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<td>The following article has been removed by request of the owner of copyright. [H. Midorikawa: &quot;Clebsch-Gordan coefficients for a tensor product representation Ad ⊗ π of a maximal compact subgroup of a real semi-simple Lie group.&quot; pp.149-175]</td>
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§1. Introduction.

We begin with a general setting. Suppose we are given a family of representations \((T^\alpha g, H)(\alpha \in \mathcal{A})\) of a Lie group \(G\) on a linear space \(H\) and that it contains "sufficiently many" representations. We consider Fourier transforms \(T^\alpha(f)\) of functions \(f \in C_0^\infty(G)\) : 
\[
T^\alpha(f) = \int_G f(g)T^\alpha g, \quad \text{ operator fields on } \mathcal{A}.
\]

If we can restore a function \(f \in C_0^\infty(G)\) from the corresponding operators \(T^\alpha(f)\) \(\alpha \in \mathcal{A}\), we may consider that the family contains important informations about "non-unitary" dual of \(G\) and we may say that we get a Paley-Wiener type theorem.

It would be desirable that the members of the family are constructed in a same space and that almost all of them are irreducible. But even for regular semidirect product groups this seems to be a difficult problem. So we permit ourselves to employ not necessarily irreducible representations and to proceed in the setting above.

In this paper we deal with a semidirect product group \(G = N \cdot W\) of a commutative normal subgroup \(N\) and a connected Lie group \(W\). We assume that \(N\) is isomorphic to a real vector space.
§2. Construction of representations.

Let \( \hat{N} \) be the dual vector space of \( N \) and \( \langle \lambda, n \rangle \); \( \lambda \in \hat{N} \), \( n \in N \), denote a dual pairing. We extend linear forms \( \langle \cdot, n \rangle \) on \( \hat{N} \) to its complexification \( N^* = \hat{N} \otimes_{\mathbb{R}} \mathbb{C} \) by complex linearity. Put \( \lambda(n) = \exp \sqrt{-1} \langle \lambda, n \rangle \). If \( \lambda \in \hat{N} \); the real part of \( N^* \), \( \lambda(n) \) is a unitary character. We construct a representation \( \pi^\lambda \) of \( G \) induced from \( N \) by \( \lambda(\cdot) \), which we realize in a space \( L^2(W, d_\tau w) \):

\[
\text{(1) } T_g^\lambda \phi(w) = \lambda(wnw^{-1}) \phi(w_0) \quad \text{for } g = nw_0.
\]

Representations \( \pi^\lambda \) are not in general irreducible. These appear in the process of the decomposition of a regular representation into irreducibles, for details see [1].

For non-unitary characters \( \lambda \in N^* \), we follow the same method as in [2]. Let \( \|w\| \) be an operator norm of the action of \( w \in W \) on \( N \); \( n \rightarrow wnw^{-1} \) in which we fix once an Euclidean norm \( \|\cdot\| \). Put \( M(w) = \max(\|w\|, \|wnw^{-1}\|) \). Clearly it holds for all \( \lambda \in N^* \).

\[
|\lambda(wnw^{-1})| \leq \exp \|\text{Im}\lambda\| \|n\| \ M(w).
\]

For every \( t \in \mathbb{R} \) consider a space \( H(t) = L^2(W, \exp [tM(w)] d_\tau w) \).

Obviously if \( s < t \), we have \( H(s) \supset H(t) \) and the inclusion is continuous. Between the spaces \( H(t) \) and \( H(-t) \) there exists a natural dual pairing

\[
\langle \phi, \psi \rangle = \int_W \phi(w) \psi(w) d_\tau w, \quad \phi \in H(t), \ \psi \in H(-t).
\]
Proposition 1. (i) Projective limit $\lim_{t} H(t)$ is a Frechet space with norms $\| \cdot \|_t$.

(ii) The dual space $H'$ of $H$ is the inductive limit: $H' = \lim_{t} H(t)$.

(iii) The expression (1) gives a representation $\pi^\lambda = (T^\lambda_g, H)$ also for every $\lambda \in \mathbb{N}^*$. It holds that

$$\| T^\lambda_g \phi \|_t \leq \| \phi \| \tau^\lambda(t; g),$$

where the seminorm $\tau^\lambda$ depends on $t$ as follows; for $g = nw$,

$$\tau^\lambda(t; g) = \begin{cases} M(w) (t + 2\|\text{Im}\lambda\|\|n\|), & \text{when } t \geq -2\|\text{Im}\lambda\|\|n\|, \\ M(w)^{-1} (t + 2\|\text{Im}\lambda\|\|n\|), & \text{when } t \leq -2\|\text{Im}\lambda\|\|n\|. \end{cases}$$

(iv) There exists an equivalence relation:

$L_w T^\lambda_g L_w^{-1} = T^{w^*\lambda}_g$ for each $w \in W$ and $\lambda \in \mathbb{N}^*$,

where $L_w$ is a left translation on $W$ and $w^*\lambda(n) = \lambda(wn w^{-1})$.

Remark. When the group $W$ is compact, $M(w)$ is bounded and so $H = L^2(W, d\tau, w)$. Euclidean, Cartan motion groups are in this case, cf. [4], [5].

Hereafter we are concerned with a non-compact case of $W$, while our method works well also for compact cases. We put the following assumption.

Assumption: for any $\alpha > 1$, the set $\{w \in W; M(w) \leq \alpha\}$ is compact.

When a non-compact group $W$ acts on $N$ through a unitary representation, this property does not hold. For example the universal covering group of 2-dimensional Euclidean motion group is this case.
For this group Paley-Wiener type theorem is given in [6] in a different way.

§4. Fourier transforms.

We take a right Haar measure on $G$ so that $d_r g = dn d_r w$.

**Proposition 2.** If a function $f \in L^1(G, d_r g)$ vanishes outside a compact set $Q_{g, \alpha} = \{g = nw, \|n\| \leq \gamma, M(w) \leq \alpha\}$, then there exists its Fourier transform $T^\lambda(f) = \int_G f(g) T^\lambda g d_r g$. We have

$$||T^\lambda(f) \phi||_L^1 \leq ||f||_{L^1} ||\phi||_L^\lambda(t),$$

where the semi-norm $\tau^\lambda(t)$ is defined as follows:

$$\tau^\lambda(t) = \tau^\lambda(t; Q_{g, \alpha}) = \begin{cases} \alpha (t + 2\gamma ||\text{Im} \lambda||), & \text{if } t \geq -2\gamma ||\text{Im} \lambda||, \\
\alpha^{-1}(t + 2\gamma ||\text{Im} \lambda||), & \text{if } t \leq -2\gamma ||\text{Im} \lambda||. \end{cases}$$

Proof is easy.

§5. Differential operators.

To each element $X$ of the Lie algebra $G$ of $G$, we attach differential operators:

$$X_f g = \frac{d}{dt} f(g \exp t X) \big|_{t=0}, X_f g = \frac{d}{dt} f(\exp(-t X) g) \big|_{t=0}.$$  

For an element $X \in W$, we define a differential operator $\partial(X)$ on $W$ as follows

$$\partial(X) \phi(w) = \frac{d}{dt} \phi(w \exp t X w^{-1}) \big|_{t=0}.$$  

To a pair $(\lambda, X)$, $\lambda \in \mathbb{N}^*$, $X \in \mathbb{N}$, we attach a multiplication operator $\lambda_X$ by a function

$$\lambda_X(w) = \frac{d}{dt} \lambda(w \cdot \exp t X \cdot w^{-1}) \big|_{t=0}.$$
The correspondence $X \mapsto X^\ell$ (or $X^r$) extends to the whole $\mathbb{U}(G)$ and the correspondence $X \mapsto \vartheta(X)$ to the whole $\mathbb{U}(W)$. For later use we define operators $\vartheta^r(X)$ and $\vartheta^\ell(X)$ as follows:

$$\vartheta^r(X) = \begin{cases} \vartheta(X) - \Delta_G(X) & \text{for } X \in W, \\ \lambda_X & \text{for } X \in N, \end{cases}$$

$$\vartheta^\ell(Y) = \begin{cases} -\vartheta(Y) & \text{for } Y \in W, \\ -\lambda_Y & \text{for } Y \in N, \end{cases}$$

where

$$\Delta_G(X) = \frac{d}{dt} \Delta_G(\exp tX) \bigg|_{t=0}$$

for $X \in W$, and $\Delta_G(g_0)$ means the modular function on $G : \Delta_G(g_0) = d_r(g_0g)/d_r g$.

Correspondences $X \mapsto \vartheta^r(X)$, $\vartheta^\ell(X)$ extend to the whole $\mathbb{U}(G)$ by associativity. Now we have

Proposition 3.

(i) $T^\lambda(X^r Y^q f) = \vartheta^r(X)^p \cdot T^\lambda(f) \cdot \vartheta^\ell(Y)^q$ for $X, Y \in W$,

(ii) $T^\lambda(X^r Y^q f) = \vartheta^r(X)^p \cdot T^\lambda(f) \cdot \vartheta^\ell(Y)^q$ for $X, Y \in N$, $p, q = 0, 1, 2, \ldots$.

As for (i), equality holds on a subspace $H_\infty$ of $H$, which consists of the functions $\psi$ on $W$ whose distribution derivatives $\vartheta(X)\psi$ also belong to $H$ for any $X \in W$. Now we have

Lemma 1. $H_\infty \subset C^\infty(W)$.

This comes from Sobolev's lemma.

Lemma 2. Every functional $F$ on $H_\infty$ has a form

$$<F, \psi> = \sum_j \int_W h_j(\psi)\vartheta(U_j)\psi(w) \, dw,$$

with a finite number of $h_j \in H'$ and $U_j \in U(W)$.

This is well known in the distribution theory.

We conclude from Propositions 1 ~ 3 and Lemmas 1, 2.
Proposition 4. Suppose a function \( f \in C^\infty_0(G) \) has a support in a compact set \( Q_{Y,\alpha} \). Then the Fourier transform \( T^\lambda = T^\lambda(f) \) has the properties 1° ~ 3° below:

1° continuity: for any \( U, V \in U(G) \), there exists a constant \( C(U, V) \) such that
\[
\| \partial_{\lambda}(U) T^\lambda \partial_{\ell}(V) \psi \|_t \leq C(U, V) \| \psi \|_{T_\lambda(t)},
\]
for any \( t \in \mathbb{R} \) and \( \psi \in H \), while \( T^\lambda(t) = T^\lambda(t; Q_{Y,\alpha}) \).

2° equivalence relation:
\[
L_w \cdot T^\lambda \cdot L_{w^{-1}} = T^{\lambda_w} \quad \text{for } w \in W \quad \text{and } \lambda \in \mathbb{N}^*.
\]

3° weak analyticity: for any \( F \in H'_\omega \), \( \phi \in H \) and \( U, V \in U(G) \)
\[
<F, \partial_{\lambda}(U) T^\lambda \partial_{\ell}(V) \phi> \quad \text{is an entire function of } \lambda \in \mathbb{N}^*.
\]

Indeed, \( \partial_{\lambda}(U) \cdot T^\lambda(f) \cdot \partial_{\ell}(V) = T^\lambda(U \cdot V \cdot f) \) by Proposition 3 (i).

The right hand side is a bounded operator on \( H \) by Proposition 2, so we can take \( C(U, V) = \|U \cdot V \cdot f\|_{L^1(G, d_\mu)} \). Proof of 3° is reduced by Lemma 2 to a special case \( <h, T^\lambda \phi> \), \( h \in H' \).

Conversely, properties 1° ~ 3° characterize the Fourier transform of \( f \).

Proposition 5. Suppose to each element \( \lambda \in \mathbb{N}^* \), there corresponds an operator \( T^\lambda \) on the space \( H \) with the properties 1° ~ 3°, where
1° is satisfied for a given scale \( T^\lambda(t) = T^\lambda(t; Q_{Y,\alpha}) \). Then there exists a unique function \( f \in C^\infty_0(G) \) such that \( T^\lambda = T^\lambda(f) \) and the support of \( f \) is contained in the given compact set \( Q_{Y,\alpha} \).

Remark. When the group \( W \) is compact, the property 3° is sufficient only for \( F \in H' \), cf. [4], [5].
Proof is quite similar to the ones given in [2], [3].


Now for a compact set $Q_\gamma,\alpha$ we consider the set $B(Q_\gamma,\alpha)$ of operator-valued functions $T = (T^\lambda)$, of $\lambda \in \mathbb{N}^*$ such that each member $T^\lambda$ is an operator on $H$, having the properties $1^\circ \sim 3^\circ$. We endow the space $B_{\gamma,\alpha}$ with a topology by the seminorms

$$
\|T\|_{U_1,U_2} = \sup_{\lambda \in \mathbb{N}^*} \sup_{t \in \mathbb{R}} \|\partial_{t}(U)^{\lambda} \phi \|_{\mathcal{L}_{\mathbb{C}}(V)} / \|\phi\|_{\mathcal{L}_{\mathbb{C}}(t)}.
$$

For compact sets $Q_1$ and $Q_2$ in $G$ such that $Q_1 \subset Q_2$, we have $B(Q_1) \subset B(Q_2)$ and the inclusion is continuous.

We can reformulate our results as follows.

Paley-Wiener type theorem. (i) Fourier transformation is a topological isomorphism from $C^\infty_0(Q_\gamma,\alpha)$ onto $B_{\gamma,\alpha}$.

(ii) If we endow $B = \cup B_{\gamma,\alpha}$ with inductive limit topology, Fourier transformation $f \rightarrow T^\alpha(f)$ is a topological isomorphism from $C^\infty_0(G)$ onto $B$.

Our result is also obtained independently by Mr Shigeru Aoki, Tokyo University.
References


