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Certain intertwining operators for exponential groups

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0. Let $G=\exp \mathfrak{g}$ be an exponential group with Lie algebra $\mathfrak{g}$, f a linear form on $\mathfrak{g}$, and let $\mathfrak{h}_i$ $(i=1,2)$ be real polarizations satisfying the Pukanszky condition of $\mathfrak{g}$ at f. Let $\rho(f, \mathfrak{h}_i, G)= \mathrm{ind}_{H_i\uparrow G}\, \chi_i$, where $H_i=\exp \mathfrak{h}_i$ and $\chi_i(\exp X)=e^{\sqrt{-1}f(X)}$ for $X\in \mathfrak{h}_i$ $(i=1,2)$. Then it is well known in the orbit theory of unitary representations that $\rho(f, \mathfrak{h}_1, G)$ is unitary equivalent to $\rho(f, \mathfrak{h}_2, G)$.

Our aime is to construct in certain cases an intertwining operator between these representations and to prove a composition formula for these operators. This is a generalization of the results due to G. Lion for nilpotent groups.

1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We denote by $K(G)$ the space of numerical continuous functions on G with compact support, dg a left Haar measure on G and by Δ_G the modular function of G : we have

$$\int_G \phi(gx^{-1})dg = \Delta_G(x)\int_G \phi(g)dg$$

for all $\phi \in K(G)$, $x\in G$.

Let H be a closed subgroup of G with Lie algebra $\mathfrak{h}$. For $h\in H$, we set

$$\delta_{G/H}(h) = \frac{\Delta_H(h)}{\Delta_G(h)} .$$

Then we have $\delta_{G/H}(\exp X)=e^{\mathrm{Trad}_{\mathfrak{g}/\mathfrak{h}} X}$ for $X\in \mathfrak{h}$.

Let $K(G, H)$ denote the space of numerical continuous functions ϕ with compact support modulo H such that

$$\phi(gh) = \delta_{G/H}(h)\phi(g) \quad (g\in G,\ h\in H) ,$$

G acts on this space by left translation.

One knows that there exists uniquely, up to a constant, the G-invariant positive linear form $\nu_{G,H}$ on $K(G, H)$, and we write

$$\nu_{G,H}(\phi) = \int_{G/H} \phi(g)d\nu_{G,H}(g) \quad (\phi \in K(G, H)) .$$

Let U be a unitary representation of H on a Hilbert space $\mathcal{H}$, and let L(U, G) be the space of $\mathcal{H}$-valued continuous function ψ on G with compact support modulo H such that

$$\psi(gh) = U(h)^{-1}\delta_{G/H}(h)^{\frac{1}{2}}\psi(g) \quad (g \in G,\ h \in H) . \qquad (*)$$

Then, for $\psi \in L(U, G)$, the function $g \mapsto \|\psi(g)\|^2$ $(g \in G)$ belongs to K(G, H) and one defines the norm N_2 in L(G, H) by

$$N_2(\psi) = \oint_{G/H} \|\psi(g)\|^2 d\nu_{G,\ H}(g) .$$

By the completion we have the Hilbert space $\mathcal{L}(U, G)$ on which we realize the unitary representation $\operatorname{ind}_{H\uparrow G} U$ of G induced by U as left translation.

2. In what follows, let G be an exponential group with Lie algebra $\mathfrak{g}$: the exponential mapping $\exp : \mathfrak{g} \longrightarrow G$ is a diffeomorphism

Let $\mathfrak{g}^*$ denote the dual space of $\mathfrak{g}$. G acts on $\mathfrak{g}^*$ by the coadjoint representations.

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is said to be subordinate to $f \in \mathfrak{g}^*$ if we have $f([\mathfrak{h}, \mathfrak{h}])=0$. We denote by $S(f, \mathfrak{g})$ the set of subalgebras of $\mathfrak{g}$ subordinate to $f \in \mathfrak{g}^*$, and set $M(f, \mathfrak{g})=\{\mathfrak{h} \in S(f, \mathfrak{g})\ ;\ \dim \mathfrak{h} =\frac{1}{2}(\dim \mathfrak{g} + \dim \mathfrak{g}_f)\}$, where $\mathfrak{g}_f=\{X \in \mathfrak{g}\ ;\ f([X, \mathfrak{g}])=0\}$.

Proposition (*L. Pukanszky* [6]). *Let* $\mathfrak{h} \in S(f, \mathfrak{g})$. *Then, the following conditions are equivalent* :

1) $H.f = f + \mathfrak{h}^{\perp}$, *where* $H = \exp \mathfrak{h}$ *and* $\mathfrak{h}^{\perp} = \{\ell \in \mathfrak{g}^*\ ;\ \ell(\mathfrak{h}) = 0\}$;

2) $f + \mathfrak{h}^{\perp} \subset G.f$ *and* $\mathfrak{h} \in M(f, \mathfrak{g})$;

3) $\mathfrak{h} \in M(f+\ell, \mathfrak{g})$ *for any* $\ell \in \mathfrak{h}^{\perp}$.

A subalgebra $\mathfrak{h} \in S(f, \mathfrak{g})$ is said to satisfy the *Pukanszky condition* if $\mathfrak{h}$ satisfies the equivalent conditions of the above propositon.

For $\mathfrak{h} \in S(f, \mathfrak{g})$, $\chi_f(\exp X)=e^{\sqrt{-1}f(X)}$ $(X \in \mathfrak{h})$ gives a unitary character of the analytic subgroup $H=\exp \mathfrak{h}$ of G corresponding to $\mathfrak{h}$. We denote by $\rho(f, \mathfrak{h}, G)$ the unitary representation $\operatorname{ind}_{H\uparrow G} \chi_f$ of G induced by χ_f, $H(f, \mathfrak{h}, G)$ the representation space of $\rho(f, \mathfrak{h}, G)$ and by $I(f, \mathfrak{g})$ the set of $\mathfrak{h} \in S(f, \mathfrak{g})$ such that $\rho(f, \mathfrak{h}, G)$ is irreducible.

Then, the following theorem is fundamental in the orbit theory.

Theorem (*P. Bernat* [1], *L. Pukanszky* [6]). *Let* f *be an element of* $\mathfrak{g}^*$.

a) $M(f, \mathfrak{g}) \supset I(f, \mathfrak{g}) \neq \phi$.

b) *For* $\mathfrak{h} \in S(f, \mathfrak{g})$, $\mathfrak{h}$ *belongs to* $I(f, \mathfrak{g})$ *if and only if* $\mathfrak{h}$ *satisfies the Pukanszky condition.*

c) *For* $\mathfrak{h}_1, \mathfrak{h}_2 \in I(f, \mathfrak{g})$, $\rho(f, \mathfrak{h}_i, G)$ $(i=1,2)$ *are equivalent.*

d) *The mapping* $f \longmapsto \rho(f, \mathfrak{h}, G)$ $(\mathfrak{h} \in I(f, \mathfrak{g}))$ *induces a bijection of the orbit space* $\mathfrak{g}^*/G$ *onto* $\hat{G}$, *the set of equivalence classes of irreducible unitary representations of* G.

3. When $\mathfrak{h}_i \in I(f, \mathfrak{g})$ $(i=1,2)$ are given, how can one construct an intertwining operator between two equivalent representations $\rho(f, \mathfrak{h}_i, G)$ $(i=1,2)$?

For this problem, M. Vergne [7] gave an idea as follows :

Suppose that all groups in question are unimodular. We put $H_i = \exp \mathfrak{h}_i$ $(i=1,2)$. Let $g \in G$ and let $\phi \in H(f, \mathfrak{h}_1, G)$, then the function $h_2 \longmapsto \phi(gh_2)\chi_f(h_2)$ on H_2 is right invariant under the subgroup $H_1 \cap H_2$ of H_2. We put formally

$$(T_{\mathfrak{h}_2\mathfrak{h}_1}\phi)(g) = \int_{H_2/H_1\cap H_2} \phi(gh_2)\chi_f(h_2)d\dot{h}_2 ,$$

where $d\dot{h}_2$ denotes a H_2-invariant measure on the homogeneous space $H_2/H_1 \cap H_2$. If this integral converges for any $g \in G$, it is clear that the function $T_{\mathfrak{h}_2\mathfrak{h}_1}\phi$ satisfies the relation (*) for H_2 and that the operator $T_{\mathfrak{h}_2\mathfrak{h}_1}$ commutes with the left translations of G.

If G is nilpotent, this idea is verified by G. Lion [5].

Theorem (*G. Lion* [5]). *Let* $G=\exp \mathfrak{g}$ *be nilpotent,* $f \in \mathfrak{g}^*$, $\mathfrak{h}_i \in I(f, \mathfrak{g})$ $(=M(f, \mathfrak{g})$ *in this case*) *and let* $H_i = \exp \mathfrak{h}_i$ $(i=1,2)$. *For any function* $\phi \in H(f, \mathfrak{h}_1, G)$ *with compact support modulo* H_1, *the integral*

$$(T_{\mathfrak{h}_2\mathfrak{h}_1}\phi)(g) = \int_{H_2/H_1\cap H_2} \phi(gh_2)\chi_f(h_2)d\dot{h}_2 ,$$

is convergent for any $g \in G$. *By continuity we can extend this operator to obtain an intertwining operator between* $\rho(f, \mathfrak{h}_1, G)$ *and* $\rho(f, \mathfrak{h}_2, G)$.

Furthermore he obtained a composition formula for these operators which are supposed to be normalized.

For an ordered triple $(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ of $\mathfrak{h}_i \in I(f, \mathfrak{g})$ $(i=1,2,3)$, one difines the Maslov index $\tau(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)$ as the signature of the quadratic form Q on the vector space $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ defined by

$$Q(x_1, x_2, x_3) = f([x_1, x_2]) + f([x_2, x_3]) + f([x_3, x_1]) .$$

Theorem (*G. Lion* [5]). *Let* $G=\exp \mathfrak{g}$ *be nilpotent,* $f \in \mathfrak{g}^*$, *and let* $\mathfrak{h}_i \in M(f, \mathfrak{g})$ $(i=1,2)$. *Then,*

$$T_{\mathfrak{h}_1 \mathfrak{h}_2} \circ T_{\mathfrak{h}_2 \mathfrak{h}_3} \circ T_{\mathfrak{h}_3 \mathfrak{h}_1} = e^{i\frac{\pi}{4}\tau(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)} \mathrm{id} ,$$

where id *denotes the identity operator on the space* $H(f, \mathfrak{h}_1, G)$.

4. Now we make some studies of intertwining operators for exponential groups. Let $G=\exp \mathfrak{g}$ be an exponential group with Lie algebra $\mathfrak{g}$ as before. Let $f \in \mathfrak{g}^*$ and let $\mathfrak{h}_i \in I(f, \mathfrak{g})$ $(i=1,2)$.

Proposition. We have

$$\operatorname{Tr} \operatorname{ad}_{\mathfrak{h}_1/\mathfrak{h}_1 \cap \mathfrak{h}_2} X + \operatorname{Tr} \operatorname{ad}_{\mathfrak{h}_2/\mathfrak{h}_1 \cap \mathfrak{h}_2} X = 0$$

for $X \in \mathfrak{h}_1 \cap \mathfrak{h}_2$.

Let $H_i=\exp \mathfrak{h}_i$ $(i=1,2)$ and let $\phi \in H(f, \mathfrak{h}_1, G)$. By the above proposition we can consider the integral

$$(T_{\mathfrak{h}_2 \mathfrak{h}_1} \phi)(g) = \nu_{H_2, H_1 \cap H_2}(\Phi_g)$$
$$= \oint_{H_2/H_1 \cap H_2} \phi(gh) \chi_f(h) \delta_{G/H_2}(h)^{-\frac{1}{2}} d\nu_{H_2, H_1 \cap H_2}(h)$$

for $g \in G$. If this integral converges, it is obvious that the function $T_{\mathfrak{h}_2 \mathfrak{h}_1} \phi$ satisfies the relation $(*)$ for H_2 and that the operator $T_{\mathfrak{h}_2 \mathfrak{h}_1}$ commutes with the left translations of G.

But, unfortunately, I cannot prove the convergence of this integral and I must put a restrictive condition :

We say that the pair $(\mathfrak{h}_1, \mathfrak{h}_2)$ satisfies the condition $\mathbb{N}$ if it satisfies at least one of the following two conditions :

1) One of the $\mathfrak{h}_i$ is contained in the normalizer of the other ;
2) One of the $\mathfrak{h}_i$ is of the form $\mathfrak{h}_i = \mathfrak{g}_f + \mathfrak{h}_i \cap \mathfrak{n}$, where $\mathfrak{n}$ denotes the maximal nilpotent ideal of $\mathfrak{g}$.

If the pair $(\mathfrak{h}_1, \mathfrak{h}_2)$ satisfies the condition $\mathbb{N}$, using the transitivity of the integral $\nu_{G, H}$ (cf. M. Duflo [2]), one can generalize the results of G. Lion.

Theorem. Let the pair $(\mathfrak{h}_1, \mathfrak{h}_2)$ *satisfy the condition* N, *and let* $\phi \in H(f, \mathfrak{h}_1, G)$ *have compact support modulo* H_1. *Then, for*

$g \in G$, *the function*

$$h \longmapsto \phi(gh)\chi_f(h)\delta_{G/H_2}(h)^{-\frac{1}{2}}$$

on H_2 *is* $\nu_{H_2}, {}_{H_1\cap H_2}$*-integrable, and the operator* $T_{\mathfrak{h}_2\mathfrak{h}_1}$;

$$(T_{\mathfrak{h}_2\mathfrak{h}_1}\phi)(g) = \oint_{H_2/H_1\cap H_2} \phi(gh)\chi_f(h)\delta_{G/H_2}(h)^{-\frac{1}{2}} d\nu_{H_2,H_1\cap H_2}(h)$$

can be extended to obtain an intertwining operator between $\rho(f, \mathfrak{h}_i, G)$ $(i=1,2)$.

Theorem. Let $\mathfrak{h}_i \in I(f, \mathfrak{g})$ $(i=1,2,3)$ *such that all pairs* $(\mathfrak{h}_i, \mathfrak{h}_j)$ $(1 \leq i < j \leq 3)$ *satisfy the condition* $\mathbb{N}$. *Suppose that all operators* $T_{\mathfrak{h}_i\mathfrak{h}_j}$ *are normalized, then*

$$T_{\mathfrak{h}_1\mathfrak{h}_2} \circ T_{\mathfrak{h}_2\mathfrak{h}_3} \circ T_{\mathfrak{h}_3\mathfrak{h}_1} = e^{i\frac{\pi}{4}\tau(\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3)} \mathrm{id}.$$

The proofs of our Proposition and Theorems are made by induction on $\dim \mathfrak{g}$, replacing a polarization $\mathfrak{h} \in I(f, \mathfrak{g})$ by $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{a}_f + \mathfrak{a} \in I(f, \mathfrak{g})$, where $\mathfrak{a}$ is a non-central minimal ideal in $\mathfrak{g}$ and $\mathfrak{a}_f = \{X \in \mathfrak{g} \ ; \ f([X, \mathfrak{a}]) = 0\}$.

Remark. If $G = \exp \mathfrak{g}$ is algebraic, any $\mathfrak{h} \in I(f, \mathfrak{g})$ is of the form $\mathfrak{h} = \mathfrak{g}_f + \mathfrak{h} \cap \mathfrak{n}$ (M. Duflo [4]). So the condition N is always satisfied in this case.

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