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Whittaker Models for Representations with Highest Weights

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Introduction

The concept of Whittaker models for irreducible admissible $(\mathfrak{g}, K)$-modules was first introduced by Jacquet-Langlands [4] in connection with the theory of automorphic forms. The existence and the uniqueness of such models were studied by many authors (see for example [1], [3], [6], [8] and [10]).

In this article we consider the class of irreducible admissible $(\mathfrak{g}, K)$-modules with highest weights (including the holomorphic discrete series) introduced by Harish-Chandra [2]. Unfortunately except for the case when $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, they cannot have Whittaker models in the usual sence (see Corollary 3.2). Hence we generalize the concept of Whittaker models (see Definition 2.1) and discuss the existence and the uniqueness of such generalized Whittaker models for irreducible admissible $(\mathfrak{g}, K)$-modules with highest weights (see Theorem 4.4).

Throughout the paper we denote the dual space of a real or complex vector space V by V^* and in addition we denote by V_c the complexification of a real vector space V. For a Lie group G with Lie algebra $\mathfrak{g}$, the action of $X \in \mathfrak{g}$ on a smooth function f on G as a left invariant vector field is denoted by $f(x; X)$ or $R(X)f(x)$ $(x \in G)$.

§1. Preliminaries

Let g be a simple Lie algebra over $\mathbb{R}$. Let $g = k + p$ be a Cartan decomposition of g. We assume that the center z of k is non-empty. Then $\dim z = 1$ and there exists an element Z in z such that $\mathrm{ad}(Z)$ on p gives a complex structure on p. We set

$$p_+ = \{X \in p_c : [Z, X] = iX\}, \quad p_- = \{X \in p_c : [Z, X] = -iX\}.$$

Then p_+ and p_- are abelian subalgebras of g_c stable under the adjoint action of k_c.

Let G_c be the simply connected complex Lie group with Lie algebra g_c. Let G and K be the analytic subgroups of G_c corresponding to g and k respectively. Then G/K has a G-invariant complex structure such that p_+ can be identified with the space of holomorphic tangent vectors at the origin of G/K.

Let t be a Cartan subalgebra of g contained in k. Let Δ be the set of non-zero roots of g_c with respect to t_c. A root α is said to be compact (resp. non-compact) if the root space g^α is in k_c (resp. p_c). Let Δ_k (resp. Δ_p) be the set of compact (resp. non-compact) roots in Δ. Then $\Delta = \Delta_k \cup \Delta_p$ (a disjoint union). Choose a system of positive roots Δ^+ in such a way that if we set $\Delta_k^+ = \Delta_k \cap \Delta^+$, $\Delta_p^+ = \Delta_p \cap \Delta^+$ then

$$p_+ = \sum_{\beta\in\Delta_p^+} g^\beta .$$

Put $\overline{\Delta}^+ = \Delta_k^+ \cup (-\Delta_p^+)$. Then $\overline{\Delta}^+$ is again a system of positive roots.

To make computations with root vectors, we fix a normalization of them. Let $B(\ ,\ )$ be the Killing form on g and let $(\ ,\ )$ be the inner product on it^* induced by the Killing form. We denote by θ the Cartan involution corresponding to $g = k + p$ and by $X \to \overline{X}$ $(X \in g_c)$ the conjugation of g_c relative to g. Select $E_\alpha \in g^\alpha$ $(\alpha \in \Delta)$ such that

$$B(E_\alpha, E_{-\alpha}) = 2/(\alpha, \alpha) \quad \text{and} \quad \theta\overline{E}_\alpha = -E_{-\alpha}.$$

Set $H_\alpha = [E_\alpha, E_{-\alpha}]$ $(\alpha \in \Delta)$. Then $H_\alpha \in it$ and $\alpha(H_\alpha) = 2$. Note that if $\alpha \in \Delta_p$, then $E_\alpha + E_{-\alpha}$, $i(E_\alpha - E_{-\alpha})$ are in g and if $\alpha \in \Delta_k$ then $E_\alpha - E_{-\alpha}$, $i(E_\alpha + E_{-\alpha})$ are also in g. Two roots α, β are said to be strongly orthogonal if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root.

Let $P_o = \{\gamma_1, \ldots, \gamma_r\}$ be the maximal set of strongly orthogonal non-compact positive roots which is constructed inductively as follows. Let γ_1 be the highest non-compact positive root. For each $j \geq 1$, let γ_{j+1} be the highest one among the non-compact positive roots strongly orthogonal to $\gamma_1, \ldots, \gamma_j$. Put

$$A_j = E_{\gamma_j} + E_{-\gamma_j} \qquad (1 \leq j \leq r)$$

and

$$a = \sum_{j=1}^r \mathbb{R}A_j.$$

Then $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{p}$. Let Σ be the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. For $\lambda \in \Sigma$, we denote the corresponding root space by $\mathfrak{n}^{\lambda}$. Choose a lexicographic order on $\mathfrak{a}^*$ relative to the basis $\{A_1, \ldots, A_r\}$ of $\mathfrak{a}$. Let Σ^+ be the set of positive roots in Σ. If we set

$$\mathfrak{n} = \sum_{\lambda \in \Sigma^+} \mathfrak{n}^{\lambda}$$

Then we get an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}.$$

For later use, we shall study about the relation between Δ and Σ and study more closer structure of $\mathfrak{n}$.

Let $\mathfrak{t}_-$ be a real linear subspace of $\mathfrak{t}_c$ defined by

$$\mathfrak{t}_- = \sum_{j=1}^{r} \mathbb{R}H_{\gamma_j}.$$

Note that $\gamma_i(H_{\gamma_j}) = 2\delta_{ij}$ and $\{\gamma_1, \ldots, \gamma_r\}$ defines a basis of $\mathfrak{t}_-^*$. Set for $1 \le i < j \le r$

$$P_{ij} = \{\beta \in \Delta_{\mathfrak{p}}^+ : \beta|_{\mathfrak{t}_-} = (\gamma_i + \gamma_j)/2\}$$

and for $1 \le i \le r$

$$P_i = \{\beta \in \Delta_{\mathfrak{p}}^+ : \beta|_{\mathfrak{t}_-} = \gamma_i/2\}.$$

Then from the results of C. C. Moore [7], it follows that

$$\Delta_{\mathfrak{p}}^+ = P_o \cup (\cup P_{ij}) \cup (\cup P_i) \qquad \text{(a disjoint union)}.$$

Let $c \in G_c$ (the Cayley transform) defined by

$$c = \exp\left\{-\frac{\pi}{4}\sum_{j=1}^{r}(E_{\gamma_j} - E_{-\gamma_j})\right\}.$$

Then $\mathrm{Ad}(c)$ induces a linear isomorphism of $\mathfrak{t}_-$ onto $\mathfrak{a}$. For each j $(1 \leq j \leq r)$, let $\lambda_j \in \mathfrak{a}^*$ such that

$$\lambda_j(A) = \gamma_j(\mathrm{Ad}(c^{-1})A) \qquad (A \in \mathfrak{a}).$$

Then the following two cases occur. Namely the set Σ^+ is either of the form

(case I) $\Sigma^+ = \{(\lambda_i - \lambda_j)/2 : 1 \leq i < j \leq r\} \cup \{(\lambda_i + \lambda_j)/2 : 1 \leq i \leq j \leq r\}$

or of the form

(case II) $\Sigma^+ = \{(\lambda_i - \lambda_j)/2 : 1 \leq i < j \leq r\} \cup \{(\lambda_i + \lambda_j)/2 : 1 \leq i \leq j \leq r\} \cup \{\lambda_i/2 : 1 \leq i \leq r\}$.

Case I occurs if and only if G/K is analytically equivalent to a tube domain.

We introduce linear subspaces $\mathfrak{n}_0$, $\mathfrak{n}_{1/2}$, $\mathfrak{n}_1$ and $\mathfrak{h}$ of $\mathfrak{g}$ by

$$\mathfrak{n}_0 = \sum_{1 \leq i < j \leq r} \mathfrak{n}^{(\lambda_i - \lambda_j)/2}$$

$$\mathfrak{n}_{1/2} = \sum_{1 \leq i \leq r} \mathfrak{n}^{\lambda_i/2},$$

$$\mathfrak{n}_1 = \sum_{1 \leq i \leq j \leq r} \mathfrak{n}^{(\lambda_i + \lambda_j)/2},$$

$$\mathfrak{h} = \mathfrak{n}_1 + \mathfrak{n}_{1/2}$$

It follows that $[\mathfrak{n}_p, \mathfrak{n}_q] \subset \mathfrak{n}_{p+q}$ for $p, q \in \{0, 1/2, 1\}$ where $\mathfrak{n}_{p+q} = (0)$ if $p + q > 1$. This implies that $\mathfrak{n}_0$, $\mathfrak{n}_1$ and $\mathfrak{h}$ are subalgebras of $\mathfrak{g}$. Clearly $\mathfrak{n}_1$ is the center of $\mathfrak{h}$ and $\mathfrak{h}$ is an ideal of $\mathfrak{n} = \mathfrak{h} + \mathfrak{n}_0$.

Let A, N_0, N_1, H and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}$, $\mathfrak{n}_0$, $\mathfrak{n}_1$, $\mathfrak{h}$ and $\mathfrak{n}$ respectively. Then N_1 is the center of H, $N = HN_0$ is the semidirect product of H and N_0 and $G = NAK$ (an Iwasawa decomposition of G).

We state the key lemma which describes the Iwasawa decomposition of $E_{-\beta}$ $(\beta \in \Delta_{\mathfrak{p}}^+)$. We denote the $\mathfrak{k}_c$-component, $\mathfrak{a}_c$-component and $\mathfrak{n}_c$-component of the Iwasawa decomposition of $X \in \mathfrak{g}_c$ by $P_{\mathfrak{k}}X$, $P_{\mathfrak{a}}X$ and $P_{\mathfrak{n}}X$ respectively.

Lemma 1.1. (i) For $\gamma_j \in P_0$ $(1 \le i \le r)$, we obtain $P_{\mathfrak{k}}E_{-\gamma_j} = -H_{\gamma_j}/2$, $P_{\mathfrak{a}}E_{-\gamma_j} = A_j/2$ and consequently $P_{\mathfrak{n}}E_{-\gamma_j} = E_{-\gamma_j} + H_{\gamma_j}/2 - A_j/2$.

(ii) If we set

$$X_j = iH_{\gamma_j}/2 - i(E_{\gamma_j} - E_{-\gamma_j})/2 \qquad \text{for} \qquad 1 \le j \le r,$$

then $X_j \in \mathfrak{n}^{\lambda_j}$ and $P_{\mathfrak{n}}E_{-\gamma_j} = -iX_j$.

(iii) If $\beta \in P_{ij}$ $(1 \le i < j \le r)$ or $\beta \in P_i$ $(1 \le i \le r)$, then $P_{\mathfrak{k}}E_{-\beta} = -[E_{\gamma_i}, E_{-\beta}]$, $P_{\mathfrak{a}}E_{-\beta} = 0$ and $P_{\mathfrak{n}}E_{-\beta} = E_{-\beta} + [E_{\gamma_i}, E_{-\beta}]$. Moreover $P_{\mathfrak{k}}E_{-\beta} \in \mathfrak{g}^{\gamma_i - \beta}$ and $\gamma_i - \beta \in \Delta_{\mathfrak{k}}^+$.

(iv) For $\beta \in P_{ij}$ $(1 \le i < j \le r)$, set

$$U_\beta = (P_{\mathfrak{n}}E_{-\beta} - [E_{\gamma_j}, P_{\mathfrak{n}}E_{-\beta}])/2$$

and

$$X_\beta = i(P_n E_{-\beta} + [E_{\gamma_j}, P_n E_{-\beta}])/2.$$

Then $U_\beta \in n_c^{(\lambda_i-\lambda_j)/2}$, $X_\beta \in n_c^{(\lambda_i+\lambda_j)/2}$, $P_n E_{-\beta} = U_\beta - iX_\beta$ and $[U_\beta, X_o] = X_\beta$. Here we put

$$X_o = \textstyle\sum_{j=1}^r X_j.$$

(v) For $\beta \in P_i$ $(1 \le i \le r)$, we get $P_n E_{-\beta} \in n_c^{\lambda_i/2} \cap n_{1/2}^-$. Here we set a complex subspace $n_{1/2}^-$ of $(n_{1/2})_c$ by

$$n_{1/2}^- = (n_{1/2})_c \cap (k_c + p_-).$$

(vi) $\{U_\beta : \beta \in P_{ij}\}$ (resp. $\{X_\beta : \beta \in P_{ij}\}$) forms a basis of $n_c^{(\lambda_i-\lambda_j)/2}$ (resp. $n_c^{(\lambda_i+\lambda_j)/2}$). Furthermore $\{P_n E_{-\beta} : \beta \in P_i$ $1 \le i \le r\}$ forms a basis of $n_{1/2}^-$.

Sketch of proof. The assertions (i) and (ii) are well known (see [9]). The former assertion in (iii) is a special case of proposition 5.2 in [5]. The latter follows from our choice of P_o. The assertion (iv) follows from direct computations with the brackets $[A, U_\beta]$, $[A, X_\beta]$ with $A \in a$ and $[U_\beta, X_o]$. (iii) implies $P_n E_{-\beta} \in k_c + p_-$. The assertion $P_n E_{-\beta} \in n_c^{\lambda_i/2}$ $(\beta \in P_i)$ is proved in the same way as in (iv). Finally the assertion (vi) follows from the linear independence of root vectors.

§2. Definition of Whittaker models

By a $(\mathfrak{g}, K)$-module we will mean a $\mathfrak{g}$-module and a K-module V such that

1) if $v \in V$, $X \in \mathfrak{g}$, $k \in K$ then $k \cdot (X \cdot v) = Ad(k)X \cdot (k \cdot v)$,

2) if $v \in V$ then $\{X \cdot v : X \in \mathfrak{k}\}$ spans a finite dimensional subspace, on which K acts continuously (hence smoothly),

3) if $X \in \mathfrak{k}$, $v \in V$ then $\frac{d}{dt} \exp tX \cdot v|_{t=o} = X \cdot v$.

Let $\hat{K}$ be the set of all equivalence classes of irreducible K-modules. A $(\mathfrak{g}, K)$-module is said to be admissible if each K-isotypic component of type τ with $\tau \in \hat{K}$ is finite dimensional.

Now we introduce the notion of a Whittaker model of an irreducible admissible $(\mathfrak{g}, K)$-module.

Let χ be a unitary representation of N on a Hilbert space $H(\chi)$. We denote by $C^\infty(G, \chi)$ the space of smooth functions f on G with values in $H(\chi)$ such that $f(ng) = \chi(n)f(g)$ for $n \in N$, $g \in G$. The right translation of f by $g \in G$ is denoted by $R(g)f$. We define the action of $X \in \mathfrak{g}$ on $C^\infty(G, \chi)$ as a left invariant vector field by

$$R(X)f(x) = \frac{d}{dt} f(x \exp tX)|_{t=o} \qquad (x \in G),$$

which we often denote by $f(x; X)$.

Let $(\tau, V_\tau) \in \hat{K}$ and let (τ^*, V^*) be the K-module contragredient to (τ, V_τ). A function f in $C^\infty(G, \chi)$ is said to be K-finite of type τ if $\{R(k)f : k \in K\}$ spans a finite dimensional subspace of $C^\infty(G, \chi)$ on which K acts according to τ. Let $C^\infty(G, \chi)_\tau$ be the subspace of those functions and let $C^\infty(G, \chi)^o$ be the subspace of all K-finite functions in $C^\infty(G, \chi)$, that is,

$$C^\infty(G, \chi)^o = \sum_{\tau\in\hat{K}} C^\infty(G, \chi)_\tau .$$

If we introduce the space $C^\infty(G, \chi, \tau)$ of smooth functions F on G with values in $H(\chi) \otimes V_\tau^*$ such that

$$F(n\, g\, k) = \chi(n) \otimes \tau^*(k^{-1})F(g) \qquad (g \in G,\ n \in N,\ k \in K)$$

and define for $v \in V_\tau$ and $F \in C^\infty(G, \chi, \tau)$

$$f_v(g) = \langle F(g), v\rangle \qquad (g \in G),$$

then clearly $f_v \in C^\infty(G, \chi)_\tau$ and all elements in $C^\infty(G, \chi)_\tau$ can be written in the form described above.

The space $C^\infty(G, \chi)^o$ is clearly a $(\mathfrak{g}, K)$-module under R.

Definition 2.1. Let (π, V) be an irreducible admissible $(\mathfrak{g}, K)$-module. We say that (π, V) has a Whittaker model of type χ if it is isomorphic to a submodule of $C^\infty(G, \chi)^o$.

We recall the definition of irreducible admissible $(\mathfrak{g}, K)$-modules with highest weights introduced in [2]. Let $\Lambda \in i\mathfrak{t}^*$ such that

(2.1) $2(\Lambda, \alpha)/(\alpha, \alpha) \in \mathbb{Z}^+$ for all $\alpha \in \Delta_{\mathfrak{k}}^+$ and

(2.2) Λ lifts to a character of T, where T is the Cartan subgroup of G corresponding to $\mathfrak{t}$.

For such Λ, Harish-Chandra showed in [2] that there exists an irreducible admissible $(\mathfrak{g}, K)$-module (π, V) which contains a non-zero vector $f_\Lambda \in V$ satisfying

1) $\pi(H)f_\Lambda = \Lambda(H)f_\Lambda$ for $H \in \mathfrak{t}_c$,

2) $\pi(E_\alpha)f_\Lambda = 0$ for $\alpha \in \overline{\Delta}^+$

Such a $(\mathfrak{g}, K)$-module is unique up to equivalence. We call it an irreducible admissible $(\mathfrak{g}, K)$-module with highest weight Λ relative to a positive root system $\overline{\Delta}^+$ and denote it simply by π_Λ.

We will consider the problem for what kind of χ π_Λ has a Whittaker model of type χ. For that purpose we will study whether there exists a non-zero function f_Λ in $C^\infty(G, \chi)^0$ such that

(2.3) $R(H)f_\Lambda = \Lambda(H)f_\Lambda$ for $H \in \mathfrak{t}_c$,

(2.4) $R(E_\alpha)f_\Lambda = 0$ for $\alpha \in \Delta_k^+$,

(2.5) $R(E_{-\beta})f_\Lambda = 0$ for $\beta \in \Delta_p^+$.

In view of (2.1) and (2.2) there exists an irreducible K-module $(\tau_\Lambda, V_\Lambda)$ with highest weight Λ relative to Δ_k^+. Let $v_\Lambda \in V_\Lambda$ be the non-zero highest weight vector. Then the first two conditions (2.3) and (2.4) imply that f_Λ is K-finite of type τ_Λ and it must be of the form

(2.6) $f_\Lambda(g) = \langle F(g), v_\Lambda \rangle$ $(g \in G)$

for some $F \in C^\infty(G, \chi, \tau_\Lambda)$. Thus our problem is reduced to find a non-zero function f_Λ of the form (2.6) which satisfies (2.5).

§3. The case of a non-degenerate character

We treat the case when χ is a non-degenerate character of N. We recall the definition of a non-degenerate character. Let η be a Lie algebra homomorphism of $\mathfrak{n}$ into $\mathbb{R}$. Then it is trivial on $[\mathfrak{n}, \mathfrak{n}]$. So if we denote by $S = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots in Σ^+, then η is uniquely determined by its restrictions η_{α_i} to $\mathfrak{n}^{\alpha_i}$ $(1 \leq i \leq r)$. We say that η is non-degenerate if all η_{α_i} are not zero. Note that every character of N is of the form

$$\chi_\eta(\exp X) = \exp 2\pi i\eta(X) \qquad (X \in \mathfrak{n})$$

with a Lie algebra homomorphism η of $\mathfrak{n}$ into $\mathbb{R}$. We say that a character χ_η of N is non-degenerate if η is non-degenerate. In view of our choice of Σ^+ in §1, we obtain that S is either of the form

(case I) $\quad S = \{(\lambda_1 - \lambda_2)/2, \ldots, (\lambda_{r-1} - \lambda_r)/2, \lambda_r\}$

or of the form

(case II) $\quad S = \{(\lambda_1 - \lambda_2)/2, \ldots, (\lambda_{r-1} - \lambda_r)/2, \lambda_r/2\}$,

Let χ_η be a character of N. Let $f_\Lambda \in C^\infty(G, \chi_\eta)^0$ such that it is of the form (2.6). Then f_Λ is completely determined by its restriction to A. Note that f_Λ satisfies $f_\Lambda(a; H) = \Lambda(H)f_\Lambda(a)$ for $H \in \mathfrak{t}_c$, $f_\Lambda(a; E_\alpha) = 0$ for $\alpha \in \Delta_{\mathfrak{k}}^+$ and

$$f_\Lambda(a; X) = 2\pi i\eta(\mathrm{Ad}(a)X)f_\Lambda(a) \qquad (X \in \mathfrak{n},\ a \in A).$$

In view of Lemma 1.1, we can reduce (2.5) to the following three equations:

(3.1) $f_\Lambda(a; A_j) = \{\Lambda(H_{\gamma_j}) - 4\pi\eta(\mathrm{Ad}(a)X_j)\}f_\Lambda(a)$ for $1 \le j \le r$,

(3.2) $\{2\pi i\eta(\mathrm{Ad}(a)U_\beta) + 2\pi\eta(\mathrm{Ad}(a)X_\beta)\}f_\Lambda(a) = 0$ for $\beta \in \cup P_{ij}$,

(3.3) $2\pi i\eta(\mathrm{Ad}(a)P_n E_{-\beta})f_\Lambda(a) = 0$ for $\beta \in \cup P_i$.

First we are concerned in (case I). Then we have only to consider (3.1) and (3.2). Since $\eta([\mathfrak{n}, \mathfrak{n}]) = (0)$, (3.1) and (3.2) are written as

(3.1)' $f_\Lambda(a; A_j) = \Lambda(H_{\gamma_j})f_\Lambda(a)$ $(1 \le j \le r - 1)$

and

$$f_\Lambda(a; A_r) = \{\Lambda(H_{\gamma_r}) - 4\pi\eta(\mathrm{Ad}(a)X_r)\}f_\Lambda(a),$$

(3.2)' $2\pi i\eta(\mathrm{Ad}(a)U_\beta)f_\Lambda(a) = 0$ for $\beta \in \bigcup_{1\le i\le r-1} P_{i\,i+1}$.

Consequently it follows that to exist a non-zero function f_Λ satisfying (3.1)', (3.2)' it is necessary and sufficient that $\eta = 0$ on each $\mathfrak{n}^{(\lambda_i-\lambda_{i+1})/2}$ $(1 \le i \le r - 1)$. If otherwise, the fact that $\{U_\beta\colon \beta \in P_{i\,i+1}\}$ is a basis of $\mathfrak{n}_c^{(\lambda_i-\lambda_{i+1})/2}$ leads that we can obtain $\eta(\mathrm{Ad}(a)U_\beta) \ne 0$ for some $\beta \in P_{i\,i+1}$. Moreover if η satisfies the condition stated above, we get, by using the properties of the Cayley transform and the fact $[A_r, X_r] = 2X_r$,

(3.4) $f_\Lambda(a) = \exp \Lambda(\mathrm{Ad}(c^{-1})\log a)\exp\{-2\pi\eta(\mathrm{Ad}(a)X_r)\}$.

Secondly we are concerned in (case II). In this case (3.1), (3.2) and (3.3) are written as

$$(3.1)'' \quad f_\Lambda(a;\, A_j) = \Lambda(H_{\gamma_j}) f_\Lambda(a) \qquad (1 \le j \le r)$$

$$(3.2)'' \quad 2\pi i \eta(\mathrm{Ad}(a)U_\beta) f_\Lambda(a) = 0 \qquad \text{for } \beta \in \bigcup_{1\le i\le r-1} P_{ii+1},$$

$$(3.3)'' \quad 2\pi i \eta(\mathrm{Ad}(a)P_n E_{-\beta}) f_\Lambda(a) = 0 \qquad \text{for } \beta \in P_r.$$

Similarly it follows that to exist a non-zero function f_Λ satisfying (3.1)", (3.2)" and (3.3)" it is necessary and sufficient that η is identically zero. If η is zero, namely, $\chi_\eta = 1$, then $f_\Lambda(a)$ is given by

$$(3.5) \qquad f_\Lambda(a) = \exp \Lambda(\mathrm{Ad}(c^{-1})\log a) \qquad (a \in A).$$

As a consequence we obtain:

Theorem 3.1. Let χ_η be a character of N and let π_Λ be the irreducible admissible $(\mathfrak{g}, K)$-module with highest weight Λ. Then the necessary and sufficient condition that π_Λ has a Whittaker model of type χ_η is that $\eta = 0$ on $\mathfrak{n}^{(\lambda_i-\lambda_{i+1})/2}$ for $1 \le i \le r - 1$ if G/K is a tube domain and $\eta = 0$ on $\mathfrak{n}$ if otherwise. Moreover when η satisfies the above condition, the Whittaker model is unique.

Remark. The uniqueness follows from the fact that the function f_Λ given by (3.4) or (3.5) is the unique solution of (3.1), (3.2), (3.3).

If χ_η is a non-degenerate character, then the condition about η stated in Theorem 3.1 does not hold except for the case $r = 1$ and (case I), namely, the case when $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$.

Corollary 3.2. Let χ_η be a non-degenerate character of N. Then except for the case $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{R})$, π_Λ cannot have a Whittaker model of type χ_η.

§4. The case of unitary representations induced from characters of N_1.

Here we treat the case when χ is the unitary representation of N induced from a character ψ_ξ of N_1, where ψ_ξ is given by

$$\psi_\xi(\exp X) = \exp 2\pi i \xi(X) \qquad (X \in \mathfrak{n}_1)$$

with $\xi \in \mathfrak{n}_1^*$. Then the representation space $H(\chi)$ is given as the Hilbert space of functions ϕ on N such that

$$(4.1) \qquad \phi(n_1 n) = \psi_\xi(n_1)\phi(n) \qquad (n_1 \in N_1,\ n \in N),$$

$$(4.2) \qquad \int_{N_1\backslash N} |\phi(n)|^2 d\dot{n} < \infty,$$

where $d\dot{n}$ is an invariant measure on $N_1\backslash N$. The representation χ is given by

$$(4.3) \qquad \chi(u)\phi(n) = \phi(nu) \qquad (u, n \in N).$$

Let $f \in C^\infty(G, \chi)^0$. Then $f(g) \in H(\chi)$ for $g \in G$, so it is a function on N. We denote the value of $f(g)$ at n by $f(n\colon g)$ and we regard it as a function on $N \times G$. Then $f(n\colon g)$ is smooth as a function of $g \in G$ and square integrable as a function of $n \in N$ mod. N_1. It follows from (4.1) and (4.3) that

(4.4) $\quad f(n_1 n: g) = \psi_\xi(n_1) f(n: g) \qquad (n_1 \in N_1,\ n \in N,\ g \in G),$

(4.5) $\quad f(n: ug) = f(nu: g) \qquad (n,\ u \in N,\ g \in G).$

Since $N = HN_o$ (a semidirect product), we can regard $f(n: g)$ as a function on $H \times N_o \times G$ so that we may write

$$f(n: g) = f(h: n_o: g)$$

if $n = hn_o$ with $h \in H$, $n_o \in N_o$.

We shall find a function f_Λ in $C^\infty(G, \chi)^o$ of the form (2.6) satisfying (2.5). Since f_Λ is K-finite of type τ_Λ it is uniquely determined by its restriction to $NA = HN_oA$. Applying Lemma 1.1, we obtain:

Lemma 4.1. In order to hold that f_Λ satisfies (2.5), it is necessary and sufficient that its restriction to HN_oA satisfies the following system of first order differential equations:

(4.6) $\quad f_\Lambda(h: n_o: a; A_j) = \{\Lambda(H_{\gamma_j}) - 4\pi\xi(Ad(n_o a)X_j)\} f_\Lambda(h: n_o: a)$

for $1 \le j \le r$,

(4.7) $\quad f_\Lambda(h: n_o; Ad(a)U_\beta: a) = -2\pi\xi(Ad(n_o a)X_\beta) f_\Lambda(h: n_o: a)$

for $\beta \in \cup P_{ij}$,

(4.8) $\quad f_\Lambda(h; W: n_o: a) = 0 \qquad$ for $W \in \mathfrak{n}^-_{1/2}$.

Proof. First we notice that $R(H)f_\Lambda = \Lambda(H)f_\Lambda$ for $H \in \mathfrak{t}_c$ and $R(E_\alpha)f_\Lambda = 0$ for $\alpha \in \Delta^+_{\mathfrak{k}}$. It follows from (4.5) that for $X \in \mathfrak{h}$

$$f_\Lambda(h: n_o: a; X) = f_\Lambda(h: \mathrm{Ad}(n_o a)X: n_o: a).$$

If $X \in \mathfrak{n}_1$, using the formula (4.4) and the fact that $\mathfrak{n}_1$ is the center of $\mathfrak{h}$, we obtain

$$f_\Lambda(h: n_o: a; X) = 2\pi i\xi(\mathrm{Ad}(n_o a)X) f_\Lambda(h: n_o: a).$$

Let $\gamma_j \in P_o$. Then it follows from Lemma 1.1 that $E_{-\gamma_j} = A_j/2 - H_{\gamma_j}/2 - iX_j$. The formula (4.6) is obtained immediately from the above argument. Similarly Lemma 1.1 (iii) and (iv) imply (4.7). For $\beta \in \cup P_i$, we obtain from Lemma 1.1 that

$$f_\Lambda(h; \mathrm{Ad}(n_o a)P_{\mathfrak{n}}E_{-\beta}: n_o: a) = 0.$$

If we use Lemma 1.1 (v) and (vi), we get (4.8) immediately.

We shall find the solutions $f_\Lambda(h: n_o: a)$ of the differential equations stated in Lemma 4.1 such that

$$(4.9) \qquad \int_{N_1\backslash H}\int_{N_o} |f_\Lambda(h: n_o: a)|^2 d\dot{h}\,dn_o < \infty,$$

where $d\dot{h}$ means an invariant measure on $N_1\backslash H$.

We define a function on $N_o \times A$ by

$$F_\Lambda(n_o: a) = \exp\{-2\pi\xi(\mathrm{Ad}(n_o a)X_o)\}\exp \Lambda(\mathrm{Ad}(c^{-1})\log a),$$

where $X_o = \sum_{j=1}^r X_j \in \mathfrak{n}_1$ and c is the Cayley transform. Since $H_{\gamma_j} = \mathrm{Ad}(c^{-1})A_j$, it is clear that

$$R(A_j)\exp \Lambda(\mathrm{Ad}(c^{-1})\log a) = \Lambda(H_{\gamma_j})\exp \Lambda(\mathrm{Ad}(c^{-1})\log a)$$

for $1 \le j \le r$. Furthermore since $[A_j, X_o] = 2X_j$,

$$R(A_j)\exp\{-2\pi\xi(Ad(n_o a)X_o)\} = -4\pi\xi(Ad(n_o a)X_j)\exp\{-2\pi\xi(Ad(n_o a)X_o)\}$$

for $1 \le j \le r$. Thus $F_\Lambda(n_o: a)$ is a solution of (4.6). Moreover $F_\Lambda(n_o: a)$ is, as a function of $a \in A$, the unique solution of (4.6) because $\{A_1, \ldots, A_r\}$ forms a basis of $\mathfrak{a}$. So we may write, by choosing a function $c(h: n_o)$ on $H \times N_o$,

$$f_\Lambda(h: n_o: a) = c(h: n_o)F_\Lambda(n_o: a).$$

Using the fact that $[U_\beta, X_o] = X_\beta$, we can easily check that $F_\Lambda(n_o: a)$ satisfies (4.7). Applying (4.7) to the above f_Λ, we obtain that $c(h: n_o)$ must satisfy

$$c(h: n_o; Ad(a)U_\beta) = 0 \qquad \text{for} \quad \beta \in \cup P_{ij}.$$

In view of Lemma 1.1, $\{U_\beta: \beta \in \cup P_{ij}\}$ forms a basis of $(\mathfrak{n}_o)_c$. This implies that $c(h: n_o)$ is a constant function of $n_o \in N_o$. So we may write

$$f_\Lambda(h: n_o: a) = c(h)F_\Lambda(n_o: a)$$

by choosing a function $c(h)$ on H. Then (4.8) implies that $c(h)$ must satisfy

$$c(h; W) = 0 \qquad \text{for} \quad W \in \mathfrak{n}^-_{1/2}.$$

Consequently we obtain:

Lemma 4.2. In order to exist a non-zero function $f_\Lambda(h: n_o: a)$ satisfying (4.6), (4.7), (4.8) and (4.9) it is necessary and sufficient that

(i) the function $\exp\{-2\pi\xi(\mathrm{Ad}(n_o a)X_o)\}$ is square integrable on N_o and

(ii) there exists a non-zero function $c(h)$ on H such that

$$(4.10) \qquad c(h;\ Z) = 2\pi i\xi(Z)c(h) \qquad \text{for } Z \in (n_1)_c,$$

$$(4.11) \qquad c(h;\ W) = 0 \qquad \text{for } W \in n^-_{1/2},$$

$$(4.12) \qquad \int_{N_1\backslash H} |c(h)|^2 d\dot{h} < \infty.$$

Moreover in the case at hand, f_Λ has a form

$$f_\Lambda(h:\ n_o:\ a) = c(h)F_\Lambda(n_o:\ a).$$

We shall decide the conditions on ξ under which (i) and (ii) hold. We define subspaces $n_{o,j}$ $(2 \le j \le r)$ of n_o by

$$n_{o,j} = \sum_{1\le p<j} n^{(\lambda_p-\lambda_j)/2}$$

Then $n_o = \sum n_{o,j}$ (a direct sum), so each element of n_o can be written as a sum $\sum U_j$ with suitable choices of $U_j \in n_{o,j}$. Note that $[n_{o,j},\ X_k] = (0)$ if $j \neq k$. In view of Lemma 1.1.4.1 in [11], we see that

$$\mathrm{Ad}(n_o)\cdot X_j = \mathrm{Ad}(\exp U_j)\cdot X_j \qquad \text{for } n_o = \exp(\textstyle\sum U_j).$$

On the otherhand we can easily check that

$$[U_j,\ X_j] \in \sum_{1\le p<j} n^{(\lambda_p+\lambda_j)/2}$$

$$[U_j,\ [U_j,\ X_j]] \in \sum_{1\le p\le q<j} n^{(\lambda_p+\lambda_q)/2},$$

and $ad(U_j)^3 \cdot X_j = 0$. Therefore we obtain

$$Ad(n_o)\cdot X_j = X_j + [U_j, X_j] + \frac{1}{2}[U_j, [U_j, X_j]].$$

If we denote $\exp \lambda_j(\log a)$ simply by a^{λ_j}, then we get $Ad(a)X_o = \sum a^{\lambda_i}\cdot X_j$ and consequently

$$Ad(n_o a)X_o = \sum_{j=1}^{r} a^{\lambda_j}\{X_j + [U_j, X_j] + \frac{1}{2}[U_j, [U_j, X_j]]\}.$$

Thus we obtain

$$\xi(Ad(n_o a)X_o) = \sum_{j=1}^{r} a^{\lambda_j}\{\xi(X_j) + \xi([U_j, X_j]) + \frac{1}{2}\xi([U_j, [U_j, X_j]])\}.$$

Note that $U_j \to \xi([U_j, X_j])$ is a linear form on $n_{o,j}$ and $U_j \to \xi([U_j, [U_j, X_j]])$ is a quadratic form on $n_{o,j}$.

Definition 4.3. Let $\xi \in n_1^*$. We say that ξ is positive semidefinite of rank $r - 1$ if the quadratic forms $\xi([U_j, [U_j, X_j]])$ on $n_{o,j}$ for $2 \le j \le r$ are all positive definite

The above discussion leads that (i) in Lemma 4.2 holds if and only if ξ is positive semidefinite of rank $r - 1$.

Next we shall study for which ξ (ii) in Lemma 4.2 holds. Since H is the simply connected 2-step nilpotent Lie group with Lie algebra $h = n_1 + n_{1/2}$, it can be identified with the group $\{(X, U): X \in n_1, U \in n_{1/2}\}$ having the multiplication law

$$(X, U)\cdot(Y, V) = (X + Y + \frac{1}{2}[U, V], U + V).$$

We recall that $n_{1/2}^- = (n_{1/2})_c \cap (k_c + p_-)$. We put $n_{1/2}^+ =$

$(\mathfrak{n}_{1/2})_c \cap (\mathfrak{k}_c + \mathfrak{p}_+)$. Then the following results are known (see [9]). The space $(\mathfrak{n}_{1/2})_c$ is the direct sum of $\mathfrak{n}^+_{1/2}$ and $\mathfrak{n}^-_{1/2}$ and moreover $\mathfrak{n}^+_{1/2} = \overline{\mathfrak{n}^-_{1/2}}$. Let J be the complex structure on $\mathfrak{n}_{1/2}$ such that $\mathfrak{n}^-_{1/2} = \{U + iJU\colon U \in \mathfrak{n}_{1/2}\}$. Since $[\mathfrak{n}^-_{1/2}, \mathfrak{n}^-_{1/2}] = (0)$, we have $[JU, JV] = [U, V]$ for $U, V \in \mathfrak{n}_{1/2}$. Let $Q(U, V)$ be the $(\mathfrak{n}_1)_c$-valued hermitian form on $\mathfrak{n}_{1/2}$ defined by

$$(4.13) \qquad Q(U, V) = ([JV, U] - i[V, U])/4.$$

Then under the isomorphism $U \to (U + iJU)/2$ of $\mathfrak{n}_{1/2}$ onto $\mathfrak{n}^-_{1/2}$ we obtain

$$(4.14) \qquad Q(W_1, W_2) = [W_1, \overline{W}_2]/2i \qquad \text{for } W_1, W_2 \in \mathfrak{n}^-_{1/2}.$$

Returning to our consideration, we define a function $c_\xi(X, U)$ on H by

$$c_\xi(X, U) = \exp 2\pi i \xi(X) \exp\{-2\pi\xi(Q(U, U))\}.$$

Then it is clear that

$$R(Z)c_\xi(X, U) = 2\pi i \xi(Z) c_\xi(X, U) \qquad \text{for} \quad Z \in (\mathfrak{n}_1)_c.$$

If we notice that $Q(U, U) = [JU, U]/4$ and $[JU, JV] = [U, V]$, we can easily check

$$R(V + iJV)c_\xi(X, U) = 0 \qquad \text{for} \quad V \in \mathfrak{n}_{1/2}.$$

Thus the function c_ξ on H satisfies (4.10) and (4.11). Furthermore we see immediately that c_ξ satisfies (4.12) if and only if $\xi(Q(U, U))$ with $U \in \mathfrak{n}_{1/2}$ is a positive definite quadratic form on $\mathfrak{n}_{1/2}$. Finally we remark that all the solutions of (4.10) and (4.11) are of the form $c_\xi(X, U)\phi(U)$, where $\phi(U)$ is an entire function on $n_{1/2}$, namely, $\phi(U; W) = 0$ for all $W \in n^-_{1/2}$. The above argument leads to the following theorem.

Theorem 4.4. Let χ be the unitary representation of N induced from a character ψ_ξ of N_1 with $\xi \in n_1^*$. Let π_Λ be the irreducible admissible (g, K)-module with highest weight Λ. Then the necessary and sufficient condition that π_Λ has a Whittaker model of type χ is that

(i) ξ is positive semidefinite of rank $r - 1$ (see Definition 4.3) and

(ii) $\xi(Q(U, U))$ is a positive quadratic form on $n_{1/2}$.

Remark (i) If G/K is a tube domain, namely $n_{1/2} = (0)$, then the condition (ii) has no contribution.

(ii) If we extend ξ to an element of h^* by letting zero on $n_{1/2}$, then in view of (4.14) we see that the condition (ii) is equivalent to saying that h^- is a positive polarization at ξ. Here h^- is a complex subalgebra of h_c given by $h^- = (n_1)_c + n^-_{1/2}$.

If we notice that f_Λ is unique as an element of $C^\infty(G, \chi)^0$, we obtain:

Corollary 4.5. If $\xi \in n_1^*$ satisfies (i) and (ii) in the above theorem, then the Whittaker model of type χ for π_Λ is unique.

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