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Tensor products for $SL_2(k)$ and the Plancherel formulae

Masao Tsuchikawa

1. Introduction. Let k be a locally compact, non-discrete, totally disconnected topological field, and $\mathcal{R}$ be a representation of $G = SL_2(k)$. We shall discuss tensor products $\mathcal{R}_1 \otimes \mathcal{R}_2$ of irreducible unitary representations of the continuous series of G and give their decompositon formulae into irreducibles. Analogous problems for some real or complex semi-simple Lie groups are discussed by many authors. For the case of $SL_2(k)$, R.P.Martin [2] discussed $\mathcal{R}_\pi \otimes \mathcal{R}$ of representations $\mathcal{R}_\pi$ of the principal series and any irreducibles $\mathcal{R}$, and gave their decomposition formulae by using Mackey's tensor product theorem, subgroup theorem, and Mackey-Anh's reciprocity theorem. The formulae are expressed as a direct integral on $\hat{G}_u$ with respect to the Plancherel measure, where $\hat{G}_u$ denotes the unitary dual of G.

Here, we give the decomposition formulae of $\mathcal{R}_{\pi_1} \otimes \mathcal{R}_{\pi_2}$, where $\mathcal{R}_{\pi_i}$ $(i = 1, 2)$ are representations of the continuous series. Our method is to use essentially only the Plancherel formula on G, and we give intertwining projections of the product space to each irreducible component. Proofs in detail will be published elsewhere.

2. Preliminaries. Let k be as above, $k^\times$ its multiplicative group, O the ring of integers in k, P the maximal ideal in O, and p an element of $k^\times$ such that $P = pO$. Let dx denote the Haar measure on the additive group k, normalized that O has measure 1. The valuation is determined by $d(ax) = |a|dx$, $a \in k^\times$, and $|0| = 0$, and put $q = |p|^{-1} = \#(O/P)$. We assume that q is odd. Put ε a primitive $(q-1)$st root of 1 in k. Any quadratic extension of k, up to isomorphism, is given by $k(\sqrt{\tau})$, where τ is an element of the set $\{\varepsilon, p, \varepsilon p\}$. For fixed τ and $z = x + \sqrt{\tau}\, y$, we define $\bar{z} =$

$x - \sqrt{\tau}\, y$, and $N_\tau(z) = z\bar{z}$. We set $k_\tau^\times = N_\tau(k(\sqrt{\tau})^\times) \subset k^\times$ and $C_\tau = N_\tau^{-1}(1) \subset k(\sqrt{\tau})$. Then $k_\tau^\times$ is a subgroup of $k^\times$ including $(k^\times)^2$ as its subgroup, $[\,k^\times : k_\tau^\times\,] = [\,k_\tau^\times : (k^\times)^2\,] = 2$, and a complete set of representatives of $k^\times/(k^\times)^2$ is given by $\{1, \varepsilon, p, \varepsilon p\}$:
$k^\times = (k^\times)^2 \cup \varepsilon(k^\times)^2 \cup p(k^\times)^2 \cup \varepsilon p(k^\times)^2$.

Let G be $SL_2(k)$ and D, N^+ and N be the following subgroups of G :

$$D = \left\{ d(a) = \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \;\middle|\; a \in k^\times \right\},$$

$$N^+ = \left\{ n^+(y) = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \;\middle|\; y \in k \right\}, \qquad N = \left\{ n(x) = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} \;\middle|\; x \in k \right\}.$$

Put $w = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Every element $g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, $\delta \neq 0$, is decomposed as

$$g = \begin{bmatrix} a^{-1} & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix} = a[g]\cdot n^+[g]\cdot n[g].$$

3. <u>Representations of the continuous series.</u> Let Ω_p be the set of unitary characters π of $k^\times$, Ω_s the set of characters of $k^\times$ of the form $\pi(x) = |x|^\lambda$, $-1 < \lambda < 0$, $\Omega_{sp} = \{\pi_{sp}\}$ where $\pi_{sp}(x) = |x|^{-1}$, $\Omega_d = \bigcup_{\tau=\varepsilon,p,\varepsilon p} \hat{C}_\tau$ where $\hat{C}_\tau$ is the set of characters π_τ of C_τ with the exception of the characters of order two, and $\Omega_{sd} = \{\pi_\varepsilon^0\}$ where π_ε^0 is the character of order two of C_ε. We set $\Omega_u = \Omega_p \cup \Omega_s \cup \Omega_{sp} \cup \Omega_d \cup \Omega_{sd}$ and $\Omega = \Omega_p \cup \Omega_{sp} \cup \Omega_d \cup \Omega_{sd}$. As shown later, corresponding to every $\omega \in \Omega_u$, irreducible unitary representations of G are naturally constructed, and by such representations non-trivial ones of G are all exhausted. So for the simplicity we roughly identify Ω_u and $\hat{G}_u$, and then use Ω_u instead of $\hat{G}_u$, and the Plancherel measure for G is supported on Ω.

The signature of $k^\times$ with respect to τ is a character of $k^\times$, which is defined as follows :

$$\mathrm{sgn}_\tau\ x = \begin{cases} 1 & x \in k_\tau^\times , \\ -1 & x \in k^\times - k_\tau^\times . \end{cases}$$

For $\pi \in \Omega_p \cup \Omega_s$, we define an irreducible representation $\mathfrak{R}_\pi = \{ T^\pi, S_\pi \}$. We identify π and the character of the group DN^+ by $\pi(d(a) n^+(y)) = \pi(a)$. Let S_π be the vector space of locally constant functions $\varphi(n(x)) = \varphi(x)$ on the group N ($\simeq k$), satisfying the condition that $\pi\rho^{-1/2}(d[nw])\varphi(n[nw])$ are again locally constant, where $\rho(x) = |x|^2$. The operator T_g^π on S_π is defined by

$$T_g^\pi\ \varphi(n(x)) = \pi\rho^{-1/2}(d[n(x)g])\ \varphi(n[n(x)g]),$$

or, more usually,

$$T_g^\pi\ \varphi(x) = \pi(\beta x + \delta)\ |\beta x + \delta|^{-1}\ \varphi\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right), \quad g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

The representation $\mathfrak{R}_\pi$ is the induced representation $ind_{DN^+}^{G}\ \pi$. For $\pi \in \Omega_p$, it is of the principal series and for $\pi \in \Omega_s$, it is of the supplementary series and they are unitary with respect to their natural inner products. The special representation $\mathfrak{R}_{-1} = \{ T^{\pi_{sp}}, S_{-1} \}$ is defined as a limiting case ($\lambda \longrightarrow -1$) of the supplementary series, where S_{-1} is the space of functions $\varphi(x)$ in $S_{\pi_{sp}}$, satisfying $\int \varphi(x)dx = 0$.

4. Tensor product representations and bilinear forms. We discuss the tensor product $\mathfrak{R}_{\pi_1} \otimes \mathfrak{R}_{\pi_2} = \{ T^{\pi_1} \otimes T^{\pi_2}, S_{\pi_1} \otimes S_{\pi_2} \}$ of the following cases:

(1) π_1, $\pi_2 \in \Omega_p$. $\mathfrak{R}_{\pi_1} \otimes \mathfrak{R}_{\pi_2}$ is unitary with respect to the inner product

$$\langle \varphi_1, \varphi_2 \rangle = \iint \varphi_1(x_1, x_2)\ \overline{\varphi_2(x_1, x_2)}\ dx_1 dx_2.$$

(2) $\pi_1 \in \Omega_s$, $\pi_2 \in \Omega_p$. $\mathfrak{R}_{\pi_1} \otimes \mathfrak{R}_{\pi_2}$ is unitary with respect to the inner product

$$< \varphi_1, \varphi_2 > = \frac{1}{\Gamma(\pi_1^{-1})} \iiint \pi_1^{-1}\rho^{-1/2}(x_1 - x_1')\, \varphi_1(x_1, x_2)\overline{\varphi_2(x_1', x_2)}dx_1dx_1'dx_2,$$

where $\Gamma(\cdot)$ is a gamma-function on k.

(3) $\pi_1, \pi_2 \in \Omega_s$. $\mathcal{R}_{\pi_1} \otimes \mathcal{R}_{\pi_2}$ is unitary with respect to the inner product

$$< \varphi_1, \varphi_2 > = \frac{1}{\Gamma(\pi_1^{-1})\Gamma(\pi_2^{-1})} \iiiint \pi_1^{-1}\rho^{-1/2}(x_1 - x_1')\, \pi_2^{-1}\rho^{-1/2}(x_2 - x_2')$$

$$\varphi_1(x_1, x_2)\overline{\varphi_2(x_1', x_2')}dx_1dx_1'dx_2dx_2'.$$

The integrals $\int dx$ are all taken over k.

Further we treat the following cases as limiting cases,

(2) $\longrightarrow$ (4), (3) $\longrightarrow$ (5) and (6) :

(4) $\pi_1 \in \Omega_{sp}$, $\pi_2 \in \Omega_p$. (5) $\pi_1 \in \Omega_{sp}$, $\pi_2 \in \Omega_s$. (6) $\pi_1, \pi_2 \in \Omega_{sp}$.

Let $\mathcal{H}(\pi_1, \pi_2)$, π_1 and $\pi_2 \in \Omega_p \cup \Omega_s$, be the space of functions $\varphi(x_1, x_2)$ in $\mathcal{S}_{\pi_1} \otimes \mathcal{S}_{\pi_2}$, vanishing on some neighborhoods of the diagonal subset of $k \times k$. And let $\mathcal{S}(G)$ be the space of locally constant and compactly supported functions f on G, and τ_g the right regular representation of G, $\tau_g f(\cdot) = f(\cdot g)$. Then we have a continuous surjective G-morphism U of $\mathcal{S}(G)$ to $\mathcal{H}(\pi_1, \pi_2)$: $Uf = \varphi$, and $\varphi(x_1, x_2)$ is given by

$$\varphi(x, \frac{1}{y} + x) = \varphi(n(x), n[wn^+(y)]n(x))$$

$$= \pi_2^{-1}\rho^{1/2}(d[wn^+(y)]) \int \pi_1^{-1}\pi_2(d(a))\, f(d(a)n^+(y)n(x))d^\times a$$

And

$$U(\tau_g f)(x_1, x_2) = T_g^{\pi_1} \otimes T_g^{\pi_2}(Uf)(x_1, x_2).$$

For $f_1, f_2 \in \mathcal{S}(G)$ we consider a sesquilinear form

$$B(f_1, f_2) = < Uf_1, Uf_2 >$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathcal{H}(\pi_1, \pi_2) \subset \mathcal{S}_{\pi_1} \otimes \mathcal{S}_{\pi_2}$. Then it is continuous and by the kernel theorem we have a distribution $h(g) \in \mathcal{S}'(G)$ such that

$$B(f_1, f_2) = \iint h(g)\, f_1(gg')\, \overline{f_2(g')}\, dg dg'.$$

Proposition 1. Corresponding to the cases (1), (2) and (3), the kernel distributions $h(g)$, related with the inner products for $\mathcal{R}_{\pi_1} \otimes \mathcal{R}_{\pi_2}$, are written as follows : for $g = d(a)n^+(y)n(x)$,

(1) $h(g) = \pi_1^{-1}\pi_2(a)\ \Delta(y)\ \Delta(x)$,

(2) $h(g) = \dfrac{1}{\Gamma(\pi_1^{-1})}\ \pi_1^{-1}\pi_2(a)\ \Delta(y)\ \pi_1^{-1}\rho^{-1/2}(x)$,

(3) $h(g) = \dfrac{1}{\Gamma(\pi_1^{-1})\ \Gamma(\pi_2^{-1})}\ \pi_1^{-1}\pi_2(a)\ \pi_2^{-1}\rho^{-1/2}(y)\ \pi_1^{-1}\rho^{-1/2}(x)$,

where $\Delta(x)$ is the delta function at 0 on k.

5. The Plancherel transform. Representations $\mathcal{R}_\pi$ of the continuous series are realized in another way called the χ-realization. It is the Fourier transform $\hat{\mathcal{R}}_\pi = \{\hat{T}^\pi, \hat{S}_\pi\}$ of $\mathcal{R}_\pi$, in which operators $\hat{T}^\pi_g$ are expressed by means of $K_\pi(g|\ u, v)$ on $k^\times \times k^\times$:

$$(\hat{T}^\pi_g\, \hat{\varphi})(u) = \int K_\pi(g|\ u, v)\, \hat{\varphi}(v)\, dv, \qquad (\ \hat{\varphi}(v) \in \hat{\mathcal{S}}_\pi \text{ or } \hat{\mathcal{S}}_{-1}, \text{ if } \pi = \pi_{sp})$$

$$\begin{aligned} K_\pi(g|\ u, v) &= \pi(a)\,|a|\ \Delta(v - a^2u) && g = d(a),\\ &= \chi(xu)\ \Delta(v - u) && g = n(x),\\ &= J_\pi(u, v) && g = w, \end{aligned}$$

where χ is the additive character of k which is trivial on O but is not trivial on P^{-1} and $J_\pi(u, v)$ is a Bessel function on k.

Let $\mathcal{S}^\times$ be the space of functions in $\mathcal{S}(k)$, vanishing on some neighborhoods of 0. Take $\pi_\tau \in \hat{C}_\tau$, and extend it a unitary character of $k(\sqrt{\tau})$. The discrete series representation $\mathcal{R}_{\pi_\tau}$ is realized on $\mathcal{S}^\times$, in which operator is also expressed by means of kernel on $k^\times \times k^\times$:

when $uv^{-1} \in k_\tau^\times$,

$$K_{\pi_\tau}(g|\ u,\ v) = \mathrm{sgn}_\tau(a)\ \pi_\tau(a)|a|\ \Delta(v - a^2u) \qquad g = d(a),$$
$$= \chi(xu)\ \Delta(v - u) \qquad g = n(x),$$
$$= a_\tau\ c_\tau\ J^d_{\pi_\tau}(\ u,\ v) \qquad g = w,$$

and when $uv^{-1} \notin k_\tau^\times$,

$$K_{\pi_\tau}(g|\ u,\ v) = 0 \qquad g \in G,$$

where a_τ and c_τ are some known real values, and

$$J^d_{\pi_\tau}(\ u,\ v) = \int_{t\bar{t} = vu^{-1}} \chi(\ ut + vt^{-1})\pi_\tau(t)d^\times t.$$

Representation $\mathcal{R}_{\pi_\tau}$ splits into the direct sum of two inequivalent irreducible ones $\mathcal{R}_{\pi_\tau}^+ = \{ T^{\pi_\tau},\ S^\times|k_\tau^\times\}$ and $\mathcal{R}_{\pi_\tau}^- = \{T^{\pi_\tau},\ S^\times|k^\times - k_\tau^\times\}$, where $S^\times|k_\tau^\times$ is the space of functions in $S^\times$ supported on $k_\tau^\times$, and similar for $S^\times|k^\times - k_\tau^\times$.

In the case of $\mathcal{R}_0 = \mathcal{R}_\pi = \{ T^{\pi_\varepsilon^0},\ S^\times\}$, the operator $T^{\pi_\varepsilon^0}$ has an analogous kernel $K_{\pi_\varepsilon^0}(g|\ u,\ v)$, moreover when $uv^{-1} \notin (k^\times)^2$ it holds that $K_{\pi_\varepsilon^0}(g|\ u,\ v) = 0$. So $\mathcal{R}_0$ splits into the sum of four inequivalent irreducible representations $\mathcal{R}_0^{+,1} = \{T^{\pi_\varepsilon^0},\ S^\times|(k^\times)^2\}$, $\mathcal{R}_0^{+,2} = \{T^{\pi_\varepsilon^0}, S^\times|\varepsilon(k^\times)^2\}$, $\mathcal{R}_0^{-,1} = \{T^{\pi_\varepsilon^0}, S^\times|p(k^\times)^2\}$, and $\mathcal{R}_0^{-,2} = \{T^{\pi_\varepsilon^0}, S^\times|\varepsilon p(k^\times)^2\}$.

Using these kernels $K_\omega(g|\ u,\ v)$, we define the Plancherel transform p of $f \in S(G)$.

$$p\ :\ f \longrightarrow \int f(g)\ K_\omega(g|\ u,\ v)dg = K_\omega(f|\ u,\ v).$$

For every $f \in S(G)$, $K_\omega(f|\ u,\ v)$ is a function on $k^\times \times k^\times \times \Omega$, and it holds that

(i) $\qquad K_\omega(\ \sigma_g f|\ u,\ v) = \int K_\omega(g|\ u,\ t)\ K_\omega(f|\ t,\ v)dt$

(ii) $\qquad K_\omega(\ f'|\ u,\ v) = K_\omega(\ f|\ v,\ u)\dfrac{\omega(v)}{\omega(u)}$

(iii) $\qquad K_\omega(\ \bar{f}|\ u,\ v) = \overline{K'_\omega(f|\ v,\ u)}$,

(iv) $\qquad K_\omega(f^*| u, v) = \overline{K_\omega(f| u, v)}\frac{\omega(v)}{\omega(u)}$,

(v) $\qquad K_{\omega^{-1}}(f| u, v) = K_\omega(f| u, v)\frac{\omega(u)}{\omega(v)}$,

where σ_g is the left regular representation, $\overline{f}$ the complex conjugate of f, $f'(g) = f(g^{-1})$, $f^* = \overline{f'}$, and $K'_\omega(g| u, v) = K_\omega(g| -u, -v)$.

We denote by $dm(\omega)$ the Plancherel measure on G (cf. [1] or [7]). The Plancherel formula gives us the equality

$$\int f_1(g)\,\overline{f_2(g)}\,dg = \int_\Omega \iint K_\omega(f_1| u, v)\,\overline{K_{\bar{\omega}}(f_2| u, v)}\,\frac{\omega(u)}{\omega(v)}\,dudv\,dm(\omega).$$

We define the Plancherel transform $K_\omega(h| u, v)$ of $h \in \mathcal{S}'(G)$, in such a way we have for $f \in \mathcal{S}(G)$

$$\int h(g)\,\overline{f(g)}\,dg = \int_\Omega \iint K_\omega(h| u, v)\,\overline{K_{\bar{\omega}}(f| u, v)}\,\frac{\omega(u)}{\omega(v)}\,dudv\,dm(\omega).$$

Then combining the equalities (i) ~ (v) and the Plancherel formula we can obtain the formula

$$(*) \qquad B(f_1, f_2) = \iint h(g)\,f_1(gg')\,\overline{f_2(g')}\,dgdg'$$

$$= \int_\Omega \iiint K_\omega(h| u, v)\,K_\omega(f_1'| t, u)\,\overline{K_{\bar{\omega}}(f_2'| t, v)}\,\frac{\omega(t)}{\omega(v)}\,dudvdt\,dm(\omega).$$

As to $K_\omega(h| u, v)$, the following theorem holds.

<u>Theorem 1.</u> Let $\pi \in \Omega_p \cup \Omega_{sp}$, then $K_\pi(h| u, v) = 0$ if $\pi_1\pi_2\pi(-1) \neq 1$. Let $\pi_\tau \in \Omega_d \cup \Omega_{sd}$, then $K_{\pi_\tau}(h| u, v) = 0$ if $\pi_1\pi_2\pi_\tau(\mathrm{sgn}_\tau)(-1) \neq 1$.

By this theorem, we know that in the integration domain of ω in ($*$), $\omega = \pi$ such that $\pi_1\pi_2\pi(-1) \neq 1$ and $\omega = \pi_\tau$ such that $\pi_1\pi_2\pi_\tau(\mathrm{sgn}_\tau)(-1) \neq 1$ disappear.

Again by this theorem, we can take the square root of the characters $(\pi_1\pi_2\pi)(x)$ and $(\pi_1\pi_2\pi_\tau(\mathrm{sgn}_\tau))(x)$ if π and π_τ are in the above restricted integration domain. So we can set

$$A_\pi(x) = (\pi_1\pi_2^{-1}\pi^{-1}\rho^{1/2})^{1/2}(x), \quad A_{\pi_\tau}(x) = (\pi_1\pi_2^{-1}\pi_\tau^{-1}(\mathrm{sgn}_\tau)\rho^{1/2})^{1/2}(x).$$

In the following, we treat these multiplicative charactere as homogeneous distributions on k.

<u>6. Results of calculations of $k_\omega(h|\ u, v)$ and the decomposition formula of the case (1).</u> Now, we give explicitly $K_\omega(h|\ u, v)$ of the case (1). For every $\omega \in \Omega$, $K_\omega(h|\ u, v)$ are obtained as the sums of four terms of homogeneous distributions of types of $A_\pi\rho^{-1/2}(u)\overline{A_\pi\rho}^{-1/2}(v)$ or $A_{\pi_\tau}\rho^{-1/2}(u)\overline{A_{\pi_\tau}\rho}^{-1/2}(v)$.

For $\omega = \pi \in \Omega_p$,

$$K_\omega(h|\ u, v) = \sum_{s=1,\varepsilon,p,\varepsilon p} A_\pi\rho^{-1/2}(\mathrm{sgn}_s)(u)\ \overline{A_\pi\rho}^{-1/2}(\mathrm{sgn}_s)(v).$$

$$(\quad (\mathrm{sgn}_1)(x) = 1\ )$$

For $\omega = \pi_{sp} \in \Omega_{sp}$,

$$K_\omega(h|\ u, v) = \sum_{s=1,\varepsilon,p,\varepsilon p} A_{\pi_{sp}}\rho^{-1/2}(\mathrm{sgn}_s)(u)\ \overline{A_{\pi_{sp}}\rho}^{-1/2}\pi_{sp}(\mathrm{sgn}_s)(v).$$

For $\omega = \pi_\tau \in \Omega_d$,

$$K_\omega(h|\ u, v) = \sum_{\alpha=\pm} A_{\pi_\tau}\rho^{-1/2}\mu^\alpha(u)\ \overline{A_{\pi_\tau}\rho}^{-1/2}\mu^\alpha(v)$$

$$+ \sum_{\alpha=\pm} A_{\pi_\tau}\rho^{-1/2}\nu^\alpha(u)\ \overline{A_{\pi_\tau}\rho}^{-1/2}\nu^\alpha(v),$$

where $\mu^\pm$ and $\nu^\pm$ are following functions :

$$\mu^+(x) = \begin{cases} 1 & x \in k_\tau^\times, \\ 0 & x \in k^\times - k_\tau^\times, \end{cases} \qquad \mu^-(x) = \begin{cases} 1 & x \in (k^\times)^2, \\ -1 & x \in k_\tau^\times - (k^\times)^2, \\ 0 & x \in k^\times - k_\tau^\times, \end{cases}$$

$$\nu^+(x) = \begin{cases} 0 & x \in k_\tau^\times, \\ 1 & x \in k^\times - k_\tau^\times, \end{cases} \qquad \nu^-(x) = \begin{cases} 0 & x \in k_\tau^\times, \\ -1 & x \in \tau'(k^\times)^2, \\ 1 & x \in \tau''(k^\times)^2, \end{cases}$$

and τ' and τ'' are elements of $\{\varepsilon, p, \varepsilon p\}$ satisfying $\tau'(k^\times)^2 \cup \tau''(k^\times)^2 = k^\times - k_\tau^\times$.

For $\omega = \pi_\varepsilon^0$,

$$K_\omega(h|\ u,\ v) = \sum_{s=1,\varepsilon,p,\varepsilon p} A_{\pi_\varepsilon^0}\rho^{-1/2}\eta_s(u)\ \overline{A_{\pi_\varepsilon^0}\rho^{-1/2}\eta_s(v)}\ ,$$

where η_s is the characteristic function of $s(k^\times)^2$.

Let $t \in k^\times$, we set when $\omega = \pi \in \Omega_p \cup \Omega_{sp}$,

$$\Phi_s(\ t|\ \pi) = \int A_\pi\rho^{-1/2}(\mathrm{sgn}_s)(u)\ K_\pi(f'|\ t,\ u)du,$$

when $\omega = \pi_\tau \in \Omega_d$,

$$\Psi^{+,\pm}(\ t|\ \pi_\tau) = \int A_{\pi_\tau}\rho^{-1/2}\mu^{\pm}(u)\ K_{\pi_\tau}(f'|\ t,\ u)du,$$

and

$$\Psi^{-,\pm}(\ t|\ \pi_\tau) = \int A_{\pi_\tau}\rho^{-1/2}\nu^{\pm}(u)\ K_{\pi_\tau}(f'|\ t,\ u)du,$$

and when $\omega = \pi_\varepsilon^0$,

$$H_s(\ t) = \int A_{\pi_\varepsilon^0}\rho^{-1/2}\eta_s(u)\ K_{\pi_\varepsilon^0}(f'|\ t,\ u)du,$$

and we set $H_1 = H^{+,1}$, $H_\varepsilon = H^{+,2}$, $H_p = H^{-,1}$, and $H_{\varepsilon p} = H^{-,2}$.

The functions Φ_s, $\Phi_s(\omega = \pi_{sp})$, $\Psi^{+,\pm}$, $\Psi^{-,\pm}$, and $H^{\pm,i}$ $(i=1,2)$ are of functions of representations $\hat{\mathcal{R}}_\pi$, $\hat{\mathcal{R}}_{-1}$, $\mathcal{R}_{\pi_\tau}^{+}$, $\mathcal{R}_{\pi_\tau}^{-}$, and $\mathcal{R}_0^{\pm,i}$ respectively. Thus we obtain a mapping

$$f \longrightarrow \left[\{\Phi_s\}_{s=1,\varepsilon,p,\varepsilon p},\{\Psi^{+,\pm}\}_{\pm},\ \{\Psi^{-,\pm}\}_{\pm},\ H^{+,1},\ H^{+,2},\ H^{-,1},\ H^{-,2}\right]_{\pi\in\Omega_p\cup\Omega_{sp},\pi_\tau\in\Omega_d}$$

$$= \Phi\ .$$

Mappings we have made of the sets of functions and the equality (i), we have a following diagram :

$$\begin{array}{ccccccc}
\varphi(x_1,x_2) & \longleftarrow & f & \longrightarrow & K_\omega(\ f'|\ u,\ v) & \longrightarrow & \Phi \\
g\downarrow & & \downarrow & & \downarrow & & \downarrow g \\
(T_g^{\pi_1} \otimes T_g^{\pi_2})\varphi & \longleftarrow & \tau_g f & \longrightarrow & K_\omega(\ \sigma_g(f')|\ u,\ v) & \longrightarrow & \Pi_g\Phi
\end{array}$$

where $\mathbb{T}_g \Phi = \left[T_g^{\pi} \Phi_s(t|\pi),\ T_g^{\pi_\tau} \Psi^{\pm,\pm}(t|\pi_\tau),\ T_g^{\pi_\varepsilon^0} H^{\pm,i}(t) \right]$.

<u>Propositon 2.</u> The kernels of the mappings $\varphi \longrightarrow f$, and $\varphi \longrightarrow \Phi$ coincide.

So the correspondence $\varphi(x_1, x_2) \longrightarrow \Phi(t|\omega)$ is bijective.

Now, from the surjectivity of operator U, formula (*), Theorem 1, the above diagram, and Propostion 2, we have a decompsition formula of the tensor product of the case (1).

<u>Theorem 2.</u> The irreducible decomposition of the tensor product $\mathcal{R}_{\pi_1} \otimes \mathcal{R}_{\pi_2}$ of representations of the principal series is given by the following formula : for $\varphi(x_1, x_2) \in \mathcal{H}(\pi_1, \pi_2)$,

$$\iint |\varphi(x_1, x_2)|^2 dx_1 dx_2 = \sum_{s=1,\varepsilon,p,\varepsilon p} \int_{\substack{\pi\in\Omega_p \\ \pi(-1)=\pi_1\pi_2(-1)}} m(\pi) \left\{ \int |\Phi_s(t|\pi)|^2 dt \right\} d\pi$$

$$+ c(\pi_1\pi_2) \sum_{s=1,\varepsilon,p,\varepsilon p} m(\pi_{sp}) \int |\Phi_s(t|\pi_{sp})|^2 |t|^{-1} dt$$

$$+ \sum_{\pm} \sum_{\substack{\pi_\tau\in\Omega_d \\ \pi_\tau(\mathrm{sgn}_\tau)(-1)=\pi_1\pi_2(-1)}} m(\pi_\tau) \left\{ \int |\Psi^{+,\pm}(t|\pi_\tau)|^2 dt + \int |\Psi^{-,\pm}(t|\pi_\tau)|^2 dt \right\}$$

$$+ d(\pi_1\pi_2)\, m(\pi_\varepsilon^0) \Big\{ \int |H^{+,1}(t)|^2 dt + \int |H^{+,2}(t)|^2 dt + \int |H^{-,1}(t)|^2 dt + \int |H^{-,2}(t)|^2 dt \Big\},$$

where $\{m(\pi)d\pi,\ m(\pi_{sp}),\ m(\pi_\tau),\ m(\pi_\varepsilon^0)\}$ is the Plancherel measure, and

$$c(\pi_1\pi_2) = \begin{cases} 1 & \pi_1\pi_2(-1) = 1, \\ 0 & \pi_1\pi_2(-1) \neq 1, \end{cases} \qquad d(\pi_1\pi_2) = \begin{cases} 1 & \pi_1\pi_2(-1) = \pi_\varepsilon^0(-1), \\ 0 & \pi_1\pi_2(-1) \neq \pi_\varepsilon^0(-1). \end{cases}$$

Remark 1. We can compare the formula with the following one which is discribed in [2].

$$\mathcal{R}_{\pi_1} \otimes \mathcal{R}_{\pi_2} \approx 4\int^{\oplus}_{\substack{\Omega_p \\ \pi(-1)=\pi_1\pi_2(-1)}} \mathcal{R}_\pi d\mu(\pi) \oplus 4c(\pi_1\pi_2)\, \mathcal{R}_{-1} \oplus 2\sum_{\substack{\pi_\tau \in \Omega_d \\ \pi_\tau(\mathrm{sgn}_\tau)(-1)=\pi_1\pi_2(-1)}} (\mathcal{R}_{\pi_\tau}^{+} \oplus \mathcal{R}_{\pi_\tau}^{-})$$

$$\oplus\, d(\pi_1\pi_2)\left\{\mathcal{R}_0^{+,1} \oplus \mathcal{R}_0^{+,2} \oplus \mathcal{R}_0^{-,1} \oplus \mathcal{R}_0^{-,2}\right\}.$$

where $\left\{\oplus d\mu(\pi),\ \oplus\sum\right\}$ is the measure equivalent to the Plancherel measure.

Remark 2. Let $\hat{\Phi}_s(x|\omega)$ be the usual Fourier transform of $\Phi_s(t|\omega)$ with respect to t. For $\pi \in \Omega_p \cup \Omega_{sp}$, the correspondence $\varphi(x_1, x_2) \longrightarrow \hat{\Phi}_s(x|\pi)$ is written in a more direct manner :

$$\hat{\Phi}_s(x|\pi) = \Gamma((\pi_1\pi_2^{-1}\pi^{-1}\rho^{1/2})^{1/2}\,\mathrm{sgn}_s)$$

$$\iint (\pi_1^{-1}\pi_2^{-1}\pi^{-1}\rho^{1/2})^{1/2}\rho^{-1/2}\mathrm{sgn}_s(x_1 - x_2)$$

$$(\pi_1^{-1}\pi_2\pi\,\rho^{1/2})^{1/2}\rho^{-/2}\mathrm{sgn}_s(x_1)\ (\pi_1\pi_2^{-1}\pi\rho^{1/2})^{1/2}\rho^{-1/2}\mathrm{sgn}_s(x_2)$$

$$\varphi(x_1 + x, x_2 + x)dx_1dx_2.$$

7. Formulae for cases (2) and (3). Analogous calculations to the case (1), we obtain the formulae for the cases (2) and (3).

Theorem 3. The irreducible decomposition of the tensor product $\mathcal{R}_{\pi_1} \otimes \mathcal{R}_{\pi_2}$ of representations of the principal and supplementary series is given by the following formula : for $\varphi(x_1, x_2) \in \mathcal{H}(\pi_1, \pi_2)$,

$$\frac{1}{\Gamma(\pi_1^{-1})}\iiint \pi_1^{-1}\rho^{-1/2}(x_1 - x_1')\,\varphi(x_1, x_2)\,\overline{\varphi(x_1', x_2)}\,dx_1dx_1'dx_2$$

$$= \sum_{s=1,\varepsilon,p,\varepsilon p} \int_{\substack{\Omega_p \\ \pi(-1)=\pi_2(-1)}} m(\pi)\left\{\int |\Phi_s(t|\pi)|^2dt\right\} d$$

$$+\, c(\pi_2) \sum_{s=1,\varepsilon,p,\varepsilon p} m(\pi_{sp})\int |\Phi_s(t|\pi_{sp})|^2|t|^{-1}dt$$

$$+ \sum_{\pm} \sum_{\substack{\pi_\tau \in \Omega_d \\ \pi_\tau(sgn_\tau)(-1)=\pi_2(-1)}} m(\pi_\tau) \quad \left\{ \int |\Psi^{+,\pm}(t|\pi_\tau)|^2 dt + \int |\Psi^{-,\pm}(t|\pi_\tau)|^2 dt \right\}$$

$$+ d(\pi_2)\ m(\pi_\varepsilon^0) \left\{ \int |H^{+,1}(t)|^2 dt + \int |H^{+,2}(t)|^2 dt + \int |H^{-,1}(t)|^2 dt + \int |H^{-,2}(t)|^2 dt \right\},$$

where $c(\pi_2) = 1$ if $\pi_2(-1) = 1$ and $c(\pi_2) = 0$ if $\pi_2(-1) \neq 1$, and $d(\pi_2) = 1$ if $\pi_2(-1) = \pi_\varepsilon^0(-1)$ and $d(\pi_2) = 0$ if $\pi_2(-1) \neq \pi_\varepsilon^0(-1)$.

The case (3), that is $\pi_1(x) = |x|^{\lambda_1}$ and $\pi_2(x) = |x|^{\lambda_2} \in \Omega_s$, is further divided into two cases :

(3a) $-1 < \lambda_1 + \lambda_2 < 0$, (3b) $-2 < \lambda_1 + \lambda_2 < -1$.

$K_\omega(h|\ u, v)$ for the cases (3a) and (3b) are obtained analogously but a little more complicated than those for the cases (1) and (2). In the cases (3a) and (3b), $K_\omega(h|\ u, v)$ are expressed by the sums of homogeneous distributions with respect to u and v, same as before, but with coefficients of products of the gamma-functions on k and on $k(\sqrt{\tau})$. We have first the formula for the case (3a), and then applying the principle of analytic continuation to (3b), we have the formula for this case.

For $\pi \in \Omega_p \cup \Omega_{sp}$, we set

$$\Phi_s'(t|\ \pi) = \Gamma(A_\pi^{-1}\rho^{1/2}(sgn_s)) \int A_\pi \rho^{-1/2}(sgn_s)(u)\ K_\pi(f'|\ t, u)du$$

and

$$\gamma_s(\pi_1, \pi_2, \pi) = \Gamma((\pi_1\pi_2\pi\rho^{1/2})^{1/2}(sgn_s))\ \Gamma((\pi_1\pi_2\pi^{-1}\rho^{1/2})^{1/2}(sgn_s)).$$

We extend the characters π_1 and π_2 of $k^\times$ to those of $k(\sqrt{\tau})^\times$ by $\pi_i^2(z) = \pi_i(N_\tau(z))$, and we denote the gamma function on $k(\sqrt{\tau})$ by $\Gamma_\tau(\cdot)$, then we set

$$g_\tau(\pi_1, \pi_2, \pi_\tau) = \Gamma_\tau(\pi_1^{-1}\pi_2\pi_\tau^{-1}\rho^{1/2}(sgn_\tau))\ \Gamma_\tau(\pi_1\pi_2\pi_\tau\rho^{1/2}(sgn_\tau)).$$

Theorem 4. The irreducible decomposition of the tensor product $\mathfrak{R}_{\pi_1} \otimes \mathfrak{R}_{\pi_2}$ of representations of the supplementary series of the case (3a) is given by the following formula : for $\varphi(x_1, x_2) \in \mathcal{H}(\pi_1, \pi_2)$,

$$\frac{1}{\Gamma(\pi_1^{-1})\,\Gamma(\pi_2^{-1})} \iiiint \pi_1^{-1}\rho^{-1/2}(x_1 - x_1')\,\pi_2^{-1}\rho^{-1/2}(x_2 - x_2')\,\varphi(x_1, x_2)\,\overline{\varphi(x_1', x_2')}\,dx_1 dx_1' dx_2 dx_2'$$

$$= \sum_{s=1,\varepsilon,p,\varepsilon p} \int_{\substack{\Omega_p \\ \pi(-1)=1}} m(\pi)\,\gamma_s(\pi_1, \pi_2, \pi)\left\{\int |\Phi_s'(t|\pi)|^2 dt\right\} d\pi$$

$$+ \sum_{s=1,\varepsilon,p,\varepsilon p} m(\pi_{sp})\,\gamma_s(\pi_1, \pi_2, \pi_{sp}) \int |\Phi_s'(t|\pi_{sp})|^2 |t|^{-1} dt$$

$$+ \sum_{\pm} \sum_{\substack{\Omega_d \\ \pi_\tau \mathrm{sgn}_\tau(-1)=1}} m(\pi_\tau)\,g_\tau(\pi_1, \pi_2, \pi_\tau) \left\{\int |\Psi^{+,\pm}(t|\pi_\tau)|^2 dt + \int |\Psi^{-,\pm}(t|\pi_\tau)|^2 dt\right\}$$

$$+ d(-1)\,m(\pi_\varepsilon^0) g_\varepsilon(\pi_1, \pi_2, \pi_\varepsilon^0)\left\{\int |H^{+,1}(t)|^2 dt + \int |H^{+,2}(t)|^2 dt + \int |H^{-,1}(t)|^2 dt + \int |H^{-,2}(t)|^2 dt\right\},$$

where $d(-1) = 1$ if $-1 \in (k^\times)^2$ and $d(-1) = 0$ if $-1 \notin (k^\times)^2$.

Theorem 5. The irreducible decomposition of the tensor product $\mathfrak{R}_{\pi_1} \otimes \mathfrak{R}_{\pi_2}$ of representations of the supplementary series of the case (3b) is given by the following formula :

for $\varphi(x_1, x_2) \in \mathcal{H}(\pi_1, \pi_2) \cap \mathcal{S} \otimes \mathcal{S}$,

$$\frac{1}{\Gamma(\pi_1^{-1})\,\Gamma(\pi_2^{-1})} \iiiint \pi_1^{-1}\rho^{-1/2}(x_1 - x_1')\,\pi_2^{-1}\rho^{-1/2}(x_2 - x_2')\,\varphi(x_1, x_2)\,\overline{\varphi(x_1', x_2')}\,dx_1 dx_1' dx_2 dx_2'$$

= the similar form to the right hand side of the formula in Theorem 5

$$+ 2q\left(1+\frac{1}{q}\right)\left(\tan^{-1}\frac{\frac{\pi}{\log q}}{|\lambda_1+\lambda_2+1|}\right)\frac{1-q^{-1}}{\log q}\;\frac{-\Gamma(\pi_1^{-1})\,\Gamma(\pi_2^{-1})}{\Gamma((\pi_1\pi_2\rho^{1/2})^{-1})}$$

$$\int|\Phi'(t|\;\pi_1\pi_2\rho^{1/2})|^2|t|^{\lambda_2+\lambda_1+1}dt,$$

where $\Phi'(t|\;\pi_1\pi_2\rho^{1/2}) = \Gamma(\pi_2\rho^{1/2})\int|u|^{-\lambda_2-1}\,K_{\pi_1\pi_2\rho^{1/2}}(f'|\;t,u)du.$

The last formula shows the in the irreducible decomposition of $\mathcal{R}_{\pi_1}\otimes\mathcal{R}_{\pi_2}$ of (3b) occurs a representation $\mathcal{R}_{\pi_1\pi_2\rho^{1/2}}$ of the supplementary series.

8 Formulae for the limiting cases (4),(5) and (6). As the limiting cases of (2) and (3b), we can obtain the decomposition formulae of tensor products of representations, at least one of these is the special representation. But in the formulae for the limiting cases of (3b), the new appeared representation of the supplementary series vanishes.

Theorem 6. The irreducible decomposition of the tensor product $\mathcal{R}_{-1}\otimes\mathcal{R}_{\pi_2}$ ($\pi_2\in\Omega_p$) is given by the following formula : for $\varphi(x_1, x_2)$ $\in\mathcal{H}(\pi_1, \pi_2)\cap S\otimes S$, satisfying $\int\varphi(x_1, x_2)dx_1 = 0,$

$$\lim_{\lambda_1\to -1}\frac{1}{\Gamma(|\;|^{-\lambda_1})}\iiint|x_1-x_1'|^{-\lambda_1-1}\varphi(x_1, x_2)\overline{\varphi(x_1', x_2)}dx_1dx_1'dx_2$$

= the right side of the formula in Theorem 3, setting $\pi_1 = \pi_{sp}$.

The irreducible decomposition of the tensor product $\mathcal{R}_{-1}\otimes\mathcal{R}_{\pi_2}$ ($\pi_2\in\Omega_s$) is given by the following formula : for $\varphi(x_1, x_2)$, same as the above,

$$\lim_{\lambda_1\to -1}\frac{1}{\Gamma(|\;|^{-\lambda_1})\,\Gamma(|\;|^{-\lambda_2})}\iiiint|x_1-x_1'|^{-\lambda_1-1}\;|x_2-x_2'|^{-\lambda_2-1}$$

$$\varphi(x_1, x_2)\overline{\varphi(x_1', x_2')}dx_1dx_1'dx_2dx_2'$$

= the right side of the formula in Theorem 4, setting $\pi_1 = \pi_{sp}$.

The irreducible decomposition of the tensor product $\mathcal{R}_{-1} \otimes \mathcal{R}_{-1}$ is given by the following formula : for $\varphi(x_1, x_2) \in \mathcal{H}(\pi_1, \pi_2) \cap S \otimes S$, satisfying $\int \varphi(x_1, x_2) dx_1 = 0$ and $\int \varphi(x_1, x_2) dx_2 = 0$,

$$\lim_{\substack{\lambda_1 \to -1 \\ \lambda_2 \to -1}} \frac{1}{\Gamma(|\ |^{-\lambda_1})\ \Gamma(|\ |^{-\lambda_2})} \iiiint |x_1 - x_1'|^{-\lambda_1 - 1}\ |x_2 - x_2'|^{-\lambda_2 - 1} \varphi(x_1, x_2)\ \overline{\varphi(x_1', x_2')} dx_1 dx_1' dx_2 dx_2'$$

= the right side of the formula in Theorem 4, setting $\pi_1, \pi_2 = \pi_{sp}$.

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Masao Tsuchikawa

Department of Mathematics
Faculty of Education
Mie University
Kamihama-cho, Tsu-shi 514
Japan