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Paley-Wiener Type Theorem for Certain Semidirect Product Groups

Shoichi ANDO

Kyoto University

§1. Introduction.

We begin with a general setting. Supose we are given a family of representations (T^{α}_{g}, H) $(\alpha \in A)$ of a Lie group G on a linear space H and that it contains "sufficiently many" representations. We consider Fourier transforms $T^{\alpha}(f)$ of functions $f \in C_0^{\infty}(G)$: $T^{\alpha}(f) = \int_G f(g) T^{\alpha}_{g} d_r g$, operator fields on A.

If we can restore a function $f \in C_0^{\infty}(G)$ from the corresponding operators $T^{\alpha}(f)$ $\alpha \in A$, we may consider that the family contains important informations about "non-unitary" dual of G and we may say that we get a Paley-Wiener type theorem.

It would be desirable that the members of the family are constructed in a same space and that almost all of them are irreducible. But even for regular semidirect product groups this seems to be a difficult problem. So we permit ourselves to employ not necessarily irreducible representations and to proceed in the setting above.

In this paper we deal with a semidirect product goup $G = N \cdot W$ of a commutative normal subgroup N and a connected Lie group W. We assume that N is isomorphic to a real vector space.

§2. Construction of representations.

Let $\hat{N}$ be the dual vector space of N and $<\lambda, n>$; $\lambda \in \hat{N}$, $n \in N$, denote a dual pairing. We extend linear forms $<\cdot, n>$ on $\hat{N}$ to its complexification $N^* = \hat{N} \otimes_R C$ by complex linearity. Put $\lambda(n) = \exp \sqrt{-1}<\lambda, n>$. If $\lambda \in \hat{N}$; the real part of N^*, $\lambda(n)$ is a unitary character. We construct a representation π^λ of G induced from N by $\lambda(\cdot)$, which we realize in a space $L^2(W, d_r w)$:

$$(1) \qquad T_g^\lambda \phi(w) = \lambda(wnw^{-1})\phi(ww_0) \quad \text{for} \quad g = nw_0.$$

Representations π^λ are not in general irreducible. These appear in the process of the decomposition of a regular representation into irreducibles, for details see [1].

For non-unitary characters $\lambda \in N^*$, we follow the same method as in [2]. Let $\|w\|$ be an operator norm of the action of $w \in W$ on N; $n \to wnw^{-1}$ in which we fix once an Euclidean norm $\|\cdot\|$. Put $M(w) = \max(\|w\|, \|w^{-1}\|)$. Clearly it holds for all $\lambda \in N^*$.

$$|\lambda(wnw^{-1})| \leq \exp\| \mathrm{Im}\lambda \| \|n\| M(w).$$

For every $t \in R$ consider a space $H(t) = L^2(W, \exp [tM(w)]d_r w)$. Obviously if $s < t$, we have $H(s) \supset H(t)$ and the inclusion is continuous. Between the spaces $H(t)$ and $H(-t)$ there exists a natural dual pairing

$$<\phi, \psi> = \int_W \phi(w) \psi(w) \, d_r w, \quad \phi \in H(t), \quad \psi \in H(-t).$$

Proposition 1. (i) Projective limit $H = \varprojlim_t H(t)$ is a Fréchet space with norms $\|\cdot\|_t$.

(ii) The dual space H' of H is the inductive limit : $H' = \varinjlim_t H(t)$.

(iii) The expression (1) gives a representation $\pi^\lambda = (T_g^\lambda, H)$ also for every $\lambda \in N^*$. It holds that

(2) $\| T_g^\lambda \phi \|_t \leqq \| \phi \|_{\tau^\lambda(t;g)}$,

where the seminorm τ^λ depends on t as follows; for $g = nw$

$$\tau^\lambda(t;\ g) = \begin{cases} M(w)\ (t + 2\|\mathrm{Im}\lambda\|\|n\|), \text{ when } t \geq -2\|\mathrm{Im}\lambda\|\|n\|, \\ M(w)^{-1}(t + 2\|\mathrm{Im}\lambda\|\|n\|), \text{ when } t \leq -2\|\mathrm{Im}\lambda\|\|n\|. \end{cases}$$

(iv) There exists an equivalence relation:

$$L_w T_g^\lambda L_{w^{-1}} = T_g^{w*\lambda} \qquad \text{for each } w \in W \text{ and } \lambda \in N^*,$$

where L_w is a left translation on W and $w*\lambda(n) = \lambda(wnw^{-1})$.

Remark. When the group W is compact, $M(w)$ is bounded and so $H = L^2(W, d_r w)$. Euclidean, Cartan motion groups are in this case, cf. [4], [5].

Hereafter we are concerned with a non-compact case of W, while our method works well also for compact cases. We put the following assumption.

Assumption : for any $\alpha \geqq 1$, the set $\{w \in W;\ M(w) \leqq \alpha\}$ is compact.

When a non-compact group W acts on N through a unitary representation, this property does not hold. For example the universal covering group of 2-dimensional Euclidean motion group is this case.

For this group Paley-Wiener type theorem is given in [6] in a different way.

§4. Fourier transforms.

We take a right Haar measure on G so that $d_r g = dn\, d_r w$.

Proposition 2. If a function $f \in L^1(G, d_r g)$ vanishes outside a compact set $Q_{\gamma,\alpha} = \{g = nw, \|n\| \leqq \gamma, M(w) \leqq \alpha\}$, then there exists its Fourier tranform $T^\lambda(f) = \int_G f(g) T_g^\lambda\, d_r g$. We have

$$\|T^\lambda(f)\phi\|_t \leqq \|f\|_{L^1} \|\phi\|_{\tau^\lambda(t)},$$

where the semi-norm $\tau^\lambda(t)$ is defined as follows:

$$\tau^\lambda(t) = \tau^\lambda(t; Q_{\gamma,\alpha}) = \begin{cases} \alpha\ (t + 2\gamma\|\mathrm{Im}\lambda\|), & \text{if } t \geqq -2\gamma\|\mathrm{Im}\lambda\|, \\ \alpha^{-1}(t + 2\gamma\|\mathrm{Im}\lambda\|), & \text{if } t \leqq -2\gamma\|\mathrm{Im}\lambda\|. \end{cases}$$

Proof is easy.

§5. Differential operators.

To each element X of the Lie algebra $\underline{\underline{G}}$ of G, we attach differential operators:

$$X_\ell f(g) = \frac{d}{dt} f(g \exp tX)\big|_{t=0}, \quad X_r f(g) = \frac{d}{dt} f(\exp(-tX)g)\big|_{t=0}.$$

For an element $X \in W$, we define a differential operator $\partial(X)$ on W as follows

$$\partial(X)\phi(w) = \frac{d}{dt}\phi(w \exp tX\, w^{-1})\big|_{t=0}.$$

To a pair (λ, X), $\lambda \in \underline{\underline{N}}^*$, $X \in \underline{\underline{N}}$, we attach a multiplication operator λ_X by a function

$$\lambda_X(w) = \frac{d}{dt}\lambda(w \cdot \exp t\, X \cdot w^{-1})\big|_{t=0}.$$

The correspondence $X \longrightarrow X_\ell$ (or X_r) extends to the whole $U(\underline{\underline{G}})$ and the correspondence $X \longrightarrow \partial(X)$ to the whole $U(\underline{\underline{W}})$. For later use we define operators $\partial_r(X)$ and $\partial_\ell(X)$ as follows

$$\partial_r(X) = \begin{cases} \partial(X) - \Delta_G(X) & \text{for } X \in \underline{\underline{W}}, \\ \lambda_X & \text{for } X \in \underline{\underline{N}}, \end{cases} \qquad \partial_\ell(Y) = \begin{cases} -\partial(Y) & \text{for } Y \in \underline{\underline{W}}, \\ -\lambda_Y & \text{for } Y \in \underline{\underline{N}}, \end{cases}$$

where

$$\Delta_G(X) = \frac{d}{dt}\Delta_G(\exp tX)\Big|_{t=0} \quad \text{for } X \in W,$$

and $\Delta_G(g_0)$ means the modular function on $G : \Delta_G(g_0) = d_r(g_0 g)/d_r g$. Correspondences $X \longrightarrow \partial_r(x)$, $\partial_\ell(X)$ extend to the whole $U(\underline{\underline{G}})$ by associativity. Now we have

Proposition 3.

(i) $T^\lambda(X_r^p Y_\ell^q f) = \partial_r(X)^p \circ T^\lambda(f) \circ \partial_\ell(Y)^q$ for $X, Y \in \underline{\underline{W}}$,

(ii) $T^\lambda(X_r^p Y_\ell^q f) = \partial_r(X)^p \circ T^\lambda(f) \circ \partial_\ell(Y)^q$ for $X, Y \in \underline{\underline{N}}$, $p,q = 0,1,2\ldots$.

As for (i), equality holds on a subspace H_∞ of H, which consists of the functions ψ on W whose distribution derivatives $\partial(X)\psi$ also belong to H for any $X \in \underline{\underline{W}}$. Now we have

Lemma 1. $H_\infty \subset C^\infty(W)$.

This comes from Soboloev's lemma.

Lemma 2. Every functional F on H_∞ has a form

$$\langle F, \psi\rangle = \sum_j \int_W h_j(w)\partial(U_j)\psi(w)\, d_r w,$$

with a finite number of $h_j \in H'$ and $U_j \in U(\underline{\underline{W}})$.

This is well known in the distribution theory.

We conclude from Propositions 1 ~ 3 and Lemmas 1,2

Proposition 4. Suppose a function $f \in C_0^\infty(G)$ has a supprot in a compact set $Q_{\gamma,\alpha}$. Then the Fourier transform $T^\lambda = T^\lambda(f)$ has the properties $1° \sim 3°$ below:

1° continuity: for any $U, V \in U(\underline{\underline{G}})$, there exists a constant $C(U, V)$ such that

$$\| \partial_r(U) \cdot T^\lambda \cdot \partial_\ell(V)\psi \|_t \le C(U, V) \| \psi \|_{\tau^\lambda(t)},$$

for any $t \in R$ and $\psi \in H$, while $\tau^\lambda(t) = \tau^\lambda(t; Q_{\gamma,\alpha})$.

2° equivalence relation:

$$L_w \cdot T^\lambda \cdot L_w{-1} = T^{w*\lambda} \quad \text{for } w \in W \text{ and } \lambda \in N^*.$$

3° weak analyticity: for any $F \in H'_\infty$, $\phi \in H$ and $U, V \in U(\underline{\underline{G}})$ $\langle F, \partial_r(U) T^\lambda \partial_\ell(V)\phi \rangle$ is an entire function of $\lambda \in N^*$.

Indeed, $\partial_r(U) \cdot T^\lambda(f) \cdot \partial_\ell(V) = T^\lambda(U_r V_\ell f)$ by Proposition 3 (i). The right hands side is a bounded operator on H by Proposition 2, so we can take $C(U, V) = \| U_r \cdot V_\ell f \|_{L^1(G,\, d_r g)}$. Proof of 3° is reduced by Lemma 2 to a special case $\langle h, T^\lambda \phi \rangle$, $h \in H'$.

Conversely, properties $1° \sim 3°$ characterize the Fourier transform of f.

Proposition 5. Suppose to each element $\lambda \in N^*$, there corresponds an operator T^λ on the space H with the properties $1° \sim 3°$, where 1° is satisfied for a given scale $\tau^\lambda(t) = \tau^\lambda(t; Q_{\gamma,\alpha})$. Then there exists a unique function $f \in C_0^\infty(G)$ such that $T^\lambda = T^\lambda(f)$ and the support of f is contained in the given compact set $Q_{\gamma,\alpha}$.

Remark. When the group W is compact, the property 3° is sufficient only for $F \in H'$, cf. [4], [5].

Proof is quite similar to the ones given in [2], [3].

§6. Paley-Wiener type theorem.

Now for a compact set $Q_{\gamma,\alpha}$ we consider the set $B(Q_{\gamma,\alpha})$ of operator-valued functions $T = (T^{\lambda})$, of $\lambda \in N^*$ such that each member T^{λ} is an operator on H, having the properties $1° \sim 3°$. We endow the space $B_{\gamma,\alpha}$ with a topology by the seminorms

$$\|T\|_{U_1,U_2} = \sup_{\lambda\in N^*} \sup_{t\in R} \sup_{\phi} \|\partial_{\pi}(U)\cdot T^{\lambda}\cdot\partial_{\ell}(V)\phi\|_t \,/\, \|\phi\|_{\tau^{\lambda}(t)}.$$

For compact sets Q_1 and Q_2 in G such that $Q_1 \subset Q_2$, we have $B(Q_1) \subset B(Q_2)$ and the inclusion is continuous.

We can reformulate our results as follows.

Paley-Wiener type theorem. (i) Fourier transformation is a topological isomorphism from $C_0^{\infty}(Q_{\gamma,\alpha})$ onto $B_{\gamma,\alpha}$.

(ii) If we endow $B = \cup B_{\gamma,\alpha}$ with inductive limit topology, Fourier transformation $f \to T^{\alpha}(f)$ is a topological isomorphism from $C_0^{\infty}(G)$ onto B.

Our result is also obtained independently by Mr Shigeru Aoki, Tokyo University.

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