi$ only if Λ contains Ng. Hence $C_\sigma(g) \cap \Xi(\mathfrak{n}) = \bigcup_\Lambda C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})$ (resp. $C_\sigma(g) \cap \Xi(\mathfrak{n})_+ = \bigcup_\Lambda C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+$), where Λ runs through all the Z-orders of $Z_\sigma(g)$ which contains Ng.

4.2. Now let's consider to classify $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+$ into $\widetilde{\widetilde{\Gamma}}$-equivalence classes. We will reduce the computation of the equivalence classes $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+ / \widetilde{\widetilde{\Gamma}}$ to the results of $3.5 \sim 3.15$. In the notation of 4.1, put

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A) = \{ x \in GL_2(F_A) \mid x^{-1} g^\sigma x \in \Xi(\mathfrak{n})_A \} .$$

For $x \in GL_2(F_A)$, we denote by $xM_2(\mathcal{O})x^{-1}$ the maximal order of $M_2(F)$ given by $xM_2(\mathcal{O})x^{-1} = \bigcap_{\mathfrak{z}} x_{\mathfrak{z}} M_2(\mathcal{O}_{\mathfrak{z}}) x_{\mathfrak{z}}^{-1}$, where $x_{\mathfrak{z}}$ is the $\mathfrak{z}$-component of x. If $Ng \notin F^\times$, put

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda) = \{ x \in GL_2(F_A) \mid x^{-1} g^\sigma x \in \Xi(\mathfrak{n})_A,\ Z_\sigma(g) \cap xM_2(\mathcal{O})x^{-1} = \Lambda \} .$$

If $Ng \in F^\times$, put

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda) = \{ x \in GL_2(F_A) \mid x^{-1} g^\sigma x \in \Xi(\mathfrak{n})_A,\ Z_\sigma(g) \cap xM_2(\mathcal{O})x^{-1} \sim \Lambda \}$$

, where $\sim$ denotes the equivalence relation (3.4.5). Here we note the following. For a quaternion D over Q, we denote by D_A the adelization of D, then for a Z-order Λ of D the type number τ of Λ is by definition

$$\tau = | D^\times \backslash D_A^\times / \prod_p N_p(\Lambda_p) \times D_\infty |$$

, where $\Lambda_p = \Lambda \otimes_Z Z_p$, $D_\infty = D \otimes_Q R$, and $N_p(\Lambda_p) = \{ x \in (D \otimes_Q Q_p)^\times \mid x^{-1} \Lambda_p x = \Lambda_p \}$. If the type number τ

is equal to one, a Z-order Λ' of D which satisfies $\Lambda'_p \sim \Lambda_p$ in $D_p = D \otimes_Q Q_p$ for all primes p also satisfies $\Lambda' \sim \Lambda$ in D.

Lemma 4.1. Let the notation and the assumption be as above.

(i) The equivalence classes $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+ / \underset{\Gamma}{\approx}$ are in one to one correspondence with the double cosets $Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+, \Lambda) / \Gamma$.

(ii) Put $\Lambda^1 = \{ x \in \Lambda \mid \det x = 1 \}$. Then the canonical map

$$Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+, \Lambda)/\Gamma \longrightarrow Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda)/GL_2(\mathcal{O})$$

is a $2/[\Lambda^\times : \Lambda^1]$ to 1 correspondence.

(iii) If $Ng \in F^\times$, we assume the type number of Λ is equal to one. Then we have the following canonical bijection.

$$Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda)/GL_2(\mathcal{O}) \simeq Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathcal{U}_F$$

Proof. The assertion (i) is obvious. (ii) By the assumption $[E : E_+] = 2^\ell$, the natural map from $Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+, \Lambda)/\Gamma$ to $Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda)/GL_2(\mathcal{O})$ is surjective. For $x \in \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+, \Lambda)$, we have

$$Z_\sigma(g)^\times x GL_2(\mathcal{O}) \cap \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_+, \Lambda) = Z_\sigma(g)^\times x \Gamma \cup Z_\sigma(g)^\times x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma .$$

And we see $Z_\sigma(g)^\times x \Gamma = Z_\sigma(g)^\times x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Gamma$ if and only if there exists $a \in \Lambda$ such that $\det a$ is totally negative, i.e. $\det a = -1$, hence if and only if $[\Lambda : \Lambda^1] = 2$. By this, we obtain (ii).

(iii) By the assumption on the class number of F, we have

$GL_2(F_A) = GL_2(F)\mathfrak{U}_F$. We see $\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)_A \cap GL_2(F) = \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda)$ for g with $Ng \notin F^\times$, and by the assumption on the type number of Λ for g with $Ng \in F^\times$, hence the map is surjective. For $x_1, x_2 \in \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n}), \Lambda)$, if there exist $\gamma \in GL_2(F)$ and $u \in \mathfrak{U}_F$ such that $\gamma x_1 u = x_2$, then $u \in GL_2(F) \cap \mathfrak{U}_F = GL_2(\mathcal{O})$. Hence $Z_\sigma(g)^\times x_1 GL_2(\mathcal{O}) = Z_\sigma(g)^\times x_2 GL_2(\mathcal{O})$, and the map is injective. Hence our assertion is proved.

Corollary 4.2.

$$|C_\sigma(g) \cap \Xi(\mathfrak{n})_+ / \tilde{\tilde{\Gamma}}| = (2/[\Lambda : \Lambda^1])\ |Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathfrak{U}_F| \quad .$$

Let ψ be the natural map from $Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathfrak{U}_F$ to $Z_\sigma(g)_A^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathfrak{U}_F$, where $Z_\sigma(g)_A$ is the adelization of $Z_\sigma(g)$. Let K be a Q-algebra and Λ be its Z-order. Put $\mathfrak{U}(\Lambda) = \prod_p \Lambda_p^\times \times K_\infty^\times$, where $\Lambda_p = \Lambda \otimes_Z Z_p$ and $K_\infty = K \otimes_Q R$. We define the class number $h(K, \Lambda)$ of Λ as the number of the double cosets $K^\times \backslash K_A^\times / \mathfrak{U}(\Lambda)$, where K_A is the adelization of K. We note that if K is a quaternion algebra and $\Lambda \sim \Lambda'$ for Z-orders Λ and Λ' of K, then $h(K, \Lambda) = h(K, \Lambda')$.

Lemma 4.3. Let the notation be as above. For a coset $\tilde{x} = Z_\sigma(g)_A^\times x\, \mathfrak{U}_F \in Z_\sigma(g)_A^\times \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathfrak{U}_F$, the number of $\psi^{-1}(\tilde{x})$ is independent of $\tilde{x}$ and is equal to $h(Z_\sigma(g), \Lambda)$.

Proof. By definition, we have $\psi^{-1}(\tilde{x}) = |Z_\sigma(g)^\times \backslash Z_\sigma(g)_A^\times x\, \mathfrak{U}_F / \mathfrak{U}_F|$.

We see

$$|Z_\sigma(g)^\times \backslash Z_\sigma(g)^\times_A x\, \mathcal{U}_F / \mathcal{U}_F| = |Z_\sigma(g)^\times \backslash Z_\sigma(g)^\times_A x\, \mathcal{U}_F x^{-1} / x\, \mathcal{U}_F x^{-1}|$$

$$= |Z_\sigma(g)^\times \backslash Z_\sigma(g)^\times_A / Z_\sigma(g)^\times_A \cap x\, \mathcal{U}_F x^{-1}| \quad .$$

From this we obtain $|\psi^{-1}(\hat{x})| = h(Z_\sigma(g), \Lambda)$.

Corollary 4.4.

$$|C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+ / \widetilde{\widetilde{\Gamma}}| = (2/[\Lambda : \Lambda^1])\, |Z_\sigma(g)^\times_A \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathcal{U}_F|$$

$$\times\, h(Z_\sigma(g), \Lambda) \quad .$$

For a prime p, put

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p) = \{ x \in GL_2(F_p) \mid x^{-1} g^\sigma x \in \Xi(\mathfrak{n})_p \} \quad .$$

Let Λ_p denote the Z_p-order $\Lambda \otimes_Z Z_p$ of $Z_\sigma(g)_p$, and put

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p, \Lambda_p) = \{ x \in GL_2(F_p) \mid x^{-1} g^\sigma x \in \Xi(\mathfrak{n})_p,\ Z_\sigma(g)_p \cap x M_2(\mathcal{O}_p) x^{-1} = \Lambda_p \} \quad .$$

Let r be the non-negative integer such that $\Xi(\mathfrak{n})_p = \Xi_p(r)$, f be the characteristic polynomial of Ng if $Ng \notin F^\times$, and a be the element Ng of F if $Ng \in F^\times$. Then $|Z_\sigma(g)^\times_p \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p, \Lambda_p)/GL_2(\mathcal{O}_p)|$ is nothing but $c_\sigma(f, r, \varphi_g^{-1}(\Lambda_p))$ or $c_\sigma(a, r, \varphi_g^{-1}(\Lambda_p))$ in the notation of §3, and is completely determined. In particular, by Prop. 3.11, 3.16, 3.29, 3.30, 3.32, we have

$$|Z_\sigma(g)^\times_p \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p, \Lambda_p)/GL_2(\mathcal{O}_p)| = 1$$

for almost all p. Since $\Xi(\mathfrak{n})_A = \prod_p \Xi(\mathfrak{n})_p \times \prod_v GL_2(F_v)$, we see

$$\mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda) \neq \phi \iff \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p, \Lambda_p) \neq \phi \quad \text{for all } p.$$

By this, we easily obtain the following.

Lemma 4.5. Let the notation be as above. Then the natural map $x \in GL_2(F_A) \longrightarrow (x_p) \in \prod_p GL_2(F_p)$ induces a bijective map

$$Z_\sigma(g)^\times_A \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_A, \Lambda)/\mathfrak{U}_F \longrightarrow \prod_p (Z_\sigma(g)^\times_p \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p, \Lambda_p)/GL_2(\mathcal{O}_p)$$

, where we denote by x_p the p-component of x considering $GL_2(F_A)$ as a subgroup of $\prod_p GL_2(F_p) \times GL_2(F_\infty)$.

Corollary 4.6. If $Ng \notin F^\times$, or $Ng \in F^\times$ and the type number of Λ is one, then we have

$$|C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{n})_+ / \underset{\Gamma}{\approx}| = (2/[\Lambda : \Lambda^1]) \prod_p |Z_\sigma(g)^\times_p \backslash \mathfrak{M}_\sigma(g, \Xi(\mathfrak{n})_p, \Lambda_p)/GL_2(\mathcal{O}_p)|$$
$$\times\ h(Z_\sigma(g), \Lambda) \quad .$$

4.3. The equivalence classes $GL_2(F)/\underset{GL_2(F)}{\approx}$ is determined by Lemma 3.4 and 3.5. For elements g with $g \notin F^\times$, we can reduce the condition (3.3.2) to a local one. Namely we can prove the following.

Lemma 4.7. Let K be a comutative $\mathbb{Q}$-algebra of rank 2. Then for x of $K^\times$ we have

$$x \in N_{K\otimes F/K}((K\otimes F)^\times) \iff x \in N_{K\otimes F_p/K\otimes \mathbb{Q}_p}((K\otimes F_p)^\times)$$

for all p.

We note the condition $x \in N_{K\otimes F_\infty/K\otimes R}((K\otimes F_\infty)^\times)$ is always satisfied, since we assume F is totally real. If both K and $K\otimes F$ are fields, this lemma is not other than Hasse's norm theorem. In general cases, we can easily reduce this lemma to Hasse's norm theorem for cyclic extensions, and we omit the proof.

Let a be an element of $F^\times$ such that $a \in N(GL_2(F)) \cap Q^\times$. Then a determines a quaternion algebra over Q as in 3.12, and we denote it by $D(a)$. Then for g with $Ng \in F^\times$, we can prove the following.

<u>Lemma 4.8</u>. Let the notation and the assumption be as above.

(i) If $\ell \neq 2$, we have $N(GL_2(F)) \cap F^\times = N_{F/Q}(F^\times)$. For an element $a \in N_{F/Q}(F^\times)$, the quaternion algebra $D(a)$ is isomorphic to $M_2(Q)$.

(ii) If $\ell = 2$, we have $N(GL_2(F)) \cap F^\times = Q^\times$. For $a \in Q^\times$, the quaternion $D(a)$ is not ramified at the archimedian prime and is ramified at a prime p if and only if $a \notin N_{F_p/Q_p}(F_p^\times)$.

<u>Proof.</u> (i) By Remark 3.8, for $g \in GL_2(F)$ such that $Ng \in F^\times$, there exists $x \in F^\times$ and $h \in GL_2(F)$ which satisfy $g = xh^{-1}\,{}^{\sigma}h$. Then $Ng = N_{F/Q}(x)$. Hence we obtain $N(GL_2(F)) \cap F^\times = N_{F/Q}(F^\times)$. The second assertion is already mentioned in Remark 3.8. (ii) For $a \in Q$, put $g = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$. Then $Ng = a$. From this we obtain $N(GL_2(F)) \cap F^\times = Q^\times$. The second assertion easily follows from the proof of Lemma 3.7.

As to the Galois cohomology group $H^1(\mathfrak{g}, E)$ of E, we have the following.

Lemma 4.9. The notation being as above, then we have

$$|H^1(\mathfrak{g}, E)| = \ell .$$

Proof. The group E_+ of totally positive units is a free abelian group of rank $\ell-1$. From the exact sequence

$$1 \longrightarrow E_+ \longrightarrow E \longrightarrow E/E_+ \longrightarrow 1$$

, we obtain the following exact sequence

$$\hat{H}^0(\mathfrak{g}, E/E_+) \longrightarrow H^1(\mathfrak{g}, E_+) \longrightarrow H^1(\mathfrak{g}, E) \longrightarrow H^1(\mathfrak{g}, E/E_+) .$$

By the assumption $[E : E_+] = 2^{\ell}$, we see easily $\hat{H}^0(\mathfrak{g}, E/E_+) = 1$ and $H^1(\mathfrak{g}, E/E_+) = 1$. Since $\mathfrak{g}$ acts on E_+ non-trivially, we see $H^1(\mathfrak{g}, E_+)$ is a cyclic group of order ℓ. Hence we obtain our result.

4.4. After these preparations, we give an explicit formula for $\operatorname{tr} T_S(T(\mathfrak{n}))$. As remarked before, we may assume there exist non-negative integers r_p such that $\Xi(\mathfrak{n})_A = \prod_p \Xi_p(r_p) \times GL_2(F_\infty)$, where $\Xi_p(r_p)$ is the union of $GL_2(\mathcal{O}_p)$-double cosets defined in 3.5. Let $\tilde{C}_v$, $\tilde{C}_e$, $\tilde{C}_h$, $\tilde{C}_p$ be as in Th.1', i.e. $\tilde{C}_i$ $(i = v, e, h, p)$ is a complete system of representatives of the set of elements of type v, e, h_a, p in the sense of 2.3 in $\Xi(\mathfrak{n})_+$ with respect to the equivalence relation $\underset{\Gamma}{\approx}$. We denote by t_v, t_e, t_h and t_p the contribution of $\tilde{C}_v$, $\tilde{C}_e$, $\tilde{C}_h$ and $\tilde{C}_p$ to $\operatorname{tr} T_S(T(\mathfrak{n}))$

respectively.

<u>4.5.</u> t_v. First we assume $\ell \neq 2$. We see $\tilde{C}_v \neq \phi$ only if r_p is even for all p, i.e. $\mathfrak{a}$ is a square of a integral ideal. This condition implies that $N\mathfrak{a}$ is a square, where $N\mathfrak{a} = |\mathcal{O}/\mathfrak{a}|$. Assume r_p is even for all p and put $N\mathfrak{a} = a^2$ with a positive rational integer a. Then by the assumption on F, $a \in N_{F/Q}(F^\times)$ and by Lemma 4.8, there exists $g \in GL_2(F)$ such that $Ng = a$. Then by Lemma 3.6 we see the set of elements of type v in $\Xi(\mathfrak{a})_+$ is $C_\sigma(g) \cap \Xi(\mathfrak{a})_+ \cup C_\sigma(-g) \cap \Xi(\mathfrak{a})_+$. It is easy to see that the contribution of $C_\sigma(-g) \cap \Xi(\mathfrak{a})_+$ to t_v is equal to that of $C_\sigma(g) \cap \Xi(\mathfrak{a})_+$. Hence by Lemma 4.9, we have

$$t_v = \frac{\kappa-1}{4\pi\ell} \sum_{g' \in C_\sigma(g) \cap \Xi(\mathfrak{a})_+/\widetilde{\approx}_\Gamma} v(H/Z_\sigma(g') \cap \Gamma)$$

$$= \frac{\kappa-1}{4\pi\ell} \sum_{\Lambda/\sim} \sum_{g' \in C_\sigma(g,\Lambda) \cap \Xi(\mathfrak{a})_+/\widetilde{\approx}_\Gamma} v(H/Z_\sigma(g') \cap \Gamma) .$$

For a Z-order Λ of $Z_\sigma(g)$, if $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{a})_+ \neq \phi$, then $\mathfrak{M}_\sigma(g, \Xi_p(r_p), \Lambda_p) \neq \phi$ for all p. By Prop. 3.39, 3.30, 3.34, if $\mathfrak{M}_\sigma(g, \Xi_p(r_p), \Lambda_p) \neq \phi$, $\varphi_g^{-1}(\Lambda_p) \sim Z_p + p^m M_2(Z_p)$ or $\begin{pmatrix} Z_p & Z_p \\ p^m Z_p & Z_p \end{pmatrix}$ for some non-negative integer m, where φ_g is a isomorphism from $D(a)_p = M_2(Q_p)$ to $Z_p(g)_p$. From this we see that the type number of Λ is one if $C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{a})_+ \neq \phi$. Hence by Cor. 4.6, we obtain

$$t_v = \frac{\kappa-1}{4\pi\ell} \sum_{\Lambda/\sim} \frac{2}{[\Lambda^\times : \Lambda^1]} \prod_p \left| Z_\sigma(g)^\times \backslash \mathfrak{M}_\sigma(g, \Xi_p(r_p), \Lambda_p)/GL_2(\mathcal{O}_p) \right|$$

$$\times\ h(Z_\sigma(g), \Lambda)\ v(H/\Lambda^1) \quad .$$

Let Λ_0 be a maximal order of $Z_\sigma(g)$ which contains Λ. Then it is known that $h(Z_\sigma(g), \Lambda_0) = 1$ and

$$h(Z_\sigma(g), \Lambda) = [\prod_p \Lambda_{0,p}^\times : \prod_p \Lambda_p^\times]/[\Lambda_0^\times : \Lambda^\times] = (\prod_p [\Lambda_{0,p}^\times : \Lambda_p^\times])/[\Lambda_0^\times : \Lambda^\times] \ .$$

We note $v(H/\Lambda_0^1) = v(H/SL_2(Z))$ and $[\Lambda_0^\times : \Lambda_0^1] = 2$. For a prime p, we denote by $\Lambda_p(m)$ and $c_{\sigma,p}(a, r_p, \Lambda)$ the Z_p-order $\Lambda(m)$ of $D(a)_p$ and the number $c_\sigma(a, r_p, \Lambda)$ given in 3.12 3.15 respectively. Then we obtain

$$t_v = \frac{\kappa-1}{4\pi\ell} \left(\sum_{0 \leq m_p \leq r_p/2} \prod_p c_{\sigma,p}(a, r_p, \Lambda_p(m_p))\ [\Lambda_p(0)^\times : \Lambda_p(m_p)^\times] \right)$$

$$\times\ v(H/SL_2(Z)) \quad .$$

Nextly we assume $\ell = 2$. We see that r_p is even for all p which decomposes in F, if $\tilde{C}_v \neq \phi$. Hence $N\mathfrak{a}$ is a square, and put $N\mathfrak{a} = a^2$ with a positive rational integer a. Then by Lemma 4.8, there exists $g \in GL_2(F)$ such that $Ng = a$. Let ε denote a unit of F such that $N_{F/Q}\varepsilon = -1$, then $N(g\varepsilon) = -a$, and $D(a)$ and $D(-a)$ are isomorphic to each other. We see the set of the elements of type v in $\Xi(\mathfrak{a})_+$ is $C_\sigma(g) \cap \Xi(\mathfrak{a})_+ \cup C_\sigma(g\varepsilon) \cap \Xi(\mathfrak{a})_+$. The contribution of $C_\sigma(g) \cap \Xi(\mathfrak{a})_+$ to t_v is equal to that of $C_\sigma(g\varepsilon) \cap \Xi(\mathfrak{a})_+$. Hence

$$t_v = \frac{\kappa-1}{4\pi \cdot 2} \sum_{g' \in C_\sigma(g) \cap \Xi(\mathfrak{n})_+ / \widetilde{\widetilde{\Gamma}}} v(H/Z_\sigma(g') \cap \Gamma)$$

We denote by $\Gamma(a)$ the unit group with the reduced norm 1 of a maximal order of $D(a)$. And we denote by $\Lambda_p(m)$ and $c_{\sigma,p}(a, r_p, \Lambda)$ the Z-order $\Lambda(m)$ of $D(a)_p$ and the number $c_\sigma(a, r_p, \Lambda)$ given in 3.12 $\sim$ 3.15. Then in the same way as above, we obtain

$$t_v = \frac{\kappa-1}{4\pi \cdot 2} \sum_{\substack{0 \leqslant m_p \leqslant [r_p/2] \\ 0 \leqslant m_d \leqslant m(a)}} \prod_p (c_{\sigma,p}(a, r_p, \Lambda(m_p)_p) [\Lambda(0)_p^\times : \Lambda(m_p)_p^\times]) \times v(H/\Gamma(a))$$

, where $m(a)$ is 0 or 1 according as $D(a)$ is a division algebra or not.

<u>4.6.</u> t_e. Let g be an element of $\Xi(\mathfrak{n})_+$ of type e and $f(X) = X^2 - sX + n$ be the characteristic polynomial of Ng. Then we see $n = N\mathfrak{n}$ and $s^2 - 4n < 0$. We denote by $S(\mathfrak{n})_e$ the set of all elements g of $GL_2(F)$ such that Ng has the characteristic polynomial $f(X) = X^2 - sX + n \in Z[X]$ with $n = N\mathfrak{n}$ and $s^2 - 4n < 0$. Then we have

$$t_e = -\frac{1}{4\ell} \sum_{g \in S(\mathfrak{n})_e / \widetilde{\widetilde{GL}}_2(F)} \frac{\eta(Ng)^{\kappa-1} - \zeta(Ng)^{\kappa-1}}{\eta(Ng) - \zeta(Ng)} (\det Ng)^{1-\frac{\kappa}{2}} \sum_{g' \in C_\sigma(g) \cap \Xi(\mathfrak{n})_+ / \widetilde{\widetilde{\Gamma}}} \frac{1}{[(Z_\sigma(g') \cap \Gamma)E : E]}$$

, where $\zeta(Ng)$ and $\eta(Ng)$ are the roots of the characteristic polynomial $f(X)$ of Ng. We denote $Z_\sigma(g') \cap \Gamma \cap E = \{\pm 1\}$ by E_Q. Then we have

$$t_e = -\frac{1}{4\ell} \sum_{g \in S(\mathfrak{a})_e / \widetilde{\widetilde{GL_2}}(F)} \frac{\eta(Ng)^{\kappa-1} - \zeta(Ng)^{\kappa-1}}{\eta(Ng) - \zeta(Ng)} (\det Ng)^{1-\frac{\kappa}{2}}$$

$$\sum_{\Lambda} \sum_{g' \in C_\sigma(g, \Lambda) \cap \Xi(\mathfrak{a})_+ / \widetilde{\widetilde{\Gamma}}} \frac{1}{[Z_\sigma(g') \cap \Gamma : E_Q]}$$

, where Λ runs through all Z-orders of $Z_\sigma(g)$ which contain Ng. By Cor. 4.6, we have

$$t_e = -\frac{1}{2\ell} \sum_{g \in S(\mathfrak{a})_e / \widetilde{\widetilde{GL_2}}(F)} \frac{\eta(Ng)^{\kappa-1} - \zeta(Ng)^{\kappa-1}}{\eta(Ng) - \zeta(Ng)} (\det Ng)^{1-\frac{\kappa}{2}}$$

$$\sum_{\Lambda} \frac{h(Z_\sigma(g), \Lambda)}{[\Lambda : E_Q]} \prod_p \left| Z_\sigma(g)_p^\times \backslash \mathfrak{M}_\sigma(g, r_p, \Lambda_p)/GL_2(\mathcal{O}_p) \right| .$$

Let $f(X) = X^2 - sX + n$ be an element of $Z[X]$ such that $n = N\mathfrak{a}$ and $s^2 - 4n < 0$. By Lemma 3.5 and Lemma 4.7 there exists $g \in GL_2(F)$ such that Ng has characteristic polynomial $f(X)$ if and only if there exists an element $g_p \in GL_2(F_p)$ such that Ng_p has the characteristic polynomial $f(X)$ for all p. For a prime p. we denote by $c_{\sigma,p}(f, r, \Lambda)$ the number $c_\sigma(f, r, \Lambda)$ defined in 3.5. Then if $f(X)$ is the characteristic polynomial of Ng for $g \in GL_2(F)$,

$$c_{\sigma,p}(f, r_p, \Lambda_p) = \left| Z_\sigma(g)_p^\times \backslash \mathfrak{M}_\sigma(g, r_p, \varphi_g(\Lambda_p))/GL_2(\mathcal{O}_p) \right| .$$

Here we denote by φ_g the natural isomorphism from $K(f)_p = Q[X]/f(X) \otimes Q_p$ to $Z_\sigma(g)_p$ given by $\varphi_g(\tilde{X}) = Ng$, where $\tilde{X}$ is the class represented by X. For a Z-order Λ of $K(f)$, if it holds $c_{\sigma,p}(f, r_p, \Lambda_p) \neq 0$ for a prime p, there exists $g_p \in GL_2(F_p)$ such that Ng_p has the characteristic polynomial $f(X)$. Hence if $c_{\sigma,p}(f, r_p, \Lambda_p) \neq 0$ for all p, there exists $g \in GL_2(F)$ such that the characteristic polynomial of Ng is $f(X)$, and we have

$$\prod c_{\sigma,p}(f, r_p, \Lambda_p) = \prod \left| Z_\sigma(g)_p^\times \backslash \mathfrak{M}_\sigma(g, r_p, \varphi_g(\Lambda_p)) / GL_2(\mathcal{O}_p) \right| .$$

For $f(X)$, put

$$(4.6.1) \qquad \omega_e(f) = (\eta^{\kappa-1} - \zeta^{\kappa-1} / \eta - \zeta)\, n^{1-\frac{\kappa}{2}}$$

, where ζ and η are the roots of $f(X)$. Then we obtain,

$$t_e = -\frac{1}{2\ell} \sum_f \omega_e(f) \sum_\Lambda \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p, \Lambda_p)$$

, where f runs through all the polynomials $X^2 - sX + n$ in $Z[X]$ which satisfy $n = N\mathfrak{n}$ and $s^2 - 4n < 0$, and Λ runs through all the Z-orders of $K(f)$ which contain $\tilde{X}$.

<u>4.7.</u> t_h. Let g be an element of $\Xi(\mathfrak{n})_+$ of type h_a, and $f(X) = X^2 - sX + n$ be the characteristic polynomial of Ng. Then $n = N\mathfrak{n}$ and $s^2 - 4n$ is a non-zero square. Let $f(X)$ be such a polynomial. For a prime p, we denote by $c_{\sigma,p}(f, r_p, \Lambda_p)$ the number $c_\sigma(f, r_p, \Lambda_p)$ defined in 3.5.

Let η and ζ be the roots of f, and put

$$\omega_h(f) = \frac{\mathrm{Min}(|\eta|, |\zeta|)^{\kappa-1}}{|\eta - \zeta|} n^{1-\frac{\kappa}{2}} \quad (4.7.2)$$

Then in the same way as in the case of t_e, we obtain

$$t_h = -\frac{1}{\ell} \sum_f \omega_h(f) \sum_\Lambda \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p, \Lambda_p)$$

, where f runs through all the polynomial $X^2 - sX + n$ in $Z[X]$ which have two distinct roots in Z and satisfy $n = N\mathfrak{a}$, and Λ runs through all Z-order of $K(f)$ which contains $\mathfrak{X}$.

4.8. t_p. If $C_p \neq \phi$, we see $\mathfrak{a}$ is a square of some integral ideal. Assume $\mathfrak{a}$ is a square, and put $N\mathfrak{a} = a^2$ with a positive integer a. Let χ_i, $1 \leqslant i \leqslant \ell$, be the characters mod. q which correspond to the extension F/Q, and χ_1 be the identity character. Since $\sum \chi_i(a) = \ell$, by Lemma 4.7 and the result of 3.12 $\sim$ 3.15, we see there exists $g \in GL_2(F)$ such that $Ng = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ and fix such an element g in the following. Let ε be an element of E with $N_{F/Q}\varepsilon = -1$. If $C_\sigma(g') \cap \Xi(\mathfrak{a})_+ \neq \phi$ for $g' \in GL_2(F)$, then it holds $C_\sigma(g') = C_\sigma(g)$ or $C_\sigma(g') = C_\sigma(\varepsilon g)$. And it is easy to see the contribution of $C_\sigma(g) \cap \Xi(\mathfrak{a})_+$ to t_p is equal to that of $C_\sigma(g\varepsilon) \cap \Xi(\mathfrak{a})_+$. Hence in the notation of Th.1', we have

$$t_p = \frac{1}{2\pi\ell} \lim_{s\to 0} \sum_{\Lambda} \sum_{g'\in C_\sigma(g,\Lambda)\cap\Xi(\mathfrak{n})_+/\tilde{\tilde{\Gamma}}} |m(g')|\lambda_1(g')^s \cdots (\lambda_1(g')\cdots\lambda_{\ell-1}(g'))^s$$

$$\times \frac{\sqrt{-1}\,\mathrm{sgn}(-A(g'))}{|A(g')|^{1+\ell s}} \exp(\pi/2\ \ell s\ \mathrm{sgn}(-A(g'))\sqrt{-1})$$

, where Λ runs through all Z-orders of $Z_\sigma(g)$ which contain Ng. For a positive integer m, we denote by $\Lambda(m)$ the Z-order of K(f) given by

(4.8.1) $$\Lambda(m) = Z + m^{-1}Z(\tilde{X} - a) \quad .$$

Then any Z-order of K(f) which contains $\tilde{X}$ is $\Lambda(m)$ for some positive integer m. Put $\mathfrak{n}\cap Z = (a'^2)$ with a positive integer a', and $a_0 = a/a'$. Then by Cor.4.6, Prop.3.11, 3.16, 3.20, 3.28, we see $C_\sigma(g, \varphi_g(\Lambda(m))\cap\Xi(\mathfrak{n})_+ \neq \phi$ only if a_0 divides m. For $\Lambda(m)$, we see $\Lambda(m)^\times = \Lambda(m)^1$ and $h(K(f), \Lambda(m)) = 1$ for any positive integer m. By the above propositions and Cor.4.6, we have

$$|C_\sigma(g, \varphi_g(\Lambda(a_0)))\cap\Xi(\mathfrak{n})_+ / \tilde{\tilde{\Gamma}}| = |C_\sigma(g, \varphi_g(\Lambda(a_0 t)))\cap\Xi(\mathfrak{n})_+ / \tilde{\tilde{\Gamma}}|$$

for any positive integer t. Now we give a complete system of representatives of $C_\sigma(g, \varphi_g(\Lambda(a_0 t)))\cap\Xi(\mathfrak{n})_+ / \tilde{\tilde{\Gamma}}$. Any class of $C_\sigma(g, \varphi_g(\Lambda(a_0)))\cap\Xi(\mathfrak{n})_+ / \tilde{\tilde{\Gamma}}$ contains an element of the form $\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \in GL_2(\mathfrak{o})$ by the assumption on F. We note $(\alpha^2) = (\delta^2) = \mathfrak{n}$ and $N_{F/Q}\alpha = N_{F/Q}\delta = a$. For such α and δ, we define two Z-submodules $Z(\alpha, \delta)$ and $B(\alpha, \delta)$ of $\mathfrak{o}$ by

$$Z(\alpha, \delta) = \left\{ x \in \mathcal{O} \;\middle|\; N(\begin{pmatrix} \alpha & x \\ 0 & \delta \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\}$$

$$B(\alpha, \delta) = \left\{ \alpha^{\sigma}x - \delta x \;\middle|\; x \in \mathcal{O} \right\} .$$

Then $Z(\alpha, \delta)$ contains $B(\alpha, \delta)$, and $|Z(\alpha, \delta)/B(\alpha, \delta)|$ is finite. Let $g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ and $g' = \begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix}$ be two elements of $C_{\sigma}(g) \cap \Xi(\mathfrak{n})_+$. Then it is easy to see that $g \underset{\Gamma}{\approx} g'$ if and only if there exist $\varepsilon_1, \varepsilon_2 \in E$ such that $\varepsilon_1\varepsilon_2$ is totally positive and it holds

$$\alpha = \varepsilon_1^{-1\sigma}\varepsilon_1\alpha' , \quad \delta = \varepsilon_2^{-1\sigma}\varepsilon_2\delta', \quad \beta - \varepsilon_1^{-1\sigma}\varepsilon_2\beta' \in B(\alpha, \delta) .$$

For $g' = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in C_{\sigma}(g, \varphi_g(\Lambda(a_0t))) \cap \Xi(\mathfrak{n})_+$ and $x \in Z(\alpha, \delta)$, we see $N(\begin{pmatrix} \alpha & \beta+x \\ 0 & \delta \end{pmatrix}) = Ng'$, hence $\begin{pmatrix} \alpha & \beta+x \\ 0 & \delta \end{pmatrix}$ is also contained in $C_{\sigma}(g, \varphi_g(\Lambda(a_0t))) \cap \Xi(\mathfrak{n})_+$. And for $\varepsilon_1, \varepsilon_2 \in E$, we have

$$Z(\varepsilon_1^{-1\sigma}\varepsilon_1\alpha, \varepsilon_2^{-1\sigma}\varepsilon_2\delta) = \varepsilon_1^{-1\sigma}\varepsilon_2 Z(\alpha, \delta)$$

$$B(\varepsilon_1^{-1\sigma}\varepsilon_1\alpha, \varepsilon_2^{-1\sigma}\varepsilon_2\delta) = \varepsilon_1^{-1\sigma}\varepsilon_2 B(\alpha, \delta)$$

From this, we see that there exist N elements $g_i = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \delta_i \end{pmatrix} \in C_{\sigma}(g, \varphi_g(\Lambda(a_0))) \cap \Xi(\mathfrak{n})_+$, $1 \leq i \leq N$, for some N such that

$$\bigcup_{1 \leq i \leq N} \; \bigcup_{x \in Z(\alpha_i, \delta_i)/B(\alpha_i, \delta_i)} \begin{pmatrix} \alpha_i & \beta_i + x \\ 0 & \delta_i \end{pmatrix}$$

is a complete system of representatives of $C_{\sigma}(g, \varphi_g(\Lambda(a_0))) \cap \Xi(\mathfrak{n})_+ / \underset{\Gamma}{\approx}$. For g_i, put

$Ng_i = \begin{pmatrix} a & b_i \\ 0 & a \end{pmatrix}$, then for $x \in Z(\alpha_i, \delta_i)$ $N\begin{pmatrix} \alpha_i & t\beta_i+x \\ 0 & \delta_i \end{pmatrix} = \begin{pmatrix} a & tb_i \\ 0 & a \end{pmatrix}$.

Since $g_i \in C_\sigma(g, \varphi_g(\Lambda(a_o))) \cap \Xi(\mathfrak{n})_+$,

$$(Q + QNg_i) \cap M_2(\mathcal{O}) = Z + a_o^{-1} Z \begin{pmatrix} 0 & b_i \\ 0 & 0 \end{pmatrix}$$

and it holds

$$(Q + Q\begin{pmatrix} a & tb_i \\ 0 & a \end{pmatrix}) \cap M_2(\mathcal{O}) = Z + (a_o t)^{-1} Z \begin{pmatrix} 0 & tb_i \\ 0 & 0 \end{pmatrix} .$$

Hence for $x \in Z(\alpha_i, \delta_i)$, the element $\begin{pmatrix} \alpha_i & t\beta_i+x \\ 0 & \delta_i \end{pmatrix}$ is contained in $C_\sigma(g, \varphi_g(\Lambda(a_o t))) \cap \Xi(\mathfrak{n})_+$. We show that

$\bigcup_{1 \leq i \leq N} \bigcup_{x \in Z(\alpha_i, \delta_i)/B(\alpha_i, \delta_i)} \begin{pmatrix} \alpha_i & t\beta_i+x \\ 0 & \delta_i \end{pmatrix}$ is a complete system

of representatives of $C_\sigma(g, \varphi_g(\Lambda(a_o t))) \cap \Xi(\mathfrak{n})_+ / \underset{\Gamma}{\approx}$. Assume

$\begin{pmatrix} \alpha_i & t\beta_i+x \\ 0 & \delta_i \end{pmatrix} \underset{\Gamma}{\approx} \begin{pmatrix} \alpha_j & t\beta_j+x' \\ 0 & \delta_j \end{pmatrix}$ for $x \in Z(\alpha_i, \delta_i)$ and

$x' \in Z(\alpha_j, \delta_j)$. Then there exist $\varepsilon_1, \varepsilon_2 \in E$ such that $\varepsilon_1\varepsilon_2 \in E_+$ $\alpha_i = \varepsilon_1^{-1}{}^\sigma\varepsilon_1\alpha_j$, $\delta_i = \varepsilon_2^{-1}{}^\sigma\varepsilon_2\delta_j$, and $t\beta_i + x - \varepsilon_1^{-1}{}^\sigma\varepsilon_2(t\beta_j + x')$ $\in B(\alpha_i, \delta_i)$. Hence $t(\beta_i - \varepsilon_1^{-1}{}^\sigma\varepsilon_2\beta_j)$ is contained in $Z(\alpha_i, \delta_i)$. Now $\beta_i - \varepsilon_1^{-1}{}^\sigma\varepsilon_2\beta_j$ is an element of $\mathcal{O}$, hence $\beta_i - \varepsilon_1^{-1}{}^\sigma\varepsilon_2\beta_j$ is also contained in $Z(\alpha_i, \delta_i)$. Put $x'' = \beta_i - \varepsilon_1^{-1}{}^\sigma\varepsilon_2\beta_j$,

then $\begin{pmatrix} \alpha_i & \beta_i-x'' \\ 0 & \delta_i \end{pmatrix} \underset{\Gamma}{\approx} \begin{pmatrix} \alpha_j & \beta_j \\ 0 & \delta_j \end{pmatrix}$, and $i = j$ by the assumption on the choice of $\begin{pmatrix}\alpha_i & \beta_i \\ 0 & \delta_i\end{pmatrix}$. If $i = j$, then it follows from the assumption $\begin{pmatrix}\alpha_i & t\beta_i+x \\ 0 & \delta_i\end{pmatrix} \underset{\Gamma}{\approx} \begin{pmatrix}\alpha_i & t\beta_i+x' \\ 0 & \delta_i\end{pmatrix}$ that $x - x'$ is contained in $B(\alpha_i, \delta_i)$, i.e. $\begin{pmatrix}\alpha_i & \beta_i+x \\ 0 & \delta_i\end{pmatrix} \underset{\Gamma}{\approx} \begin{pmatrix}\alpha_i & \beta_i+x' \\ 0 & \delta_i\end{pmatrix}$.

In notice of the fact that $\left|C_\sigma(g, \varphi_g(\Lambda(a_0)))\cap\Xi(\mathfrak{n})_+ / \underset{\Gamma}{\approx}\right|$ $= \left|C_\sigma(g, \varphi_g(\Lambda(a_0 t)))\cap\Xi(\mathfrak{n})_+ / \underset{\Gamma}{\approx}\right|$, we obtain our assertion.

By definition, we have $\lambda_j(\begin{pmatrix}\alpha_i & t\beta_i+x \\ 0 & \delta_i\end{pmatrix}) = \lambda_j(g_i)$, $\left|m\begin{pmatrix}\alpha_i & t\beta_i+x \\ 0 & \delta_i\end{pmatrix}\right|$ $= |m(g_i)|$, $A(\begin{pmatrix}\alpha_i & t\beta_i+x \\ 0 & \delta_i\end{pmatrix}) = t\, A(g_i)$. Since $|m(g_i)| = |b_i/a_0|$ and $A(g_i) = b_i/a$, $|m(g_i)/A(g_i)| = a/a_0$. Hence we obtain

$$t_p = \frac{1}{2\pi\ell} \lim_{s\to 0} \sum_{1\leqslant i\leqslant N} Z(\alpha_i, \delta_i)/B(\alpha_i, \delta_i) \left(\frac{a}{a_0}\right)^{1+\ell s} \frac{\sqrt{-1}\ \mathrm{sgn}(-A(g_i))}{|m(g_i)|^{\ell s}}$$

$$\times\ \lambda_1(g_i)^s \cdots (\lambda_1(g_i)\cdots\lambda_{\ell-1}(g_i))^s \exp(\pi/2\ \ell s\ \mathrm{sgn}(-A(g_i))\sqrt{-1}) \sum_{t=1}^{\infty} \frac{1}{t^{1+\ell s}},$$

It is easy to see that we may take $g_i' = \begin{pmatrix}\alpha_i & -\beta_i \\ 0 & \delta_i\end{pmatrix}$ in place of g_i, and that $|m(g_i)| = |m(g_i')|$, $\lambda_j(g_i) = \lambda_j(g_i')$, and $A(g_i) = -A(g_i')$. Hence we have

$$t_p = \frac{1}{4\pi\ell} \lim_{s\to 0} \sum_{1\leq i\leq N} Z(\alpha_i, \delta_i)/B(\alpha_i, \delta_i) \left(\frac{a}{a_0}\right)^{1+\ell s} \frac{\sqrt{-1}\,\mathrm{sgn}(A(g_i))}{|m(g_i)|^{\ell s}}$$

$$\times \lambda_1(g_i)^s \cdots (\lambda_1(g_i)\cdots\lambda_{\ell-1}(g_i))^s \Big\{\exp(\pi/2\ell s\ \mathrm{sgn}A(g_i)\sqrt{-1})$$

$$- \exp(\pi/2\ell s\ \mathrm{sgn}(-A(g_i))\sqrt{-1})\Big\} \times \sum_{t=1} \frac{1}{t^{1+\ell s}}$$

$$= -\frac{1}{4\ell}\frac{a}{a_0} \sum_{1\leq i\leq N} |Z(\alpha_i, \delta_i)/B(\alpha_i, \delta_i)| \quad .$$

By definition, we have $\sum_{1\leq i\leq N} |Z(\alpha_i, \delta_i)/B(\alpha_i, \delta_i)|$ $= |C_\sigma(g, \varphi_g(\Lambda(a_0))) \cap \Xi(\sigma)_+ / \underset{\Gamma}{\approx}|$. The characteristic polynomial of Ng is $f(X) = (X - a)^2$, hence by Cor. 4.6 , we obtain

$$t_p = -\frac{1}{2\ell}\cdot\frac{a}{a_0}\prod_p c_{\sigma,p}(f, r_p, \Lambda(a_0)_p)$$

, where $f = (X - a)^2$.

<u>4.9.</u> Thus we obtain the following theorem.

<u>Theorem 2.</u> Let F and σ be as in 4.1. For a prime p, let $\Xi_p(r)$, $\Lambda_p(m)$, $c_{\sigma,p}(f, r, \Lambda)$ and $c_{\sigma,p}(a, r, \Lambda)$ be $\Xi(r)$, $\Lambda(m)$, $c_\sigma(f, r, \Lambda)$ and $c_\sigma(a, r, \Lambda)$ in § 3 for Q_p and F_p , respectively. Let r_p be the non-negative integers such that $\Xi(\sigma)_A = \prod_p \Xi_p(r_p) \times GL_2(F_\infty)$. If κ is even and $\geqslant 4$, the trace $\mathrm{tr}\ T_S(T(\sigma))$ is given by the following formula.

(4.9.1) $\quad \mathrm{tr}\, T_S(T(\mathfrak{a})) = t_v + t_e + t_h + t_p$

, where t_v, t_e, t_h and t_p are given as follows.

(1) If $N\mathfrak{a}$ is not a square,

$$t_v = 0 \quad .$$

If $N\mathfrak{a}$ is a square, put $N\mathfrak{a} = a^2$, and let $D(a)$ be as in 4.3, then

$$t_v = \frac{\kappa-1}{4\pi\ell}\left(\sum_{\substack{0\leq m_p\leq[r_p/2],\ p\neq q\\ 0\leq m_q\leq m(a)}} \prod_p \left(c_{\sigma,p}(a,\ r_p,\ \Lambda_p(m_p))\,[\Lambda_p(0)^\times : \Lambda_p(m_p)^\times]\ \right)\right) \times\ v(H/\Gamma(a)) \quad .$$

Here $\Gamma(a)$ is the group of all units of a maximal order of $D(a)$ with the reduced norm 1, and $m(a)$ is 0 or 1 according as $D(a)$ is ramified at the prime q or not.

(2) t_e. We have

$$t_e = -\frac{1}{2\ell}\sum_f \omega_e(f) \sum_\Lambda \frac{h(K(f),\ \Lambda)}{[\Lambda^\times : E_Q]} \prod_p c_{\sigma,p}(f,\ r_p,\ \Lambda_p) \quad .$$

Here f runs through all the polynomial $X^2 - sX + n$ in $Z[X]$ such that $n = N\mathfrak{a}$ and $s^2 - 4n < 0$. For f, $K(f) = Q[X]/(f(X))$ and $\omega_e(f)$ is given by (4.6.1). Λ runs through all Z-orders of $K(f)$ which contain the element $\tilde{X}$ of $K(f)$ represented by X. For a prime p, $\Lambda_p = \Lambda\otimes_Z Z_p$, and $h(K(f),\ \Lambda)$ is the class number of Λ defined in 4.2.

(3) t_h. Let $K(f)$, Λ_p, and $h(K(f), \Lambda)$ be as in (2). Then

$$t_h = -\frac{1}{\ell} \sum_f \omega_h(f) \sum_\Lambda \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p, \Lambda_p) \quad .$$

Here f runs through all the polynomial $X^2 - sX + n$ in $Z[X]$ such that $n = N\mathfrak{a}$ and $f(X)$ has distinct two roots in Q, and Λ runs all the Z-orders of $Z_\sigma(g)$ which contains $\tilde{X}$. $\omega_h(f)$ is given by (4.7.2).

(4) If $\mathfrak{a}$ is not a square,

$$t_p = 0 \quad .$$

If $\mathfrak{a}$ is a square, put $N\mathfrak{a} = a^2$ with a positive integer a. Then we have

$$t_p = -\frac{1}{2\ell} \bar{a} \prod_p c_{\sigma,p}(f, r_p, \Lambda(a/\bar{a})_p)$$

, where $f(X) = (X - a)^2$, and $\bar{a}$ is a positive integer such that $\mathfrak{a} \cap Z = (\bar{a}^2)$. $\Lambda(a/\bar{a})$ is the Z-order of $K(f)$ given by (4.8.1) for $m = a/\bar{a}$ and $\Lambda(a/\bar{a})_p = \Lambda(a/\bar{a}) \otimes_Z Z_p$.

<u>4.10</u>. We will rewrite the formula (4.9.1) in Th.2 for later use with some remarks.

(1) t_v. Assume $N\mathfrak{a}$ is a square, and put $N\mathfrak{a} = a^2$ with a positive integer a. First assume $\ell \neq 2$. Let α be an element of Z_p. For a prime $p \neq q$ and a non-negative integer r, put

$$c_{\sigma,p}(\alpha,\ r) = \begin{cases} \sum_{m\geq 0} c_{\sigma,p}(\alpha,\ r,\ \Lambda_p(m))\ [\Lambda_p(0)^\times : \Lambda_p(m)^\times] \\ \qquad\qquad\qquad\qquad , \ \alpha \in N(GL_2(F_p)) \\ 0 \qquad\qquad\qquad\qquad , \ \text{otherwise} \end{cases}$$

and for $p = q$, put

$$c_{\sigma,q}(\alpha,\ 0) = \begin{cases} \frac{1}{\ell} \sum_{m\geq 0} c_{\sigma,q}(\alpha,\ 0,\ \Lambda_q(m))\ [\Lambda_q(0)^\times : \Lambda_q(m)^\times] \\ \qquad\qquad\qquad\qquad , \ \alpha \in N(GL_2(F_q)) \\ 0 \qquad\qquad\qquad\qquad , \ \text{otherwise} \end{cases}.$$

We note for $p \neq q$

(4.10.1) $\qquad c_{\sigma,p}(\alpha,\ r) = c_{\sigma,p}(\alpha u,\ r) \qquad$ for any $u \in Z_p^\times$.

And for $p = q$

(4.10.2) $\qquad c_{\sigma,q}(\alpha,\ 0) = c_{\sigma,q}(\alpha u,\ 0) \qquad$ for any $u \in N(\mathcal{O}_{\mathfrak{q}}^\times)$

, where $\mathfrak{q}$ is the prime factor of q in F.

Using $c_{\sigma,p}(\alpha,\ r)$, we can write t_v in the following form

$$t_v = \frac{\kappa-1}{4\pi} \prod_p c_{\sigma,p}(a,\ r_p)\ v(H/SL_2(Z))$$

, and $c_{\sigma,p}(a,\ r_p)$ is given explicitly as follows. If p decomposes in F, by Prop. 3.29 we have

(4.10.3) $\qquad c_{\sigma,p}(a,\ r_p) = 1$.

If p remains prime in F, taking notice of the fact that

$[\Lambda_p(0)^\times : \Lambda_p(m)^\times] = |PGL_2(Z/p^m Z)|$, we see by Prop. 3.32,

(4.10.4) $$c_{\sigma,p}(a, r_p) = 1 + \sum_{1 \leq m \leq r_p/2} p^{\ell m-(\ell-1)}(p^{(\ell-1)} - 1) \quad .$$

For $p = q$, since $[\Lambda(0)^\times : \Lambda(1)^\times] = q + 1$, by Prop. 3.34 we have

(4.10.5) $$c_{\sigma,q}(a, 0) = 1 + \frac{(\ell-1)(q+1)}{2} \; .$$

Next assume $\ell = 2$. Let α be an element of Z_p. For a prime $p \neq q$ and a non-negative integer r, put

$$c_{\sigma,p}(\alpha, r) = \begin{cases} \sum_{m \geq 0} c_{\sigma,p}(\alpha, r, \Lambda_p(m))[\Lambda_p(0)^\times : \Lambda_p(m)^\times] \\ \qquad\qquad \text{if } \alpha \in N(GL_2(F_p)) \text{ and } r \text{ is even.} \\ \sum_{m \geq 0} c_{\sigma,p}(\alpha, r, \Lambda_p(m))[\Lambda_p(0)^\times : \Lambda_p(m)^\times](p - 1) \\ \qquad\qquad \text{if } \alpha \in N(GL_2(F_p)) \text{ and } r \text{ is odd.} \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

For $p = q$, we see $\sum \chi_i(\alpha) = 0$ if and only if $D(\alpha)$ is a division algebra, and put

$$c_{\sigma,q}(\alpha, 0) = \begin{cases} \frac{1}{2}\sum_{0 \leq m \leq 1} c_{\sigma,q}(\alpha, 0, \Lambda_q(m))[\Lambda_q(0)^\times : \Lambda_q(m)^\times] \\ \qquad\qquad \text{if } \alpha \in N(GL_2(F_q)) \text{ and } \sum \chi_i(\alpha) \neq 0 \, . \\ \frac{1}{2} c_{\sigma,q}(\alpha, 0, \Lambda_q(0))(q - 1) \\ \qquad\qquad \text{if } \alpha \in N(GL_2(F_q)) \text{ and } \sum \chi_i(\alpha) = 0 \, . \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

We note $c_{\sigma,p}(\alpha, 0)$ and $c_{\sigma,q}(\alpha, 0)$ also satisfies the relation (4.10.1) and (4.10.2). If the discriminant of $D(a)$ is $(p_1 \dots p_n)^2$ with distinct primes p_i, then $v(H/\Gamma(a)) = \prod(p_i - 1)\ v(H/SL_2(Z))$. Hence we see

$$t_v = \frac{\kappa-1}{4\pi}\prod_p c_{\sigma,p}(a, r_p)\ v(H/SL_2(Z))$$

, and $c_{\sigma,p}(a, r_p)$ is given explicitly as follows. If p decomposes in F, by Prop. 3.29, we have

$$(4.10.6) \qquad c_{\sigma,p}(a, r_p) = 1 \quad .$$

If p remains prime in F, we see that for a positive integer m $[\Lambda(0)^{\times} : \Lambda(m)^{\times}] = p^{2m-1}(p - 1)$ (resp. $= p^{2m}$) if r_p is even (resp. if r_p is odd). Hence by Prop. 3.30, we have

$$(4.10.7) \qquad c_{\sigma,p}(a, r_p) = \begin{cases} 1 + \sum\limits_{1\leq m\leq r_p/2} p^{2m-1}(p - 1) & \text{if } r_p \text{ is even} \\ (p - 1)\sum\limits_{0\leq m\leq [r_p/2]} p^{2m} & \text{if } r_p \text{ is odd} \end{cases} .$$

For $p = q$, let χ_2 be the non-trivial character mod. q corresponding to F, then by Prop. 3.35, we see

$$(4.10.8) \qquad c_{\sigma,q}(a, 0) = \chi_2(a)(1 + \frac{q+1}{2}\chi_2(a)) \quad .$$

(2) t_e. Let $f(X) = X^2 - sX + n$ be a polynomial in $Z[X]$ such that $n = N\pi$ and $s^2 - 4n < 0$, and Λ_0 be the maximal order of $K(f)$. Put $h(K(f)) = h(K(f), \Lambda_0)$, then for

Z-order Λ of $K(f)$,

$$(4.10.9) \qquad h(K(f), \Lambda) = \frac{h(K(f))}{[\Lambda_0^\times : \Lambda^\times]} \prod_p [\Lambda_{0,p}^\times : \Lambda_p^\times] \quad .$$

Let $\bar{f}(X) = X^2 - \bar{s}X + \bar{n}$ be a polynomial in $Z_p[X]$, and for a non-negative integer m, let $\Lambda_p(m)$ be the Z_p-order $\Lambda_K(m)$ given by (3.9.2) and (3.9.3) for $K = K(\bar{f}) \otimes_Q Q_p$. For a prime $p \neq q$ and a non-negative integer r, put

$$c_{\sigma,p}(\bar{f}, r) = \sum_{m \geq 0} c_{\sigma,p}(\bar{f}, r, \Lambda_p(m)) [\Lambda_p(0)^\times : \Lambda_p(m)^\times]$$

and for $p = q$, put

$$c_{\sigma,q}(\bar{f}, r) = \frac{1}{\ell} \sum_{m \geq 0} c_{\sigma,q}(\bar{f}, r, \Lambda_q(m)) [\Lambda_q(0)^\times : \Lambda_q(m)^\times] \quad .$$

For $u \in Z_p^\times$, put $\bar{f}_u(X) = u^{-2}\bar{f}(uX)$. Then we see for $p \neq q$

$$(4.10.10) \qquad c_{\sigma,p}(\bar{f}, r) = c_{\sigma,p}(\bar{f}_u, r) \qquad \text{for any } u \in Z_p^\times$$

and for $p = q$

$$(4.10.11) \qquad c_{\sigma,q}(\bar{f}, r) = c_{\sigma,q}(\bar{f}_u, r) \qquad \text{for any } u \in N(\mathcal{O}_\mathfrak{q}^\times)$$

By the definition of $c_{\sigma,p}(\bar{f}, r)$, we have

$$t_e = -\frac{1}{2} \sum_f \omega_e(f) \frac{h(K(f))}{[\Lambda_0^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p)$$

, where f runs through the same set as in (2) of Th. 2 and

$\omega_e(f)$ is given by (4.6.1). The number $c_{\sigma,p}(f, r_p)$ can be given in more explicit form by using the result of §3, but we note here only the following as to $c_{\sigma,q}(f, 0)$. We denote by $\left\{\frac{\Lambda}{p}\right\}$ the symbol given as follows. Let K be a Q_p-algebra of type a), b), or c) in Remark 3.2, and Λ its Z_p-order. If Λ is the maximal order, we set

$$\left\{\frac{\Lambda}{p}\right\} = 1, -1, 0,$$

according as K is of type a), b), or c). If Λ is not the maximal order, put

$$\left\{\frac{\Lambda}{p}\right\} = 1 .$$

Let δ be the integer such that $Z_q[\tilde{X}] = \Lambda_q(\delta)$, then by Prop. 3.25, 3.26, 3.27,

$$(4.10.12) \qquad c_{\sigma,q}(f, 0) = \sum_{0 \leq m \leq \delta} \left(1 + \frac{1 + \left\{\frac{\Lambda_q(m)}{q}\right\}}{2} \sum_{i \neq 1} \frac{\chi_i(\alpha) + \chi_i(\beta)}{2}\right) \times [\Lambda_q(0)^{\times} : \Lambda_q(m)^{\times}] ,$$

where α and β are the roots of the equation $f(X) \equiv 0 \mod. q$.

(3) t_h. Let $f(X) = X^2 - sX + n$ be a polynomial in $Z[X]$ such that $n = N\mathfrak{a}$ and $f(X)$ has distinct two roots in Q, and Λ_0 be the maximal order of $K(f)$. Put $h(K(f)) = h(K(f), \Lambda_0)$. For a non-negative integer m, let $\Lambda_p(m)$ be the Z_p-order $\Lambda_K(m)$ in 3.9 for $K = K(f) \otimes Q_p$ as in (2). Let $c_{\sigma,p}(f, r)$ be as

in (2). Since $h(K(f), \Lambda_c) = 1$, and it holds the relation (4.10.3) also in this case, we have as in the same way as above.

$$t_h = - \sum_f \frac{\omega_h(f)}{[\Lambda_0^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p)$$

, where f runs through the same set as in (3) of Th. 2, and $\omega_h(f)$ is given by (4.7.1). For $p = q$, we note the following.

$$(4.10.13) \qquad c_{\sigma,q}(f, 0) = \left(1 + \sum_{i \neq 1} \frac{\lambda_i(\alpha) + \lambda_i(\beta)}{2}\right) \sum_{0 \leqslant m \leqslant \delta} [\Lambda_q(0)^\times : \Lambda_q(m)^\times]$$

, where δ is the integer such that $Z_q[\hat{X}] = \Lambda_q(\delta)$, and α and β are the roots of the equation $f(X) \equiv 0 \bmod. q$.

(4) t_p. Assume σ is a square and put $N\sigma = a^2$, and $f(X) = (X - a)^2$. By Prop. 3.11, 3.20,

$$\bar{a} \prod_{p \neq q} c_{\sigma,p}(f, 0, \Lambda(a/\bar{a})_p) = a \quad .$$

By Prop. 3.28 we obtain

$$t_p = - \frac{1}{2} (1 + \sum_{i \neq 1} \chi_i(a))\, a = - \frac{\ell}{2} a$$

, since $\chi_i(a) = 1$ for all i, $1 \leqslant i \leqslant \ell$.

Thus we obtain the following.

<u>Theorem 2'</u>. Let the notation and the assumption be as in Th. 2 and let $c_{\sigma,p}(\alpha, r)$, $c_{\sigma,p}(f, r)$ be as above. The trace $\operatorname{tr} T_S(T(\sigma))$ is given by

$$\mathrm{tr}\, T_S(T(\alpha)) = t_v + t_e + t_h + t_p$$

, where t_v, t_e, t_h and t_p are given as follows.

(1) t_v. If $N\alpha$ is not a square, $t_v = 0$. If $N\alpha$ is a square, put $N\alpha = a^2$ with a positive integer a, then we have

$$t_v = \frac{\kappa - 1}{4\pi} \prod_p c_{\sigma,p}(\alpha, r_p)\, v(H/SL_2(Z))$$

(2) t_e. Let $\omega_e(f)$ be as in (2) of Th. 2. Λ_o be the maximal order of $K(f)$, and $h(K(f))$ be its class number. Then we have

$$t_e = -\frac{1}{2} \sum_f \omega_e(f) \frac{h(K(f))}{[\Lambda_o^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p)$$

, where f runs through the same set as in (2) of Th. 2.

(3) t_h. Let $\omega_h(f)$ be as in (3) of Th. 2 and Λ_o be as in (2). Then we have

$$t_h = - \sum_f \frac{\omega_h(f)}{[\Lambda_o^\times : E_Q]} \prod_p c_{\sigma,p}(f, r_p)$$

, where f runs the same set as in (3) of Th. 2.

(4) t_p. If α is not a square, $t_p = 0$. If α is a square, put $N\alpha = a^2$ with a positive integer a. Then we have

$$t_p = -\frac{\ell}{2} a \quad .$$

§5. Main result

5.1. In this section, we shall prove our main result Th.3 using the result of §3 and §4. We use the same notation as in §1 and §4. In particular F is a totally real algebraic number field which satisfies the condition (1), (2), (3) and (4) of 4.1. Let $R(\mathfrak{U}_F, GL_2(F_A))$ be the Hecke ring with respect to $GL_2(F_A)$ and $\mathfrak{U}_F$ as in §1, and $R^0(\mathfrak{U}_F, GL_2(F_A))$ be its subring generated by the double cosets $\mathfrak{U}_F \alpha \mathfrak{U}_F$ with $\alpha \in GL_2(F_A)$ such that $\alpha_{\mathfrak{q}} \in GL_2(\mathcal{O}_{\mathfrak{q}})$, where $\alpha_{\mathfrak{q}}$ is the $\mathfrak{q}$-component of α, and $\mathfrak{q}$ is the prime factor of the conductor q. Let $R(\mathfrak{U}_Q, GL_2(Q_A))$ be the Hecke ring with respect to $GL_2(Q_A)$ and $\mathfrak{U}_Q$, where $\mathfrak{U}_Q = \prod_p GL_2(Z_p) \times GL_2(R)$. We denote as above by $R^0(\mathfrak{U}_Q, GL_2(Q_A))$ the subring of $R(\mathfrak{U}_Q, GL_2(Q_A))$ generated by the double cosets $\mathfrak{U}_Q \alpha \mathfrak{U}_Q$ with $\alpha \in GL_2(Q_A)$ such that $\alpha_q \in GL_2(Z_q)$, where α_q is the q-component of α. Now let's define a homomorphism λ from $R(\mathfrak{U}_F, GL_2(F_A))$ to $R(\mathfrak{U}_Q, GL_2(Q_A))$ in the following way. For a prime iedal $\mathfrak{z}'$ of F, let $T(\mathfrak{z}'^m)$ be as in 1.3, and let $T(\mathfrak{z}', \mathfrak{z}')$ denote the double coset $\mathfrak{U}_F \alpha \mathfrak{U}_F$ such that $\alpha_{\mathfrak{z}} \in GL_2(\mathcal{O}_{\mathfrak{z}})$ for prime ideals $\mathfrak{z} \neq \mathfrak{z}'$ and $\alpha_{\mathfrak{z}'} = \begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$, where π is a prime element of $\mathcal{O}_{\mathfrak{z}'}$. For a prime p', we denote by $T(p'^m)$ the sum of all $\mathfrak{U}_Q \alpha \mathfrak{U}_Q$ such that the right $M_2(Z)$-ideal $\bigcap_p \alpha_p M_2(Z_p)$ is integral and of the norm p'^m, and by $T(p', p')$ the double coset $\mathfrak{U}_Q \alpha \mathfrak{U}_Q$ such that $\alpha_p \in GL_2(Z_p)$

for $p \neq p'$ and $\alpha_{p'} = \begin{pmatrix} p' & 0 \\ 0 & p' \end{pmatrix}$ for $p = p'$. We define an element $U(\mathfrak{a})$ for an integral ideal $\mathfrak{a}$. For a prime ideal $\mathfrak{p}$, following [11], we put

$$U(\mathfrak{p}^m) = \begin{cases} 2T(\mathfrak{o}) & , m = 0 \\ T(\mathfrak{p}) & , m = 1 \\ T(\mathfrak{p}^m) - N\mathfrak{p}T(\mathfrak{p}, \mathfrak{p})T(\mathfrak{p}^{m-2}) & , m \geqslant 2 \end{cases}$$

and for a integral ideal $\mathfrak{a}$, we put

$$U(\mathfrak{a}) = \prod_i U(\mathfrak{p}_i^{e_i})$$

, where $\mathfrak{a} = \prod_i \mathfrak{p}_i^{e_i}$. For a positive integer a, we define an element $U(a)$ of $R(\mathfrak{N}_Q, GL_2(Q_A))$ as above. Namely for a prime p, put

$$U(p^m) = \begin{cases} 2T(1) & , m = 0 \\ T(p) & , m = 1 \\ T(p^m) - pT(p, p)T(p^{m-2}) & , m \geqslant 2 \end{cases}$$

and for a positive integer a, put

$$U(a) = \prod_i U(p_i^{e_i})$$

, where $a = \prod_i p_i^{e_i}$. Then we see $U(\mathfrak{p}^m)$ (resp. $U(p^m)$) satisfies the following relation.

$$(5.1.1) \qquad U(\mathfrak{p}^m)U(\mathfrak{p}^n) = U(\mathfrak{p}^{m+n}) + (N\mathfrak{p}T(\mathfrak{p}, \mathfrak{p}))^n U(\mathfrak{p}^{m-n})$$

$$(\text{resp. } U(p^m)U(p^n) = U(p^{m+n}) + (pT(p, p))^n U(p^{m-n}) \;)$$

for $m \geq n \geq 1$.

If we put $\lambda(T(\mathfrak{p}, \mathfrak{p})) = T(N\mathfrak{p}, N\mathfrak{p})$ and $\lambda(U(\mathfrak{p}^m)) = U(N\mathfrak{p}^m)$ for a prime ideal $\mathfrak{p}$ of F, then we see that λ can be extended uniquely to a ring homomorphism from $R(\mathcal{U}_F, GL_2(F_A))$ to $R(\mathcal{U}_Q, GL_2(Q_A))$ and then $\lambda(R^o(\mathcal{U}_F, GL_2(F_A))) \subset R^o(\mathcal{U}_Q, GL_2(Q_A))$.

5.2. In § 1 we defined a representation T_S of $R(\mathcal{U}_F, GL_2(F_A))$ in the space $SS_\kappa(\Gamma)$. We will consider the other spaces of cusp forms of one variable and the representations of $R^o(\mathcal{U}_F, GL_2(F_A))$ in those spaces.

We consider the spaces of cusp forms $S_\kappa(SL_2(Z))$ and $S_\kappa(\Gamma_0(q), \chi_i)$, $i \geq 2$, given as follows. We denote by $S_\kappa(SL_2(Z))$ the space of all holomorphic functions on H which satisfies the followings ; (i) $f(gz) = (cz+d)^\kappa f(z)$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z)$, (ii) $f(z)$ vanishes at all cusps of $SL_2(Z)$. Put

$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \;\middle|\; c \equiv 0 \bmod . q \right\}$ and denote by $S_\kappa(\Gamma_0(q), \chi_i)$ for $i \geq 2$ the space of all holomorphic functions on H which satisfies (i) $f(gz) = \chi_i(a)(cz+d)^\kappa f(z)$ for all $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(q)$ and (ii) $f(z)$ vanishes at all cusps of $\Gamma_0(q)$. Put $GL_2(Q)_+ = \{ g \in GL_2(Q) \mid \det g > 0 \}$, and for a function $f(z)$ on H and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(Q)_+$, put

$$f|[g] = \frac{f(z)}{(cz+d)^{\kappa}} (\det g)^{\kappa/2} \quad .$$

The Hecke ring $R(\mathcal{U}_Q, GL_2(Q_A))$ acts on $S_\kappa(SL_2(Z))$ in the following way. For a double coset $\mathcal{U}_Q \alpha \mathcal{U}_Q$ with $\alpha \in GL_2(Q_A)$, let $\mathcal{U}_Q \alpha \mathcal{U}_Q \cap GL_2(Q)_+ = \bigcup_{\nu=1}^{d} \alpha_\nu \Gamma$ be a disjoint union. For $f \in S_\kappa(SL_2(Z))$, put

$$T_1(\mathcal{U}_Q \alpha \mathcal{U}_Q) f = \sum_{\nu=1}^{d} f|[\alpha_\nu^{-1}] \quad .$$

Then by linearlity T_1 can be extended to a homomorphism from $R(\mathcal{U}_Q, GL_2(Q_A))$ to the ring of endomorphisms of $S_\kappa(SL_2(Z))$. To define an action of $R^0(\mathcal{U}_Q, GL_2(Q_A))$ on $S_\kappa(\Gamma_0(q), \chi_i)$, we put $\overline{\mathcal{U}}_Q = \prod_{p \neq q} GL_2(Z_p) \times \Lambda_q^\times \times GL_2(R)$, where $\Lambda_q = \begin{pmatrix} Z_q & Z_q \\ qZ_q & Z_q \end{pmatrix}$ and we consider the Hecke ring $R(\overline{\mathcal{U}}_Q, GL_2(Q_A))$. If we denote by $R^0(\overline{\mathcal{U}}_Q, GL_2(Q_A))$ the subring of $R(\overline{\mathcal{U}}_Q, GL_2(Q_A))$ generated by the double cosets $\overline{\mathcal{U}}_Q \alpha \overline{\mathcal{U}}_Q$ such that $\alpha_q \in \Lambda_q^\times$, then $R^0(\overline{\mathcal{U}}_Q, GL_2(Q_A))$ and $R^0(\mathcal{U}_Q, GL_2(Q_A))$ are isomorphic to each other by the correspondence $\overline{\mathcal{U}}_Q \alpha \overline{\mathcal{U}}_Q \longrightarrow \mathcal{U}_Q \alpha \mathcal{U}_Q$. Assume $\mathcal{U}_Q \alpha \mathcal{U}_Q$ corresponds to $\overline{\mathcal{U}}_Q \alpha \overline{\mathcal{U}}_Q$, and let $\overline{\mathcal{U}}_Q \alpha \overline{\mathcal{U}}_Q \cap GL_2(Q)_+ = \bigcup_{\nu=1}^{d} \alpha_\nu \Gamma_0(q)$ be a disjoint union. For $f \in S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$, put

$$T_i(\mathfrak{n}_Q \alpha \mathfrak{n}_Q) f = \sum_{\nu=1}^{d} \chi_i(a_\nu) f \,|\, [\alpha_\nu^{-1}]$$

, where $\alpha_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu \end{pmatrix}$. Then by linearlity T_i can be extended to a homomorphism from $R^0(\mathfrak{n}_Q, GL_2(Q_A))$ to the ring of endomorphisms of $S_\kappa(\Gamma_0(q), \chi_i)$. Hence connecting T_i with λ, we obtain representations of $R^0(\mathfrak{n}_F, GL_2(F_A))$ in the spaces $S_\kappa(SL_2(Z))$ and $S_\kappa(\Gamma_0(q), \chi)$. It is known $T_1(e)$ for $e \in R(\mathfrak{n}_Q, GL_2(F_A))$ (resp. $T_i(e)$, $i \geqslant 2$, for $e \in R^0(\mathfrak{n}_Q, GL_2(F_A))$) is a normal operator in the space $S_\kappa(SL_2(Z))$ (resp. $S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$), and $S_\kappa(SL_2(Z))$ (resp. $S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$) has a basis consisting of common eigen-functions for all $T_1(e)$, $e \in R(\mathfrak{n}_Q, GL_2(Q_A))$ (resp. $T_i(e)$, $i \geqslant 2$, $e \in R^0(\mathfrak{n}_Q, GL_2(Q_A))$).

5.3. In 5.3 and 5.4, we will give formulas for $\operatorname{tr} T_S(U(\mathfrak{a}))$ and $\operatorname{tr} T_i(\lambda(U(\mathfrak{a})))$. For a prime ideal $\mathfrak{z} \neq \mathfrak{q}$, $T_S(U(\mathfrak{z})) = T_S(U({}^\sigma\mathfrak{z}))$ and $T_i(\lambda(U(\mathfrak{z}))) = T_i(\lambda(U({}^\sigma\mathfrak{z})))$ for i, $1 \leqslant i \leqslant \ell$, hence it is enough to calculate $\operatorname{tr} T_S(U(\mathfrak{a}))$ and $\operatorname{tr} T_i(\lambda(U(\mathfrak{a})))$ for integral ideals $\mathfrak{a}$ such that $\mathfrak{a}$ is prime to $\mathfrak{q}$ and is devided by at most one prime factor of p in F for any prime $p \neq q$. In the following we assume $\mathfrak{a}$ satisfies the above condition and let r_p be the integers such that $\Xi(\mathfrak{a})_A = \prod_p \Xi_p(r_p) \times GL_2(F_\infty)$. For a prime ideal $\mathfrak{z}$, $\operatorname{tr} T_1(\lambda(U(\mathfrak{z}^m)))$

is already given in [11].

To deduce $\mathrm{tr}\, T_S(U(\mathfrak{a}))$ from the formula in Th. 2', first we prove the following Lemma 5.1. For a polynomial $f(X) = X^2 - sX + n$ in $Z[X]$ and a positive integer N, we denote by $f_N(X)$ the polynomial $N^{-2}f(NX)$. For a prime p, we call f primitive at p if f_p is not contained in $Z[X]$.

<u>Lemma 5.1</u>. Let the notation be as above and as in §4 and for a non-negative integer r, $c_{\sigma,p}(\alpha, r)$ and $c_{\sigma,p}(f, r)$ be as in 4.10. For a prime p different from q, let $\mathfrak{z}$ denote a prime factor of p in F.

(i) Assume $N\mathfrak{a}$ is a square, and put $N\mathfrak{a} = a^2$ with a positive integer a.

(a) For $p \neq q$ with $r_p \geqslant 1$, we have

$$c_{\sigma,p}(a, r_p) - N\mathfrak{z}\, c_{\sigma,p}(N\mathfrak{z}^{-1}a, r_p-2)$$

$$= \begin{cases} 1 - p & , r_p \text{ is even} \\ -(1 - p) & , r_p \text{ is odd} \end{cases}$$

, where we set $c_{\sigma,p}(N^{-1}a, r_p-2) = 0$ if $r_p-2 < 0$, or $N\mathfrak{z}^{-1}a \notin Z_p$.

(b) For $p \neq q$ with $r_p = 0$, we have

$$c_{\sigma,p}(a, 0) = 1$$

(c) For $p = q$, we have

$$c_{\sigma,q}(a, 0) = \begin{cases} 1 + \frac{q+1}{2} \sum_{i \neq 2} \chi_i(a) & , \ell \neq 2 \\ \chi_2(a)\left(1 + \frac{q+1}{2}\chi_2(a)\right) & , \ell = 2 \end{cases} .$$

(ii) Let $f(X) = X^2 - sX + n$ be a polynomial in $Z[X]$ such that $n = N\mathfrak{a}$ and $s^2 - 4n \neq 0$.

(a) For $p \neq q$ with $r_p \geqslant 1$, we have

$$c_{\sigma,p}(f, r_p) - N\mathfrak{z}\, c_{\sigma,p}(f_{N\mathfrak{z}}, r_p-2) = \begin{cases} 1, & \text{if } K(f)\otimes Q_p \simeq Q_p \oplus Q_p \\ & \text{and } f \text{ is primitive at } p. \\ 1 - \left(\dfrac{K(f)/Q}{p}\right), & \text{otherwise}, \end{cases}$$

where we set $c_{\sigma,p}(f_{N\mathfrak{z}}, r_p-2) = 0$ if $r_p-2 < 0$ or $f_{N\mathfrak{z}} \notin Z_p[X]$, and $\left(\dfrac{K(f)/Q}{p}\right) = 1, -1, 0,$ according as $K(f)_p = K(f)\otimes Q_p$ is of type a), b), or c) in Remark 3.2.

(b) For $p \neq q$ with $r_p = 0$, we have

$$c_{\sigma,p}(f, 0) = \sum_{K(f)_p \supset \Lambda \supset Z_p[\tilde{X}]} [\Lambda_0^{\times} : \Lambda^{\times}],$$

where Λ_0 is the maximal order of $K(f)_p$ and Λ runs through all Z_p-orders of $K(f)_p$ which contain $\tilde{X}$.

(c) For $p = q$, let Λ_0 be the maximal order of $K(f)_q$, and α, β be the two roots of the equation $f(X) \equiv 0 \mod. q$. Then we have,

$$c_{\sigma,q}(f, 0) = \sum_{K(f)_q \supset \Lambda \supset Z_q[\tilde{X}]} \left(1 + \frac{1}{2}\left(1 + \left\{\frac{\Lambda}{q}\right\}\right)\left(\sum_{i\neq 1} \frac{\chi_i(\alpha)+\chi_i(\beta)}{2}\right)\right)[\Lambda_0^{\times} : \Lambda^{\times}],$$

where Λ runs through all Z_q-orders of $K(f)_q$ which contain $\tilde{X}$.

Proof. (i) The assertion (c) is nothing but the formula (4.10.12). The assertion (b) easily follows from Prop. 3.29, 3.30. We prove (a). The case where $r_p = 1$ can take place only for $\ell = 2$. For $\ell = 2$ and $r_p = 1$, by (4.10.7) we have

$$c_{\sigma,p}(a, 1) = -(1-p) \quad .$$

Assume $r_p \geqslant 2$. If p decomposes in F, by (4.10.3)

$$c_{\sigma,p}(a, r_p) = c_{\sigma,p}(N\mathfrak{z}^{-1}a, r_p-2) = 1$$

, hence we have

$$c_{\sigma,p}(a, r_p) - N\mathfrak{z}\, c_{\sigma,p}(N\mathfrak{z}^{-1}a, r_p-2) = 1 - p \quad .$$

If p remains prime in F, by (4.10.7) we have

$$c_{\sigma,p}(a, r_p) = \begin{cases} 1 + \sum_{1 \leqslant m \leqslant r_p/2} p^{\ell m-(\ell-1)}(p^{(\ell-1)} - 1) & , r_p \text{ even} \\ (p-1) \sum_{0 \leqslant m \leqslant [r_p/2]} p^{\ell m} & , r_p \text{ odd} \end{cases}$$

$$c_{\sigma,p}(N\mathfrak{z}^{-1}a, r_p-2) = \begin{cases} 1 + \sum_{1 \leqslant m \leqslant (r_p-2)/2} p^{\ell m-(\ell-1)}(p^{(\ell-1)} - 1) & , r_p \text{ even} \\ (p-1) \sum_{0 \leqslant m \leqslant [(r_p-2)/2]} p^{\ell m} & , r_p \text{ odd} \end{cases} ,$$

Hence we have

$$c_{\sigma,p}(a, r_p) - N\mathfrak{z}\, c_{\sigma,p}(N\mathfrak{z}^{-1}a, r_p-2) = \begin{cases} 1 - p & , r_p \text{ even} \\ -(1-p) & , r_p \text{ odd} \end{cases} .$$

Thus (i) is proved. (ii) The assertion (c) easily follows from (4.10.12) and (4.10.13), and the assertion (b) follows from Prop. 3.11, 3.16. We prove the assertion (a).

Let δ_1 and δ_2 be the non-negative integers defined in 3.9 for $k = Q_p$, that is to say, δ_1 is the maximal integer such that $p^{-\delta_1}\tilde{X}$ is integral, and δ_2 is the integer such that $Z_p[p^{-\delta_1}\tilde{X}] = \Lambda_p(\delta_2)$, where $\Lambda_p(0)$ is the maximal order of $K(f)_p$ and $\Lambda_p(m) = Z_p + p^m\Lambda_p(0)$ for a non-negative integer m. The polynomial $f(X)$ is primitive at p if and only if $\delta_1 = 0$, and $Z_p[\tilde{X}] = \Lambda_p(\delta_1+\delta_2)$. We note that if $f(X)$ is primitive at p, then $\delta_1 = \delta_2 = 0$. For if $f(X)$ is primitive at p, then we see $K(f)_p \simeq Q_p \oplus Q_p$, or $K(f)_p$ is a ramified extension of Q_p and $v_p(n) = 1$. In the former case, $Z_p[\tilde{X}] = \Lambda_p(0)$, and in the latter case, $\tilde{X}$ is a prime element in $K(f)_p$, hence $Z_p[\tilde{X}]$ is equal to the maximal order $\Lambda_p(0)$ also in this case. And we have proved $\delta_1 = \delta_2 = 0$. This shows that our assertion holds if $f(X)$ is primitive, since $f_{N\mathfrak{z}} \notin Z_p[X]$ and $c_{\sigma,p}(f, r, \Lambda_p(0)) = 1$. Hence in the following, we assume $f(X)$ is not primitive. We will prove our assertion according to the type of F_p. First assume F_p is the direct product of ℓ-copies of Q_p, then $N\mathfrak{z} = p$ and since $f(X)$ is not primitive, $f_p(X) \in Z_p[X]$. By Prop. 3.11, we have

$$c_{\sigma,p}(f, r_p) = \sum_{0 \leq m \leq \delta_1+\delta_2} [\Lambda_p(0)^\times : \Lambda_p(m)^\times]$$

and

$$c_{\sigma,p}(f_p, r_p-2) = \begin{cases} \sum_{0 \leq m \leq (\delta_1-1)+\delta_2} [\Lambda_p(0)^\times : \Lambda_p(m)^\times] & , r_p \geq 2 \\ 0 & , r_p = 1 \end{cases},$$

If $r_p = 1$, $f(X)$ is primitive at p. Our assertion has been proved in this case, and we assume $r_p \geqslant 2$. Since $[\Lambda_p(0)^\times : \Lambda_p(m)^\times] = p^m\left(1 - \frac{1}{p}\left(\frac{K(f)/Q}{p}\right)\right)$ for a positive integer m, we obtain

$$c_{\sigma,p}(f,\ r_p) - N\mathfrak{F}\, c_{\sigma,p}(f_{N\mathfrak{F}},\ r_p-2) = 1 - \left(\frac{K(f)/Q}{p}\right).$$

Nextly we assume F_p is the unramified extension of Q_p with $[F_p : Q_p] = \ell$. First assume $r_p = 1$. If $\left(\frac{K(f)/Q}{p}\right) = 1$, $c_{\sigma,p}(f,\ 1)=0$, or $f(X)$ is primitive, and it holds our assertion. The case where $\left(\frac{K(f)/Q}{p}\right) = -1$ can occur only if $\ell = 2$. For $\ell = 2$, we see $c_{\sigma,p}(f,\ 1) = 2 = 1 - \left(\frac{K(f)/Q}{p}\right)$ by Prop. 3.18. If $\left(\frac{K(f)/Q}{p}\right) = 0$, then $\delta_1 = \frac{\ell-1}{2}$ for $\ell \neq 2$ and $\delta_1 = 2$ for $\ell = 2$. By Prop. 3.19, we see $c_{\sigma,p}(f,\ 1) = 1 = 1 - \left(\frac{K(f)/Q}{p}\right)$.

Now let's consider the case where $r_p \geqslant 2$. If $\left(\frac{K(f)/Q}{p}\right) = 1$ and $f(X)$ does not satisfies the condition in i) of Prop. 3.17, then we see $c_{\sigma,p}(f,\ r_p) = c_{\sigma,p}(f_{N\mathfrak{F}},\ r_p-2) = 0$, and our assertion holds in this case. Hence in the following, we assume f satisfies the condition in i) of Prop. 3.17 if $\left(\frac{K(f)/Q}{p}\right) = 1$. Under this assumption, $f_{N\mathfrak{F}}(X)$ is integral if $f(X)$ is not primitive at p. First assume it holds neither of the following two conditions ; (1) $\ell = 2$ and $\left(\frac{K(f)/Q}{p}\right) = -1$, (2) ℓr_p is odd and $\left(\frac{K(f)/Q}{p}\right) = 0$. Then by Prop. 3.17, 3.18, and 3.19, we have

$$c_{\sigma,p}(f, r_p) = 1 + \sum_{1 \leq m \leq \delta_1/\ell} N\mathfrak{z}^m\left(1 - \frac{1}{N\mathfrak{z}}\left(\frac{K(f)/Q}{p}\right)\right)$$
$$+ (N\mathfrak{z})^{\delta_1/\ell} \sum_{1 \leq m \leq \delta_2} p^m \left(1 - \frac{1}{p}\left(\frac{K(f)/Q}{p}\right)\right)$$

and

$$c_{\sigma,p}(f_{N\mathfrak{z}}, r_p-2) = 1 + \sum_{1 \leq m \leq (\delta_1-\ell)/\ell} N\mathfrak{z}^m \left(1 - \frac{1}{N\mathfrak{z}}\left(\frac{K(f)/Q}{p}\right)\right)$$
$$+ (N\mathfrak{z})^{(\delta_1-\ell)/\ell} \sum_{1 \leq m \leq \delta_2} p^m \left(1 - \frac{1}{p}\left(\frac{K(f)/Q}{p}\right)\right) .$$

From these formulas, we obtain

$$c_{\sigma,p}(f, r_p) - N\mathfrak{z}\, c_{\sigma,p}(f_{N\mathfrak{z}}, r_p-2) = 1 - \left(\frac{K(f)/Q}{p}\right) .$$

In the case (i), since $[\Lambda_p(0)^\times : \Lambda_p(m)^\times] = p^m(1 + 1/p)$ for a positive integer m, we have by Prop. 3.18

$$c_{\sigma,p}(f, r_p) = \begin{cases} \delta_1 + 1 + \sum_{1 \leq m \leq (\delta_1-1)/2} (\delta_1 - 2m + 1)N\mathfrak{z}^m(1 - 1/N\mathfrak{z}) & , \delta_1 \text{ is odd} \\ \delta_1 + 1 + \sum_{1 \leq m \leq \delta_1/2} (\delta_1 - 2m + 1)N\mathfrak{z}^m(1 - 1/N\mathfrak{z}) \\ \quad + N\mathfrak{z}^{\delta_1/2} \sum_{1 \leq m \leq \delta_2} p^m(1 + 1/p) & , \delta_1 \text{ is even} \end{cases}$$

and

$$c_{\sigma,p}(f_p,\ r_p-2) = \begin{cases} (\delta_1-2)+1+\sum_{1\leq m\leq((\delta_1-2)+1)/2}((\delta_1-2)-2m+1)N\mathfrak{z}^m(1-1/N\mathfrak{z}) & ,\ \delta_1-2 \text{ is odd} \\ (\delta_1-2)+1+\sum_{1\leq m\leq(\delta_1-2)/2}((\delta_1-2)-2m+1)N\mathfrak{z}^m(1-1/N\mathfrak{z}) & \\ \quad + N\mathfrak{z}^{(\delta_1-2)/2}\sum_{1\leq m\leq\delta_2}p^m(1+\frac{1}{p}) & ,\ \delta_1 \text{ is even} \end{cases}$$

Since we have

$$\sum_{1\leq m\leq[\delta_1/2]}(\delta_1-2m+1)N\mathfrak{z}^m(1-1/N\mathfrak{z}) - N\mathfrak{z}\sum_{1\leq m\leq[(\delta_1-2)/2]}((\delta_1-2)-2m+1)N\mathfrak{z}^m(1-1/N\mathfrak{z})$$

$$= (\delta_1-1)(N\mathfrak{z}-1)$$

, we obtain

$$c_{\sigma,p}(f,\ r_p) - N\mathfrak{z}\, c_{\sigma,p}(f_{N\mathfrak{z}},\ r_p) = 2 = 1-\left(\frac{K(f)/Q}{p}\right) .$$

Now we consider the case (ii). In this case by Prop. 3.19 we have

$$c_{\sigma,p}(f,\ r_p) = 1+\sum_{1\leq m\leq(2\delta_1+1-\ell)/2\ell}N\mathfrak{z}^m$$

and

$$c_{\sigma,p}(f_{N\mathfrak{z}},\ r_p-2) = 1+\sum_{1\leq m\leq(2(\delta_1-\ell)+1-\ell)/2\ell}N\mathfrak{z}^m$$

, hence we obtain

$$c_{\mathfrak{r},p}(f,\ r_p) - N\mathfrak{z}\, c_{\mathfrak{r},p}(f_{N\mathfrak{z}},\ r_p-2) = 1 = 1 - \left(\frac{K(f)/Q}{p}\right) .$$

Thus (ii) is proved completely.

By the above lemma, we can deduce the following formula for $\mathrm{tr}\ T_S(U(\mathfrak{a}))$.

Proposition 5.2. Let the notation and the assumption be as above. Let $\mathfrak{a}$ be an integral ideal of F such that $\mathfrak{a}$ is prime to $\mathfrak{q}$ and is divided by at most one prime factor of p for any prime p. Assume κ is even and $\geqslant 4$, then $\mathrm{tr}\ T_S(U(\mathfrak{a}))$ for $\mathfrak{a} \neq \mathfrak{O}$ (resp. $\mathfrak{a} = \mathfrak{O}$) is given by the following formula.

$$\mathrm{tr}\ T_S(U(\mathfrak{a})) = t_v + t_e + t_h + t_p$$

$$(\text{resp. } \frac{1}{2}\,\mathrm{tr}\ T_S(U(\mathfrak{a})) = t_v + t_e + t_h + t_p\ )$$

, where t_v, t_e, t_h and t_p are given as follows.

(1) t_v. For a positive integer N, put $\delta(N) = 1$ or 0 according as N is a square or not. Then we have

$$t_v = \delta(N\mathfrak{a})\,\frac{\kappa-1}{4\pi}\, v(H/SL_2(Z)) \prod_{p\,|\,N\mathfrak{a}} (1 - p) \left(1 + \frac{q+1}{2} \sum_{i\neq 1} \chi_i(\sqrt{N\mathfrak{a}})\right)$$

(2) t_e. Let $\left\{\frac{\ }{q}\right\}$ be as in Th. 2, $\omega_e(f)$ be as in (2) of Th. 2, and α, β be the roots of the equation $f(X) \equiv 0 \bmod. q$. Then we have

$$t_e = -\frac{1}{2}\sum_f \omega_e(f) \prod_{\substack{p \mid N\mathfrak{a} \\ \left(\frac{K(f)/Q}{p}\right) \neq 1}} \left(1 - \left(\frac{K(f)/Q}{p}\right)\right)$$

$$\times \sum_{K(f) \supset \Lambda \supset Z[\tilde{X}]} \left(1 + \left(\frac{1 + \left\{\frac{\Lambda q}{q}\right\}}{2}\right)\left(\sum_{i \neq 1} \frac{\chi_i(\alpha)+\chi_i(\beta)}{2}\right)\right) \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]}$$

, where f runs through all polynomials $X^2 - sX + n \in Z[X]$ which satisfy (i) $s^2 - 4n < 0$, $n = N\mathfrak{a}$, (ii) $f(X)$ is primitive at every prime p such that $\left(\frac{K(f)/Q}{p}\right) = 1$, and Λ runs through all Z-orders of $K(f)$ which satisfy (i) $\Lambda \supset Z[\tilde{X}]$, (ii) $\Lambda_p = \Lambda \otimes_Z Z_p$ is the maximal order of $K(f)_p$ for all primes p which divide $N\mathfrak{a}$.

(3) t_h. Let $\omega_h(f)$ be as in (3) of Th. 2, and let α and β be the roots of the equation $f(X) \equiv 0 \mod. q$. Then we have

$$t_h = \begin{cases} 0 & , \mathfrak{a} = \mathfrak{o} \\ -\sum_f \omega_h(f)\left(1 + \sum_{i \neq 1} \frac{\chi_i(\alpha)+\chi_i(\beta)}{2}\right) \sum_{K(f) \supset \Lambda \supset \tilde{X}} \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} & , \text{otherwise} \end{cases}$$

, where f runs through all the polynomials $X^2 - sX + n \in Z[X]$ which satisfy (i) $s^2 - 4n$ is a non-zero square , (ii) $f(X)$ is primitive at all p which divide $N\mathfrak{a}$, and Λ runs through all Z-orders of $K(f)$ which satisfy (i) $\Lambda \supset Z[\tilde{X}]$, (ii) Λ_p is the maximal order of $K(f)_p$ for all primes p which divide $N\mathfrak{a}$.

(4) t_p. We have

$$t_p = \begin{cases} -\frac{\ell}{2} & , \mathfrak{a} = \mathfrak{o} \\ 0 & , \text{otherwise} \end{cases} ,$$

Proof. For $\mathfrak{a} = \mathfrak{o}$, we note that any polynomial $f(X) = X^2 - sX + n \in Z[X]$ with $n = N\mathfrak{o} = 1$ is primitive at all primes, and we can easily verify our assertion for $\mathfrak{a} = \mathfrak{o}$ by Th. 2' and the result of § 3. For an integral ideal $\mathfrak{a} \neq \mathfrak{o}$, put $\mathfrak{a} = \prod_{i=1}^{N} \mathfrak{z}_i^{e_i}$ with prime ideals $\mathfrak{z}_i$ of F and positive integers e_i. We denote by p_i the prime which divide $N\mathfrak{z}_i$, then $p_i \neq p_j$ if $i \neq j$ by the assumption on $\mathfrak{a}$. We denote by I the set of indeces of $\mathfrak{z}_i$'s, i.e. $I = \{1, \ldots, N\}$, and for a subset J of I, let p(J) denote the set of primes $\{p_i \mid i \in J\}$. For a subset J of I, we denote the integral ideal $\prod_{i \in J} \mathfrak{z}_i$ also by J. Then by the definition of $U(\mathfrak{a})$,

$$U(\mathfrak{a}) = \sum_{J \subset I} (-1)^{|J|} NJT(J, J)T(\mathfrak{a}J^{-2}) .$$

Here we put $T(\mathfrak{a}J^{-2}) = 0$ if $\mathfrak{a}J^{-2}$ is not integral ,and $T(J, J) = \prod_{i \in J} T(\mathfrak{z}_i, \mathfrak{z}_i)$. Hence we have

$$\operatorname{tr} T_S(U(\mathfrak{a})) = \sum_{J \subset I} (-1)^{|J|} NJ \operatorname{tr} T_S(T(\mathfrak{a}J^{-2})) .$$

We denote the contribution of the terms t_v (resp. t_e, t_h, t_p) in Th. 2' to $\operatorname{tr} T_S(U(\mathfrak{a}))$ also by t_v (resp. t_e, t_h, t_p).

(1) t_v. For a subset J of I, $N\mathfrak{a}$ is a square if and only if $N(\mathfrak{a}J^{-2})$ is a square. Hence if $N\mathfrak{a}$ is not a square, $t_v = 0$. Assume $N\mathfrak{a}$ is a square, and put $N\mathfrak{a} = a^2$ with a positive integer a. Then by Th. 2' we have

$$t_v = \frac{\kappa-1}{4\pi} v(H/SL_2(Z)) \sum_{J\subset I} (-1)^{|J|} NJ \prod_{p\notin p(J)} c_{\sigma,p}(aNJ^{-1}, r_p)$$

$$\times \prod_{p\in p(J)} c_{\sigma,p}(aNJ^{-1}, r_p-2) \quad .$$

Here we put $c_{\sigma,p}(aNJ^{-1}, r_p-2) = 0$ if $aNJ^{-1}\notin Z_p$ or $r_p-2<0$. By (4.10.1) we have for $p\notin p(J)$, $\neq q$,

$$c_{\sigma,p}(aNJ^{-1}, r_p) = c_{\sigma,p}(a, r_p)$$

and for $p = q$ by (4.10.2) and the assumption on F, we have

$$c_{\sigma,q}(aNJ^{-1}, 0) = c_{\sigma,q}(a, 0) \quad .$$

For $p\in p(J)$, let $\mathfrak{z}$ denote a prime factor of p, then by (4.10.1)

$$c_{\sigma,p}(aNJ^{-1}, r_p) = c_{\sigma,p}(aN\mathfrak{z}^{-1}, r_p) \quad .$$

Hence we see

$$t_v = \frac{\kappa-1}{4\pi} v(H/SL_2(Z)) \sum_{J\subset I} (-1)^{|J|} NJ \prod_{p\notin p(J)} c_{\sigma,p}(a, r_p)$$

$$\times \prod_{p\in p(J)} N\mathfrak{z}\, c_{\sigma,p}(aN\mathfrak{z}^{-1}, r_p-2)$$

$$= \frac{\kappa-1}{4\pi} v(H/SL_2(Z)) \prod_{p \notin p(I)} c_{\sigma,p}(a, r_p)$$

$$\times \prod_{p \in p(I)} \left(c_{\sigma,p}(a, r_p) - N\mathfrak{z}\, c_{\sigma,p}(aN\mathfrak{z}^{-1}, r_p - 2) \right) .$$

For $p \notin p(I)$, $r_p = 0$, hence by (i) (b) of Lemma 5.1,

$$\prod_{\substack{p \notin p(I) \\ p \neq q}} c_{\sigma,p}(a, r_p) = 1 ,$$

We note for $\ell = 2$, $\chi_2(a)$ is 1 or -1 according as the ordinary of the set $\{ p \mid r_p \text{ odd} \}$ is even or odd. Hence by (i) of Lemma 5.1 we have

$$c_{\sigma,q}(a, 0) \prod_{p \in p(I)} \left(c_{\sigma,p}(a, r_p) - N\mathfrak{z}\, c_{\sigma,p}(aN\mathfrak{z}^{-1}, r_p - 2) \right)$$

$$= \prod_{p \in p(I)} (1 - p) \times \left(1 + \frac{q+1}{2} \sum_{i \neq 1} \chi_i(a) \right) .$$

Thus we obtain

$$t_v = \frac{\kappa-1}{4\pi} v(H/SL_2(Z)) \prod_{p \in p(I)} (1 - p) \left(1 + \frac{q+1}{2} \sum_{i \neq 1} \chi_i(a) \right) .$$

(2) t_e. Let $\omega_e(f)$ be as in (2) of Th.2, then $\omega_e(f) = \omega_e(f_N)$ for all positive integers N. By the same argument as above and the relations (4.10.10) and (4.10.11), we see in the notation of Th.2',

$$(5.3.1) \qquad t_e = -\frac{1}{2} \sum_f \omega_e(f) \prod_{p \notin p(I)} c_{\sigma,p}(f, r_p)$$

$$\times \prod_{p \in p(I)} \left(c_{\sigma,p}(f, r_p) - N\mathfrak{z}\, c_{\sigma,p}(f_{N\mathfrak{z}}, r_p - 2) \right) \times \frac{h(K(f))}{[\Lambda_0^\times : E_Q]}$$

, where f runs through all the polynomials $f(X) = X^2 - sX + n$ in $Z[X]$ which satisfy $n = N\mathfrak{n}$ and $s^2 - 4n < 0$, and we set $c_{\sigma,p}(f_{N\mathfrak{z}}, r_p - 2) = 0$ if $f_{N\mathfrak{z}}$ is not integral or $r_p - 2 < 0$. By Lemma 5.1, (2), we have

$$(5.3.2) \qquad \prod_{p \in p(I)} \left(c_{\sigma,p}(f, r_p) - N\mathfrak{z}\, c_{\sigma,p}(f_{N\mathfrak{z}}, r_p - 2) \right)$$

$$= \begin{cases} \displaystyle\prod_{\substack{p \in p(I) \\ \left(\frac{K(f)/Q}{p}\right) \neq 1}} \left(1 - \left(\frac{K(f)/Q}{p} \right) \right) & \text{, if } f \text{ is primitive at all } p \text{ with } \left(\frac{K(f)/Q}{p}\right) = 1 \\ 0 & \text{, otherwise} \end{cases}$$

, and by (4.10.9) and Lemma 5.1, (2), we have

$$(5.3.3) \qquad \prod_{p \notin p(I)} c_{\sigma,p}(f, r_p) = \sum_{K(f) \supset \Lambda \supset Z[\tilde{X}]} \left(1 + \left(\frac{1 + \left\{\frac{\Lambda q}{q}\right\}}{2} \right) \right.$$

$$\left. \times \left(\sum_{i \neq 1} \frac{\chi_i(\alpha) + \chi_i(\beta)}{2} \right) \right) \times \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} \times \frac{[\Lambda_0^\times : E_Q]}{h(K(f))}$$

, where α and β are the roots of the equation $f(X) \equiv 0 \bmod. q$, and Λ runs through all Z-orders of $K(f)$ which satisfy (i) $\Lambda \ni \tilde{X}$ and (ii) Λ_p is the maximal order of $K(f)_p$ for all p dividing N. By (5.3.1), (5.3.2) and (5.3.3), we obtain our assertion for t_e.

(3) t_h. Let $\omega_h(f)$ be as in (3) of Th. 2, then it holds also in this case that $\omega_h(f) = \omega_h(f_N)$ for any positive

integer N. We can prove our assertion for t_h in the same way as for t_e, and we omit the details.

(4) t_p. If $\mathfrak{n}$ is not a square, the ideal $\mathfrak{n}J^{-1}$ is not a square for any subset J of I, and we see $t_p = 0$ in this case. If $\mathfrak{n}$ is a square, it holds $r_p \geq 2$ for all $p \in p(I)$. By the same way as above, we obtain

$$t_p = -\frac{\ell}{2} \prod_{i \in I} (N\mathfrak{z}_i^{e_i} - N\mathfrak{z}\, N\mathfrak{z}_i^{e_i - 1})$$

$$= 0 \ .$$

Thus our proposition is proved completely.

5.4. Let N be a positive integer, the explicit formula for $\mathrm{tr}\, T_i(T(N))$ is known by M. Eichler [3], [4], [5] and H. Hijikata [8]. We quote the result of them in a convenient form. Let p be a prime and f be a polynomial $f(X) = X^2 - sX + n \in Z_p[X]$ with $s^2 - 4n \neq 0$. For a non-negative integer m, we denote by $\Lambda_p(m)$ the order $\Lambda_K(m)$ in 3.9 for $K = Q_p[X]/(f)$, and by v_p the valuation of Q_p given by $v_p(p) = 1$. For a non-negative integer r, put for $p \neq q$,

$$c_p(f, r) = \begin{cases} \displaystyle\sum_{K_p \supset \Lambda_p(m) \supset Z_p[\tilde{X}]} [\Lambda_p(0)^\times : \Lambda_p(m)^\times] & , \text{ if } v_p(n) = r \\ 0 & , \text{ otherwise} \end{cases}$$

, where $K_p = Q_p[X]/(f)$. For $p = q$, let α, β be the roots of the equation $f(X) \equiv 0 \bmod. q$, and put

$$c_q(f,\ 0) = \begin{cases} \sum_{K_q \supset \Lambda_q(m) \supset Z_q[\tilde{X}]} [\Lambda_q(0)^\times : \Lambda_q(m)^\times] & \text{, if } v_q(n)=0 \text{ and } i=1 \\ \sum_{K_q \supset \Lambda_q(m) \supset Z_q[\tilde{X}]} \left(1 + \left\{\frac{\Lambda_q(m)}{q}\right\}\right) \frac{\left(\chi_i(\alpha) + \chi_i(\beta)\right)}{2} [\Lambda_q(0)^\times : \Lambda_q(m)^\times] & \text{, if } v_q(n)=0 \text{ and } 2 \leqslant i \leqslant \ell \\ 0 & \text{, otherwise} \end{cases}$$

, where $K_q = Q_q[X]/(f)$. Then we have by [3], [4], [5] and [8],

Theorem 5.3. Let the notation be as above. Let N be a positive integer prime to q, and put $s_p = v_p(N)$. Then we have

$$\operatorname{tr} T_i(T(N)) = t_v + t_e + t_h + t_p$$

, where t_v, t_e, t_h and t_p are given as follows.

(1) $$t_v = \begin{cases} \delta(\sqrt{N}) \frac{\kappa-1}{4\pi} v(H/SL_2(Z)) & , i = 1 \\ \delta(\sqrt{N}) \frac{\kappa-1}{4\pi} \chi_i(\sqrt{N})(q+1) v(H/SL_2(Z)) & , 2 \leqslant i \leqslant \ell \end{cases}$$

, where $\delta(\sqrt{N}) = 1$ or 0 according as N is a square or not.

(2) $$t_e = -\frac{1}{2} \sum_f \omega_e(f) \prod_p c_p(f,\ s_p) \frac{h(K(f))}{[\Lambda_0^\times : E_Q]}$$

, where f runs through all polynomials $f(X) = X^2 - sX + n$ such that $n = N\sigma$ and $s^2 - 4n < 0$.

(3) $t_h = -\sum_f \omega_h(f) \prod_p c_p(f,\ s_p) \dfrac{1}{[\Lambda_0^\times : E_Q]}$

, where f runs through all polynomials $f(X) = X^2 - sX + n \in Z[X]$ such that $n = N\mathfrak{a}$ and $s^2 - 4n$ is a non-zero square.

(4) $t_p = \begin{cases} -\delta(\sqrt{N})\,\dfrac{\sqrt{N}}{2} & ,\ i = 1 \\ -\delta(\sqrt{N})\,\chi_i(\sqrt{N})\,\sqrt{N} & ,\ 2 \leqslant i \leqslant \ell \end{cases}$.

To deduce the formula for $\operatorname{tr} T_i(\lambda(U(\mathfrak{a})))$ from that for $\operatorname{tr} T_i(T(N))$, we prove the following.

Lemma 5.4. Notation being as above, let $f(X)$ be a polynomial $f(X) = X^2 - sX + n \in Z[X]$ such that $n = N\mathfrak{a}$ and $s^2 - 4n \neq 0$. Then

(1) For $p \neq q$ with $s_p \geqslant 1$, we have

$$c_p(f,\ s_p) - p\, c_p(f_p,\ s_p - 2)$$

$$= \begin{cases} 1 & ,\ \text{if } \left(\dfrac{K(f)/Q}{p}\right) = 1 \text{ and } f \text{ is primitive at } p. \\ 1 - \left(\dfrac{K(f)/Q}{p}\right) & ,\ \text{otherwise} \end{cases}$$

, where $f_p = p^{-2} f(pX)$ and we put $c_p(f_p,\ s_p - 2) = 0$ if $f_p \notin Z[X]$ or $s_p - 2 < 0$.

(2) For $p \neq q$ with $s_p = 0$ and a positive integer u prime to p

$$c_p(f, 0) = c_p(f_u, 0) = \sum_{K_p \supset \Lambda_p(m) \supset Z_p[\tilde{X}]} [\Lambda_p(0)^{\times} : \Lambda_p(m)^{\times}] .$$

(3) For $p = q$ and a positive integer u prime to q,

$$c_q(f, 0) = \chi_i(u)\, c_q(f_u, 0) .$$

We note $[\Lambda_p(0)^{\times} : \Lambda_p(m)^{\times}] = p^m \left(1 - \frac{1}{p}\left(\frac{K(f)/Q}{p}\right)\right)$ for a positive integer m. Then we can prove our assertion in the similar way as Lemma 5.1, and omit the proof.

Using the above lemma, we can prove the following in the same way as Prop. 5.2.

Proposition 5.5. Let the notation be as above. Then, for $\mathfrak{a} \neq \mathfrak{O}$ (resp. $\mathfrak{a} = \mathfrak{O}$) we have

$$\operatorname{tr} T_i(\lambda(U(\mathfrak{a}))) = \operatorname{tr} T_i(U(N\mathfrak{a})) = t_v + t_e + t_h + t_p$$

$$\left(\text{resp. } \frac{1}{2} \operatorname{tr} T_i(\lambda(U(\mathfrak{O}))) = t_v + t_e + t_h + t_p\right)$$

, where t_v, t_e, t_h and t_p are given as follows.

(1) $$t_v = \begin{cases} \delta(\sqrt{N\mathfrak{a}})\, \frac{\kappa-1}{4\pi}\, v(H/SL_2(Z)) & , i = 1 \\ \delta(\sqrt{N\mathfrak{a}})\, \frac{\kappa-1}{4\pi}\, \chi_i(\sqrt{N\mathfrak{a}})(q+1) v(H/SL_2(Z)) & , 2 \leq i \leq \ell \end{cases} .$$

(2) $$t_e = -\frac{1}{2}\sum_f \omega_e(f) \prod_{\substack{p \mid N\mathfrak{a} \\ \left(\frac{K(f)/Q}{p}\right) \neq 1}} \left(1 - \left(\frac{K(f)/Q}{p}\right)\right)$$

$$\times \begin{cases} \displaystyle\sum_{\Lambda} \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} & , \ i = 1 \\ \displaystyle\sum_{\Lambda} \left(1 + \left\{\frac{\Lambda_q}{q}\right\}\right)\left(\frac{\chi_i(\alpha)+\chi_i(\beta)}{2}\right) \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} & , \ 2 \leq i \leq \ell \end{cases}$$

, where f (resp. Λ) runs through the same set as in (2) of Prop.5.2 and we denote by α and β the roots of the equation $f(X) \equiv 0 \bmod. q$.

(3) $$t_h = -\sum_f \omega_h(f) \times \begin{cases} \displaystyle\sum_{\Lambda} \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} & , \ i = 1 \\ \displaystyle\sum_{\Lambda} \left(\chi_i(\alpha)+\chi_i(\beta)\right) \frac{h(K(f), \Lambda)}{[\Lambda^\times : E_Q]} & , \ 2 \leq i \leq \ell \end{cases}$$

, where f (resp. Λ) runs the same set as in (3) of Prop.5.2, and α and β being as above.

(4) $$t_p = \begin{cases} -\frac{1}{2} & , \ \mathfrak{a} = \mathfrak{o} \ , \quad i = 1 \\ -1 & , \ \mathfrak{a} = \mathfrak{o} \ , \quad 2 \leq i \leq \ell \\ 0 & , \ \text{otherwise} \end{cases}$$

5.5. By Prop.5.2 and 5.5, we obtain the following.

Theorem 5.6. Let the notation and the assumption be as above. Then we have

$$(5.5.1) \qquad \mathrm{tr}\ T_S(e) = \mathrm{tr}\ T_1(\lambda(e)) + \frac{1}{2}\sum_{i=2}^{\ell} \mathrm{tr}\ T_i(\lambda(e))$$

for e of $R^0(\mathfrak{N}_F, GL_2(F_A))$.

Proof. Let $\mathfrak{n}$ be an integral ideal which is prime to $\mathfrak{q}$ and is divided by at most one prime factor of p for every prime $p \neq q$. Then for $U(\mathfrak{n}) \in R^0(\mathfrak{N}_F, GL_2(F_A))$ with such $\mathfrak{n}$, our assertion is a direct consequence of Prop.5.2 and 5.5. As remarked before, for a prime ideal $\mathfrak{p} \neq \mathfrak{q}$, it holds,

$$T_S(U(\mathfrak{p}^m)) = T_S(U({}^{\sigma}\mathfrak{p}^m)), \qquad T_i(\lambda(U(\mathfrak{p}^m))) = T_i(\lambda(U({}^{\sigma}\mathfrak{p}^m)))$$

and

$$T_S(T(\mathfrak{p}, \mathfrak{p})) = \mathrm{id}, \qquad T_i(\lambda(T(\mathfrak{p}, \mathfrak{p}))) = \mathrm{id},$$

By this and (5.1.1), we see our assertion holds for $U(\mathfrak{n})$ with an integral ideal $\mathfrak{n}$ prime to $\mathfrak{q}$. If we put $T(\mathfrak{b}, \mathfrak{b}) = \prod T(\mathfrak{p}_i, \mathfrak{p}_i)^{e_i}$ for a fractional ideal $\mathfrak{b} = \prod \mathfrak{p}_i^{e_i}$, then any element of $R^0(\mathfrak{N}_F, GL_2(F_A))$ can be written as a Z-linear combination of $T(\mathfrak{b}, \mathfrak{b})U(\mathfrak{n})$'s with integral ideals $\mathfrak{n}$ prime to $\mathfrak{q}$ and fractional ideals $\mathfrak{b}$. Hence the relation (5.5.1) holds for all e of $R^0(\mathfrak{N}_F, GL_2(F_A))$.

Remark 5.7. The formula (5.5.1) is a generalization of the formula (21) in [9]. In fact we assume $\ell = 2$, and denote by $\hat{\Gamma}$ the group generated by Γ and T_σ. Let χ (resp. $\hat{\chi}$) be the arithmetic genus of the surface $H \times H / \Gamma$ (resp. $H \times H / \hat{\Gamma}$). Then the formula (21) reads as follows.

$$\hat{\chi} = \frac{1}{2}\left(\chi - \left[\frac{q-1}{24}\right]\right) .$$

By the way, $\dim S_2(\Gamma) = \chi - 1$, $\dim \mathfrak{S}_2(\Gamma) = (\chi-1) - 2(\hat{\chi}-1)$, and $\frac{1}{2}\dim S_2(\Gamma_0(q), \chi) = 1 + \left[\frac{q-29}{24}\right]$ for $\chi = \left(\frac{}{q}\right)$. Hence the above formula is equivalent to

$$\dim \mathfrak{S}_2(\Gamma) = \frac{1}{2}\dim S_2(\Gamma_0(q), \chi) .$$

Here we note $\dim S_2(SL_2(Z)) = 0$. On the other hand our formula (5.5.1) for $e = T(\theta)$ and $\ell = 2$ asserts

$$\dim \mathfrak{S}_\kappa(\Gamma) = \dim S_\kappa(SL_2(Z)) + \frac{1}{2}\dim S_\kappa(\Gamma_0(q), \chi)$$

if κ even and $\geqslant 4$.

<u>5.6</u>. Since $T_S(e)$ (resp. $T_i(\lambda(e))$) for $e \in R^0(\mathfrak{N}_F, GL_2(F_A))$ is a normal operator in the space $\mathfrak{S}_\kappa(\Gamma)$ (resp. $S_\kappa(SL_2(Z))$ or $S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$), they generate a commutative semi-simple algebra over C. Hence the formula (5.5.1) in Prop.5.6 implies that the two spaces $2\mathfrak{S}_\kappa(\Gamma)$ and $2S_\kappa(SL_2(Z)) \oplus (\bigoplus_{i\geqslant 2} S_\kappa(\Gamma_0(q), \chi_i))$ are isomorphic to each other as $R^0(\mathfrak{N}_F, GL_2(F_A))$ - modules, where for a space S, we denote by 2S the direct product $S \oplus S$ of two copies of S. Moreover we can prove the following.

<u>Theorem 3</u>. The notation being as above, let F be a totally real field which satisfies the conditions (i), (ii), (iii) and (iv) in 4.1, and assume κ is even and $\geqslant 4$. Then

there exists a subspace S of $\bigoplus_{i\geq 2} S_\kappa(\Gamma_0(q), \chi_i)$ which is stable under the action of $R^0(\mathcal{H}_F, GL_2(F_A))$, and satisfies

(5.6.1) $\quad 2S \simeq \bigoplus S_\kappa(\Gamma_0(q), \chi_i)$

and

(5.6.2) $\quad SS_\kappa(\Gamma) \simeq S_\kappa(SL_2(Z)) \oplus S$

as $R^0(\mathcal{H}_F, GL_2(F_A))$-modules. Moreover we may assume S has a basis consisting of common eigen-functions for all e of $R^0(\mathcal{H}_Q, GL_2(Q_A))$.

<u>Proof.</u> For $\ell \geq 3$, it is easy to give such S. In fact, for a function $f(z)$ on H, put

$$W_q f(z) = f(\tau z) z^{-\kappa} q^{-\frac{\kappa}{2}}$$

with $\tau = \begin{pmatrix} 0 & -1 \\ q & 0 \end{pmatrix}$. We assume $\chi_{\frac{\ell-1}{2}+i} = \overline{\chi}_i$ for $2 \leq i \leq \frac{\ell+1}{2}$

Then it is known ([13], Th.B, [19], Prop. 3.55) that W_q induces anisomorphism between $S_\kappa(\Gamma_0(q), \chi_i)$ and $S_\kappa(\Gamma_0(q), \chi_{(\ell-1)/2+i})$, $2 \leq i \leq (\ell+1)/2$, and that W_q satisfies

$$W_q T_i(T(n)) = \chi_i(n) T_{(\ell-1)/2+i}(T(n)) W_q \qquad , 2 \leq i \leq \frac{\ell+1}{2}$$

for a positive integer n prime to q and

$$W_q^2 = 1 \quad .$$

Hence for a positive integer n prime to q, it holds

$$W_q T_i(U(n)) = \chi_i(n) T_{(\ell-1)/2+i}(U(n)) W_q$$

and

$$W_q T_i(\lambda(U(\pi)) = T_{(\ell-1)/2+i}(\lambda(U(\pi)))W_q$$

, since $\chi_i(N\pi) = 1$ by the assumption on F. If we put

$$S = \bigoplus_{i=2}^{\frac{\ell+1}{2}} S_\kappa(\Gamma_0(q), \chi_i)$$

, we see easily S satisfies (5.6.1) and (5.6.2), and it is obvious S has a basis consisting of common eigen-functions for all $e \in R^0(\mathfrak{N}_Q, GL_2(Q_A))$. For $\ell = 2$, we note that $\frac{1}{2}\dim S_\kappa(\Gamma_0(q), \chi_2)$ is an integer, since

$$\frac{1}{2}\dim S_\kappa(\Gamma_0(q), \chi_2) = \dim \mathbb{S}_\kappa(\Gamma) - \dim S_\kappa(SL_2(Z)) .$$

, and that if $f \in S_\kappa(\Gamma_0(q), \chi_2)$ is a common eigen-function for all T(e) with $e \in R^0(\mathfrak{N}_Q, GL_2(F_A))$, then $W_q f$ also has the same property. For, W_q induces an automorphism of $S_\kappa(\Gamma_0(q), \chi_2)$ of order 2 and satisfies

(5.6.3) $$W_q T_2(T(n)) = \chi_2(n) T_2(T(n)) W_q$$

for a positive integer n prime to q. (c.f. [19], Prop. 3.55) We will show there exists a basis $\{h_i\}$, $1 \leqslant i \leqslant \dim S_\kappa(\Gamma_0(q), \chi_2)$, which consists of common eigen-functions for all T(e) with $e \in R^0(\mathfrak{N}_Q, GL_2(Q_A))$, and satisfies $W_q h_i = h_{d+i}$, $1 \leqslant i \leqslant d$, where $d = \frac{1}{2}\dim S_\kappa(\Gamma_0(q), \chi_2)$. If this is shown, the subspace

S of $S_\kappa(\Gamma_0(q), \chi_2)$ spanned by f_i's , $1 \leqslant i \leqslant \frac{1}{2} \dim S_\kappa(\Gamma_0(q), \chi_2)$, satisfies the conditions (5.6.1), (5.6.2) in our theorem, since by (5.6.3) it holds $W_q T_2(\lambda(U(\mathfrak{a}))) = T_2(\lambda(U(\mathfrak{a})))W_q$ for any integral ideal $\mathfrak{a}$. Let $\{f_i\}$, $1 \leqslant i \leqslant \dim \mathbb{S}_\kappa(\Gamma)$, be a basis consisting of common eigen-functions for all $T_S(e)$ with $e \in R(\mathfrak{n}_F, GL_2(F_A))$, and let $C[f_i]$ be the one-dimensional subspace of $\mathbb{S}_\kappa(\Gamma)$ generated by f_i . We note the following, which holds also for $\ell \neq 2$. If two spaces $C[f_i]$ and $C[f_j]$ are isomorphic to each other as $R^0(\mathfrak{n}_F, GL_2(F_A))$-modules, then by ([13], Th. 2) there exists a constant c such that $f_i = cf_j$. Hence any two $R^0(\mathfrak{n}_F, GL_2(F_A))$-modules $C[f_i]$ and $C[f_j]$ are not isomorphic to each other if $i \neq j$. Let $\{g_i\}$, $1 \leqslant i \leqslant \dim S_\kappa(SL_2(Z))$, (resp. $\{h_i\}$, $1 \leqslant i \leqslant \dim S_\kappa(\Gamma_0(q), \chi_2)$) be a basis of $S_\kappa(SL_2(Z))$ (resp. $S_\kappa(\Gamma_0(q), \chi_2)$) consisting of common eigen-functions for all $T_1(e)$ with $e \in R(\mathfrak{n}_Q, GL_2(Q_A))$ (resp. $T_2(e)$ with $e \in R^0(\mathfrak{n}_Q, GL_2(Q_A))$). Since $2\mathbb{S}_\kappa(\Gamma) \simeq 2S_\kappa(SL_2(Z)) \oplus S_\kappa(\Gamma_0(q), \chi_2)$ and $C[f_i] \not\simeq C[f_j]$ for $i \neq j$, we may assume by replacing indeces,

$$C[f_i] \simeq C[g_i] \qquad , 1 \leqslant i \leqslant \dim S_\kappa(SL_2(Z))$$

and

$$\begin{cases} 2\,C[f_{s+i}] \simeq C[h_i, h_{d+i}] & , 1 \leqslant i \leqslant \dim S_\kappa(\Gamma_0(q), \lambda_2) \\ C[h_i] \simeq C[h_{d+i}] \simeq C[f_i] & \end{cases}$$

as $R^0(\mathfrak{N}_F, GL_2(F_A))$-modules, where $s = \dim S_\kappa(SL_2(Z))$, $d = \frac{1}{2}\dim S_\kappa(\Gamma_0(q), \chi_2)$ and $C[h_i, h_{d+i}]$ is the space spanned by h_i and h_{d+i}. We show h_{d+i} is a constant multiple of $W_q h_i$. First we assume h_i and $W_q h_i$ are linearly independent. Since $S_\kappa(\Gamma_0(q), \chi_2)$ has a basis consisting of new forms in the sense of Atkin-Lehner-Miyake, $W_q h_i$ is a constant multiple of h_j for some j. But as $R^0(\mathfrak{N}_F, GL_2(F_A))$-modules we have

$$C[W_q h_i] \simeq C[h_i] \simeq C[h_{d+i}]$$

$$\not\simeq C[h_j] \quad , j \neq i, d+i$$

, hence $W_q h_i$ is a constant multiple of h_{d+i}. Next assume $W_q h_i = ch_i$ with a constant c. If $W_q h_{d+i}$ and h_{d+i} are linearly independent, we can show in the same way as above that $W_q h_{d+i} = ch_i$, with a constant c and $W_q h_i = c^{-1}h_{d+i}$. Hence we assume that $W_q h_i$ and $W_q h_{d+i}$ are constant multiples of h_i and h_{d+i} respectively. Let $h_j(z) = \sum_{n=1}^{\infty} c_j(n)\, e^{2\pi inz}$ be the Fourier expansion of $h_j(z)$, and $a_j(n)$ be the eigen-value of $h_j(z)$ for $T(n)$ with n prime to q, then it holds

$$c_j(n) = n^{\kappa/2-1}\chi_2(n)a_j(n) \quad .$$

By (5.6.3), we obtain

$$c_j(p) = \chi_2(p)c_j(p) \qquad \text{for} \quad j = i,\ d+i \quad .$$

Hence we have

$$c_i(p) = c_{d+i}(p) = 0$$

for all $p \neq q$ with $\chi_2(p) = -1$. For $p \neq q$ with $\chi_2(p) = 1$, We have $T_2(\lambda(U(\mathfrak{p}))) = T_2(U(p)) = T_2(T(p))$, where $\mathfrak{p}$ is a prime factor of p. From this it follows that $c_i(p) = c_{d+i}(p)$ for all $p \neq q$ with $\chi_2(p) = 1$, since $C[h_i] \simeq C[h_{d+i}] \simeq C[f_{s+i}]$, hence we obtain $c_i(p) = c_{d+i}(p)$ for all $p \neq q$. By ([13], Th. 3, Cor. 2) this implies $h_i = ch_{d+i}$ with a constant c, and this contradicts to the assumption on the choice of $\{h_i\}$. Hence it has been proved that $W_q h_i = ch_{d+i}$ with a constant c. By multiplying suitable constants, we obtain a basis of $S_\kappa(\Gamma_0(q), \chi_2)$ which satisfies the conditions mentioned above. Thus our theorem is proved completely.

As a corollary of the above proof for $\ell = 2$, we have

<u>Corollary 1.</u> Let q be a prime such that $q \equiv 1 \bmod. 4$. Assume the class number of $Q(\sqrt{q})$ is one. For an even positive integer κ larger than 2, let $f \in S_\kappa(\Gamma_0(q), \chi)$ be a common eigen-function of $T(n)$ with the eigen-value $a(n)$ for all

positive integers n prime to q, where χ denotes the quadratic residue symbol mod. q. Then the field K generated by all $a(n)$ over $\mathbb{Q}$ is a totally imaginary quadratic extension of a totally real field.

Proof. The above assertion for $\kappa = 2$ is contained in Th.7.16, of [19]. For $\kappa \geqslant 4$, by the above proof it is seen that there exists a prime p such that $c(p) \neq 0$ and $\chi(p) = -1$. If we denote by $\overline{a(p)}$ the complex conjugation of $a(p)$, then $a(p)$ satisfies $a(p) = \chi(p)\overline{a(p)}$ (c.f. Prop.3.56, [19]), hence $\overline{a(p)} = -a(p)$. This implies K is not totally real. From this fact, our assertion easily follows by the same argument as in p183 ∼ 185, of [19].

The assertion in this corollary is stated in [20] under a more general condition.

Now we interpret Th.3 in terms of Fourier coefficients. Let $g(z) \in S_\kappa(SL_2(\mathbb{Z}))$ (resp. $S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$) be a common eigen-function for all $T(n)$ (resp. $T(n)$ with n prime to q). We have the Fourier expansion of $g(z)$ given by

$$g(z) = \sum_{n=1}^{\infty} c(n)\, e^{2\pi i n z} .$$

By multiplying a constant, we may assume $c(1) = 1$. If we denote by $a(n)$ the eigen-value of $g(z)$ for $T(n)$, then we have

$$(5.6.4) \qquad c(n) = n^{\kappa/2-1} \chi_i(n)^{-1} a(n)$$

for all n (prime to q if $g(z) \in S_\kappa(\Gamma_0(q), \chi_i)$, $i \geq 2$), where for $g \in S_\kappa(SL_2(Z))$, we put $\chi_1(n) = 1$. From the sequence $\{c(n)\}$ we define another sequence $\{C(\mathfrak{a})\}$ for integral ideals $\mathfrak{a}$ prime to $\mathfrak{q}$. For $\mathfrak{a} = \mathfrak{o}$, put $C(\mathfrak{o}) = c(1) = 1$, and for a prime ideal $\mathfrak{p} \neq \mathfrak{q}$, define

$$C(\mathfrak{p}_1) = \ldots\ldots = C(\mathfrak{p}_\ell) = c(p) \qquad , \text{ if } (p) = \mathfrak{p}_1 \cdots \mathfrak{p}_\ell$$

$$C(\mathfrak{p}) = c(N\mathfrak{p}) - p^{\kappa-1} \overline{\chi_i(p)}\, c(N\mathfrak{p}p^{-2}) \qquad , \text{ if } (p) = \mathfrak{p} .$$

For $m \geq 2$, define $C(\mathfrak{p}^m)$ inductively by

$$C(\mathfrak{p}^m) - N\mathfrak{p}^{\kappa-1} C(\mathfrak{p}^{m-2}) = c(N\mathfrak{p}^m) - p^{\kappa-1} \overline{\chi_i(p)}\, c(N\mathfrak{p}^m p^{-2}) .$$

Then we see $C(\mathfrak{p}^m)$ satisfies

$$C(\mathfrak{p}^m) = C(\mathfrak{p})C(\mathfrak{p}^{m-1}) - N\mathfrak{p}^{\kappa-1} C(\mathfrak{p}^{m-2}) .$$

Lastly for $\mathfrak{a} = \prod \mathfrak{p}_i^{e_i}$, put

$$C(\mathfrak{a}) = \prod_i C(\mathfrak{p}_i^{e_i}) .$$

For $\ell = 2$ and $\mathfrak{a}$ prime to $\mathfrak{q}$, this rule for defining $C(\mathfrak{a})$ from

$c(n)$ is nothing but the rule given in [2] and [14].

Corollary 2. Let the notation and the assumption be as in Th. 3.

(i) Let $f(z) \in \mathfrak{S}_\kappa(\Gamma)$ be a common eigen-function for all $T(\mathfrak{n})$ with Fourier coefficients $C_f(\mathfrak{n})$ such that $C_f(\mathcal{O}) = 1$. Then there exists a common eigen-function $g(z)$ for all $T(n)$ (with n prime to q, if $g(z) \in S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$) in $S_\kappa(SL_2(Z))$ or $S_\kappa(\Gamma_0(q), \chi_i)$ such that the Fourier coefficients $C_f(\mathfrak{n})$ for $\mathfrak{n}$ prime to $\mathfrak{q}$ are identical with $C(\mathfrak{n})$ defined from the Fourier coefficients $c(n)$ of $g(z)$ in the above way.

(ii) Let $g(z) \in S_\kappa(SL_2(Z))$ (resp. $\in S_\kappa(\Gamma_0(q), \chi_i)$, $i \geqslant 2$) be a common eigen-function for all $T(n)$ (resp. with n prime to q) with the Fourier expansion given as follows

$$g(z) = \sum_{n=1}^{\infty} c(n)\, e^{2\pi i n z} \qquad , \; c(1) = 1 \quad .$$

Define $C(\mathfrak{n})$ for $\mathfrak{n}$ prime to $\mathfrak{q}$ in the above way from $c(n)$, then there exists a unique common eigen-function $f(z) \in \mathfrak{S}_\kappa(\Gamma)$ for all $T(\mathfrak{n})$ such that the Fourier coefficients $C_f(\mathfrak{n})$ of $f(z)$ are given by $C(\mathfrak{n})$ for all $\mathfrak{n}$ prime to $\mathfrak{q}$.

(iii) In (ii), if two common eigen-functions g_1 and g_2 for all $T(n)$ (with n prime to q for $g_i \in S_\kappa(\Gamma_0(q), \chi_j)$, $j \geqslant 2$)

correspond to the same element of $\mathfrak{S}_\kappa(\Gamma)$, then g_1 and g_2 are contained in $S_\kappa(SL_2(Z))$ and $g_1 = cg_2$ with a constant c, or g_1 and g_2 are contained in $\bigoplus_{i \geq 2} S_\kappa(\Gamma_0(q), \chi_i)$ and $g_1 = cg_2$ or $g_1 = cW_q g_2$ with a constant c.

<u>Proof</u>. Let $g(z)$ be an element of $S_\kappa(SL_2(Z))$ or $S_\kappa(\Gamma_0(q), \chi_i)$ which is a common eigen-function for all $T(n)$ ($(n, q) = 1$) with eigen-values $a(n)$, and let $f(z)$ be an element of $\mathfrak{S}_\kappa(\Gamma)$ which is a common eigen-function for all $T(\mathfrak{n})$ with eigen-values $a(\mathfrak{n})$. If $C[f] \simeq C[g]$ as $R^0(\mathfrak{n}_F, GL_2(F_A))$-modules, then

$$a(\mathcal{O}) = a(1) = 1$$

$$a(\mathfrak{z}) = \begin{cases} a(p) & \text{if } (p) = \mathfrak{z}_1 \cdots \mathfrak{z}_\ell \\ a(N\mathfrak{z}) - \chi_i(p)pa(N\mathfrak{p}\, p^{-2}) & \text{if } (p) = \mathfrak{z} \end{cases}$$

, since $T_i(T(p, p))g = \chi_i(p)g$. For $m \geq 2$, we have

$$a(\mathfrak{p}^m) - N\mathfrak{p}\, a(\mathfrak{z}^{m-2}) = a(N\mathfrak{z}^m) - \chi_i(p)pa(N\mathfrak{p}^m p^{-2})$$

In notice of the relation (5.6.4) (resp. (1.3.2)) between $c(n)$ and $a(n)$ (resp. $C_f(\mathfrak{n})$ and $a(\mathfrak{n})$), we easily obtain our assertions (i) and (ii) by Th.3. As noted in the proof of Th.3, $\mathfrak{S}_\kappa(\Gamma)$ is a direct product of one dimensional simple $R^0(\mathfrak{n}_F, GL_2(F_A))$-modules which are not isomorphic to each other, and a common eigen-function $g(z) \in S_\kappa(SL_2(Z))$, or $S_\kappa(\Gamma_0(q), \chi_i)$ for all $T(n)$ (n prime to q if $g(z) \in S_\kappa(\Gamma_0(q), \chi_i)$) is a new form in the sense of Atkin-Lehner-Miyake. Hence our assertion follows from the proof of Th. 3.

5.7. In the correspondence given in (ii) of Th. 3, Cor. 2, our theorem does not give any imformation on the Fourier coefficient $C_f(\mathfrak{q})$. But it seems that there exists some relation $C_f(\mathfrak{q})$ and $c(q)$. For $\ell = 2$, the results of [2] and [14] shows that $C_f(\mathfrak{q})$ is related to $c(q)$ in the following way. Let $g(z)$ be as in (ii) of Cor. 2. For a prime ideal $\mathfrak{z} = \mathfrak{q}$, define $C(\mathfrak{z}^m)$ as above, and for $\mathfrak{q}$, as in [2] and [14], put

$$C(\mathfrak{q}) = \begin{cases} c(q) & \text{if } g \in S_\kappa(SL_2(Z)) \\ c(q) + \overline{c(q)} & \text{if } g \in S_\kappa(\Gamma_0(q), \chi_2) \end{cases}$$

, where $\overline{c(q)}$ denotes the complex conjugate of $c(q)$. For $m \geqslant 2$, define $C(\mathfrak{q}^m)$ inductively by

$$C(\mathfrak{q}^m) = C(\mathfrak{q}^{m-1})C(\mathfrak{q}) - N\mathfrak{q}^{\kappa-1}C(\mathfrak{q}^{m-2}) \quad .$$

For $\mathfrak{n} = \prod_i \mathfrak{z}_i^{e_i}$, put

$$C(\mathfrak{n}) = \prod_i C(\mathfrak{z}_i^{e_i}) \quad .$$

Then we can prove the following by the method of Miyake [13].

Proposition 5.8. Let F and κ be as in Th. 3, and g be as in (ii) of Th. 3, Cor. 2. If g corresponds to $f \in \mathfrak{S}_\kappa(\Gamma)$ in the correspondence given in (ii) of Th. 3, Cor. 2, then the Fourier coefficients $C_f(\mathfrak{a})$ of f is given by $C(\mathfrak{a})$ for all $\mathfrak{a}$. In other words, the function f on $H \times H$ given by the Fourier series

$$f(z) = \sum_{\substack{\mathfrak{a} = (\mu) \\ \frac{\mu}{\sqrt{q}} \gg 0}} C(\mathfrak{a}) \sum_{\varepsilon \in E_+} \exp 2\pi\sqrt{-1} \left(\frac{\varepsilon\mu}{\sqrt{q}} z_1 + {}^{\sigma}\!\left(\frac{\varepsilon\mu}{\sqrt{q}}\right) z_2 \right)$$

belongs to $\mathfrak{S}_\kappa(\Gamma)$, hence to $S_\kappa(\Gamma)$.

Proof. Assume g is in correspondence with $f(z) \in \mathfrak{S}_\kappa(\Gamma)$. We consider the following two Dirichlet series

$$D_f(s) = \sum \frac{C_f(\mathfrak{a})}{N\mathfrak{a}^s}$$

$$D(s) = \sum \frac{C(\mathfrak{a})}{N\mathfrak{a}^s}$$

, where $C_f(\mathfrak{a})$ are the Fourier coefficients of $f(z)$. They have the Euler products

$$D_f(s) = \prod_{\mathfrak{z}} (1 - C_f(\mathfrak{z}) N\mathfrak{z}^{-s} + N\mathfrak{z}^{\kappa-1-2s})^{-1}$$

$$D(s) = \prod_{\mathfrak{z}} (1 - C(\mathfrak{z})N\mathfrak{z}^{-s} + N\mathfrak{z}^{\kappa-1-2s})^{-1}$$

for $\operatorname{Re} s > N$ with some N. Put

$$D_f^*(s) = q^s (2\pi)^{-s} \Gamma(s)^2 D_f(s)$$

$$D^*(s) = q^s (2\pi)^{-s} \Gamma(s)^2 D(s) \quad ,$$

Then it is known that $D_f^*(s)$ satisfies the functional equation ([7])

$$D_f^*(\kappa - s) = D_f^*(s)$$

, and by [2] and [14] , we have

$$D^*(\kappa - s) = D^*(s) \quad .$$

Comparing the above two functional equations, we obtain

$$\frac{1 - C(\mathfrak{q})N\mathfrak{q}^{-s} + N\mathfrak{q}^{\kappa-1-2s}}{1 - C_f(\mathfrak{q})N\mathfrak{q}^{-s} + N\mathfrak{q}^{\kappa-1-2s}} = \frac{1 - C(\mathfrak{q})N\mathfrak{q}^{s-\kappa} + N\mathfrak{q}^{2s-\kappa-1}}{1 - C_f(\mathfrak{q})N\mathfrak{q}^{s-\kappa} + N\mathfrak{q}^{2s-\kappa-1}}$$

, since $C_f(\mathfrak{z}) = C(\mathfrak{z})$ for $\mathfrak{z} \neq \mathfrak{q}$. From this, we see $C(\mathfrak{q}) = C_f(\mathfrak{q})$, and $C(\mathfrak{a}) = C_f(\mathfrak{a})$ for all $\mathfrak{a}$.

Note. We can prove a little more general result in the same way as in this paper. We can consider $S_\kappa(\Gamma_0(q),\chi)$ not only as a $R^0(\mathfrak{U}_Q, GL_2(Q_A))$-module but also as a $R(\mathfrak{U}_Q, GL_2(Q_A))$-module. In fact, for $e \in R^0(\mathfrak{U}_Q, GL_2(Q_A))$, define the action of e as before. For q, let $\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\Gamma_0(q) = \bigcup_{\nu=1}^{d} \alpha_\nu \Gamma_0(q)$ be a disjoint union, and put for $g \in S_\kappa(\Gamma_0(q),\chi)$,

$$g|[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\Gamma_0(q)] = \sum_\nu \overline{\chi}(d_\nu)\frac{(\det \alpha_\nu)^{\kappa/2}}{(-c_\nu z + a_\nu)^\kappa} g(\alpha_\nu^{-1} z) ,$$

where $\alpha_\nu = \begin{pmatrix} a_\nu & b_\nu \\ c_\nu & d_\nu\end{pmatrix}$. And we define the action of $T(q)$ and $T(q,q)$ on $S_\kappa(\Gamma_0(q),\chi)$ by

$$T(q)g = g|[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\Gamma_0(q)] + g|[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\Gamma_0(q)]^*$$

$$T(q,q)g = g .$$

Here $[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\Gamma_0(q)]^*$ denotes the adjoint operator of $[\Gamma_0(q)\begin{pmatrix} q & 0 \\ 0 & 1\end{pmatrix}\Gamma_0(q)]$ with respect to the Petersson inner product. If we denote this action also by T_χ, $S_\kappa(\Gamma_0(q),\chi)$ can be viewed as a $R(\mathfrak{U}_F, GL_2(F_A))$-module by $T_\chi \circ \lambda$, and we can prove

Theorem. There exists a subspace S of $\bigoplus_\chi S(\Gamma_0(q),\chi)$ such that

$$SS_\kappa(\Gamma) \simeq S_\kappa(SL_2(Z)) \oplus S$$

$$(\text{and } \bigoplus_\chi S_\kappa(\Gamma_0(q),\chi) \simeq S \oplus S)$$

as $R(\mathfrak{U}_F, GL_2(F_A))$-modules, where in $\bigoplus_\chi$, runs through all characters of order ℓ of $(Z/qZ)^\times$.

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