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WEIL'S REPRESENTATIONS AND SIEGEL'S MODULAR FORMS

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INTRODUCTION

The purpose of this report is to present an explicit construction of a Siegel modular form of genus 2, which is a common-eigenfunction of Hecke operators, from a pair of elliptic modular forms or from a Hilbert modular form over a real quadratic field, as an application of Weil's representations.

T.Shintani [18] successfully applied Weil's representations to a construction of modular cusp forms of half integral weight. Afterwards many authors employed Weil's representations for constructions of automorphic forms with Euler products, in various cases. Especially R.Howe [8] has given a fairly general frame work called "dual reductive pairs". In this report, we shall exclusively be concerned with the case of the Weil representations of the symplectic group of genus 2 associated with quaternary positive definite quadratic forms for the construction of Siegel modular forms of genus 2. Even in this particular case, we shall encounter a few important problems and conjectures, which would be suggestive for the development of general theory. Here we only mention the following problem of global nature. Our Siegel modular forms are written as linear combinations of theta series (cf. (23)). As an inevitable obstacle which lies in such a construction, it is difficult to know whether the constructed modular form does vanish or not. However we can at least show that several non-zero Siegel modular cusp forms arise by our construction in every prime level (cf.Theorem 6). We formulate a precise conjecture for the non-vanishing property of our construction in the case of the prime level (§4). In §5, we shall propose a characterization of the image of our construction, which can be regarded as a preliminary stage for the application of the Selberg trace formula to resolve the above mensioned difficulty. Most ot the results will be stated without proofs. The full details will appear elsewhere.

Notation. For an associative ring R with a unit, $R^{\times}$ denotes the group of invertible elements of R. We denote by $M(m,R)$ the set of m x m-matrices with entries in R. For a matrix A, ${}^{t}A$ denotes the transpose of A, and $\sigma(A)$ denotes the trace of A if A is a square matrix. The diagonal matrix with diagonal elements $d_1, d_2, \cdots, d_n$ is denoted by $[d_1, d_2, \cdots, d_n]$. If R is commutative, we put $GL(m,R) = M(m,R)^{\times}$ and assume that the group of R-valued points $Sp(m,R)$ of the symplectic group of genus m is given explicitly by $Sp(m,R) = \left\{ x \in GL(2m,R) \mid {}^{t}xwx = w \right\}$, where $w = \begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$ and 1_m and 0_m denote the identity and the zero matrix in $M(m,R)$ respectively. For a positive integer N, we put $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$ and $\tilde{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2,\mathbb{Z}) \mid c \equiv 0 \bmod N \right\}$, where, in the second equality, $c \equiv 0 \bmod N$ means that $c \in M(2,\mathbb{Z})$ is congruent to the zero matrix modulo N. The space of elliptic modular forms(resp. modular cusp forms) of weight k with respect to $\Gamma_0(N)$ is denoted by $G_k(\Gamma_0(N))$(resp. $S_k(\Gamma_0(N))$). The space of Siegel modular forms(resp. modular cusp forms) of genus 2 and of weight k with respect to $\tilde{\Gamma}_0(N)$ is denoted by $\tilde{G}_k(\tilde{\Gamma}_0(N))$(resp. $\tilde{S}_k(\tilde{\Gamma}_0(N))$). Let k be a global field and v be a place of k. Then k_v denotes the completion of k at v. For an algebraic group G defined over k, G_A denotes the adelization of G and G_v denotes the group of k_v-rational points of G. For $g \in G_A$, g_v denotes the v-component of g and g_f(resp. g_∞) denotes the finite(resp. the infinite) component of g. For a quasi-character χ of $k_A^{\times}$, χ_v denotes the quasi-character of $k_v^{\times}$ which is naturally obtained from χ. We denote by ∞ the archimedean place of $\mathbb{Q}$. For a commutative field F and a quaternion algebra D over F, N, Tr and $*$ denote the reduced norm, the reduced trace and the main involution of D respectively. By $\mathbb{H}$, we denote the division ring of Hamilton quaternions. For a locally compact abelian group G, $\mathcal{S}(G)$

denotes the space of Schwarz-Bruhat functions on G. For $z \in \mathbb{C}$, we set $e(z) = \exp(2\pi\sqrt{-1}z)$.

§1. Construction of automorphic forms via Weil's representations

Let F be a totally real algebraic number field of degree m and D be a totally definite quaternion algebra over F. Let R be an order of D. We put $R_v = R \otimes_{\mathcal{O}} \mathcal{O}_v$ for every finite place of F, where $\mathcal{O}$ and $\mathcal{O}_v$ are maximal orders of F and F_v respectively. For every place v of F, we define a subgroup K'_v of $D_v^\times$ by $K'_v = R_v^\times$ if v is finite, and $K'_v = \mathbb{H}^\times$ if v is infinite. We put $K' = \prod_v K'_v$, which is considered as a subgroup of $D_A^\times$. Let ι_0 be an injective homomorphicm of $\mathbb{H}$ into $M(2,\mathbb{C})$ as algebras over $\mathbb{R}$. For a non-negative integer n, let ξ_n denote the symmetric tensor representation of $GL(2,\mathbb{C})$ of degree n; ξ_n: $GL(2,\mathbb{C}) \longrightarrow GL(n+1,\mathbb{C})$. We set $\sigma_n(g) = (\xi_n \circ \iota_0)(g)\, N(g)^{-n/2}$. Let $(n_1,\cdots,n_m)$ be an m-tuple of non-negative integers and V be the representation space of $\sigma_{n_1} \otimes \cdots \otimes \sigma_{n_m}$. Let $Z \cong F_A^\times$ be the center of $D_A^\times$ and ω be a character of Z. By $S(R,n_1,\cdots,n_m,\omega)$, we denote the vector space of all V-valued functions φ on $D_A^\times$ which satisfy the following conditions (A)$\sim$(C).

(A) $\varphi(\gamma g) = \varphi(g)$ for any $\gamma \in D^\times$, $g \in D_A^\times$.

(B) $\varphi(gk) = (\sigma_{n_1} \otimes \cdots \otimes \sigma_{n_m})(k_\infty)\varphi(g)$ for any $k \in K'$, $g \in D_A^\times$.

(C) $\varphi(gz) = \omega(z)\varphi(g)$ for any $z \in Z$, $g \in D_A^\times$.

If the class number h_F of F in the narrow sense is 1, we have $S(R,n_1,\cdots,n_m,\omega) = \{0\}$ if $\omega \neq \omega_0$, where ω_0 is the trivial character of Z. Hence if $h_F = 1$, we assume that $\omega = \omega_0$ and abbreviate $S(R,n_1,\cdots,n_m,\omega_0)$ to $S(R,n_1,\cdots,n_m)$. We define the action of Hecke operators on φ as follows. Let v be a finite place of F at which D splits. We assume that R_v is a maximal order of D_v. We fix a splitting $D_v \cong M(2,F_v)$ so that R_v is mapped onto $M(2,\mathcal{O}_v)$. Let ϖ be a prime element of F_v and let $R_v^\times \begin{pmatrix} 1 & 0 \\ 0 & \varpi \end{pmatrix} R_v^\times = \bigcup_s h_s R_v^\times$ be a disjoint union. For $\varphi \in S(R, n_1, \cdots, n_m, \omega)$, we put

(1) $\qquad (T'(v)\varphi)(h) = \sum_s \varphi(h\, \iota_v(h_s))$,

where ι_v denotes the natural injection of $D_v^\times$ into $D_A^\times$. Clearly $T'(v)\varphi \in S(R, n_1, \cdots, n_m, \omega)$.

In this report, we shall exclusively consider the case where $F = \mathbb{Q}$ or $[F:\mathbb{Q}] = 2$. If $F = \mathbb{Q}$, we put $X = D \oplus D$, $Y = D$ and define the action ρ of $D^\times \times D^\times$ on X by $\rho(g_1, g_2)(x_1, x_2) = (g_1^* x_1 g_2, g_1^* x_2 g_2)$. We put $H = \{(a,b) \in D^\times \times D^\times \mid N(a) = N(b) = 1\}$. Then H is an algebraic group over $\mathbb{Q}$ which acts on X through ρ as an group of isometries. If F is real quadratic, we assume that $D = D_0 \otimes_{\mathbb{Q}} F$ with a definite quaternion algebra D_0 over $\mathbb{Q}$. Let σ denote the extension of the non-trivial automorphism of F over $\mathbb{Q}$ to the semi-automorphism of D. We have $(x^\sigma)^* = (x^*)^\sigma$ for $x \in D$. We put $Y = \{x \in D \mid x^\sigma = x^*\}$ and $X = Y \oplus Y$. We define the action of $D^\times$ on X by $\rho(g)(x_1, x_2) = ((g^\sigma)^* x_1 g, (g^\sigma)^* x_2 g)$. We put $H = \{a \in D^\times \mid N(a) = 1\}$. Then H is an algebraic group over F which acts on X as an group of isometries. We call the former situation Case (I) and the latter one Case (II). Let χ be the character of $\mathbb{Q}_A^\times/\mathbb{Q}^\times$ which corresponds to F by class field theory if we are in Case (II) and let χ be the trivial character of $\mathbb{Q}_A^\times/\mathbb{Q}^\times$ if we are in Case (I).

Let G be the symplectic group of genus 2. We take an additive character ψ of $\mathbb{Q}_A/\mathbb{Q}$ such that $\psi_\infty(x) = e(x)$, $x \in \mathbb{R}$ and $\psi_p(x) = e(-Fr(x))$, $x \in \mathbb{Q}_p$ for every rational prime p, where $Fr(x)$ denotes the fractional part of x. For every place v of $\mathbb{Q}$, we have the so called Weil representation π_v of G_v realized on $\mathcal{S}(X_v)$ which is characterized by the following conditions (i)$\sim$(iii).(cf.Weil [19], Yoshida [21]).

(i) $(\pi_v(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}) f)(x_1, x_2) = \psi_v(\sigma(u \begin{pmatrix} N(x_1) & Tr(x_1 x_2^*)/2 \\ Tr(x_1 x_2^*)/2 & N(x_2) \end{pmatrix}))$

$\qquad \times f(x_1, x_2)$,

(ii) $(\pi_v(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix})f)(x_1,x_2) = \chi_v(\det a)|\det a|_v^2 f((x_1,x_2)a)$,

(iii) $(\pi_v(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})f)(x_1,x_2) = \gamma_v f^*(x_1,x_2)$,

where $(x_1,x_2)\in Y_v \oplus Y_v$, $|\ |_v$ denotes the absolute value of $\mathbb{Q}_v$ and f^* is the Fourier transform of f with respect to the self-dual measure on X_v. (γ_v is a certain complex number of absolute value 1. We have $\gamma_v = 1$ for every place v of $\mathbb{Q}$ if we are in case (I)). The global Weil representation π of G_A realized on $\mathcal{S}(X_A)$ is defined as follows. For $f\in\mathcal{S}(X_A)$ of the form $f = \prod_v f_v$, $f_v\in\mathcal{S}(X_v)$ such that f_p is equal to the characteristic function of $R_p \oplus R_p$(resp. $U_p \oplus U_p$) for almost all p, we put $\pi(g)f = \prod_v \pi_v(g_v)f_v$, $g\in G_A$ if we are in Case (I) (resp. Case (II)), where $U_p = \{x\in R \mid x^\sigma = x^*\}\otimes_{\mathbb{Z}} \mathbb{Z}_p$ for the Case (II). Then, extending by continuity, we obtain the representation π.

To construct an automorphic form on G_A, first we assume that we are in Case (I). Take $\varphi_1\in S(R,n_1)$, $\varphi_2\in S(R,n_2)$ and let $V_i\cong\mathbb{C}^{n_i+1}$ be the representation space of σ_{n_i}, i = 1,2. Then $\varphi = \varphi_1\otimes\varphi_2$ defines a $V = V_1\otimes V_2$-valued function on $D_A^\times \times D_A^\times$. We take $f_p\in\mathcal{S}(X_p)$ as the characteristic function of $R_p \oplus R_p$ for every rational prime p and take any $f_\infty\in\mathcal{S}(X_\infty)\otimes V$. (The choice of f_∞ will be clarified in §3). Let $\langle\ ,\ \rangle$ be the inner product in V such that $\sigma_{n_1}\otimes\sigma_{n_2}|\mathbb{H}^{(1)}\times\mathbb{H}^{(1)}$ is unitary with respect to $\langle\ ,\ \rangle$, where $\mathbb{H}^{(1)} = \{x\in\mathbb{H}^\times | N(x) = 1\}$. We set

$$(2)\quad \Phi_f(g) = \int_{H_{\mathbb{Q}}\backslash H_A} \Big\langle \sum_{x\in X_{\mathbb{Q}}} (\pi(g)f)(\rho(h)x),\ \varphi(h)\Big\rangle\, dh\ .$$

Now suppose that we are in Case (II). Take $\varphi\in S(R,n_1,n_2,\omega)$ and let V be the representation space of $\sigma_{n_1}\otimes\sigma_{n_2}$. We take $f_p\in\mathcal{S}(X_p)$ as the characteristic function of $U_p \oplus U_p$ for every rational prime p and take any $f_\infty\in\mathcal{S}(X_\infty)\otimes V$. We set

$$(3)\quad \Phi_f(g) = \int_{H_F\backslash H_A} \Big\langle \sum_{x\in X_{\mathbb{Q}}} (\pi(g)f)(\rho(h)x),\ \varphi(h)\Big\rangle\, dh\ .$$

In (2) and (3), dh denotes invariant measures on $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}$ and $H_F\backslash H_{\mathbb{A}}$ respectively, and $G_{\mathbb{A}}$ acts on $\mathcal{S}(X_{\mathbb{A}}) \otimes V$ through the first factor. The integrals in (2) and (3) exist since $H_{\mathbb{Q}}\backslash H_{\mathbb{A}}$ and $H_F\backslash H_{\mathbb{A}}$ are compact and the integrands are continuous functions of h. By virtue of proposition 5 of Weil [19], one can see that Φ_f is a left $G_{\mathbb{Q}}$-invariant continuous function on $G_{\mathbb{A}}$.

For every rational prime p, let $\check{R}_p$ and $\check{U}_p$ be the dual lattices of R_p and U_p respectively. Let $(p^{-\ell(p)})$, $\ell(p) \geqslant 0$ be the $\mathbb{Z}_p$-ideal generated by norms of all elements of $\check{R}_p$ or $\check{U}_p$, according to the cases (I) and (II). Define an open compact subgroup $K_p^{(\ell(p))}$ of $Sp(2, \mathbb{Q}_p)$ by $K_p^{(\ell(p))} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbb{Z}_p) \,\middle|\, c \equiv 0 \bmod p^{\ell(p)} \right\}$, and define a representation M_p of $K_p^{(\ell(p))}$ by $M_p(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \chi_p(\det d)$.

<u>Proposition 1</u>. We have $\pi_p(k)f_p = M_p(k)f_p$ for any $k \in K_p^{(\ell(p))}$.

We set $K_p = K_p^{(\ell(p))}$ and $K_f = \prod_p K_p$. We define a representation M_f of K_f by $M_f = \otimes_p M_p$. By Proposition 1, we have

(4) $\Phi_f(gk) = M_f(k)\Phi_f(g)$ for any $g \in G_{\mathbb{A}}$, $k \in K_f$.

§2. Results on Hecke operators

Let $\widetilde{G}$ be the group of symplectic similitude of genus 2, which is considered as an algebraic group over $\mathbb{Q}$. We assume that for any commutative field k which contains $\mathbb{Q}$, the group $\widetilde{G}_k$ of all k-rational points of $\widetilde{G}$ is given explicitly by $\widetilde{G}_k = \{ g \in GL(4,k) \mid {}^t g w g = m(g)w, m(g) \in k^\times \}$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in GL(4,k)$. To define the action of Hecke operators on Φ_f, we must extend Φ_f to a suitable automorphic form on $\widetilde{G}_{\mathbb{A}}$. Let M be a subgroup of $\widetilde{G}_{\mathbb{A}}$ which consists of all elements $\nu \in \widetilde{G}_{\mathbb{A}}$ such that $\nu_v = [1, 1, \mu_v, \mu_v]$ with $\mu_v \in \mathbb{Z}_v^\times$ if v is a finite place and $\nu_v = [\mu_v, \mu_v, \mu_v, \mu_v]$ with $\mu_v \in \mathbb{R}_+^\times$ if v is the infinite place. By virtue of the decomposition $\mathbb{Q}_{\mathbb{A}}^\times = \mathbb{Q}^\times \cdot \prod_p \mathbb{Z}_p^\times \cdot \mathbb{R}_+^\times$, every $g \in \widetilde{G}_{\mathbb{A}}$ can be written as $g = \gamma g_1 \nu$ with $\gamma \in G_{\mathbb{Q}}$, $g_1 \in G_A$, $\nu \in M$. We put

(5) $\quad \widetilde{\Phi}_f(\gamma g_1 \nu) = \Phi_f(g_1)$.

One can verify easily that a well-defined function $\widetilde{\Phi}_f$ on $\widetilde{G}_{\mathbb{Q}} \backslash \widetilde{G}_A$ is obtained by (5). The restriction of $\widetilde{\Phi}_f$ to G_A coincides with Φ_f. For a rational prime p, we put $\widetilde{G}_{\mathbb{Z}_p} = \widetilde{G}_p \cap GL(4, \mathbb{Z}_p)$. If $K_p = K^{(\ell(p))}$, we set $\widetilde{K}_p = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| g \in G_{\mathbb{Z}_p},\ c \equiv 0 \bmod p^{\ell(p)} \right\}$, and define a representation $\widetilde{M}_p$ of $\widetilde{K}_p$ by $\widetilde{M}_p\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \chi_p(\det d)$. We put $\widetilde{K} = \prod_p \widetilde{K}_p$ and $\widetilde{M}_f = \bigotimes_p \widetilde{M}_p$. Then we have

(6) $\quad \widetilde{\Phi}_f(gk) = \widetilde{M}_f(k)\widetilde{\Phi}_f(g)$ for any $g \in \widetilde{G}_A$, $k \in \widetilde{K}_f$.

Let p be a rational prime such that $\widetilde{K}_p = \widetilde{G}_{\mathbb{Z}_p}$. For a double coset $\widetilde{K}_p B \widetilde{K}_p$, $B \in \widetilde{G}_p$ and for any function Ψ on $\widetilde{G}_A$ which satisfies (6), we put

(7) $\quad ((\widetilde{K}_p B \widetilde{K}_p)\Psi)(g) = \sum_i \Psi(g\, \iota_p(g_i))$,

where $\widetilde{K}_p B \widetilde{K}_p = \bigcup_i g_i \widetilde{K}_p$ (disjoint union) and ι_p denotes the natural injection of $\widetilde{G}_p$ into $\widetilde{G}_A$. We can see that $(\widetilde{K}_p B \widetilde{K}_p)\Psi$ also satisfies (6). The double coset $\widetilde{K}_p [p^{d_1}, p^{d_2}, p^{e_1}, p^{e_2}] \widetilde{K}_p$, $d_1 + e_2 = d_2 + e_1$, is denoted by $T(p^{d_1}, p^{d_2}, p^{e_1}, p^{e_2})$.

To state the results on Hecke operators, first let us assume that we are in Case(I). For each rational prime ℓ, let N_ℓ be the image of $R_\ell^\times$ under the reduced norm. We have $N_\ell = \mathbb{Z}_\ell^\times$ for almost all ℓ. Set $N = \prod_\ell N_\ell$ and let $\mathbb{Q}_A^\times = \bigcup_i \mathbb{Q}^\times a_i (N \times \mathbb{R}_+^\times)$ be a double coset decomposition such that $(a_i)_\infty = 1$ for every i. Take $\widetilde{a}_i \in D_A^\times$ so that the reduced norm of $\widetilde{a}_i$ is a_i. We may assume that $(\widetilde{a}_i)_\infty = 1$ and that $(\widetilde{a}_i)_\ell = 1$ if $N_\ell = \mathbb{Z}_\ell^\times$. We set $f_{i,j}(x) = f(\rho(\widetilde{a}_i, \widetilde{a}_j)x)$ for $x \in X_A$ and $\varphi_{i,j}(h) = \varphi(h(\widetilde{a}_i, \widetilde{a}_j))$ for $h \in D_A^\times \times D_A^\times$. Let $\Phi_f^{(i,j)}$ be the function on $G_{\mathbb{Q}} \backslash G_A$ defined by (2) using $f_{i,j}$ and $\varphi_{i,j}$ instead of f and φ. Let $\widetilde{\Phi}_f^{(i,j)}$ be the extension of $\Phi_f^{(i,j)}$ to G_A defined by (5). We see that $\widetilde{\Phi}_f^{(i,j)}$ satisfies (6). We put $\Phi_f^* = \sum_i \sum_j \widetilde{\Phi}_f^{(i,j)}$.

Then we have

Theorem 1. Let p be an odd prime at which D splits. We assume that R_p is a maximal order of D_p and $\varphi_i \in S(R, n_i)$ are eigenfunctions of $T'(p)$, $i = 1,2$. Put $T'(p)\varphi_i = \lambda_i \varphi_i$, $i = 1,2$. Then we have

(8) $\quad T(1,1,p,p)\Phi_f^* = p(\bar{\lambda}_1 + \bar{\lambda}_2)\Phi_f^*$,

(9) $\quad T(1,p,p,p^2)\Phi_f^* = \{(p^2-1) + p\bar{\lambda}_1\bar{\lambda}_2\}\Phi_f^*$,

where $-$ denotes the complex conjugation.

Now let us assume that we are in Case (II). For each finite place v of F, let N_v be the image of $R_v^\times$ under the reduced norm. Set $N = \prod_v N_v$ and let $F_A^\times = \bigcup_i F^\times a_i(N \times \mathbb{R}_+^\times \times \mathbb{R}_+^\times)$ be a double coset decomposition. We may assume that the idele norm of a_i is 1 and that $(a_i)_{\infty_1} > 0$, $(a_i)_{\infty_2} > 0$. Let $\tilde{a}_i \in D_A^\times$ be an element whose reduced norm is a_i. We may assume that $(\tilde{a}_i)_\infty \in \mathbb{H}^\times \times \mathbb{H}^\times$ belongs to the center of $\mathbb{H}^\times \times \mathbb{H}^\times$. We set $f_i(x) = f(\rho(\tilde{a}_i)x)$ for $x \in X_A$ and $\varphi_i(h) = \varphi(h\tilde{a}_i)$ for $h \in D_A^\times$. Let $\Phi_f^{(i)}$ be the function on $G_{\mathbb{Q}}\backslash G_A$ defined by (3) using f_i and φ_i instead of f and φ. Then $\Phi_f^{(i)}$ satisfies (4). Let $\tilde{\Phi}_f^{(i)}$ be the extension of $\Phi_f^{(i)}$ to $\tilde{G}_A$ defined by (5). We see that $\tilde{\Phi}_f^{(i)}$ satisfies (6). We put $\Phi_f^* = \sum_i \tilde{\Phi}_f^{(i)}$ and assume that

(α) $R^\sigma = R$.

Then we have

Theorem 2. Let p be an odd rational prime which is unramified in F. We assume that D_0 splits at p. If p remains prime in F, we assume that R_p is a maximal order of D_p and φ is an eigenfunction of $T'(p)$. Put $T'(p)\varphi = \lambda\varphi$. Then we have

(10) $\quad T(1,1,p,p)\Phi_f^* = 0$,

(11) $\quad T(1,p,p,p^2)\Phi_f^* = -\{(p^2+1) + p\bar{\lambda}\}\Phi_f^*$,

where $-$ denotes the complex conjugation. If p decomposes into two prime divisors v_1 and v_2 in F, we assume that R_{v_i} is a maximal order of D_{v_i} and that φ is an eigenfunction of $T'(v_i)$ for $i = 1,2$. Put

$T'(v_i)\varphi = \lambda_i\varphi$ for $i = 1,2$. Then we have

(12) $\quad T(1,1,p,p)\overline{\Phi}^{*}_{f} = p(\omega_{v_1}(p)\overline{\lambda}_1 + \omega_{v_2}(p)\overline{\lambda}_2)\overline{\Phi}^{*}_{f}$,

(13) $\quad T(1,p,p,p^2)\overline{\Phi}^{*}_{f} = \left\{(p^2-1) + p\,\overline{\lambda}_1\overline{\lambda}_2\right\}\overline{\Phi}^{*}_{f}$.

We shall sketch a proof of (11). For simplicity, we assume that $(\widetilde{a}_i)_p = 1$. Let $\widetilde{K}_p[1,p,p,p^2]\widetilde{K}_p = \bigcup g_i\widetilde{K}_p$ be a disjoint union such that $m(g_i) = p^2$. We put $f'_p = \sum_i \pi_p([p^{-1},p^{-1},p^{-1},p^{-1}]g_i)f_p$, $f' = \prod_{v\neq p} f_v \times f'_p$ and $f'_i(x) = f'(\rho(\widetilde{a}_i)x)$. Let $\Phi^{(i)}_{f'}$ denote the function on G_A defined by (2) using f'_i and φ_i instead of f and φ, and let $\widetilde{\Phi}_{f'}^{(i)}$ denote the extension of $\Phi_{f'}^{(i)}$ to $\widetilde{G}_A$ defined by (5). Then one can see that $T(1,p,p,p^2)\widetilde{\Phi}^{(i)}_{f}(g) = \widetilde{\Phi}_{f'}^{(i)}(g)$ for $g\in\widetilde{G}_A$. We fix a splitting $D_p \cong M(2,F_p)$ such that R_p is mapped onto $M(2,\mathcal{O}_p)$, where $\mathcal{O}_p$ is the maximal order of F_p. Then we can prove a local relation of Hecke operators;

$$f'_p(x\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}) = -p\Big[\sum_v f_p(\rho(\begin{pmatrix} p & v\\ 0 & 1\end{pmatrix})x) + f_p(\rho(\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix})x)\Big] \tag{14}$$
$$- (p^2+1)f_p(x\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix})\,,$$

where v extends over a complete set of representatives of $\mathcal{O}_p$ mod p. Let $\{h_s\}$ denote the set of elements of R_p; $\begin{pmatrix} p & v\\ 0 & 1\end{pmatrix}$, $v\in\mathcal{O}_p$, v mod p and $\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}$. By (14), we get $T(1,p,p,p^2)\widetilde{\Phi}^{(i)}_{f}(g) = -(p^2+1)\widetilde{\Phi}^{(i)}_{f}(g)$ $- p\sum_s \int_{H_F\backslash H_A}\langle \sum_{x\in X_{\mathbb{Q}}} (\pi(g)f_i)(\rho(h_s)\rho(h)x\begin{pmatrix} p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}),\ \varphi_i(h)\rangle\, dh$, if $g\in G_A$. Let ϖ be the element of $F^{\times}_A$ such that $\varpi_p = p$ and that $\varpi_v = 1$ if $v \neq p$. We take $\gamma\in F^{\times}$ so that $\gamma^{-1}a_i\varpi = a_j nr$ with $n\in N$, $r\in \mathbb{R}^{\times}_{+}\times\mathbb{R}^{\times}_{+}$. Then γ is totally positive. Hence there exists a $\widetilde{\gamma}\in D^{\times}$ such that $N(\widetilde{\gamma}) = \gamma$. We have $\Xi(g) = \sum_s\int_{H_F\backslash H_A}\langle\sum_{x\in X_{\mathbb{Q}}}(\pi(g)f_i)(\rho(h_s)$ $\rho(h)x\begin{pmatrix} p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}),\varphi_i(h)\rangle\, dh = \sum_s\int_{H_F\backslash\widetilde{\gamma}^{-1}H_A h_1\widetilde{a}_i}\langle\sum_{x\in X_{\mathbb{Q}}}(\pi(g)f)$ $(\rho(h')x\begin{pmatrix} p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}),\ \varphi(h'h_s^{-1})\rangle\, dh'$, where dh' denotes a suitable invariant measure on $H_F\backslash\widetilde{\gamma}^{-1}H_A h_1\widetilde{a}_i$. (Note that $H_F\backslash\widetilde{\gamma}^{-1}H_A h_s\widetilde{a}_i$ does not depend on s). We can see that $\sum_s \varphi(h'h_s^{-1}) = \omega_p(p^{-1})\lambda\varphi(h')$ and that $\omega_p(p^{-1}) = 1$. We have $H_F\backslash\tilde{\gamma}^{-1}H_A h_1\tilde{a}_i = H_F\backslash H_A\delta\tilde{a}_j$ with $\delta\in\prod_v R_v^\times \times \mathbb{H}^\times \times \mathbb{H}^\times$. We get

$$\Xi(g) = \bar{\lambda}\int_{H_F\backslash H_A}\left\langle\sum_{x\in X_{\mathbb{Q}}}(\pi(g)f_j)(\rho(h\delta)x\begin{pmatrix}p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}),\ \varphi(h\delta)\right\rangle dh$$

$$= \bar{\lambda}\int_{H_F\backslash H_A}\left\langle\sum_{x\in X_{\mathbb{Q}}}(\pi(g)f_j)(\rho(\delta_\infty)\rho(h)x\begin{pmatrix}p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}),\ \varphi_j(h\delta_\infty)\right\rangle dh,$$

if $g\in G_A$, $g_f = 1$. We can write $\delta_\infty = (p^{-k_1}\delta_1, p^{-k_2}\delta_2)$ with δ_1, $\delta_2 \in\mathbb{H}^{(1)}$, $k_1,k_2\in\mathbb{R}$ such that $k_1 + k_2 = 1$. We have

$$\Xi(g) = \bar{\lambda}\int_{H_F\backslash H_A}\left\langle\sum_{x\in X_{\mathbb{Q}}}(\pi(g)f_j)(\rho(p^{-k_1},p^{-k_2})_\infty\rho(h)x\begin{pmatrix}p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}),\ \varphi_j(h(p^{-k_1},p^{-k_2})_\infty)\right\rangle dh.$$

We can verify that $\varphi_j(h(p^{-k_1},p^{-k_2})_\infty) = \varphi_j(h)$ and that $\sum_{x\in X_{\mathbb{Q}}}(\pi(g)f_j)(\rho(p^{-k_1},p^{-k_2})_\infty\rho(h)x\begin{pmatrix}p^{-1} & 0\\ 0 & p^{-1}\end{pmatrix}) = \sum_{x\in X_{\mathbb{Q}}}(\pi(g)f_j)(\rho(h)x)$. Hence we obtain

$$(15)\qquad T(1,p,p,p^2)\tilde{\Phi}_f^{(i)}(g) = -(p^2+1)\tilde{\Phi}_f^{(i)}(g) - p\bar{\lambda}\tilde{\Phi}_f^{(j)}(g),$$

if $g\in G_A$ and $g_f = 1$. Since $\tilde{\Phi}_f^{(i)}$ and $\tilde{\Phi}_f^{(j)}$ satisfies (6), (15) holds for any $g\in\tilde{G}_A$. Hence (11) follows immediately.

§3. Translation into the classical terminology

In order to obtain a Siegel modular form from Φ_f^*, we must choose $f_\infty\in\mathcal{S}(X_\infty)\otimes V$ appropriately. Let W_n be the space of all functions p on $\mathbb{H}$ such that $p(a+bi+cj+dk) = q(b,c,d)$, where 1,i,j,k are the standard quaternion basis, $a,b,c,d\in\mathbb{R}$ and q is a homogeneous polynomial of degree n with complex coefficients. We put $(\tau_n(g)p)(x) = p(g^*xg)$ for $g\in\mathbb{H}^\times$, $x\in\mathbb{H}$. Then τ_n defines a representation of $\mathbb{H}^\times$ on W_n. We have

$$(16)\qquad \tau_n|\mathbb{H}^{(1)} \cong \begin{cases}(\sigma_{2n}\oplus\sigma_{2n-4}\oplus\cdots\oplus\sigma_0)|\mathbb{H}^{(1)} & \text{if } n \text{ is even,}\\ (\sigma_{2n}\oplus\sigma_{2n-4}\oplus\cdots\oplus\sigma_2)|\mathbb{H}^{(1)} & \text{if } n \text{ is odd.}\end{cases}$$

Let W_n^* be the subspace of W_n consisting of all functions in W_n which transform according to σ_{2n}. We can naturally identify π_∞ with the Weil representation of $G_\infty \cong Sp(2,\mathbb{R})$ realized on $\mathcal{S}(\mathbb{H} \oplus \mathbb{H})$. ($\pi_\infty$ is characterized by (i)$\sim$(iii) with $(x_1,x_2) \in \mathbb{H} \oplus \mathbb{H}$ and $\chi_\infty = 1$, $\gamma_\infty = 1$). Let K_∞ be the standard maximal compact subgroup of G_∞ defined by $K_\infty = \left\{ g = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \ g \in G_\infty \right\} \cong U(2,\mathbb{C})$.

<u>Proposition 2</u>. For $p \in W_n^*$, define $f \in \mathcal{S}(\mathbb{H} \oplus \mathbb{H})$ by $f(x_1,x_2) = p(x_1^* x_2)\exp(-2\pi(N(x_1) + N(x_2)))$. Then we have

$$\pi_\infty\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right) f = \det(A + B\sqrt{-1})^{n+2} f \quad \text{for every} \quad \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_\infty .$$

By virtue of Proposition 2, we can choose $f_\infty \in \mathcal{S}(X_\infty) \otimes V$ so that the following conditions (17)$\sim$(19) are satisfied.

(17) $\pi_\infty\left(\begin{pmatrix} A & B \\ -B & A \end{pmatrix}\right) f_\infty = \det(A + B\sqrt{-1})^{n+2} f_\infty$ for any $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_\infty$.

(18) $f_\infty(\rho(g_1,g_2)x) = (\sigma_0(g_1) \otimes \sigma_{2n}(g_2)) f_\infty(x)$ for any $x \in \mathbb{H} \oplus \mathbb{H}$ and $(g_1,g_2) \in \mathbb{H}^{(1)} \times \mathbb{H}^{(1)}$.

(19) Each component of f_∞ has the form as in Proposition 2.

Hereafter we assume

(β) $n_1 = 0$ and $n_2 = 2n$

and that f_∞ is chosen as above. We set $M_\infty(k) = \det(A + B\sqrt{-1})^{n+2}$ for $k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K_\infty$. For $K = \prod_v K_v$, we define a representation M of K by $M = \bigotimes_v M_v$. By (4) and Proposition 2, we have

(20) $\Phi_f^*(gk) = M(k)\Phi_f^*(g)$ for any $g \in G_A$, $k \in K$.

Let $\mathfrak{H}$ be the Siegel upper half space of genus 2. For $g \in G_\infty$, let $\tilde{g} \in G_A$ be the adele such that $\tilde{g}_f = 1$ and $\tilde{g}_\infty = g$. We define a function J on $\mathfrak{H}$ by

(21) $J(g \cdot \mathring{i}) = \Phi_f^*(\tilde{g})(\det(c\mathring{i} + d))^{n+2}$,

where $\mathring{i} = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$ and $g = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. We put $\Gamma = G_\mathbb{Q} \cap \prod_p K_p$. Then we have $\Gamma = \tilde{\Gamma}_0(N)$, where $N = \prod_p p^{\ell(p)}$. We define a character

M_Γ of Γ by $M_\Gamma(\gamma) = \prod_p M_p(\gamma)$. Since Φ_f^* is left $G_{\mathbb{Q}}$-invariant and satisfies (20), we have

$$(22) \qquad J(\gamma z) = M_\Gamma(\gamma) J(z) \det(cz+d)^{n+2} ,$$

for any $\gamma \in \Gamma$ and $z \in \mathfrak{H}$, where $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$.

The explicit form of J is given as follows. Suppose that we are in Case (I). We put $K^* = (K' \times K') \cap H_A$ and let $H_A = \bigcup_{\ell=1}^{h} H_{\mathbb{Q}} y_\ell K^*$ be a double coset decomposition of H_A. We assume that $(y_\ell)_\infty = 1$, $1 \leq \ell \leq h$. We put $e_\ell = |H_{\mathbb{Q}} \cap y_\ell K^* y_\ell^{-1}|$ and $f_\infty(x) = \begin{pmatrix} p_1(x_1^* x_2) \\ \vdots \\ p_{2n+1}(x_1^* x_2) \end{pmatrix} \exp(-2\pi(N(x_1) + N(x_2))$, $x = (x_1, x_2) \in \mathbb{H} \oplus \mathbb{H}$. Define a V-valued function P on $\mathbb{H}$ by $P(x) = \begin{pmatrix} p_1(x) \\ \vdots \\ p_{2n+1}(x) \end{pmatrix}$. Let ι be the isometric embedding of $D_{\mathbb{Q}}$ into $\mathbb{H}$ derived from the algebra injection $D_{\mathbb{Q}} \hookrightarrow \mathbb{H}$. Set $S = \prod_p (R_p \oplus R_p)$, which is the support of $\prod_p f_p$. Then we have

$$(23) \qquad \begin{aligned} J(z) = \mathrm{vol}(K^*) \sum_{i,j} \sum_{\ell=1}^{h} \sum_{x \in X_{\mathbb{Q}} \cap \rho(y_\ell(\tilde{a}_i, \tilde{a}_j))^{-1} S} & \Big\langle P(\iota(x_1^* x_2)) \\ & e\Big(\sigma\Big(\begin{pmatrix} N(x_1) & \mathrm{Tr}(x_1 x_2^*)/2 \\ \mathrm{Tr}(x_1 x_2^*)/2 & N(x_2) \end{pmatrix} z\Big)\Big),\ \varphi(y_\ell(\tilde{a}_i, \tilde{a}_j))\Big\rangle / e_\ell , \end{aligned}$$

where $\mathrm{vol}(K^*)$ denotes the volume of K^* measured by dh. We can get a similar formula to (23) for the Case (II) under the assumption (α). By virtue of (23), we can see that J(z) is a holomorphic function on $\mathfrak{H}$. Namely we have

<u>Theorem 3</u>. J(z) is a holomorphic Siegel modular form of weight n+2 which satisfies (22).

The classical definition of the action of the Hecke operator $T(p^{d_1}, p^{d_2}, p^{e_1}, p^{e_2})$ on J is as follows. We assume that $\ell(p) = 0$ and put $k = n+2$. Let $\Gamma\{p^{d_1}, p^{d_2}, p^{e_1}, p^{e_2}\}\Gamma = \bigcup_i \Gamma\gamma_i$ be a disjoint union. We put

(24) $$(T(p^{d_1},p^{d_2},p^{e_1},p^{e_2})J)(z) = (p^{d_1+e_2})^{2k-3} \sum_i M_\Gamma(\gamma_i)J(\gamma_i z) \cdot \det(c_i z+d_i)^{-k},$$

where $\gamma_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}$ (cf. Andrianov [1], Matsuda [11]).

Then we can translate Theorem 1 and 2 into the following form.

Theorem 4. Suppose that we are in Case (I) and let the assumptions be the same as in Theorem 1. We have $T(1,1,p,p)J = p^{k-2}(\overline{\lambda}_1 + \overline{\lambda}_2)J$ and $T(1,p,p,p^2)J = p^{2k-6}\left\{(p^2-1) + p\overline{\lambda}_1\overline{\lambda}_2\right\} J$.

By Theorem 4, the p-factor $L_p(s,J)$ of the L-function attached to J in the classical sense is given by

(25) $$L_p(s,J) = \prod_{i=1}^{2} (1 - \overline{\lambda}_i p^{k-2-s} + p^{2k-3-2s})^{-1}.$$

Theorem 5. Suppose that we are in Case (II) and let the assumptions be the same as in Theorem 2. If p remains prime in F, we have $T(1,1,p,p)J = 0$, $T(1,p,p,p^2)J = -p^{2k-6}\left\{(p^2+1) + p\overline{\lambda}\right\} J$. If p decomposes in F, we have $T(1,1,p,p)J = p^{k-2}(\omega_{v_1}(p)\overline{\lambda}_1 + \omega_{v_2}(p)\overline{\lambda}_2)J$, $T(1,p,p,p^2)J = \left\{(p^2-1) + p\overline{\lambda}_1\overline{\lambda}_2\right\} J$.

Let $L_p(s,J)$ be the p-factor of the L-function attached to J in the classical sense. If p decomposes in F, we have

(26) $$L_p(s,J) = \prod_{i=1}^{2} (1 - \overline{\lambda}_i \omega_{v_i}(p)p^{k-2-s} + p^{2k-3-2s})^{-1}.$$

If p remains prime in F, we have

(27) $$L_p(s,J) = (1 - \overline{\lambda} p^{2k-4-2s} + p^{4k-6-4s})^{-1}.$$

Concerning the question when J is a cusp form, we can prove(see also Proposition 4),

Proposition 3. If $n>0$, J is a cusp form.

Remark. The assumption (β) and the corresponding choice of $f_\infty \in \mathcal{S}(X_\infty) \otimes V$ is necessary because; (i) we must choose an f_∞ so that it transforms according to a one-dimensional representation under K_∞, to obtain Siegel modular forms of genus 2 with the usual automorphic factor; (ii) the assumption (β) is required for the coincidence of the Γ-factor in the functional equation of the L-function attached to J with that in [1] and [11], taking account of the results in §2.

In general, there arises a question: Find an $f_\infty \in \mathcal{S}(X_\infty) \otimes V$ which transforms according to $\sigma_{n_1} \otimes \sigma_{n_2}$ under the action of K'_∞ through ρ and which transforms according to a prescribed higher dimensional representation (which depends on n_1 and n_2) under the action of K_∞ through π_∞. If this purely archimedean question is solved, we will be able to construct a Siegel modular form with more general automorphic factor from any pair of $\varphi_1 \in S(R,n_1)$ and $\varphi_2 \in S(R,n_2)$(resp. any $\varphi \in S(R,n_1,n_2,\omega)$) if we are in Case ($\mathrm{I}$) (resp. Case ($\mathrm{II}$)).

§4. The case of the prime level

In this section, we shall consider the simplest case and examine our construction in detail. Namely we assume that we are in Case (I) and that D ramifies only at p and ∞, where p is a fixed prime number. Let R be a maximal order of D and let $D_A^\times = \bigcup_{i=1}^{H} D^\times y_i (\prod_\ell R_\ell^\times \times \mathbb{H}^\times)$ be a double coset decomposition of $D_A^\times$. Note that $N_\ell = \mathbb{Z}_\ell^\times$ for every ℓ. We may assume that the reduced norm of y_i is 1 and that $(y_i)_\infty = 1$ for $1 \leq i \leq H$. For $1 \leq i,j \leq H$, we define a lattice L_{ij} of D by $L_{ij} = D \cap y_i (\prod R_\ell) y_j^{-1}$. Note that L_{ii} is a maximal order of D. We put $R_i = L_{ii}$ and $e_i = |R_i^\times|$. Let $S_k^0(\Gamma_0(p))$ be the space of new forms in $S_k(\Gamma_0(p))$. Assume that $\varphi\ (\neq 0) \in S(R,2m)$ satisfies $T'(\ell)\varphi = \lambda(\ell)\varphi$ for every $\ell \neq p$, where m is any non-negative integer. Then there exists a cusp form $f(\neq 0) \in S_{2m+2}^0(\Gamma_0(p))$ such that $T(\ell)f = \lambda(\ell)\ell^m f$ for every $\ell \neq p$ if $m>0$, and vice versa. If $m = 0$, there exists a modular form $f \in G_2(\Gamma_0(p))$ such that $T(\ell)f = \lambda(\ell)f$ for every $\ell \neq p$, and vice versa. Here $T(\ell)$ denotes the Hecke operator which acts on $G_{2m+2}(\Gamma_0(p))$. These results follow from the well-known work of M.Eichler on the representability of modular forms by theta series.

If f satisfies the above condition, let us call that f corresponds to φ. (f is unique up to constant multiple). We take $\varphi_1(\neq 0) \in S(R,0)$ and $\varphi_2(\neq 0) \in S(R,2n)$. We assume that φ_1 and φ_2 are common-eigenfunctions of $T'(\ell)$, $\ell \neq p$. Put

$$(28) \quad \tilde{\vartheta}_{ij}(z) = \sum_{(x,y)\in L_{ij}\oplus L_{ij}} P(\iota(x^*y))e(\sigma(\begin{pmatrix} N(x) & Tr(xy^*)/2 \\ Tr(xy^*)/2 & N(y) \end{pmatrix} z)),$$

$$z \in \mathfrak{H},$$

$$(29) \quad F(\varphi_1, \varphi_2) = \sum_{i=1}^{H} \sum_{j=1}^{H} \langle \tilde{\vartheta}_{ij}, \ \varphi_1(y_i) \otimes \varphi_2(y_j) \rangle / e_i e_j .$$

Let $f_1 \in G_2(\Gamma_0(p))$ and $f_2 \in G_{2n+2}(\Gamma_0(p))$ be the elliptic modular forms which correspond to φ_1 and φ_2 respectively. Let $L(s,f_1)$ and $L(s,f_2)$ be the Euler products in the classical sense attached to f_1 and f_2 respectively. Then, (23) and Theorem 3 show that the Euler product $L(s,F(\varphi_1, \varphi_2))$ attached to $F(\varphi_1, \varphi_2)$ is equal to $L(s-n,f_1)L(s,f_2)$ up to the 2 and p-factors if $F(\varphi_1, \varphi_2) \neq 0$. ($L(s,F(\varphi_1, \varphi_2))$ is defined by $\prod_{\ell \neq 2,p} L_\ell(s,F(\varphi_1, \varphi_2))$). Suppose that we have taken φ_1 as a constant function on $D_A^\times$. Then f_1 is an Eisenstein series of $G_2(\Gamma_0(p))$ and we have $L(s,f_1) = \zeta(s)\zeta(s-1)(1-p^{1-s})$, where $\zeta(s)$ denotes the Riemann zeta function. For such φ_1, the Euler product of $F(\varphi_1, \varphi_2)$ has a similar form to the examples of Kurokawa[9]. For n = 0, we have the following criterion for $F(\varphi_1, \varphi_2)$ to be a cusp form.

<u>Proposition 4</u>. If n = 0, $F(\varphi_1, \varphi_2)$ is a cusp form if and only if φ_2 is not a constant multiple of φ_1.

Here the main question arises: For which pair (φ_1, φ_2), $F(\varphi_1, \varphi_2)$ does not vanish? Hereafter we shall be concerned with this question. Let ϖ_p be a prime element of D_p. We set $S^+(R,2m) = \{\varphi \in S(R,2m) \mid \varphi(g\,\iota_p(\varpi_p)) = \varphi(g)$ for any $g \in D_A^\times\}$, $S^-(R,2m) = \{\varphi \in S(R,2m) \mid \varphi(g\,\iota_p(\varpi_p)) = -\varphi(g)$ for any $g \in D_A^\times\}$, where ι_p denotes the natural injection of $D_p^\times$ into $D_A^\times$. We have

$$(30) \qquad S(R,2m) = S^+(R,2m) \oplus S^-(R,2m) \text{ (direct sum)} .$$

Proposition 5. If $\varphi_1 \in S^{\pm}(R,0)$ and $\varphi_2 \in S^{\mp}(R,2n)$, we have $F(\varphi_1, \varphi_2) = 0$.

It seems natural to conjecture the converse. Namely

Conjecture. If n is even and $\varphi_1 \in S^{\pm}(R,0)$, $\varphi_2 \in S^{\pm}(R,2n)$, then $F(\varphi_1, \varphi_2)$ would not vanish.

At present, we can only prove that several non-vanishing cusp forms arise by our construction (except for some numerical evidences). We put $y_i \iota_p(\varpi_p) = \gamma y_{j(i)} \delta$ with $\gamma \in D^{\times}$ and $\delta \in \prod_{\ell} R_{\ell}^{\times} \times \mathbb{H}^{\times}$ for every y_i, $1 \leqslant i \leqslant H$. The map $i \longrightarrow j(i)$ induces a permutation of order 2 on H letters. If $i = j(i)$ (resp. $i \neq j(i)$), let us call y_i of the first kind (resp. second kind).

Theorem 6. Let $\varphi_1 \in S(R,0)$ be a non-zero common-eigenfunction of $T'(\ell)$, $\ell \neq p$. We assume that $\varphi_1(y_i) \neq 0$ for some y_i which is of the first kind. We assume that n is even and that $n \geqslant 4$ if $p = 2$. Then there exists $\varphi_2 \in S(R,2n)$ which is a common-eigenfunction of $T'(\ell)$, $\ell \neq p$ such that $F(\varphi_1, \varphi_2) \neq 0$.

Let U(resp. 2V) be the number of y_i's of the first kind(resp. second kind). We have $U + 2V = H$, $U + V = T$, where T is the type number of D. A constant function $\varphi_1 (\neq 0) \in S(R,0)$ satisfies the condition of Theorem 6. Moreover one can see easily that there exist at least $U = 2T - H$ linearly independent $\varphi \in S(R,0)$ such that $\varphi(y_i) \neq 0$ for some y_i which is of the first kind. We note that (cf. A.Pizer[12] for example)

$$U = \begin{cases} h_p/2 & \text{if } p \equiv 1 \bmod 4, \\ 2h_p & \text{if } p \equiv 3 \bmod 8, \\ h_p & \text{if } p \equiv 7 \bmod 8, \end{cases} \tag{31}$$

if $p \geqslant 5$. Here h_p denotes the class number of $\mathbb{Q}(\sqrt{-p})$.

§5. A characterization

Hereafter we fix an odd prime p and the definite quaternion algebra D over $\mathbb{Q}$ whose discriminant is p^2 and assume that we are in Case (I). For $n \in \mathbb{Z}$, we set $\chi(n) = 0$ if $p|n$ and $\chi(n) = (\frac{n}{p})$ if $p \nmid n$. In §4, we constructed a correspondence η_1: $S(R_1, 0) \times S(R_1, 2n) \longrightarrow \widetilde{G}_{n+2}(\widetilde{\Gamma}_0(p))$ which "preserves" Euler products, where R_1 is a maximal order of D. The image of η_1 has the following property.

Proposition 6. Let R be any order of D. For $\varphi_1 \in S(R, 0)$ and $\varphi_2 \in S(R, 2n)$, define J by (23). Let $J(z) = \sum_N a_J(N)e(\sigma(Nz))$ be the Fourier expansion of J, where N extends over all positive semi-definite half integral symmetric matrices. Then we have $a_J(N) = 0$ if $\chi(-\det 2N) = 1$.

Proof. Put $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$, $a, b, c \in \mathbb{Z}$ and assume that $a_J(N) \neq 0$. Then there must exist $x, y \in D$ such that $N(x) = a$, $N(y) = c$, $xy^* + yx^* = b$. Assume $x \neq 0$ and put $t = x^{-1}y$. We have $N(t) = a^{-1}c$, $t + t^* = a^{-1}b$. We may assume that $b^2 - 4ac \neq 0$. Then $\mathbb{Q}(t)$ is isomorphic to the imaginary quadratic field $\mathbb{Q}(\sqrt{b^2 - 4ac})$. Therefore we must have $\chi(b^2 - 4ac) = -1$ or 0. If $y \neq 0$, we can argue similarly. If $x = y = 0$, we have $a = b = c = 0$ and $\chi(-\det 2N) = 0$. This completes the proof.

A simple consideration about the dimension shows that η_1 can not be surjective if n is sufficiently large. To clarify the nature of our conjecture about the characterization, let us first introduce the twisting operator. For $F \in \widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$, let $F(z) = \sum_N a_F(N)e(\sigma(Nz))$ be the Fourier expansion of $F(z)$. We put

$$(32) \qquad (QF)(z) = \sum_N a_F(N)\chi(-\det 2N)e(\sigma(Nz)) .$$

Proposition 7. The operator Q induces an endomorphism of $\widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$.

Proof. For $F \in \widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2, \mathbb{R})$, put $F|[\gamma]_k = F(\gamma z)\det(cz+d)^{-k}$. Then we have $F|[\gamma_1\gamma_2]_k = (F|[\gamma_1]_k)|[\gamma_2]_k$. Take $\varepsilon \in \mathbb{Z}$ so that $\chi(\varepsilon) = -1$. We put

$$Q_1F = \sum_{u=0}^{p-1}\sum_{v=0}^{p-1} F\left|\left[\begin{pmatrix} 1 & U(u,v)/p \\ 0 & 1 \end{pmatrix}\right]_k\right. , \quad Q_2F = \sum_{u=0}^{p-1}\sum_{v=0}^{p-1} F\left|\left[\begin{pmatrix} 1 & \varepsilon U(u,v)/p \\ 0 & 1 \end{pmatrix}\right]_k\right.$$

where $U(u,v) = \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}$. We also put $Q_3F = \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \sum_{w=0}^{p-1} F \Big| \left[\begin{pmatrix} 1 & V(u,v,w)/p \\ 0 & 1 \end{pmatrix}\right]_k$, where $V(u,v,w) = \begin{pmatrix} u & v \\ v & w \end{pmatrix}$. We first show that Q_1, Q_2 and Q_3 induce endomorphisms of $\widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$. Take any $\begin{pmatrix} a & b \\ p^2c & d \end{pmatrix} \in \widetilde{\Gamma}_0(p^2)$. Since a mod $p \in GL(2,\mathbb{Z}/p\mathbb{Z})$, we can find $U' \in M(2,\mathbb{Z})$ so that ${}^tU' = U'$, $aU' \equiv Ud \bmod p$, where $U = U(u,v)$. Then we have

$$(F \Big| \left[\begin{pmatrix} 1 & U/p \\ 0 & 1 \end{pmatrix}\right]_k) \Big| \left[\begin{pmatrix} a & b \\ p^2c & d \end{pmatrix}\right]_k = F \Big| \left[\begin{pmatrix} 1 & U'/p \\ 0 & 1 \end{pmatrix}\right]_k.$$

Since $d \equiv {}^ta^{-1} \bmod p$, we can take U' in the form $U' = U(u',v')$ and the map $U \to U'$ induces a bijection on the set of integral matrices of the form $\begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix}$ taken up to modulo p. Hence we get $Q_1F \big| [\gamma]_k = Q_1F$ for any $\gamma \in \widetilde{\Gamma}_0(p^2)$. By virtue of the criterion that $F \in \widetilde{G}_k(\widetilde{\Gamma}_0(p^2))$ is a cusp form if and only if $\det(\mathrm{Im}(z))^{k/2}F(z)$ is bounded on $\mathfrak{H}$, we see immediately that Q_1F is a cusp form, where $\mathrm{Im}(z)$ denotes the imaginary part of $z \in \mathfrak{H}$. For Q_2 and Q_3, we can use similar arguments. For $a,b,c \in \mathbb{Z}$, define a character sum $G(a,b,c)$ by $G(a,b,c) = \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} e((au^2+buv+cv^2)/p)$. By a standard evaluation using Gaussian sums, we get

$$(33)\quad G(a,b,c) = \begin{cases} p\chi(b^2-4ac) & \text{if } b^2-4ac \not\equiv 0 \bmod p, \\ \chi(a)pG & \text{if } b^2-4ac \equiv 0 \bmod p \text{ and } a \not\equiv 0 \bmod p, \\ \chi(c)pG & \text{if } b^2-4ac \equiv 0 \bmod p \text{ and } c \not\equiv 0 \bmod p, \\ p^2 & \text{if } a \equiv b \equiv c \equiv 0 \bmod p, \end{cases}$$

where $G = \sum_{u=0}^{p-1} \chi(u)e(u/p) = \sqrt{(-1)^{(p-1)/2}p}$. Using (33), we get $QF = (Q_1F + Q_2F)/2p - Q_3F/p^2$, hence our assertion.

We define subspaces V_+, V_- and Y of $\widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$ by

$$V_+ = \left\{F \in \widetilde{S}_k(\widetilde{\Gamma}_0(p^2)) \,\middle|\, a_F(N) = 0 \text{ if } \chi(-\det 2N) = 1\right\},$$
$$V_- = \left\{F \in \widetilde{S}_k(\widetilde{\Gamma}_0(p^2)) \,\middle|\, a_F(N) = 0 \text{ if } \chi(-\det 2N) = -1\right\},$$
$$Y = \left\{F \in S_k(\widetilde{\Gamma}_0(p^2)) \,\middle|\, a_F(N) = 0 \text{ if } p \nmid \det 2N\right\}.$$

It is obvious that $Y = V_+ \cap V_-$. Let W_+ (resp. W_-) be the orthogonal complement of Y in V_+ (resp. V_-) with respect to the Petersson inner product (cf. Maaß [10]) in $\widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$. For a positive integer m such that $p \nmid m$, let $T(m)$ be the Hecke operator which acts on $\widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$. (cf. [1], [11]).

Lemma 1. V_+, V_- and Y are stable under the action of the Hecke operator $T(m)$ for $p \nmid m$.

Proof. It is sufficient to show that V_+, V_- and Y are stable under all $T(\ell^\delta)$, where ℓ is a rational prime different from p and δ is a positive integer. Then our assertion follows immediately from proposition 1 of Andrianov [1], noting that his result holds also for our case without any modification.

The following Lemma can also be proven using proposition 1 of [1].

Lemma 2. Assume that $F \in \widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$ is a common-eigenfunction of $T(m)$ for $p \nmid m$. Put $T(m)F = \lambda_F(m)F$. Then we have $T(m)QF = \lambda_F(m)QF$.

Take any $F \in \widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$. It is clear that $F + QF \in V_-$ and $F - QF \in V_+$. Hence we have an orthogonal decomposition

(34) $$\widetilde{S}_k(\widetilde{\Gamma}_0(p^2)) = W_+ \oplus Y \oplus W_- .$$

With respect to the Petersson inner product in $\widetilde{S}_k(\widetilde{\Gamma}_0(p^2))$, $T(m)$, $(m,p) = 1$ are mutually commutative self-adjoint operators. Hence we can take a basis of W_+(resp. Y, W_-) so that every element of the basis is a common-eigenfunction of $T(m)$, $(m,p) = 1$.

Proposition 8. For every positive integer m such that $(m,p) = 1$, we have $-\mathrm{Trace}(T(m)\circ Q \mid S_k(\widetilde{\Gamma}_0(p^2))) = \mathrm{Trace}(T(m)\mid W_+) - \mathrm{Trace}(T(m)\mid W_-)$.

Proof. It is clear that $QY = 0$. Let $F_1, \cdots, F_t$ (resp. $H_1, \cdots, H_u$) be a basis of W_+(resp. W_-) which consists of common-eigenfunctions of $T(m)$, $(m,p) = 1$. Put $T(m)F_i = \lambda_i(m)F_i$ and $T(m)H_j = \mu_j(m)H_j$. Clearly we have $-QF_i - F_i \in Y$. Put $G = -QF_i - F_i$. Then we get $-(T(m)\circ Q)F_i = \lambda_i(m)F_i + T(m)G$ and $T(m)G \in Y$ by Lemma 1. Similarly we have $-(T(m)\circ Q)H_j = -\mu_j(m)H_j + L$ with $L \in Y$. Hence our assertion follows immediately.

Let $K = \mathbb{Q}_p(\sqrt{p})$ be a ramified quadratic extension of $\mathbb{Q}_p$. We set $B = \left\{ \begin{pmatrix} \alpha & \beta \\ u\beta^\tau & \alpha^\tau \end{pmatrix} \middle| \alpha, \beta \in K \right\}$, where τ denotes the generator of $\mathrm{Gal}(K/\mathbb{Q}_p)$ and $u \in Z_p^\times$ is a quadratic non-residual element modulo p. Then B has a structure of the division quaternion algebra over $\mathbb{Q}_p$. Hence $B \cong D_p$. Let $\mathcal{O}$ be the ring of integers of K and let $\mathfrak{P} = (\sqrt{p})$ be the maximal ideal of $\mathcal{O}$. For non-negative integer r, set

$$M_{r+1} = \left\{ \begin{pmatrix} \alpha & \beta \\ u\beta^\tau & \alpha^\tau \end{pmatrix} \middle| \alpha \in \mathcal{O}, \beta \in \mathfrak{P}^r \right\} . \tag{35}$$

Then M_{r+1} is an order of D_p. Especially M_1 is the maximal order of D_p and M_2 is an order of "level p^2" of D_p, which was first studied in A. Pizer [13] (cf. also Hijikata-Pizer-Shemanske [7] for more general cases). Let R_{r+1} be an order of D such that $(R_{r+1})_\ell$ is a maximal order of D_ℓ if $\ell \neq p$ and that $(R_{r+1})_p \cong M_{r+1}$.

Our results in §2 and §3 give a correspondence η_2: $S(R_2,0) \times S(R_2,2n) \longrightarrow \widetilde{S}_{n+2}(\widetilde{\Gamma}_0(p^2))$ which preserves Euler products if $n > 0$. We have $\mathrm{Im}\, \eta_2 \subseteq V_+$ by Proposition 6. Let Z be the orthogonal projection of $\mathrm{Im}\, \eta_2$ to W_+. This orthogonal projection commutes with the action of $T(m)$, $(m,p) = 1$, by Lemma 1. In particular Z is stable under the action of $T(m)$, $(m,p) = 1$. Let W_+' be the orthogonal complement of Z in W_+. We conjecture the following characterization (C) of Z.

(C) Let $F_1, \cdots, F_v$ (resp. $H_1, \cdots, H_u$) be a basis of W_+' (resp. W_-) which consists of common-eigenfunctions of $T(m)$, $(m,p) = 1$. Then $v = u$ and $\{F_i\}$ and $\{H_j\}$ are in one-to-one correspondence in such a way that F_i and H_i have the same eigenvalue for every $T(m)$, $(m,p) = 1$.

Thus we expect that the trace of $T(m) \circ Q | \widetilde{S}_{n+2}(\widetilde{\Gamma}_0(p^2))$ would be expressed in terms of the traces of Hecke operators on certain subspaces of $S(R_2,0)$ and of $S(R_2,2n)$.

Remark. This "characterization" is somewhat similar to that of elliptic modular cusp forms which correspond to L-functions with Grössencharacters of an imaginary quadratic field. In the elliptic modular case, the trace formula was first applied to the twisting operator by Shimura [16] and was exploited further by Saito-Yamauchi [14].

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