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A DUALITY THEOREM FOR FACTOR SPACES

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Let G be a locally compact group and H its closed subgroup. We denote the left cosets space H\G by X . The purpose of this paper is to extend the Pontrjagin-Tannaka duality theorem for groups (see [3]) to factor spaces X .

In 1966, N. Iwahori and M. Sugiura [2] gave a notion of "representations of X " for the case G is a compact Lie group. And they proved a duality relation which holds between the categories of such factor spaces and of families of these "representations".

After their works, in this paper we shall give an analogous definition of "representations" for general pair (G,H) , and consider a duality property for these categories, which is essentially similar to so-called weak duality for the case of groups. We call this property I-S duality.

The biggest difference between the duality theories for factor spaces and for groups is as follows. As is well-known, the group duality is always valid, but for factor spaces, the I-S duality doesn't hold in general. In addition, a necessary condition (we call it (P-1)) for our

I-S duality leads us to some, even somewhat strict, structural restriction for the closed subgroup H in the pair (§6). We have not able to determine yet a satisfactory criterion for the validity of I-S duality. But we give a sufficient condition (P-3) for it (§5, Theorem 1). In the case that G is a Lie group, this is necessary at the same time.

In §1, we set up our definitions and give the notion of I-S duality. And in these words, our main aim can be stated as "to investigate for what pair (G,H) I-S duality holds".

In §2, we consider the key separating properties which play important rolls for our theory, and establish some relations between them.

§3 supplies tools for the proof of our duality, and using this we define an important subgroup (the core subgroup) in G (§4).

§5 is the main part of this paper. In this section we give the main theorem (Theorem 1) which gives our duality.

§6 is devoted to discuss that the requirement of I-S duality deduces a strict structural restriction for the subgroup H.

§1. Description of the problem

Notations.

G : a locally compact group.

H : a closed subgroup of G (for simplicity, we assume $H \neq G$).

Hereafter we write such a pair by (G,H) .

$X \equiv H\backslash G$.

$\tilde{g} \equiv \pi(g)$: the canonical image of $g \in G$ in X .

G operates on X as a transformation group, $X \ni x \mapsto xg \in X$.

$C_o(X)$: the space of all complex valued continuous functions with compact supports on X .

$\Omega \equiv \{$ unitary representation ω of $G \}$. (We can avoid the set theoretical difficulties by bounding the dimensions of representations by some sufficiently large cardinal number.)

$\omega \equiv \{ \mathcal{H}(\omega), T_g(\omega) \}$. Here $\mathcal{H}(\omega)$ is the space of representation ω , and $T_g(\omega)$ are the representation operators.

$\mathcal{H}_H(\omega) \equiv \{$ H-invariant vector v in $\mathcal{H}(\omega) \}$.

Obviously $\mathcal{H}_H(\omega)$ is a closed subspace of $\mathcal{H}(\omega)$.

$H_v \equiv \{ g \in G \mid T_g(\omega)v = v \}$ for a vector $v \in \mathcal{H}(\omega)$.

$H_\omega \equiv \{ g \in G \mid T_g(\omega)v = v$ for any $v \in \mathcal{H}_H(\omega) \}$ for a representation $\omega \in \Omega$.

It is easy to see H_v and H_ω are closed subgroups of G .

$\sigma \equiv \mathrm{Ind}_H^G 1_H = \{ \mathcal{H}(\sigma), T_g(\sigma) \}$: the representation of G induced from the trivial representation 1_H of H .

Definition 1. A representation of X is a pair $\{\omega, \psi\}$ of a unitary representation ω of G , and a map ψ from X to $\mathcal{H}(\omega)$ such that

(1-1) $\psi(xg) = T_{g^{-1}}(\omega)(\psi(x))$ for any $x \in X$, $g \in G$.

Lemma 1. For any representation $\{\omega, \psi\}$ of X ,

(1-2) $T_h(\omega)(\psi(\tilde{e})) = \psi(\tilde{e}h^{-1}) = \psi(\tilde{e})$ for any $h \in H$.

That is, $\psi(\tilde{e}) \in \mathcal{H}_H(\omega)$.

Proof. Trivial from (1-1) in Definition 1.

Conversely, for an $\omega \in \Omega$ and a $v \in \mathcal{H}_H(\omega)$, if we define a vector valued function ψ on X by

(1-3) $\psi(\tilde{g}) = T_{g^{-1}}(\omega)v$,

then the following is valid.

Lemma 2. The pair $\{\omega, \psi\}$ is a representation of X .

Proof. It is easy to see that ψ satisfies (1-1).

By Lemmata 1 and 2, giving a representation $\{\omega, \psi\}$ of X is equivalent to giving a pair (ω, v) of a representation ω of G and a vector v in $\mathcal{H}_H(\omega)$. Therefore, hereafter, we use the notation

$$\psi = (\omega, v) \tag{1-4}$$

to show a representation $\{\omega, \psi\}$ of X such that $v = \psi(\tilde{e}) \in \mathcal{H}_H(\omega)$, following the convenience.

We show the set of all representations ψ of X , by Ψ .

Definition 2. For two representations $\psi_j = (\omega_j, v_j)$ $(j = 1, 2)$ of X ,

1) $\psi_1 \underset{U}{\sim} \psi_2$, ψ_1 is equivalent to ψ_2 by U , if

(1) ω_1 is equivalent to ω_2 with the intertwining operator U , and

(2) $$Uv_1 = v_2 ,$$

2) $\psi_1 \oplus \psi_2 = (\omega_1 \oplus \omega_2 , v_1 \oplus v_2)$ (direct sum),

3) $\psi_1 \otimes \psi_2 = (\omega_1 \otimes \omega_2 , v_1 \otimes v_2)$ (tensor product).

Lemma 3. For any $x \in X$, and any $\psi, \psi_1, \psi_2 \in \Psi$,

1) $\psi_1 \underset{U}{\sim} \psi_2 \implies U(\psi_1(x)) = \psi_2(x)$,

2) $(\psi_1 \oplus \psi_2)(x) = \psi_1(x) \oplus \psi_2(x)$,

3) $(\psi_1 \otimes \psi_2)(x) = \psi_1(x) \otimes \psi_2(x)$,

4) $||\psi(x)|| = ||\psi(\tilde{e})||$.

Proof. Applying (1-2) to Definition 2, we obtain 1) ∿ 3) easily. And 4) follows from (1-2) immediately. q.e.d.

In a similar way as in the case of group duality theory, we define our notion of "birepresentation" over Ψ .

Definition 3. A vector field $\upsilon \equiv \{ u(\psi) \}$ over Ψ is called a birepresentation over Ψ when υ takes its value $u(\psi)$ in $\mathcal{H}(\omega)$ for $\psi = (\omega, v)$ and

1) $\psi_1 \underset{U}{\sim} \psi_2 \implies U(u(\psi_1)) = u(\psi_2)$,

2) $u(\psi_1 \oplus \psi_2) = u(\psi_1) \oplus u(\psi_2)$ for $\psi_1, \psi_2 \in \Psi$,

3) $u(\psi_1 \otimes \psi_2) = u(\psi_1) \otimes u(\psi_2)$ for $\psi_1, \psi_2 \in \Psi$.

4) there exists a common finite number M such that

$$||u(\psi)|| \leq M\,||\psi(\tilde{e})|| = M\,||v|| \qquad \text{for any } \psi \in \Psi \ .$$

Moreover, we call s-birepresentation, if a birepresentation $\upsilon = \{u(\psi)\}$ satisfies the following additional condition.

5) $u(\psi) \neq 0$ for any $\psi = (\omega, v) \in \Psi$ ($v \neq 0$).

Lemma 3 means that for any $x \in X$, the vector field $\upsilon_x \equiv \{\psi(x)\}$ gives a s-birepresentation over Ψ .

The zero vector field $0 \equiv \{0(\psi) = 0\}_{\psi \in \Psi}$ is also a birepresentation. We call it the trivial birepresentation. In §5 , we shall give an example of non-trivial birepresentation which is not the form of υ_x for any $x \in X$.

Now we can state a duality property, which we shall discuss in this paper.

[I-S duality] For any s-birepresentation $\upsilon \equiv \{u(\psi)\}$ over Ψ , there exists a unique element x in X such that $\upsilon = \upsilon_x$, that is, $u(\psi) = \psi(x)$ for any $\psi \in \Psi$.

Our main problem is as follows.

Problem. For what pair (G,H), does I-S duality holds?

Lemma 4. Under the assumptions 1) and 3) of Definition 3, the constant M mentioned in 4) can be take as $M = 1$.

Proof. If there exists an $\varepsilon > 0$ and ψ such that $||u(\psi)|| > (1+\varepsilon)||\psi(\tilde{e})||$, from 3)

$$||u(\overset{m\otimes}{\Pi}\psi)|| = ||\overset{m\otimes}{\Pi}\ u(\psi)|| = ||u(\psi)||^{m} > (1+\varepsilon)^{m}||\psi(\tilde{e})||^{m} = (1+\varepsilon)^{m}||\overset{m\otimes}{\Pi}\psi(\tilde{e})||.$$

This contradicts 4).

Example 1. When G is a compact Lie group, I-S duality holds by the results of Iwahori and Sugiura[2].

Example 2. When H is a normal subgroup of G, by Lemma 1, we can restrict ourselves to representations of the factor group $H\backslash G$. This reduction leads us easily to the equivalency of I-S duality for $H\backslash G$ as a factor space and the group duality as a factor group. That is, I-S duality holds in this case too.

Example 3. Put $G = SL(2,\mathbb{C})$, the group of 2×2-matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant one on the complex field $\mathbb{C}$. And put $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$, the subgroup of all upper triangular matrices.

It is well-known that for any irreducible representation ω of G, the restriction $\omega|_H$ of ω to H is irreducible too. This asserts the trivial representation 1_G of G is the only irreducible $\omega \in \Omega$ which has non-trivial H-invariant vectors, i.e., $\mathcal{H}_H(\omega) \neq 0$. This means, for any $\psi = (\omega, v)$ in Ψ, $\psi(\tilde{g}) = \psi(\tilde{e})$ for any $g \in G$.

Therefore representations on $H\backslash G$ do not separate elements of $X = H\backslash G$. So I-S duality fails in this case.

Example 4. We must remark that I-S duality is a duality for factor spaces but not for homogeneous spaces. In other words, I-S duality depends not only on the structure of homogeneous space $H\backslash G$, but also on the pair of groups (G,H) . The following example given by Prof. T. Hirai shows this fact.

Put $G_1 = SL(2,\mathbb{C})$, $H_1 = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right\}$, just as in Example 3. And put $G_2 = SU(2)$ and $H_2 = SU(1)$. Then obviously $H_1\backslash G_1 \simeq H_2\backslash G_2$ as homogeneous spaces.

By Example 3, I-S duality fails for the left hand side. However since $SU(2)$ is compact, Iwahori and Sugiura's result (Example 1) assures I-S duality for the right hand side.

§2. Separating conditions

Definition 4. We introduce the following different separating conditions for the pair (G,H) .

(P-0) There exists an $\omega \in \Omega$ such that $H_\omega \neq G$.

(P-1) $H = \cap H_\omega$, where ω runs over Ω .

(P-2) $H = H_\sigma$. That is, H-invariant vectors of $\sigma = \mathrm{Ind}_H^G 1_H$ separate the point $\tilde{e}$ from other points in X .

(P-3) There exists a fundamental system of neighborhoods of $\tilde{e}$ in X , consisting of H-invariant sets.

Lemma 5. (P-2) $\Rightarrow$ (P-1) $\Rightarrow$ (P-0) .

Proof. This is trivial from the definitions.

Lemma 6. I-S duality $\Rightarrow$ (P-1) .

Proof. If I-S duality holds, representations $\psi \in \Psi$ of X must separate each points of X . Therefore for any $\tilde{g} \neq \tilde{e}$ in X , there exists a $\psi = (\omega,v)$ such that $\psi(\tilde{g}) = T_{g^{-1}}(\omega)\psi(\tilde{e}) \neq \psi(\tilde{e})$, i.e., $g \notin H_v$. This leads us to (P-1) . q.e.d.

The following property is important.

Proposition 7. If (G,H) is a (P-1) pair, there exists a non-trivial G-invariant measure on X .

This Proposition 7, excludes Example 3 from candidates for (I-S) pair.

To prove Proposition 7, we prepare some supplementary lemmata which are also useful in later §'s.

Lemma 8. Let X_1 be a locally compact space, and $\mathcal{F} \equiv \{F_\alpha\}$ is a family of closed sets in X_1 , satisfying $\cap F_\alpha = \{x\}$ $(x \in X)$.

Then for any compact set C_1 in X_1 and any neighborhood V_1 of x in X_1 , there exists a finite subset $\{F_j\}_{1 \leq j \leq N}$ in $\mathcal{F}$,such that

$$C_1 \cap (\bigcap_j^N F_j) \subset V_1 .$$

Proof. We can assume V_1 is open without loss of generality. Then $C_1 - V_1$ is compact, and $\{(F_\alpha)^c\}$ is its open covering.

Thus we can take a finite open covering,

$$\bigcup_j (F_j)^c \supset C_1 - V_1 .$$

This means the conclusion.

Corollary. For a (P-1) pair (G,H) , any compact set C in X , and any neighborhood V of $\tilde{e}$ in X , there exists a finite family of H-invariant open sets $\{F_j\}_{1 \leq j \leq N}$ such that

$$C \cap (\bigcap_j^N F_j) \subset V .$$

Proof. For any $v \in \mathcal{H}_H(\omega)$ and $\varepsilon > 0$, put

(2-1) $$E(\varepsilon,v) \equiv \{g \in G \mid |\langle v,v\rangle - \langle T_g(\omega)v,v\rangle| \leq \varepsilon \},$$

(2-2) $$F(\varepsilon,v) \equiv \pi(E(\omega,v)) .$$

Since v is H-invariant,

(2-3) $$HE(\varepsilon,v)H = E(\varepsilon,v) .$$

By the definitions, $E(\varepsilon,v)$ is a neighborhood of e, therefore $F(\varepsilon,v)$ is an H-invariant neighborhood of $\tilde{e}$ in X. And the assumption (P-1) assures

$$(2\text{-}4) \qquad H = \cap E(\varepsilon,v) \qquad (\omega \in \Omega,\ v \in \mathcal{H}_H(\omega),\ \varepsilon > 0),$$

$$(2\text{-}5) \qquad \{\tilde{e}\} = \cap F(\varepsilon,v).$$

Put $X \equiv X_1$, $\mathcal{F} \equiv F(\varepsilon,v)$, $C = C_1$ in Lemma 8, and we obtain a finite family $\{F_j \equiv F(\varepsilon_j,v_j)\}$ such that

$$C \cap (\bigcap_j F_j) \subset V.$$

<u>Lemma 9.</u> For a (P-1) pair (G,H), any compact set C_o in H, and any neighborhood V of $\tilde{e}$ in $X = H\backslash G$, there exists a neighborhood W of $\tilde{e}$ such that $W \subset V$ and $WC_o \subset W$.

<u>Proof.</u> We may assume V is compact. In Corollary of Lemma 8, put $C = VC_o$ and $W = C \cap (\bigcap_j F_j) \subset V$. Then $WC_o \subset VC_o = C$, and $WC_o \subset WH \subset (\bigcap_j F_j)H = (\bigcap_j F_j)$. Therefore $WC_o \subset C \cap (\bigcap_j F_j) = W$.

<u>Lemma 10.</u> Assume that for any $h_1 \in H$ and any neighborhood V of $\tilde{e}$, there exists a neighborhood W of $\tilde{e}$ in V such that

$$Wh_1 \subset W.$$

Then there exists a non-trivial G-invariant measure in $X = H\backslash G$.

<u>Proof.</u> Let Δ_G, Δ_H be the modular functions for Haar measures on G and H respectively. A. Weil's criterion ([6] p 45) shows, the existence of G-invariant measure on X is equivalent to

$$(2\text{-}6) \qquad \delta(h) \equiv \Delta_G(h) / \Delta_H(h) = 1 \qquad \text{for any} \quad h \in H.$$

So if there is no non-trivial G-invariant measure on X, for some $h_1 \in H$, $\delta(h_1) > 8$. Let ξ be a positive continuous function such that $\xi(hg) = \delta(h)\xi(g)$ for any $h \in H$ and $g \in G$, then there exists a quasi-invariant measure μ on X satisfying

$$(d\mu(gg_1)/d\mu(g)) = \xi(gg_1)/\xi(g) \qquad \text{(see [1]).}$$

Take an open relative compact neighborhood W_1 of e in G such that,

$$(\xi(e)/2) < \xi(g) < 2\xi(e) \quad \text{for any } g \text{ in } W_1 .$$

Put $\quad V_1 \equiv W_1 \cap h_1 W_1 h_1^{-1}$ and $V \equiv \pi(V_1) \ni \tilde{e}$.

From the assumption, there exists a neighborhood W of $\tilde{e}$ in V and

(2-7) $$Wh_1^{-1} \subset W .$$

Evidently $\quad \pi^{-1}(W) \subset \pi^{-1}(V) = HV_1$. Thus

$$0 < \mu(Wh_1^{-1}) = \int_W d\mu(\tilde{g}h_1) = \int_W (\xi(gh_1)/\xi(g))d\mu(\tilde{g}) < +\infty .$$

Any element g in $\pi^{-1}(W)$ can be written as hg_1 $(h \in H,\ g_1 \in V)$.

$$\begin{aligned}(\xi(gh_1)/\xi(g)) &= (\xi(hg_1h_1)/\xi(hg_1)) = (\xi(g_1h_1)/\xi(g_1)) \\ &= (\xi(h_1h_1^{-1}g_1h_1)/\xi(g_1)) = \delta(h_1)(\xi(h_1^{-1}gh_1)/\xi(g_1)) \\ &> 8((\xi(e)/2)/\ 2\xi(e)) = 2 .\end{aligned}$$

Finally we obtain

$$\mu(Wh_1^{-1}) > 2\int_W d\mu(\tilde{g}) = \mu(W) > 0 .$$

This contradicts (2-7) .

Proof of Proposition 7. It is sufficient to see that for a (P-1) pair the assumption in Lemma 10 is satisfied. This is a direct result of Lemma 9 for the case $C = \{h_1\}$.

Lemma 11. $\quad$ (P-3) $\Rightarrow$ (P-2) .

Proof. (P-3) assures the existence of an H-invariant neighborhood W of $\tilde{e}$ in an arbitrary given neighborhood V of $\tilde{e}$. This supplies the assumption in Lemma 10, thus there exists a non-trivial G-invariant measure μ on X . Therefore the induced representation $\sigma = \mathrm{Ind}_H^G 1_H$ is realized on $L^2_\mu(X)$ as

$$T_g(\sigma)f(x) = f(xg) \quad \text{for any} \quad f \in L^2_\mu(X) \equiv H(\sigma) \ .$$

The family of characteristic functions χ_E of H-invariant compact neighborhoods E of $\tilde{e}$ gives a family of H-invariant vectors in $H(\sigma)$ which separates $\tilde{e}$ in X from other points. q.e.d.

Based on the result of Proposition 7, hereafter we assume that there exists a G-invariant measure μ on X .

If E is an H-invariant measurable set in X , $\pi^{-1}(E)$ is a set of type HE_1H in G for some measurable set E_1 . Put

$$E^{-1} \equiv \pi(HE_1{}^{-1}H) = \{\tilde{g} \in H\backslash G \mid \tilde{g}^{-1} \in E\} \ .$$

Obviously E^{-1} is an H-invariant measurable set in X .

Analogously for two H-invariant sets E_1 and E_2 in X , we can define their product by

$$E_1E_2 \equiv \pi(\pi^{-1}(E_1)\pi^{-1}(E_2)).$$

If an H-invariant set E is compact, there exists a compact set F in G such that $HFH = HF$ and $E = \pi(F)$. This concludes that for compact H-invariant sets E_1 , E_2 , the product $E_1E_2 = \pi(HF_1F_2H) = \pi(F_1F_2)$ is also compact in X .

Lemma 12. The nullities of E and E^{-1} with respect to μ are equivalent.

Proof. From the relation between nullities on μ and on Haar measure τ on G , we get

$$\mu(E) = 0 \Longleftrightarrow \tau(\pi^{-1}(E)) = 0 \Longleftrightarrow \tau((\pi^{-1}(E))^{-1}) = 0$$

$$\Longleftrightarrow \mu(E^{-1}) = 0 \ .$$

q.e.d.

By the reason of Lemma 12, for any H-invariant μ-measurable function f on X, we can define an H-invariant μ-measurable function

$$f^*(\tilde{g}) \equiv \overline{f(\tilde{g}^{-1})} .$$

Definition 5. An H-invariant set E in X is called *symmetric* if

$$E = E^{-1} .$$

And an H-invariant μ-measurable function f is called *symmetric* if

$$f^* = f .$$

For a μ-measurable function f_1 and an H-invariant μ-measurable function f_2 on X, if the following integral on the right hand side has a meaning, we write,

$$\langle T_{\tilde{g}}\, f_1,\ f_2\rangle \equiv \int_X f_1(xg)\ \overline{f_2(x)}\ d\mu(x) .$$

This function is μ-measurable, and if f_1 is H-invariant, it is H-invariant as a function of $\tilde{g}$. We put

$$||f_j||_p \equiv \left(\int_X |f_j(x)|^p\ d\mu(x)\right)^{1/p} \quad (j = 1,\ 2\ ;\ 1 \leq p).$$

Lemma 13.

(2-8) $$|\langle T_{\tilde{g}}\, f_1,\ f_2\rangle| = ||f_1||_2\ ||f_2||_2 \quad \text{for } \tilde{g} \in X .$$

(2-9) $$||\langle T_{\tilde{g}}\, f_1,\ f_2\rangle||_1 = ||f_1||_1||f_2{}^*||_1 \quad \text{for } f_1,\ f_2{}^* \in L^1{}_\mu(X) .$$

(2-10) $$||\langle T_{\tilde{g}}\, f_1,\ f_2\rangle||_2 = ||f_1||_2||f_2||_1{}^{1/2}||f_2{}^*||_1{}^{1/2}$$
$$\text{for } f_1 \in L^2{}_\mu(X) \text{ and } f_2,\ f_2{}^* \in L^1{}_\mu(X) .$$

(2-11) $$||\langle T_{\tilde{g}}\, f_1,\ f_2\rangle||_2 = ||f_2{}^*||_2||f_1||_1$$
$$\text{for } f_1 \in L^1{}_\mu(X) \text{ and } f_2{}^* \in L^2{}_\mu(X) .$$

Proof. (2-8) is given by Cauchy-Schwarz's inequality directly.

$$||\langle T_{\tilde{g}}\, f_1,\ f_2\rangle||_1 = \int_X \left|\int_X f_1(xg)\ \overline{f_2(x)}\ d\mu(x)\right|\ d\mu(\tilde{g})$$

$$\leq \int_X \int_X |f_1(xg)\ \overline{f_2(x)}| d\mu(x) d\mu(\tilde{g}) = \int_X \int_X |f_1(x)|\ |f_2(xg^{-1})|\ d\mu(x) d\mu(\tilde{g})$$

$$= \int_X \int_X |\ f_1(\tilde{g}_1)||f_2{*}(\tilde{g}g_1^{-1})|\ d\mu(\tilde{g})\ d\mu(\tilde{g}_1) = \int_X |f_1(\tilde{g}_1)|\ ||f_2{*}||_1\ d\mu(\tilde{g}_1)$$

$= ||f_1||_1\ ||f_2{*}||_1$. This shows (2-9) .

$$||<T_{\tilde{g}}\ f_1,\ f_2>||_2^{\,2} = \int_X |\int_X f_1(xg)\ \overline{f_2(x)}\ d\mu(x)|^{\,2}\ d\mu(\tilde{g})$$

$$\leq \int_X (\int_X |f_1(xg)|^{\,2} |f_2(x)| d\mu(x)) (\int_X |f_2(x)|\ d\mu(x)) d\mu(\tilde{g})$$

$$= ||f_2||_1 \int_X |f_1(x)|^{\,2} \int_X |f_2(xg^{-1})|\ d\mu(\tilde{g}) d\mu(x) = ||f_2||_1\ ||f_2{*}||_1\ ||f_1||_2^{\,2} .$$

This is (2-10) .

$$||<T_{\tilde{g}}\ f_1,\ f_2>||_2^{\,2} \leq \int_X (\int_X |f_1(xg)| d\mu(x)) (\int_X |f_1(x)||f_2(xg^{-1})|^{\,2} d\mu(x)) d\mu(\tilde{g})$$

$$= ||f_1||_1 \int_X |f_1(\tilde{g}_1)|\ (\int_X |f_2{*}(\tilde{g}g_1^{-1})|^2 d\mu(\tilde{g})) d\mu(\tilde{g}_1)$$

$= (||f_1||_1\ ||f_2{*}||_2\ )^2$. Thus (2-11) is shown .

Definition 6. Define the following conditions.

(A-1) There exists a compact H-invariant neighborhood of $\tilde{e}$ in X .

(A-1') There exists an ω in Ω , and a non-zero v in $\mathcal{H}_H(\omega)$ and $1 \leq p < +\infty$ such that

$$\xi(\tilde{g}) \equiv <T_g(\omega)v,\ v> \in L^p_{\mu}(X) .$$

(A-2) X is locally connected.

(A-3) There exists a normal closed subgroup N of G in H such that the factor group N\H is generated by a compact set.

Proposition 14. (P-1) + (A-1) $\Longleftrightarrow$ (P-3) .

Proof. ($\Longleftarrow$) Trivial.

($\Longrightarrow$) Let V be a neighborhood assumed in (A-1) . The proof of Lemma 9 gives a fundamental system of neighborhoods of $\tilde{e}$,

$$\{ F(\varepsilon,v) \cap V \mid \omega \in \Omega \ , v \in \mathcal{H}_H(\omega) \ , \varepsilon > 0 \},$$

which is proposed in (P-3) .

Definition 7. We introduce other conditions.

(A-1") There exists an H-invariant symmetric μ-finite μ-positive set in X .

(A-1"') There exists a symmetric continuous f in $\mathcal{H}_H(\sigma) \cap L^1_\mu(X)$.

Lemma 15. (A-1'), (A-1") and (A-1"') are all equivalent.

Proof. (A-1') $\Longrightarrow$ (A-1") . The set $E \equiv \{x \mid |\xi(x)| > \xi(\tilde{e})/2\}$ is an example of the set for (A-1") .

(A-1") $\Longrightarrow$ (A-1"') . Let E be a set given in (A-1"), then its characteristic function χ_E is in $\mathcal{H}_H(\sigma)$ and by (2-9) the function $f(\tilde{g}) = \langle T_g(\sigma)\chi_E, \chi_E\rangle$ is the one asked in (A-1"') .

(A-1"') $\Longrightarrow$ (A-1') . (A-1"') is a special case of (A-1') .

Proposition 16. (P-1) + (A-1') $\Longrightarrow$ (P-2) .

Proof. Take the set E given in the first step of the proof of Lemma 15 and $F(\varepsilon,v)$ in the proof of Proposition 14. Next construct the family of the sets

$$\{ F \equiv F(\varepsilon,v) \cap E \mid \omega \in \Omega \ , v \in \mathcal{H}_H(\omega) \ , \varepsilon > 0 \}.$$

The family of vectors $\{\chi_F\}$ in $\mathcal{H}_H(\sigma)$ separates $\tilde{e}$ in X , that is, (P-2) is satisfied.

Proposition 17. (P-1) + (A-2) $\Longrightarrow$ (P-3) .

Proof. We assume X is locally connected. Let V be given relative compact open neighborhood of $\tilde{e}$ in X . Put $C = \overline{V}$, and adapt Corollary of Lemma 8, then there exist finite H-invariant open sets $\{F_j\}_{1 \leq j \leq N}$ such that $W \equiv C \cap (\bigcap_j F_j) \subset V$. Because of locally connectedness of X, the connected component W_o $(\ni \tilde{e})$ of W is a neighborhood of $\tilde{e}$. For any h in H, $W_o h$ is connected and

$$V \cap W_o h \subset C \cap W_o h \subset C \cap (\bigcap_j F_j)h = C \cap (\bigcap_j F_j) = W \subset V.$$

This asserts $V \cap W_o h = C \cap W_o h$ and this set is a relatively open and relatively closed in the connected set $W_o h$. Since this set contains $\tilde{e}$, it is non-void, therefore is equal to $W_o h$. That is,

$$\tilde{e} \in W_o h = V \cap W_o h \subset W .$$

Thus we obtain an H-invariant neighborhood W_o in V, and the condition (P-3) is proved.

Corollary. If G is a Lie group, for a pair (G,H), (P-1) is equivalent to (P-3).

Proof. In this case $H\backslash G$ is locally connected. So by Proposition 17, it is direct.

Proposition 18. (P-1) + (A-3) $\Rightarrow$ (P-3).

Proof. Because $H\backslash G \sim (N\backslash H)\backslash(N\backslash G)$, we may assume $N = \{e\}$. Let C_o be the compact set generating H. By Lemma 9, for given compact neighborhood V of $\tilde{e}$ in X, we get a neighborhood W of $\tilde{e}$ in V such that $WC_o \subset W$. Repeating adaptation of this relation leads us to $WC_o{}^n \subset W$ for any n. And lastly we obtain

$$\overline{WH} = \overline{\bigcup_n WC_o{}^n} \subset \overline{W} .$$

§3. Approximate identity and operator T_E.

At first we remark that if H-invariant f_1, f_2 are in $L^2_\mu(X)$, they can be considered as elements in $\mathcal{H}_H(\sigma)$ and

$$< T_{\tilde g} f_1, f_2 > = < T_g(\sigma) f_1, f_2 > .$$

Lemma 19. For (P-3) pair (G,H), any $k \in C_0(X)$ and $\mathcal{E} > 0$, there exists an H-invariant neighborhood V of $\tilde e$ such that

$$| k(xg) - k(\tilde g)| < \varepsilon \qquad \text{for any } x \in V \text{, and any } g \in G .$$

Proof. Since k is continuous, for any $g \in G$, there exists a neighborhood $V(g)$ of e in G such that

$$| k(\tilde e g_1 g) - k(\tilde g)| < \varepsilon /2 \qquad \text{for any } g_1 \in V(g) .$$

By (P-3) assumption, there exists a symmetric neighborhood $W(g)$ such that $W(g)^2 \subset V(g)$ and

$$\text{(3-1)} \qquad HW(g)H = HW(g) .$$

Therefore we can determine $W(g)$ depending only on the H-coset which contains g. Thus we show it by $W(\tilde g)$.

Take a finite covering $[k] \subset \overset{N}{\cup}\, \tilde e W(g_j) g_j$, and put $W = \overset{N}{\cap}\, W(g_j)$ and $V \equiv \pi(W) = \tilde e W$. This V is the asked one.

In fact, any $\tilde g \in [k]$ is written as $\tilde g = \tilde e g_0 g_j$ for some j and $g_0 \in W(g_j)$. Similarly for any $x \in V$, $xg = \tilde e g' g_0 g_j$. Here $g' \in W \subset W(g_j)$ so $g' g_0 \in (W(g_j))^2 \subset V(g_j)$.

$$\begin{aligned} | k(xg) - k(\tilde g)| &= |k(\tilde e g' g_0 g_j) - k(\tilde g_j)| + |k(\tilde g_j) - k(\tilde e g_0 g_j)| \\ &= \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

Let $\tilde g \notin [k]$ satisfying $\tilde e W g \cap [k] \neq \phi$, by the symmetricity of W, for any $g_1 \in HWg \cap \pi^{-1}([k])$, $g \in HWg_1$. That is, $\tilde g \in \tilde e W g_1 = V g_1$. This means $|k(\tilde g_1)| < \varepsilon$.

Proposition 20. For a (P-3) pair (G,H) there exists an approximate identity $\{\theta_\alpha\}$ in $L^1_\mu(X) \cap L^\infty_\mu(X)$ with respect to

$\langle T_{\tilde{g}}f, \theta_\alpha\rangle$ in $L^2_\mu(X)$. That is for any $f \in L^2_\mu(X)$,

$$\lim_\alpha \langle T_{\tilde{g}}f, \theta_\alpha\rangle = f \qquad \text{in } L^2_\mu(X) .$$

Proof. Let $\mathcal{V} \equiv \{V(\alpha)\}$ be a fundamental system of neighborhoods of $\tilde{e}$ in X , consisting of symmetric H-invariant sets. Put

$$\theta_\alpha \equiv \mu(V(\alpha))^{-1}\chi_{V(\alpha)} .$$

For arbitrary given $f \in L^2_\mu(X)$ and $\varepsilon > 0$, select a $k \in C_o(X)$ such that $||f - k||_2 < \varepsilon/3$. Then evidently $||T_g(\sigma)(f - k)||_2 < \varepsilon/3$.

By Lemma 19, for some $V \in \mathcal{V}$,

$|k(xg) - k(g)| < \varepsilon/3M^{1/2}$ for any $x \in V$, and any $g \in G$.

And $k(xg) = 0$ for any $x \in V$, and any $g \notin \pi^{-1}(V)\,\pi^{-1}([k])$. Where $M = (\mu(V\pi^{-1}([k])))^{1/2}$. Therefore, for $\theta_V \equiv (\mu(V))^{-1}\chi_V$,

$$|\langle T_{\tilde{g}}k, \theta_V\rangle - k(\tilde{g})| = |\int_X (\mu(V)^{-1}\chi_V(\tilde{g}_1)(k(\tilde{g}_1 g) - k(\tilde{g}))\, d\mu(\tilde{g}_1)|$$

$$\leq \mu(V)^{-1}\int_V |k(\tilde{g}_1 g) - k(\tilde{g})|\, d\mu(\tilde{g}_1) \leq \varepsilon/3M^{1/2}.$$

And $k(\tilde{g}) = \langle T_{\tilde{g}}k, \theta_V\rangle = 0$ for $\tilde{g} \notin V\pi^{-1}([k])$.

$$||\langle T_{\tilde{g}}f, \theta_V\rangle - f||_2 \leq ||\langle T_{\tilde{g}}f - T_{\tilde{g}}k, \theta_V\rangle||_2 +$$

$$+ ||\langle T_{\tilde{g}}k, \theta_V\rangle - k||_2 + ||k - f||_2$$

$$= ||T_g f - T_g k||_2\, ||\theta_V{}^*||_1^{1/2}\, ||\theta_V||_1^{1/2} + (\int |\langle T_{\tilde{g}}k, \theta_V\rangle - k(\tilde{g})|^2 d\mu(g))^{1/2}$$

$$+ \varepsilon/3$$

$$= \varepsilon/3 + ((\varepsilon^2/9M)M)^{1/2} + \varepsilon/3 = \varepsilon .$$

Proposition 21. Let f and f^* be in $\mathcal{H}_H(\sigma)$. For given $\varepsilon > 0$, there exists a symmetric H-invariant neighborhood V of $\tilde{e}$ in X , such that $||\, f - \langle T_{\tilde{g}}\theta_V, f^*\rangle\, ||_2 < \varepsilon$, here $\theta_V = \mu(V)^{-1}\chi_V$.

Proof. We assume $f \neq 0$ without loss of generality. The existence of such an f assures that of a symmetric H-invariant μ-finite μ-positive set

$E_1 \equiv \{\tilde{g} \mid |f| > c\} \cup \{\tilde{g} \mid |f^*| > c\}$ in X for some $c > 0$, and again we consider the set

$$E \equiv \{\tilde{g} \in X \mid \mu(E_1 g \cap E_1) > (\mu(E_1)/2)\} .$$

As is easily shown, E is a symmetric μ-finite H-invariant open neighborhood of $\tilde{e}$.

On the other hand, the set $V_1 \equiv \{\tilde{g} \mid \|f - T_g(\sigma)f\|_2^2 < \varepsilon\}$ gives also an H-invariant symmetric neighborhood of $\tilde{e}$.

Put $V \equiv V_1 \cap E$, then

$$\|f - \langle T_{\tilde{g}}\theta_V, f^*\rangle\|_2^2 = \|f - \int_X \theta_V(\tilde{g}_1) f((\tilde{g}_1 \tilde{g}^{-1})^{-1})\, d\mu(\tilde{g}_1)\|_2^2$$

$$\leq \int_X [\int_X |\theta_V(\tilde{g}_1)|\, |f(\tilde{g}) - f(\tilde{g}g_1^{-1})|\, d\mu(\tilde{g}_1)\,]^2 d\mu(\tilde{g})$$

$$\leq \int_X [\int_X |\theta_V(\tilde{g}_1)|\, d\mu(\tilde{g}_1)]\, [\int_X |\theta_V(\tilde{g}_1)|\, |f(\tilde{g}) - f(\tilde{g}g_1^{-1})|^2 \times$$

$$\times\, d\mu(\tilde{g}_1) d\mu(\tilde{g})$$

$$= \|\theta_V\|_1 \int_X \theta_V(\tilde{g}_1)\, [\int_X |f(\tilde{g}) - f(\tilde{g}g_1^{-1})|^2\, d\mu(g)]\, d\mu(\tilde{g}_1)$$

$$= \int_X \theta_V(\tilde{g}_1)\, \|f - T_{g_1^{-1}} f\|_2^2\, d\mu(\tilde{g}_1) \leq \varepsilon .$$

Definition 8. For an H-invariant symmetric μ-finite set E in $H\backslash G$, we consider the operator T_E on $L^2_\mu(H\backslash G)$ as follows. For any f in $L^2_\mu(H\backslash G) = \mathcal{H}(\sigma)$,

$$(T_E f)(\tilde{g}) \equiv \int_{H\backslash G} f(xg)\chi_E(x)\, d\mu(x) = \langle T_{\tilde{g}} f, \chi_E\rangle .$$

Proposition 22. i) T_E is a bounded symmetric operator on $\mathcal{H}(\sigma)$.

ii) $T_E T_g(\sigma) = T_g(\sigma) T_E$ for any $g \in G$.

iii) Take an H-invariant μ-finite μ-positive set F in $H\backslash G$, and consider the H-invariant subspace $\mathcal{K} \equiv L^2_\mu(F)$ in $\mathcal{H}(\sigma)$.

Then the restriction $T_{\mathcal{K}} \equiv T_E|_{\mathcal{K}}$ is an operator of Hilbert-Schmidt type from $L^2_\mu(F)$ into $L^2_\mu(H\backslash G)$.

Proof. i) From (2-11),

$$\|\langle T_g f, \chi_E\rangle\|_2 \le \|f\|_2\, \|\chi_E\|_1 = \|f\|_2\, \mu(E).$$

This shows the boundedness of T_E.

$$\langle T_E f, k\rangle = \int \Big[\int f(xg)\chi_E(x)\, d\mu(x) \Big]\, \overline{k(\tilde g)}\, d\mu(\tilde g)$$

$$= \iint f(\tilde g_1)\, \overline{\chi_E(\tilde g_1 g^{-1}) k(\tilde g)}\, d\,\mu(\tilde g)\, d\mu(\tilde g_1)$$

$$= \int f(\tilde g_1)\, \Big[\int \overline{\chi_E(\tilde g g_1^{-1}) k(\tilde g)\, d\mu(\tilde g)} \Big]\, d\mu(\tilde g_1)$$

$$= \langle f, T_E k\rangle \qquad \text{for any} \quad f, k \in L^2_\mu(H\backslash G).$$

That is the symmetricity of T_E.

ii) $T_E(T_{g_1}(\sigma)f)(\tilde g) = \langle T_{\tilde g g_1} f, \chi_E\rangle = T_{g_1}(\sigma)(T_E f)(\tilde g)$.

iii) We take an orthonormal base $\{f_\alpha\}$ in $\mathcal{K} = L^2_\mu(F)$. And put $P_{\mathcal{K}}$ the projection from $\mathcal{H}(\sigma)$ onto $\mathcal{K}$. Evidently $P_{\mathcal{K}}$ is the operator multiplying the characteristic function χ_F.

$$\sum_\alpha |(T_E f_\alpha)(g)|^2 = \sum_\alpha |\langle T_g f_\alpha, \chi_E\rangle|^2 = \sum_\alpha |\langle f_\alpha, T_{g^{-1}}(\sigma)\chi_E\rangle|^2$$

$$= \|P_{\mathcal{K}} T_{g^{-1}}(\sigma)\chi_E\|_2^2 = \int_X \chi_F(x)\, \chi_E(xg^{-1})\, d\mu(x).$$

Thus

$$|||T_{\mathcal{K}}|||^2 \equiv \sum_\alpha \|T_E f_\alpha\|_2^2 = \sum_\alpha \int_X |(T_E f_\alpha)(g)|^2\, d\mu(\tilde g)$$

$$= \int_X \int_X \chi_F(x)\, \chi_E(xg^{-1})\, d\mu(x)\, d\mu(\tilde g) = \mu(F)\mu(E) < +\infty.$$

Lemma 23. For a (P-3) pair (G,H),

$$\bigcap_E (T_E^{-1}(0)) = \{0\}.$$

Here E runs all H-invariant symmetric μ-finite sets.

Proof. For (P-3) pair, a fundamental system $\{E_\alpha\}$ of neighborhoods of $\tilde e$ in X, consisting of H-invariant symmetric μ-finite sets, exists. As in the proof of Proposition 20, the family of functions

$\theta_\alpha = (\mu(E_\alpha))^{-1}\chi_{E_\alpha}$ consists an approximate identity for $L^2{}_\mu(X)$.

That is,

$$0 = \lim_\alpha (\mu(E_\alpha))^{-1} T_{E_\alpha} f = \lim_\alpha (\mu(E_\alpha))^{-1} \langle T_{\tilde{g}} f, \chi_{E_\alpha} \rangle = \lim_\alpha \langle T_{\tilde{g}} f, \theta_\alpha \rangle$$
$$= f , \qquad \text{for any } f \text{ in } \cap (T_E^{-1}(0)) \subset \cap_\alpha (T_{E_\alpha}^{-1}(0)) .$$

§4 Core subgroup.

We introduce the following notations.

$$\mathcal{S} \equiv \{E \subset X \mid \text{H-invariant measurable and } 0 < \mu(E) < +\infty \}.$$
$$\mathcal{S}_1 \equiv \{E \in \mathcal{S} \mid E^{-1} \in \mathcal{S} \} .$$

$\mathcal{S}$ and $\mathcal{S}_1$ may be void in general. But the following is trivial by the definition.

Lemma 24. $\mathcal{S}_1 \neq \phi \Longleftrightarrow$ (A-1″) .

Proposition 25. If $\mathcal{S}_1 \neq \phi$, there exists the smallest open subgroup G_o in G containing H and

(4-1) $\mu(E \cap \pi(G_o)) = \mu(E)$ for any $E \in \mathcal{S}_1$.

Proof. 1) If $E \in \mathcal{S}_1$, its characteristic function χ_E is H-invariant, and both of χ_E and $\chi_E{}^*$ are in $L^1{}_\mu(X)$. Lemma 13 assures that the continuous function

(4-2) $\beta(E, \tilde{g}) \equiv \langle T_{\tilde{g}}\chi_E, \chi_E \rangle = \mu(Eg \cap E)$

is in $L^1{}_\mu(X)$. It is also H-invariant symmetric and $\beta(E, \tilde{e}) = \mu(E) > 0$. Therefore for some $\varepsilon > 0$, the set

$$F \equiv F(\varepsilon) \equiv \{x \in X \mid \beta(E, x) > \varepsilon \}$$

is H-invariant symmetric open μ-finite μ-positive, and contains $\tilde{e}$.

2) Now for any $E \in \mathcal{S}_1$, take $\tilde{E} \equiv E \cup F$ and

$$\beta(\tilde{E}, \tilde{g}) \equiv \mu(\tilde{E}g \cap \tilde{E}) \geq \mu(Fg \cap E) ,$$

$$F(E,\ \varepsilon) \ \equiv \{x \in X \mid \ \beta(\tilde{E},\ x) > \varepsilon\} \quad (\varepsilon > 0)\ .$$

Then the set $G_o \ \equiv \ \cup\pi^{-1}(F(E,\varepsilon))$ (the join runs over the set of pairs $(E,\varepsilon) \in \mathcal{S}_1\times(0,\infty))$ is the asked one.

Indeed, evidently G_o is open as a join of open sets $\pi^{-1}(F(E,\varepsilon))$, and contains $H = \pi^{-1}(\tilde{e})$. For any $g_j \in \pi^{-1}(F(E_j,\varepsilon_j)) \subset G_o$ $(j = 1,\ 2)$, the set $F(E_1,\varepsilon_1)g_1^{-1} \cap F(E_2,\varepsilon_2)g_2^{-1} \ni \tilde{e}$ is a non-void open set in X. Thererfore, if we put $\tilde{F} \equiv F(E_1,\varepsilon_1) \cup F(E_2,\varepsilon_2)$, there exists an $\varepsilon > 0$ such that

$$\begin{aligned}\beta(\tilde{F},\ g_1^{-1}g_2) &= < T_{\tilde{g_1^{-1}g_2}}\ \chi_{\tilde{F}},\ \chi_{\tilde{F}} > = \mu(\tilde{F}g_1^{-1}g_2 \cap \tilde{F})\\ &\geq \ \mu(F(E_1,\varepsilon_1)g_1^{-1}g_2 \cap F(E_2,\varepsilon_2))\\ &= \ \mu(F(E_1,\varepsilon_1)g_1^{-1} \cap F(E_2,\varepsilon_2)g_2^{-1}) \ > \varepsilon.\end{aligned}$$

This means $g_1^{-1}g_2 \in \pi^{-1}(F(\tilde{F},\varepsilon)) \subset G_o$, i.e., G_o is an open subgroup.

3) Next we show the relation (4-1). For this, it is sufficient to see that for any $E \in \mathcal{S}_1$, $\mu(E \cap (F_E)^c) = 0$. Here

$$F_E \equiv \{\ x \in X \mid \beta(\tilde{E},\ x) > 0\ \} = \bigcup_{\varepsilon>0} F(E,\ \varepsilon) \subset \pi(G_o)\ .$$

If not, there exists a compact C in $E \cap (F_E)^c$ such that $\mu(C) > 0$. Take a finite covering by open sets Fg_j's as

$$C \subset \bigcup^{N} Fg_j \qquad (\pi(g_j) \in C)\ .$$

Since $\tilde{g}_j = \pi(g_j) \in C \subset (F_E)^c$,

$$\begin{aligned}\mu(C) = \mu(E \cap C) &\leq \mu(E \cap \bigcup^{N} Fg_j) \leq \sum^{N} \mu(E \cap Fg_j)\\ &\leq \sum^{N} \beta(\tilde{E},\ \tilde{g}_j) = 0\ .\end{aligned}$$

That is a cotradiction.

4) G_o must be the smallest. In fact, all $F(E,\varepsilon)$ is open and in $\mathcal{S}_1$. So $\pi^{-1}(F(E,\varepsilon))$ is contained in the group which is stated in this proposition.

<u>Definition 9.</u> We call G_o given in Proposition 25, the <u>core</u>

subgroup of (G,H) and write $X_o = \pi(G_o)$.

Lemma 26. For a (P-3) pair (G,H) , S_1 is non-void, that is, the core subgroup G_o exists.

Proof. As a consequence of propositions in §2, we get (P-3) $\Rightarrow$ (A-1) $\Rightarrow$ (A-1') . And Lemma 24 leads us to the result.

Lemma 27. If $S_1 \neq \phi$ and G is connected, $G = G_o$.

Proof. Since G_o is an open subgroup of G , $G_o = G$.

Example 5. If H is compact, the pair (G,H) always satisfies $G_o = G$.

Indeed, for any $g \in G$ and any relative compact open neighborhood V of g in G , the set $W = HVH \cup HV^{-1}H$ is also relative compact and open. Thus $\pi(W)$ is in S_1 and $G_o \supset W \quad g$, i.e., $G_o = G$. And by the same reason, (G,H) is a (P-3) pair.

Example 6. When $G \equiv \mathrm{Lor}'(2)$ (2-dimensional inhomogeneous Lorentz group) and $H \equiv \mathrm{Lor}(2)$ (2-dimensional Lorentz group), then $H\backslash G \simeq R^2$, and G-invariant measure μ on it is just the Lebesgue measure. The group H operates on it as Lorentz transformations. So any H-invariant open set has infinite measure.

That is, this is a case of $S_1 = \phi$, its core subgroup doesn't exist. Easily shown that the pair (G,H) is not even (P-0).

Example 7. However if we introduce the discrete topology in G given in Example 6, the G-invariant measure on $H\backslash G$ must be the point mass. There is a unique μ-finite H-invariant set $\{\tilde{e}\}$ in it. This gives an example for which G_o ($= H$) exists but is not equal to G . And since $H\backslash G$ is discrete, this pair (G,H) is also (P-3) .

Example 8. Consider discrete additive groups $D_j \simeq Z$ $(-\infty < j < +\infty)$ and $G_1 \equiv \prod D_j$ with discrete topology. Let $G_2 = \{s_n\}_n$ be the group of automorphisms s_n on G_1 given by

$$s_n \ : \ G_1 \ni (\ldots , \overset{j}{x_j}, \ldots) \mapsto (\ldots , \overset{j}{x_{j-n}}, \ldots) \in G_1 .$$

Construct the semidirect product $G \equiv G_2 \ltimes G_1$ with discrete topology, and take the discrete abelian subgroup $H \equiv \prod_j D_j$ $(1 \leq j < +\infty)$

Then any element in $X = H\backslash G$ is parametrized by

$$w(n,x) = (s_n, \ x = (\cdots ,x_{-1}, x_0)) \ .$$

the H-orbit passing through $w(n,x)$ has isotropy subgroup in H according to n as follows.

1) H for $n \leq 0$. Mass of the orbit = 1 .

2) $\prod_j D_j$ $(n < j < +\infty)$ for $n > 0$.
Mass of the orbit = $+ \infty$.

It is easy to see that the inverse of the orbit corresponding to n is the one corresponding to $-n$. This shows $G_0 = G_1 \neq G$, and gives an example such that there exists an H-invariant μ-finite μ-positive set which is not contained in X_0 $(= \pi(G_1))$. And the discreteness of $H\backslash G$ leads us to (P-3) property of the pair (G,H) .

Example 9. An example of pair, which is (P-1) but not (P-2) , is given by a restricted direct product as follows.

Let (A_j, B_j) $(1 \leq j < + \infty)$ be (P-3) pairs. Assume that there are compact open subgroups K_j of A_j , which are not contained in each core subgroups A_j^o of (A_j, B_j) . This includes that the sets $K_j - A_j^o$ is open in A_j , and the B_j-invariant canonical image of $(K_j - A_j^o)B_j$ in $X_j = B_j\backslash A_j$ has infinite mass.

Take the restricted direct product $G = \prod{}' A_j$ with respect to

$\{K_j\}$, that is, for an element $g = (g_1, g_2, \cdots)$ in G, $g_j \in K_j$ except finite j's, and the topology of G is given by the one of a compact neighborhood ΠK_j of e as an ordinary product of compact groups K_j.

Put $H = \Pi' B_j$ the restricted direct sum with respect to $\{K_j \cap B_j\}$, then H is a closed subgroup of G, and $X = H\backslash G = \Pi' X_j$. The restricted direct sum of the last term is taken with respect to $K_j \cap B_j \backslash K_j = \pi_j(K_j)$.

Under this situation, for any finite set $F = \{j\}$ of indices, we consider the finite direct product $G_F = \Pi A_j$, $H_F = \Pi B_j$, $X_F = \Pi X_j$ ($= H_F\backslash G_F$) (each product is taken for $j \in F$). Then representations of $H_F\backslash G_F$ is considered as representations of $H\backslash G$ in natural way, which separate the image $\tilde{e}_F$ of e in X_F from other points. Running F, we obtain a separating family of representations of $H\backslash G$. That is, this pair (G,H) is (P-1).

On the other hand, a G-invariant measure μ is given by

$\mu = \prod_j \mu_j$, where μ_j is the A_j-invariant measure on X_j, normalized as $\mu_j(\pi_j(K_j)) = 1$. And any neighborhood of $\tilde{e}$ in X contains a set of the form

$$\pi(\prod_{j<N} E_j \times \prod_{j\geq N} K_j) \qquad \text{for some } N .$$

Here $\pi_j(E_j)$ are relative compact open B_j-invariant sets in $B_j\backslash A_j$ respectively. This set contains the open set

$$\pi(\prod_{j<N} E_j \times (K_N - A_N) \times \prod_{j>N} K_j) .$$

And the smallest H-invariant set containing this set also contains the set

$$E = \pi\left(\prod_{j<N} E_j \times (K_N - A_N^o)B_N \times \prod_{j>N} K_j \right).$$

$$\mu(E) = \prod_{j<N} \mu_j(E_j) \times \mu_N(\, (K_N - A_N^o)B_N) \times \mu(\prod_{j>N} K_j) .$$

This concludes that any H-invariant neighborhood of $\tilde{e}$ in X has infinite mass, therefore the pair (G,H) is not (P-2) .

A concrete example of this case is given as follows. Let $\{S\} = \{b^n\}_{-\infty < n < +\infty}$ be a discrete multiplicative group, and K is the automorphism group $\{e, a\}$ on S , given by

$$K \ni a \;:\; S \ni b^n \mapsto a(b) \equiv b^{-n} \in S .$$

Put $B_o = K \ltimes S$ the discrete semi-direct product group. And consider the group B of inner automorphisms on B_o with discrete topology. Take again the semi-direct product $A = B \ltimes B_o$. We adopt as (A_j, B_j, K_j) in the above arguments the replicas of the same triplet (A, B, K) . Since the factor space B\A is discrete, the pair (A,B) is a (P-3) pair, and its core subgroup is $A_o = B$. Thus we obtain the result.

§5 Duality theorem.

In this §, we shall prove one of our main results as follows.

<u>Theorem 1.</u> For any (P-3) pair (G,H) , I-S duality holds.

To show Theorem 1 , we prepare a series of lemmata.

<u>Lemma 28.</u> For a fixed $\omega \in \Omega$ and H-invariant vectors $v_j \in \mathcal{H}_H(\omega)$, let $\psi_j \equiv (\omega, v_j)$ $(j = 1, 2)$ and $\psi_o \equiv (\omega, av_1 + bv_2)$ $(a, b \in C)$ in Ψ .

Then for any birepresentation $\upsilon \equiv \{u(\omega)\}$ over Ψ , (5-1) $$u(\psi_o) = a\,u(\psi_1) + b\,u(\psi_2)\ .$$

Proof. If $a = b = 0$, by 4) of Definition 3 , $u(\psi_o) = 0$ and (5-1) is trivial. Therefore using the symmetricity, we can assume a is non-zero.

In $H(\omega \oplus \omega) = H(\omega) \oplus H(\omega)$, consider two subspaces

$$V_1 \equiv \{v \oplus (\overline{b}/\overline{a})v \mid v \in H(\omega)\}\ ,$$

$$V_2 \equiv \{(b/a)v \oplus (-v) \mid v \in H(\omega)\}\ ,$$

then $H(\omega \oplus \omega) = V_1 \oplus V_2$ gives a direct sum decomposition of $\omega \oplus \omega$, the both components of which are equivalent to ω by intertwining operators U_1 and U_2 respectively. Direct calculations show that the componets of vector $w_1 \oplus w_2$ in $H(\omega \oplus \omega)$ are brought by U_j's to

(5-2) $$\bar{a}\ (aw_1 + bw_2)(|a|^2 + |b|^2)^{-1},$$

(5-3) $$a\ (\bar{b}w_1 - \bar{a}w_2)(|a|^2 + |b|^2)^{-1}$$

in $H(\omega)$ respectively. We write $c_o \equiv \bar{a}\ (|a|^2 + |b|^2)^{-1}$.

Applying (5-2) and 1), 2) of Definition 3 , to the cases $w_j = v_j$ and $w_j = u(\psi_j)$, we obtain

(5-4) $$u((\omega,\ c_o(av_1 + bv_2))) = c_o(au(\psi_1) + bu(\psi_2))\ .$$

Substituting $v = v_1 = v_2$, for any $c \neq 0$, we get

(5-5) $$u((\omega,\ cv)) = cu((\omega,\ v))\ .$$

From (5-4) and (5-5) , (5-1) follows.

Lemma 29. For $\psi \equiv (\omega,\ v) \in \Psi$, let H_o be the closed subspace of $H(\omega)$ spanned by $\{\ T_g(\omega)v \mid g \in G\ \}$.

Then for any birepresentation $\upsilon = \{u(\psi)\}$ over Ψ , $u(\psi) \in H_o$.

Proof. Consider the direct sum decomposition $\omega = \omega_1 \oplus \omega_2$

according to $\mathcal{H}(\omega) \equiv \mathcal{H}_o \oplus \mathcal{H}_o^{\perp}$, and representations $\psi_1 = (\omega_1, v)$, $\psi_2 = (\omega_2, 0)$. Then by the definition $\psi = \psi_1 \oplus \psi_2$ and

$$u(\psi) = u(\psi_1) \oplus u(\psi_2) = u(\psi_1) \oplus 0 \in \mathcal{H}(\omega_1) = \mathcal{H}_o \,.$$

q.e.d.

Lemmata 28 and 29 show that any birepresentation $\upsilon = \{u(\psi)\}$ over Ψ gives a family of operators

$$(5\text{-}6) \qquad U(\omega) : \mathcal{H}_H(\omega) \ni v \mapsto U(\omega)v \equiv u(\psi) \in \mathcal{H}(\omega)$$

for $\psi = (\omega, v) \in \Psi$. And 4) of Definition 3 assures that these operators are all uniformly bounded by one.

Hereafter we study about this operator. And the proof of Theorem 1 is done in very similar way as in the case of group duality.

Corollary of Lemma 29. If $U(\omega)v \neq 0$ for a $v \in \mathcal{H}_H(\omega)$,

$$< T_g(\omega)U(\omega)v, v > \not\equiv 0 \,.$$

Proof. Because of Lemma 29, the vector $U(\omega)v$ is contained in the space spanned by $\{T_g(\omega)v\}$.

Lemma 30. We fix a complete orthonormal system $\{w_\alpha\}_\alpha$ in $\mathcal{H}_H(\omega)$, and consider the linear operator given by

$$B_\omega : \mathcal{H}(\omega) \otimes \mathcal{H}(\sigma) \ni v \otimes f \to \{<T_g(\omega)v, w_\alpha> f(\tilde{g})\}_\alpha \in \sum{}^{\oplus}\mathcal{H}(\sigma) \,.$$

Then this operator is a bounded intertwining operator from the space of $\omega \otimes \sigma$ into the one of $\sum^{\oplus}\sigma$.

Proof. Write P the projection on $\mathcal{H}(\omega)$ onto the space $\mathcal{H}_H(\omega)$, then easily, B_ω is considered as the operator

$$\tilde{B}_\omega : L^2_\mu(X, \mathcal{H}(\omega)) \ni v(x) \mapsto (PT_g(\omega)v(\tilde{g})) \in L^2_\mu(X, \mathcal{H}_H(\omega)) \,.$$

$$||B_\omega v||^2 = \int_X ||PT_g(\omega)v(\tilde{g})||^2 \, d\mu(\tilde{g}) \le \int_X ||T_g(\omega)v(\tilde{g})||^2 \, d\mu(\tilde{g})$$

$$= \int_X ||v(\tilde{g})||^2 \, d\mu(\tilde{g}) = ||v||^2 \,.$$

Thus the operator norm $||B_\omega||$ is bounded by one. And the intertwining property is direct from the form of B_ω .

Corollary. For a fixed complete orthonormal system $\{k_\alpha\}_\alpha$ in $\mathcal{H}(\sigma)$, the operator given by

$$B : \mathcal{H}(\sigma) \otimes \mathcal{H}(\sigma) \ni f_1 \otimes f_2 \to \{< T_g(\sigma)f_1, k > f_2(\tilde{g})\}_\alpha \quad \in \sum^{\oplus}\mathcal{H}(\sigma) .$$

is a bounded intertwinging operator from $\sigma \otimes \sigma$ into $\sum^{\oplus} \sigma$.

Proof. A special case of Lemma 30.

Lemma 31. For arbitrary given birepresentation $\upsilon \equiv \{u(\psi)\}$ over Ψ , the corresponding operators $U(\omega)$ from $\mathcal{H}_H(\omega)$ to $\mathcal{H}(\omega)$ and $U(\sigma)$ from $\mathcal{H}_H(\sigma)$ to $\mathcal{H}(\sigma)$ satisfy

$$\text{(5-7)} \quad < T_g(\omega)(U(\omega)v_1), v_2 >(U(\sigma)f)(\tilde{g}) = $$

$$= [U(\sigma)(<T.(\omega)v_1, v_2>f)](\tilde{g}) \quad \text{in } \mathcal{H}(\sigma)$$

for any $v_1, v_2 \in \mathcal{H}_H(\omega)$, and any $f \in \mathcal{H}_H(\sigma)$.

Proof. Applying Definition 3 to the definition of B_ω , we obtain

$$\{< T_g(\omega)(U(\omega)v_1), w_\alpha> U(\sigma)f\}_\alpha = B_\omega(U(\omega)v_1 \otimes U(\sigma)f)$$

$$= (\sum^{\oplus}U(\sigma))B_\omega(v_1 \otimes f) = \{U(\sigma)(<T.(\omega)v_1, w_\alpha> f)\}_\alpha .$$

Compare the α-components of both sides and from the arbitrariness of $\{w_\alpha\}_\alpha$, replace w_α by v_2 . Then we get the result.

Corollary. $< T_g(\sigma)(U(\sigma)f_1), f_2>(U(\sigma)f_3)(\tilde{g}) =$

$$\text{(5-8)} \quad = [U(\sigma)(<T.(\sigma)f_1, f_2>f_3)](\tilde{g}) \quad \text{in } \mathcal{H}(\sigma) ,$$

for any $f_1, f_2, f_3 \in \mathcal{H}_H(\sigma)$.

Proof. A special case of Lemma 31.

Lemma 32. Let (G,H) be (P-3), then for any $f_1 \in H_H(\sigma) \cap L^\infty_\mu(X)$ and any $f_3 \in H_H(\sigma)$,

(5-9) $$(U(\sigma)f_1)(U(\sigma)f_3) = U(\sigma)(f_1 f_3) \quad \text{in } H(\sigma) .$$

Proof. We substitute an approximate identity $\{\theta_\alpha\}_\alpha$ given in Proposition 20, into f_2 of (5-8). If it is necessary, taking a subsequence, we get

$$(U(\sigma)f_1)(U(\sigma)f_3) = \lim <T_g(\sigma)(U(\sigma)f_1),\ \theta_j>U(\sigma)f_3$$
$$= U(\sigma)(\lim <T_g(\sigma)f_1,\ \theta_j>f_3) = U(\sigma)(f_1 f_3) .$$

Lemma 33. For any H-invariant μ-finite E in X, there exists a Borel set $U(E)$ and

(5-10) $$U(\sigma)\chi_E = \chi_{U(E)} \quad \text{in } H(\sigma).$$

Proof. Put $f_1 = f_2 = \chi_E$ in (5-9), then

$$(U(\sigma)\ \chi_E)^2 = U(\sigma)\chi_E \qquad \text{a.e..}$$

That is, $U(\sigma)\chi_E$ must be a characteristic function of some measurable set $U(E)$.

Corollary 1.

(5-11) $$\mu(U(E)) \leq \mu(E) .$$

Proof. " $||U(\sigma)|| \leq 1$ ", leads us to

$$\mu(U(E)) = ||\chi_{U(E)}||_2^{\,2} \leq ||\chi_E||_2^{\,2} = \mu(E) .$$

Corollary 2. For any $f \in H_H(\sigma)$ such that $f \geq 0$,

(5-12) $$U(\sigma)f(x) \geq 0 \qquad \text{a.e..}$$

Proof. It is true for step functions. And for general case, we take their limit in $L^2_\mu(x)$.

Lemma 34. If there exists a non-zero $v \in H_H(\omega)$ such that $U(\omega)v = 0$, we get

$$(5\text{-}13) \qquad U(\sigma)f = 0 \quad \text{for any} \quad f \in H_H(\sigma) \cap L^2_\mu(X_o) \ .$$

Proof. From (5-7) , we obtain

$$(5\text{-}14) \qquad U(\sigma)(\langle T.(\omega)v,\ v\rangle\ f)(\tilde{g}) = 0 \quad \text{a.e..}$$

Since $v \neq 0$, for any neighborhood V of $\tilde{e}$ which is contained in a set of type $\{ \tilde{g} \mid \langle T_g(\omega)v,\ v \rangle > \varepsilon \}$, we can choose an f as $\langle T_g(\omega)v,\ v\rangle\ f(g) \geq 1$ on V . Thus by (5-12) , we get $U(\sigma)(\chi_V) = 0$. Consequently for an approximate identity $\{\theta_V\}$ given in Proposition 21 , we obtain $U(\sigma)(\theta_V) = 0$.

Using (5-8), for an f in $H_H(\sigma)$ such that $f^* \in H_H(\sigma)$,

$U(\sigma)(\langle T_g(\sigma)\theta_V,\ f^*\rangle\ f) = \langle T_g(\sigma)U(\sigma)\theta_V,\ f^*\rangle\ U(\sigma)f = 0$.

Take the limit of left side, we get

$$(U(\sigma)f)^2 = U(\sigma)(f^2) = 0 \ .$$

q.e.d.

This Lemma 34 states an ideal-like property of $\Psi_o \equiv (\sigma,\ H_H(\sigma) \cap L^2_\mu(X_o))$. That is, a birepresentation is an s-birepresentation, if it does not vanish on a element of Ψ_o .

Thus in the following of this § , we assume that $U(\sigma)$ is a non-zero operator on $H_H(\sigma) \cap L^2_\mu(X_o)$.

Lemma 35. Let E be a compact H-invariant neighborhood of $\tilde{e}$. Then i) $\mu(E) = \mu(U(E))$ and ii) there exists a $\tilde{g}_E$ in X such that $U(E) \subset E^2 g_E$.

Proof. Consider the function

$$\theta(\tilde{g}) \equiv (\mu(E))^{-1} \langle T_{\tilde{g}}\chi_E\ ,\ \chi_{E^2}\rangle \in H_H(\sigma) \cap L^1_\mu(X) \cap L^\infty_\mu(X) \ .$$

Then $\theta(x) \leq 1$ for any x in X, and $E_1 \equiv \{x \mid \theta(x) = 1\} \supset \mathring{E}$. Repeating application of (5-8), we get for $n \geq 2$,

$$(U(\sigma)\theta)^n(\tilde{g}) = U(\sigma)(\theta^n)(\tilde{g}) = (\mu(E))^{-n+1}(< T_{\tilde{g}}\chi_{U(E)}, \chi_{E^2}>)^{n-1}(U(\sigma)\theta)(\tilde{g}) .$$

That is, for $\tilde{g} \in [U(\sigma)\theta]$, and $n \geq 1$,

$$(\mu(E))^{-n}(<T_{\tilde{g}}\chi_{U(E)} , \chi_{E^2}>)^n = (U(\sigma)\theta)^n(\tilde{g}) = U(\sigma)(\theta^n)(\tilde{g}) .$$

Take the limit in $n \to \infty$. Then $\theta^n \to \chi_{E_1}$ in $\mathcal{H}_H(\sigma)$, so the left hand side must converge to $\chi_{U(E_1)} \neq 0$. This results the existence of g_E such that

$$1 = (\mu(E))^{-1}< T_{\tilde{g}_E} \chi_{U(E)}, \chi_{E^2} > = (\mu(E))^{-1}\mu(U(E){g_E}^{-1} \cap E^2)$$
$$\leq (\mu(E))^{-1}\mu(U(E)) .$$

Combining Corollary 1 of Lemma 33, we get $\mu(E) = \mu(U(E))$ and $U(E) \subset E^2 g_E$.

Since the set E^2 is H-invariant, g_E is determined as H-coset wise.

Lemma 36. Let (G,H) be a (P-3) pair, and $\{E_\alpha\}$ be a fundamental family of H-invariant symmetric compact neighborhoods of $\tilde{e}$. We take $\tilde{g}_\alpha$ for E_α given in Lemma 35.

Then $\{g_\alpha\}$ converges to some x_o in X.

Proof. By Lemma 33, if $E_\alpha \supset E_\beta$ then $U(E_\alpha) \supset U(E_\beta)$. Thus from Lemma 35, ${E_\beta}^2 g_\beta \supset U(E_\beta) \subset U(E_\alpha) \subset {E_\alpha}^2 g_\alpha$. This shows $\tilde{g}_\beta \in {E_\beta}^2 {E_\alpha}^2 g_\alpha$ for $\beta \quad \alpha$. Therefore $\{\tilde{g}_\alpha\}$ gives a Cauchy net, and has a limit point x_o in X. q.e.d.

Now we put an assumption that (G,H) is (P-3).

Lemma 37. For $f \in \mathcal{H}_H(\sigma) \cap L^2_\mu(X_o)$,

$$(U(\sigma)f) = T_{{g_o}^{-1}}(\sigma)f \qquad (\tilde{g}_o = x_o) . \tag{5-15}$$

Proof. For $\{E_\alpha\}$, as in Lemma 36, $U(E_\alpha)g_\alpha^{-1} \subset E_\alpha^2$. But by similar arguments as in Proposition 21, $\{(\mu(E_\alpha^2))^{-1}\chi_{E_\alpha^2}\}$ is an approximate identity in $H(\sigma)$. And this is same for the family $\{(\mu(E_\alpha))^{-1}T_{g_\alpha}(\sigma)\chi_{U(E_\alpha)}\}$ and $\{(\mu(E_\alpha))^{-1}\chi_{E_\alpha}\}$.

We take the limits of both sides of

$$\begin{aligned} <T_{\tilde{g}}[(\mu(E_\alpha))^{-1}T_{g_\alpha}(\sigma)\chi_{U(E_\alpha)}] , f^*> &= (\mu(E_\alpha))^{-1}\, U(\sigma)(< T.\ \chi_{E_\alpha} , f^*>)(\tilde{g}g_\alpha) \\ &= (\mu(E_\alpha))^{-1}\, T_{g_\alpha}(\sigma)U(\sigma)(<T.\ \chi_{E_\alpha} , f^*>)(\tilde{g}) , \end{aligned}$$

and get $$f = T_{g_o}(\sigma)\, U(\sigma)f .$$

That is the result.

Lemma 38. For any $\psi = (\omega, v) \in \Psi$,

(5-16) $$u(\psi) = U(\omega)v = T_{g_o^{-1}}(\omega)v .$$

Proof. From (5-7),

$$\begin{aligned} <T_g(\omega)U(\omega)v,\ v>f(\tilde{g}g_o^{-1}) &= <T_g(\omega)U(\omega)v,\ v>(U(\sigma)f(\tilde{g})) \\ &= U(\sigma)(<T.(\omega)v,\ v>f)(\tilde{g}) = <T_{gg_o^{-1}}(\omega)v,\ v > f(\tilde{g}g_o^{-1}), \end{aligned}$$

for any f in $H_H(\omega) \cap L^2_\mu(X_o)$.

Let f be continuous, and put $g = g_o$, then

$$< T_{g_o}(\omega)U(\omega)v,\ v> = ||v||^2 .$$

From the boundedness $||U(\omega)|| \leq 1$ and $||T_{g_o}||= 1$, we get

$$U(\omega)v = T_{g_o^{-1}}(\omega)v .$$ q.e.d.

And this completes the proof of Theorem 1.

We state the remark after Lemma 34, as a proposition here again.

Proposition 39. If a birepresentation is non-trivial on $\Psi_o \equiv (\sigma, H_H(\sigma) \cap L^2_\mu(X_o))$, it is not zero for any (ω, v) such that $v \neq 0$.

<u>Example 11.</u> We can show that the concrete example stated in the after half part of Example 9, is (P-1) but not (I-S) . Here we sketch a proof of this fact.

All irreducible representations of the group $B_o = K \ltimes S \ (= \{e, a\} \ltimes \{b^n\})$ are as follows.

1) $\omega_o = \mathbb{1}$: the trivial representation.

2) ω_- : the lifting up of the character χ_- $(\chi_-(a) = -1)$ of K .

3) $\omega_\lambda = \{\mathbb{C}^2 , U^\lambda_x\} \simeq \omega_{\bar{\lambda}}$ $(|\lambda| = 1$ and $\lambda \neq 1$)

such that, $U_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $U_b = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

In general, let $A_o \equiv B_1 \ltimes B_2$ be a semi-direct product where $B_1 \simeq B_2$ and B_1 operates as inner automorphism group of B_2 . Then all irreducible representations $\mathcal{D} \equiv \{\mathcal{H}, U_{(x,y)}\}$ of A_o are given by any pair of factor representations $\mathcal{D}_j \equiv \{\mathcal{H}, \tilde{V}^j{}_x\}$, $(j = 1, 2)$ such that $\{\tilde{V}^1{}_x\}' \supset \{\tilde{V}^2{}_y\}$ and $\{\tilde{V}^1{}_x\}' \cap \{\tilde{V}^2{}_y\}' = \mathbb{C}I$, as

$$U_{(x,\ e)} = \tilde{V}^1{}_x \tilde{V}^2{}_x \quad \text{and} \quad U_{(e,y)} = \tilde{V}^2{}_y \quad \text{for} \quad (x, y) \in A_o .$$

Moreover, if $B_1 (\simeq B_2)$ is type I group , the factor representations $\mathcal{D}_j$ $(j = 1, 2)$ must be multiples of irreducible $\omega_j = \{\mathcal{H}_j, V^j{}_x\}$ respectively, and

$$\{\mathcal{H}, U_{(x,y)}\} = \{\mathcal{H}_1 \otimes \mathcal{H}_2 , V^1{}_x \otimes V^2{}_{xy}\} .$$

We write this representation of A_o by $\omega_1 \ltimes \omega_2$.

This representation $\omega_1 \ltimes \omega_2$ has a non-trivial B_1-invariant vector if and only if the representation $\omega_1 \otimes \omega_2$ of B_1 has it, i.e.,

(1) $\omega_1{}^* = \omega_2$ and (2) $\dim \omega_1 < + \infty$.

And this B_1-invariant vector is unique up to constant factor.

After Proposition 39, for our I-S duality we can replace the considion 5) in Definition 3 of s-birepresentation by weaker ones as follows.

5') There exists a $\psi \in \Psi_o$ such that $u(\psi) \neq 0$.

Or the same thing,

5'') $U(\sigma)$ is a non-zero operator on $\mathcal{H}_H(\sigma) \cap L^2_\mu(X_o)$.

Example 10. In [3], we have Examples 3 ~ 5 of non-trivial birepresentations which do not correspond to any element of G in group duality.

In the similar way, our Example 8 in this paper gives an example of a pair (G,H) which has non-trivial birepresentations not corresponding to any $x \in X$.

In fact, by G. W. Mackey's method [5], irreducible representations of G are exausted by

(1) ζ : The lift up of characters of the abelian subgroup G_2 to G.

(2) ω_χ : $\mathrm{Ind}_{G_1}^{G} \chi$ (χ are non-trivial characters of G_1).

The one-dimensional representations ζ are trivial on G_1, so (ζ, v)'s $(v \in \mathcal{H}(\zeta))$ give representations of X. And it is easy to see the family of representations being disjoint to all type (1) representations, constructs a prime ideal in similar sense as [4].

Thus we can construct a birepresentation which is zero on this prime ideal and is v for (ζ, v) of type (1) representations.

Remark. Ψ_o is an ideal too, in this Example 8. But it is shown that this ideal is not prime.

Applying this to our example $A = B \ltimes B_o$, all irreducible representations of A with normalized B-invariant vector v are given as,

(1) $\mathcal{D}_o \equiv \omega_o \ltimes \omega_o = \mathbb{1}_A$, $v = v_o$.

(2) $\mathcal{D}_- \equiv \omega_- \ltimes \omega_-$, $v = v_-$.

(3) $\mathcal{D}_\lambda \equiv \omega_\lambda \ltimes \omega_\lambda$, $v = v_\lambda$.

The operation of the subgroup K on these B-invariant vector v are,

(1) $U_{(e,a)} v_o = v_o$,

(2) $U_{(e,a)} v_- = -v_-$,

(3) $U_{(e,a)} v_\lambda \perp v_\lambda$.

Now consider the restricted direct product $G = \Pi' A_j$ with respect to $\{K_j\}$. For any given N , G can be considered as a direct product

$$G = A_1 \times A_2 \times A_3 \times \cdots \times A_N \times (\textstyle\prod_{j>N}{}' A_j) .$$

Because all A_j are type I , all irreducible representations $\tilde{\mathcal{D}} = \{ \tilde{H}, \tilde{U}_g \}$ of G are of the form of an outer tensor product

$$\tilde{\mathcal{D}} = \mathcal{D}_1 \hat{\otimes} \mathcal{D}_2 \hat{\otimes} \cdots \hat{\otimes} \mathcal{D}_N \hat{\otimes} \mathcal{D}_N'$$

of irreducible representations $\mathcal{D}_j$ of A_j and $\mathcal{D}_N'$ of $\prod_{j>N}{}' A_j$. We assume that $\tilde{\mathcal{D}}$ has a normalized H-($= \Pi' B_j$-)invariant vector v . By the continuity, we can choose an M such that

$$||U_k v - v|| < 1 \quad \text{for any} \quad k \in \prod_{j\leq M}\{e_j\} \times \prod_{j>M} K_j .$$

We shall show that $\mathcal{D}_j = \mathbb{1}_{A_j}$ for any $j > M$.

In fact, since v is invariant with respect to the subgroups

$$\tilde{B}_j \equiv \{\{e_1\} \times \{e_2\} \times \cdots \times \{e_{j-1}\} \times B_j \times \{e_{j-1}\} \times \cdots \} ,$$

and from the uniqueness of normalized B_j-invariant vector v_j for $\mathcal{D}_j$, v is written as

$$v = v_1 \otimes v_2 \otimes \cdots \otimes v_N \otimes v_N' ,$$

here v_N' is a normalized $\prod_{j>N}{}' B_j$-invariant vector.

If there exists an $N > M$ such that $\mathcal{D}_N \neq \mathbb{1}_{A_N}$, for the element

$$k_o = (e_1, e_2, \dots, e_{N-1}, a, e_{N-1}, \dots) \in \prod_{j\leq M}\{e_j\} \times \prod_{j>M} K_j ,$$

$$\begin{aligned} \|\tilde{U}_{k_o} v - v\| &= \|v_1\| \times \|v_2\| \times \dots \times \|v_{N-1}\| \times \\ &\quad \times \|U_{(e,a)}v_N - v_N\| \times \|v_N'\| \\ &= \|U_{(e,a)}v_N - v_N\| \geq \sqrt{2} > 1 . \end{aligned}$$

That is a contradiction.

Thus we conclude that there exists an M such that $\tilde{\mathcal{D}}$ is considered as a representation of the factor group

$$A_1 \times A_2 \times \dots \times A_M \simeq (G / \prod_{j>M}{}' A_j) ,$$

and is shown as

$$\tilde{\mathcal{D}} \simeq \prod_{j=1}^{M}{}^{\hat{\otimes}} \mathcal{D}_j \quad \text{and} \quad v = \prod_{j=1}^{M}{}^{\otimes} v_j .$$

Next we select $g_j \in A_j$ and $\notin K_j$, put $x_j = \pi(g_j) \in B_j \setminus A_j$ for each j. Consider the map u,

$$\mathcal{H}_H(\tilde{\mathcal{D}}) \ni v = v_1 \otimes v_2 \otimes \dots \otimes v_M \overset{u}{\mapsto} U_{g_1}v_1 \otimes U_{g_2}v_2 \otimes \dots \otimes U_{g_M}v_M \in \mathcal{H}(\mathcal{D}).$$

From the form of v, it is easy to see the family $\{u((\tilde{\mathcal{D}}, v))\}$ defines a s-birepresentation over Ψ. But by the definition of restricted direct product, there exists no element in $H\backslash G \simeq \prod'(B_j \backslash A_j)$ corresponding to this s-birepresentation. That is, (I-S) duality fails for this pair (G,H).

§6. Structure of groups in a (P-1) pair.

As is shown in the previous sections, (P-1) property of a pair (G,H) plays an important role for the validity of I-S duality. In this §, we investigate conditions under which (P-1) axiom is satisfied and we get some structural conditions for the subgroup H .

Definition 10. We call the closed subgroup

(6-1) $$H^{\sim} \equiv \cap H_{\omega} \quad (\omega \text{ runs in } \Omega),$$

the P-closure of H for the pair (G,H) .

Lemma 40. (i) $H^{\sim} \supset H$.

(ii) $H^{\sim} \neq G \iff (G,H)$ is (P-0) .

(iii) $H^{\sim} = H \iff (G,H)$ is (P-1) .

(iv) $H^{\sim} = \cap H_{\omega}$ (ω runs in $\hat{G}$, i.e., the set of all equivalence classes of irreducible unitary representations of G)

(v) $\mathcal{H}_H(\omega) = \mathcal{H}_{H^{\sim}}(\omega)$ for any $\omega \in \Omega$.

(vi) $(H^{\sim})^{\sim} = H^{\sim}$.

(vii) If (G,H) is (P-0) , that is, if $H^{\sim} \neq G$, $(G,H^{\sim})$ is (P-1) .

(viii) For an other closed subgroup $H_1 (\supseteq H)$, $H_1^{\sim} \supseteq H^{\sim}$.

(ix) If (G,H) is (P-0) , $H^{\sim}$ is the smallest closed subgroup of G containing H for which (G,H) is (P-1) .

Proof. The assertions (i) (ii) and (iii) are immediate from definitions. And from Gel'fand-Raikov's theorem, (iv) is shown directly.

From (i), $\mathcal{H}_H(\omega) \supseteq \mathcal{H}_{H^{\sim}}(\omega)$. But any v in $\mathcal{H}_H(\omega)$ is $H^{\sim}$-invariant by the definition, so is contained in $\mathcal{H}_{H^{\sim}}(\omega)$. This is (v) . (iii) and (v) give (vi) and (vii) soon.

If $H_1 \supseteq H$, we get $\mathcal{H}_{H_1}(\omega) \subseteq \mathcal{H}_H(\omega)$, and (viii) also. Moreover if (G,H_1) is (P-1) , $H_1 = H_1^{\sim} \supseteq H^{\sim}$. This shows $H^{\sim}$ is the smallest one, i.e., (ix) . q.e.d.

This Lemma 40 shows that to a (P-0) pair (G,H) we can construct a (P-1) pair $(G,H^{\sim})$ uniquely. Furthermore we have a way to construct a (P-3) pair as follows.

Proposition 41. For a (P-1) pair (G,H), there exists an open subgroup H_o of H such that (G,H_o) is (P-3).

Proof. Take a compact neighborhood U of e in H, and put H_1 the open subgroup of H generated by U. Let H_o be the P-closure of H_1 for the pair (G,H_1). Then obviously $H_o \supseteq H_1$ and H_o is an open subgroup of H.

By Lemma 9, for any μ-finite neighborhood V of $\tilde{e}$ in X ($= H_o\backslash G$), there exists an open U-invariant neighborhood W of $\tilde{e}$ such that $\overline{W} \subseteq V$. Since W is U-invariant, it is H_1-invariant too. The characteristic function χ_W gives an H_1-invariant vector in the representation space $L^2_{\mu_o}(H_o\backslash G)$ of $\mathrm{Ind}_{H_o}^{G} 1\!\!1_{H_o}$ (μ_o : invariant measure over $H_o\backslash G$). Lemma 40 (v) shows, χ_W is invariant to the P-closure H_o of H_1. Therefore for any $h \in H_o$, $\mu_o(Wh \,\Delta\, W) = 0$. This means $Wh \subseteq \overline{W}$, so $\overline{Wh} \subseteq \overline{W}$, thus $\overline{Wh} = \overline{W}$. Therefore we obtain an H_o-invariant neighborhood $\overline{W}$ of $\tilde{e}$ in an arbitrary given V. This concludes that (G,H_o) is (P-3).

Lemma 42. For a (P-3) pair (G,H), let τ be the subrepresentation of H which is the restriction of $\sigma|_H$ to subspace $L^2_\mu(X_o)$. Then τ is a discrete direct sum of finite dimensional representations of H.

Proof. By the proof of Proposition 25, we can divide X_o to a disjoint sum $\sum_\alpha F_\alpha$, of H-invariant μ-finite μ-positive symmetric sets F_α , for instance $\bigcap\limits_j^N F(E_j, \varepsilon_j)$ etc. According to this division, we can write the direct sum decomposition,

$$L^2_\mu(X_o) = \sum_\alpha^\oplus \mathcal{H}_\alpha \quad (\mathcal{H}_\alpha = L^2_\mu(F_\alpha)),$$

in which each $L^2_\mu(F_\alpha)$ is H-invariant.

By Proposition 22, for any $E \in \mathcal{S}_1$ and α , $T_{E,\alpha} \equiv T_E|\mathcal{H}_\alpha$ is an operator of Hilbert-Schmidt type commuting with T_h's $(h \in H)$. Using the trace class operator $(T_{E,\alpha})^*T_{E,\alpha}$, we obtain a decomposition of τ on $(T_{E,\alpha}{}^{-1}(0))^\perp$ to a direct sum of finite dimensional subrepresentations. Repeating the same steps to $T_{E,\alpha}{}^{-1}(0)$ we reach the result as a maximal decompositon, by Lemma 23.

<u>Lemma 43.</u> The kernel N of representation τ in H in Lemma is given by

(6-2) $$N = \cap\, gHg^{-1} \qquad (g \text{ runs in } G_o)$$

Particularly N is a normal subgroup of G_o , contained in H .

<u>Proof.</u> The kernel of τ is characterized as

$$\begin{aligned} N &= \{h \in H \mid T_h v = v , \ \text{any } v \in L^2_\mu(X_o)\} \\ &= \{h \in H \mid xh = x , \ \text{any } x \in X_o\} \\ &= \{h \in H \mid Hgh = Hg , \ \text{any } g \in G_o\} \\ &= \{h \in H \mid ghg^{-1} \in H , \ \text{any } g \in G_o\} = \bigcap_{g \in G_o} gHg^{-1} . \end{aligned}$$

<u>Proposition 44.</u> The connected component of $\tilde{e}$ in N\H is isomorphic to $\mathbb{R}^n \times K$ for some n and some compact group K .

<u>Proof.</u> Let $\tau = \sum\limits^\oplus \tau_\alpha$ be the decomposition given in Lemma 42, i.e., the spaces $\mathcal{H}_\alpha$ of each components τ_α are all finite dimensional. Thus the group N\H has a faithfull representation in a compact

group $\prod_{\alpha} \mathcal{U}(\mathcal{H}_{\alpha})$. Here $\mathcal{U}(\mathcal{H}_{\alpha})$ show the groups of all unitary matrices on $\mathcal{H}_{\alpha}$. Apply the following A. Weil's lemma and the result is obtained.

Weil's lemma [6]. A connected locally compact group which is represented faithfully in a compact group, is isomorphic to $\mathbb{R}^n \times K$ for some n and some compact group K .

Summarizing the above arguments, we obtain the following results about the structure of groups in a (P-0) pair.

Proposition 45. Let (G,H) be a (P-0) pair, and H_1 be the connected component of e in H . Then there exists a normal subgroup N_1 in H_1 and m such that

(6-3) $$N_1\backslash H_1 \simeq \mathbb{R}^m \times \text{(compact group)} .$$

Proof. The connected group H_1 is contained in the connected component in P-closure $H^{\sim}$ of H . Therefore H_1 is also contained in the component of the open subgroup H_o of $H^{\sim}$ in Proposition 41, for which (G,H_o) is (P-3) . Thus by Proposition 44, there exists a normal subgroup N , the kernel of τ , and $N\backslash H_o$ is faithfully represented in a compact group.

Put $N_1 \equiv N \cap H_1$, then the imbedding $N_1\backslash H_1 \to N\backslash H_o$ is continuous and an algebraic isomorphism. That is, $N_1\backslash H_1$ is also faithfully represented in a compact group. Again we can apply the Weil's lemma and get the required result.

Corollary 1. If H in a (P-0) pair (G, H) is connected, there exists a normal subgroup N in H and m such that

(6-4) $$N\backslash H \simeq \mathbb{R}^m \times \text{(compact group)} .$$

Proof. The case $H = H_1$ in Proposition 45.

Corollary 2. In Corollary 1, if G is connected, N is a normal subgroup of G itself.

Proof. The case that G is just equal to the core subgroup G_o of (G,H).

Corollary 3. In Corollary 2, moreover if one of G and H has no non-trivial normal subgroup,

$$H \simeq \mathbb{R}^m \times \text{(compact group)} \qquad \text{for some } m.$$

Proof. From the assumption, N must be the trivial subgroup, i.e., $\{e\}$. q.e.d.

Conversely we consider the case when $N\backslash H$ has the structure as (6-4).

In the case $N\backslash H$ is compact, as is shown in Example 5, the pair (G,H) is (P-0), even (P-3).

However Example 6 gives an example that "$N\backslash H \simeq \mathbb{R}^m$" does not result (P-0) property in general.

Example 12. We have a non-abelian example for which $H \simeq R$ and the pair (G,H) is (P-3), in so-called "Mautner's group".

$$G = \left\{ \begin{pmatrix} e^{it} & 0 & z_1 \\ 0 & e^{i\alpha t} & z_2 \\ 0 & 0 & 1 \end{pmatrix} \middle| -\infty < t < +\infty,\ z_1, z_2 \in \mathbb{C} \right\},$$

$$H = \left\{ \begin{pmatrix} e^{it} & 0 & 0 \\ 0 & e^{i\alpha t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| -\infty < t < +\infty \right\}$$

(α: irrational in $\mathbb{R}$).

In fact, in the space $H\backslash G \simeq \mathbb{C} \times \mathbb{C}$, the H-invariant sets $V_\varepsilon \equiv \{(z_1, z_2) \mid |z_1|, |z_2| < \varepsilon\}$ give a fundamental family of neighborhoods of $(0, 0)$.

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