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The Plancherel formula for Sp(n, $\mathbb{R}$)

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Introduction

Let G be a connected real semi-simple Lie group, with finite center and acceptable. Let $\mathfrak{g}$ be its Lie algebra. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Let K be the maximal compact subgroup of G corresponding to $\mathfrak{k}$. Let P = MAN be a Langlands decomposition of a cuspidal parabolic subgroup P of G. Then there exists the θ-invariant Cartan subalgebra $\mathfrak{h}$ satisfying $\mathfrak{p} \cap \mathfrak{h} = \mathfrak{a}$ (the Lie subalgebra corresponding to A). Let H be the Cartan subgroup corresponding to $\mathfrak{h}$. Then $B = H \cap K$ is a compact Cartan subgroup of M. Let ω be a square integrable irreducible unitary representation of M, and ν a purely imaginary valued regular linear form of $\mathfrak{a}$. We extend $\omega \otimes \nu$ as a representation of P naturally ($\omega\otimes\nu$ (man) = $\omega(m)e^{\nu(\log a)}$, $m \in M$, $a \in A$, $n \in N$) and define the induced representation of G as follows:

$$\pi(\omega, \nu) = \operatorname*{Ind}_{P\uparrow G} \omega\otimes\nu$$

Then $\pi(\omega, \nu)$ is an irreducible unitary representation of G. The representations so obtained are called of non-degenerate principal series. Set $\Theta(\omega, \nu)$ = its character. Then $\Theta(\omega, \nu)$ is a locally summable function on G and a real analytic function on G′ (the set of all regular elements in G). In this paper, we are concerned with the groups $G_n \simeq$ Sp(n, $\mathbb{R}$). First we express $\Theta(\omega, \nu)$ on G'_n explicitly. Next, we investigate fundamental functions which construct $\Theta(\omega,\nu)$. We use the Parseval formula for the reduced forms of these functions, and get the Plancherel formula for G_n. When the author gave this result at the occasion of Summer Institute on group representations and harmonic analysis held at Research Institute for Mathematical Sciences, Kyoto University, 1979, Professor T. Hirai pointed out that the explicit form of the Plancherel measure for $Sp(n, \mathbb{R})$ has been given by R. A. Herb without proof. In this paper we give our method to reach the result. The author expresses his hearty thanks to him.

§1. Structures of Cartan subgroups and Weyl groups

1.1. put

$$(1) \qquad I_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}, \quad J_n = \begin{pmatrix} 1_n & 0_n \\ 0_n & -1_n \end{pmatrix},$$

where 1_n and 0_n denote the identity matrix and zero matrix of order n respectively. The group $Sp(n, \mathbb{C})$ or $Sp(n, \mathbb{R})$ is defined respectively as the group of complex or real matrices of order 2n satisfying

$$^t g \, I_n g = I_n ,$$

where $^t g$ denotes the transposed matrix of g. $Sp(n, \mathbb{R})$ is a real form of $Sp(n, \mathbb{C})$. In this paper we always treat the following group isomorphic to $Sp(n, \mathbb{R})$:

$$(2) \qquad G_n = \left\{ g \in Sp(n, \mathbb{C}) \; ; \; g^* J_n g = J_n \right\} \qquad (g^* = {}^t\bar{g}).$$

Put $G = G_n$ and let $\mathfrak{g}$ be its Lie algebra. Let θ be the Cartan involution of $\mathfrak{g}$ given by $\theta X = J_n X J_n$ ($X \in \mathfrak{g}$), and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition of $\mathfrak{g}$. Let K be the maximal compact subgroup corresponding to $\mathfrak{k}$.

1.2. Let k, ℓ and m be non negative integers satisfying $k+2\ell+m = n$. The following θ-invariant Cartan subgroups $H_{k,\ell}$ $(k+2\ell \leq n)$ form a maximal set of Cartan subgroups which are not conjugate to each other under inner automorphisms of G. They are given as $H_{k,\ell} = {}^{\oplus}H_{k,\ell}\,{}^{\ominus}H_{k,\ell}$, where ${}^{\oplus}H_{k,\ell} = H_{k,\ell} \cap K$ and ${}^{\ominus}H_{k,\ell} = H_{k,\ell} \cap \exp\mathfrak{p}$ consist of the elements of the following forms respectively:

(3)

$$
\begin{aligned}
{}^{\oplus}H_{k,\ell} : h_+ = \operatorname{diag}(&e^{\sqrt{-1}\phi_1}, e^{\sqrt{-1}\phi_2}, \ldots, e^{\sqrt{-1}\phi_k};\ e^{\sqrt{-1}\theta_1}, e^{-\sqrt{-1}\theta_1}, e^{\sqrt{-1}\theta_2}, e^{-\sqrt{-1}\theta_2}, \ldots, e^{\sqrt{-1}\theta_\ell},\\
&e^{-\sqrt{-1}\theta_\ell};\ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m;\ e^{-\sqrt{-1}\phi_1}, e^{-\sqrt{-1}\phi_2}, \ldots, e^{-\sqrt{-1}\phi_k};\ e^{-\sqrt{-1}\theta_1},\\
&e^{\sqrt{-1}\theta_1}, e^{-\sqrt{-1}\theta_2}, e^{\sqrt{-1}\theta_2}, \ldots, e^{-\sqrt{-1}\theta_\ell}, e^{\sqrt{-1}\theta_\ell};\ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)\ ;
\end{aligned}
$$

$$
{}^{\ominus}H_{k,\ell} : h_- = \left(\begin{array}{c|c|c|c|c|c}
1_k & & & 0_k & & \\ \hline
 & C & & & D & \\ \hline
 & & \begin{matrix} \operatorname{ch} t_1 & & \\ & \operatorname{ch} t_2 & \\ & \ddots & \\ & & \operatorname{ch} t_m \end{matrix} & & & \begin{matrix} \operatorname{sh} t_1 & & \\ & \operatorname{sh} t_2 & \\ & \ddots & \\ & & \operatorname{sh} t_m \end{matrix} \\ \hline
0_k & & & 1_k & & \\ \hline
 & D & & & C & \\ \hline
 & & \begin{matrix} \operatorname{sh} t_1 & & \\ & \operatorname{sh} t_2 & \\ & \ddots & \\ & & \operatorname{sh} t_m \end{matrix} & & & \begin{matrix} \operatorname{ch} t_1 & & \\ & \operatorname{ch} t_2 & \\ & \ddots & \\ & & \operatorname{ch} t_m \end{matrix}
\end{array}\right),
$$

$$
C = \begin{pmatrix} \operatorname{ch} \tau_1 1_2 & & \\ & \operatorname{ch} \tau_2 1_2 & \\ & \ddots & \\ & & \operatorname{ch} \tau_\ell 1_2 \end{pmatrix}, \quad D = \begin{pmatrix} \operatorname{sh} \tau_1 j_2 & & \\ & \operatorname{sh} \tau_2 j_2 & \\ & \ddots & \\ & & \operatorname{sh} \tau_\ell j_2 \end{pmatrix}, \quad j_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

where ϕ_p , θ_q , τ_q , $t_r \in R$, $\varepsilon_r = \pm 1$, $\operatorname{diag}(a_1, a_2, \ldots, a_{2n})$ denotes a diagonal matrix with diagonal elements a_1, a_2, ..., a_{2n} and the blank spaces must be filled with zeros.

Put $h = h_+ h_- \in H_{k,\ell}$ and we arrange its eigenvalues as $(\delta_1, \delta_2, \ldots, \delta_n ;\ \delta_1^{-1}, \delta_2^{-1}, \ldots, \delta_n^{-1})$, where

$$
(4)\qquad \begin{cases} \delta_p = e^{\sqrt{-1}\,\phi_p} & (1 \le p \le k), \\ \delta_{k+2q-1} = e^{z_q},\ \delta_{k+2q} = e^{\bar{z}_q} \ \ (z_q = \tau_q + \sqrt{-1}\,\theta_q) & (1 \le q \le \ell), \\ \delta_{k+2\ell+r} = \varepsilon_r\, e^{t_r} & (1 \le r \le m). \end{cases}
$$

We use $\delta = (\delta_1, \delta_2, \dots, \delta_n)$ as the co-ordinates of h. And we put

$$
(5)\qquad \begin{aligned}
\Delta^{k,\ell}(h) &= \prod_{1\le p<q\le n} ((\delta_p + \delta_p^{-1}) - (\delta_q + \delta_q^{-1})) \prod_{1\le p\le n} (\delta_p - \delta_p^{-1}), \\
\Delta_R^{k,\ell}(h) &= \prod_{1\le p\le \ell} (1 - e^{-2\tau_p}) \prod_{1\le p\le m} (1 - e^{-2t_p}) \prod_{1\le p<q\le m} (1 - \varepsilon_p \varepsilon_q e^{-t_p - t_q})(1 - \varepsilon_p \varepsilon_q e^{-t_p + t_q}), \\
\Delta_I^{k,\ell}(h) &= \prod_{1\le p<q\le k} (1 - e^{-\sqrt{-1}(\phi_p + \phi_q)}) \prod_{1\le p\le k} (1 - e^{-2\sqrt{-1}\,\phi_p}) \prod_{1\le p\le \ell} (1 - e^{-2\sqrt{-1}\,\theta_p}), \\
\varepsilon_R^{k,\ell}(h) &= \operatorname{sgn} \Delta_R^{k,\ell}(h).
\end{aligned}
$$

To make n clear, we denote sometimes $\Delta^{k,\ell}$ by ${}^n\Delta^{k,\ell}$. Let $H'_{k,\ell}$, $H'_{k,\ell}(R)$, and $H'_{k,\ell}(I)$ be the subset of $H_{k,\ell}$ defined by $\Delta^{k,\ell}(h) \ne 0$, $\Delta_R^{k,\ell}(h) \ne 0$, and $\Delta_I^{k,\ell}(h) \ne 0$ respectively. Let $W^{k,\ell}$ be the Weyl group of $(G, H_{k,\ell})$. For $w \in W^{k,\ell}$, we define $\varepsilon(w)$, $\varepsilon'(w)$ as follows:

$$
(6)\qquad \begin{aligned}
\varepsilon_R^{k,\ell}(wh)\ \Delta^{k,\ell}(wh) &= \varepsilon(w)\ \varepsilon_R^{k,\ell}(h)\ \Delta^{k,\ell}(h), \\
\varepsilon_R^{k,\ell}(wh) &= \varepsilon'(w)\ \varepsilon_R^{k,\ell}(h).
\end{aligned}
$$

$W^{k,\ell}$ is generated by the elements w listed below. Denote by S_q the symmetric group of order q.

operations of $w \in W^{k,\ell}$	$\varepsilon(w)$	$\varepsilon'(w)$
any permutation $w \in S_k$ of $e^{\sqrt{-1}\phi_1}, e^{\sqrt{-1}\phi_2}, \dots, e^{\sqrt{-1}\phi_k}$	sgn(w)	1
any permutation of $z_1, z_2, \cdots, z_\ell$ or of the pair $(\tau_1, \theta_1), (\tau_2, \theta_2), \cdots, (\tau_\ell, \theta_\ell)$	1	1
$\tau_r \to -\tau_r \quad (1 \le r \le \ell)$	1	-1
$\theta_r \to -\theta_r \quad (1 \le r \le \ell)$	-1	1
any permutation $w \in S_m$ of $\varepsilon_1 e^{t_1}, \varepsilon_2 e^{t_2}, \cdots, \varepsilon_m e^{t_m}$	$(\varepsilon_1 \varepsilon_{w(1)})^m (\varepsilon_2 \varepsilon_{w(2)})^{m-1} \cdots (\varepsilon_m \varepsilon_{w(m)})$	sgn(w) $\varepsilon(w)$
$t_r \to -t_r \quad (1 \le r \le m)$	1	-1

Let $d_{k,\ell}\bar{g}$ be a left invariant measure on $\bar{G}^{k,\ell} = G/H_{k,\ell}$. For $f \in C_0^\infty(G)$ (the space of all C^∞-functions on G with compact support), we define the Harish-Chandra transform of f as follows:

$$(7) \qquad K_f^{k,\ell}(h) = \varepsilon_R^{k,\ell}(h) \operatorname{conj}(\Delta^{k,\ell}(h)) \int_{\bar{G}^{k,\ell}} f(ghg^{-1})\, d_{k,\ell}\bar{g} \quad (h \in H'_{k,\ell}).$$

Every $K_f^{k,\ell}$ $(k+2\ell \leq n)$ has the following properties.

(i) $K_f^{k,\ell}$ on $H'_{k,\ell}$ can be extended to an infinitely differentiable function on $H'_{k,\ell}(I)$. The restriction of the extended function on any connected component of $H'_{k,\ell}(I)$ may be considered as an infinitely differentiable function on its closure. ([1(a)], Lemma 40)

(ii) $K_f^{k,\ell}(wh) = \varepsilon(w) K_f^{k,\ell}(h) \quad (w \in W^{k,\ell},\ h \in H'_{k,\ell}(I))$

For δ_j, we denote the corresponding differential operator by X_j, i.e.,

$$(8) \quad \begin{cases} \dfrac{\partial}{\partial \tilde{\phi}_p} = \dfrac{1}{\sqrt{-1}} \dfrac{\partial}{\partial \phi_p}, \dfrac{\partial}{\partial z_q} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial \tau_q} + \dfrac{\partial}{\partial \tilde{\theta}_q}\right), \dfrac{\partial}{\partial \bar{z}_q} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial \tau_q} - \dfrac{\partial}{\partial \tilde{\theta}_q}\right) \quad \left(\dfrac{\partial}{\partial \tilde{\theta}_q} = \dfrac{1}{\sqrt{-1}} \dfrac{\partial}{\partial \theta_q}\right), \\ \dfrac{\partial}{\partial t_r}. \end{cases}$$

Let us consider the polynomial

$$L(y_1, y_2, \ldots, y_n) = \prod_{1 \leq p < q \leq n} (y_p + y_q)(y_p - y_q) \prod_{1 \leq p \leq n} y_p$$

of n variables and put the differential operator

$$(9) \qquad L^{k,\ell}(X_1, X_2, \ldots, X_n) = \prod_{1 \leq p < q \leq n} (X_p + X_q)(X_p - X_q) \prod_{1 \leq p \leq n} X_p.$$

Define the Haar measure dh on $H_{k,\ell}$ as follows:

$$(10) \qquad dh = d\phi_1\, d\phi_2 \ldots d\phi_k\, d\theta_1\, d\theta_2 \ldots d\theta_\ell\, d\tau_1\, d\tau_2 \ldots d\tau_\ell\, dt_1\, dt_2 \ldots dt_m.$$

Let $H^p_{k,\ell}$ $(0 \leq p \leq m)$ be connected components of $H_{k,\ell}$ defined by

$$(11) \qquad H^p_{k,\ell} = \{h \in H_{k,\ell} \,;\, \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_p = 1,\ \varepsilon_{p+1} = \varepsilon_{p+2} = \ldots = \varepsilon_m = -1\}.$$

Then $\{ H^p_{k,\ell} \,;\, k+2\ell \leq n,\ 0 \leq p \leq m\}$ from a maximal system of representatives which are not conjugate to each other under inner automorphism of G. We put,

$$(12) \qquad F^{k,\ell}_p = \{ h \in H^p_{k,\ell} \,;\, \tau_r > 0 \ (1 \leq r \leq \ell),\ t_1 > t_2 > \ldots > t_p > 0,\ t_{p+1} > t_{p+2} > \ldots > t_m > 0 \}.$$

Notations 1. Let $\ell_1, \ell_2, \ldots, \ell_p$ be non negative integers such that $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_p \geq 0$ and $\nu_j = \pm 1$ $(1 \leq j \leq p)$. For $\delta_1 = e^{\sqrt{-1}\phi_1}$, $\delta_2 = e^{\sqrt{-1}\phi_2}$, $\ldots$, $\delta_p = e^{\sqrt{-1}\phi_p}$ $(\phi_j \in \mathbb{R})$, we put

$$\begin{aligned}
\text{(a-1)} \quad & \xi_p(\delta_1, \delta_2, \ldots, \delta_p; \nu_1\ell_1, \nu_2\ell_2, \ldots, \nu_p\ell_p) \\
&= \det(\nu_i\, \delta_j^{\nu_i \ell_i}) \quad (1 \leq i,\ j \leq p) \\
&= |\nu_1 \delta^{\nu_1\ell_1},\ \nu_2 \delta^{\nu_2\ell_2}, \ldots, \nu_p \delta^{\nu_p\ell_p}| \quad \delta = \delta_1, \delta_2, \ldots, \delta_p, \\
{}^{+}\xi_p(*; \ell_1, \ell_2, \ldots, \ell_p) &= \sum_{\nu_1, \nu_2, \ldots, \nu_p} \xi_p(*; \nu_1\ell_1, \nu_2\ell_2, \ldots, \nu_p\ell_p)
\end{aligned}$$

Let ℓ_1, ℓ_2 be non negative integers such that $\ell_1 \geq \ell_2 \geq 0$ and $\nu_j = \pm 1$ $(j = 1, 2)$. For $\delta_1 = e^{z}$, $\delta_2 = e^{\bar{z}}$ $(z \in \mathbb{C})$, we put

$$\text{(b-1)} \quad \eta_2(\delta_1, \delta_2; \nu_1\ell_1, \nu_2\ell_2) = \begin{vmatrix} -\delta_1^{-\ell_1} & -\nu_1\nu_2\delta_1^{-\nu_1\nu_2\ell_2} \\ -\delta_2^{-\ell_1} & -\nu_1\nu_2\delta_2^{-\nu_1\nu_2\ell_2} \end{vmatrix},$$

For $\delta_1 = \varepsilon e^{t_1}$, $\delta_2 = \varepsilon e^{t_2}$ $(t_1, t_2 \in \mathbb{R},\ \varepsilon = \pm 1)$, we put

$$\text{(c-1)} \quad \zeta_2(\delta_1, \delta_2; \nu_1\ell_1, \nu_2\ell_2) = \begin{vmatrix} -\delta_1^{-\ell_1} & -\delta_1^{-\ell_2} \\ -\delta_2^{-\ell_1} & -\nu_1\nu_2(\delta_2^{-\ell_2} + \delta_2^{\ell_2}) + \delta_2^{\ell_2} \end{vmatrix},$$

For $\delta_1 = \varepsilon e^{t_1}$, $\delta_2 = -\varepsilon e^{t_2}$, $(t_1, t_2 \in \mathbb{R})$, we put (d-1) $\chi(\delta_1, \delta_2; \nu_1\ell_1, \nu_2\ell_2) = \begin{vmatrix} -\delta_1^{-\ell_1} & -\delta_1^{-\ell_2} \\ -\delta_2^{-\ell_1} & -\delta_2^{-\ell_2} \end{vmatrix}$,

We put $^+\eta_2$, $^+\zeta_2$, and $^+\chi$ as follows :

$$^+\kappa(*; \ell_1, \ell_2) = \sum_{\nu_1, \nu_2} \kappa(*; \nu_1\ell_1, \nu_2\ell_2),$$

where κ expresses η_2, ζ_2, and χ respectively. For $\ell_1 \geq 0$, $\nu_1 = \pm 1$ and $\delta_1 = \varepsilon_1 e^{t_1}$, we put

(d′-1) $\chi(\delta_1; \nu_1\ell_1) = -\delta_1^{-\ell_1}$,

$^+\chi(\delta_1; \ell_1) = \sum_{\nu_1} \chi(\delta_1; \nu_1\ell_1)$.

Let m be a non negative integer and $\xi \in \mathbb{R}$, and set $\lambda = \frac{m+\sqrt{-1}\xi}{2}$ $\bar{\lambda} = \frac{m-\sqrt{-1}\xi}{2}$. For $\delta_1 = e^z$, $\delta_2 = e^{\bar{z}}$ ($z = \tau + \sqrt{-1}\theta$, τ, $\theta \in \mathbb{R}$), we put

(e-1) $H_2(\delta_1, \delta_2; \lambda, \bar{\lambda}) = (e^{-\sqrt{-1}m\theta} - e^{\sqrt{-1}m\theta})(e^{\sqrt{-1}\xi\tau} + e^{-\sqrt{-1}\xi\tau})$,

For $\delta_1 = \varepsilon e^{t_1}$, $\delta_2 = \varepsilon e^{t_2}$, we put

(f-1) $Z_2(\delta_1, \delta_2; \lambda, \bar{\lambda}) = 2 \begin{vmatrix} e^{\bar{\lambda}t_1} & e^{\bar{\lambda}t_2} + e^{-\bar{\lambda}t_2} \\ -e^{-\lambda t_1} & e^{\lambda t_2} + e^{-\lambda t_2} \end{vmatrix} \varepsilon^{\lambda+\bar{\lambda}}$.

Let $\lambda_j = (\epsilon_j, \sqrt{-1}\rho_j)$, $\epsilon_j = 0, 1$, $\rho_j \in \mathbb{R}$ $(1 \leq j \leq p)$. For $\delta_1 = \varepsilon_1 e^{t_1}$, $\delta_2 = \varepsilon_2 e^{t_2}$, .. , $\delta_p = \varepsilon_p e^{t_p}$ ($\varepsilon_j = \pm 1$, $t_j \in \mathbb{R}$), we put

(g-1) $\Xi_p(\delta_1, \delta_2, \ldots, \delta_p; \lambda_1, \lambda_2, \ldots, \lambda_p) = \sum_{\sigma \in S_p} \prod_{j=1}^{p} (e^{\sqrt{-1}\rho_{\sigma(j)} t_j} + e^{-\sqrt{-1}\rho_{\sigma(j)} t_j}) \varepsilon_j^{\epsilon_{\sigma(j)}}$.

§2. Characters of non degenerate principal series

Fix non negative integers k, ℓ, m $(k+2\ell+m = n)$. We take non negative integers k', ℓ', m', j such that $k'+2\ell' \leq k+2\ell$, $k' \leq k$, $0 \leq j \leq m'$ $(k'+2\ell'+m' = n)$. And we take an integer p such that $0 \leq p \leq \mathrm{Min}([\frac{k-k'}{2}], \ell')$. Let $\ell_1, \ell_2, \ldots, \ell_k$ be non negative integers such that $\ell_1 \geq \ell_2 \geq \ldots \geq \ell_k \geq 0$ and let $\nu_1, \nu_2, \ldots, \nu_k$ be 1 or -1. Let $m_1, m_2, \ldots, m_\ell$ be non negative integers and $\xi_i \in \mathbb{R}$ $(1 \leq i \leq \ell)$. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ be 0 or 1 and $\rho_i \in \mathbb{R}$ $(1 \leq i \leq m)$ We define ordered sets of different integers, satisfying the following conditions :

$$(13) \quad \begin{cases} A = (a_1, a_2, \ldots, a_{k'}), & a_1 < a_2 < \ldots < a_{k'}, \\ B = (b_1, b_2, \ldots, b_p), & \\ C = (c_1, c_2, \ldots, c_p), & b_i < c_i \quad (1 \leq i \leq p). \end{cases}$$

For an integer j' satisfying $\mathrm{Max}(j-m'+a, 0) \leq j' \leq \mathrm{Min}(j, a)$ $(a = k-k'-2p)$, put

$$J = [j'/2], \quad I = [(k-k'-2p-j')/2],$$

and define

$$(14) \quad \begin{cases} P = (p_1, p_2, \ldots, p_{J+I}), & \\ Q = (q_1, q_2, \ldots, q_{J+I}), & p_i < q_i \quad (1 \leq i \leq J+I). \end{cases}$$

We prepare two symbols r and s. The symbol r denotes an integer when $2J = j'-1$, and the empty set ϕ when $2J = j'$. The symbol s denotes an integer when $2I = k-k'-2p-j'-1$, and ϕ when $2I = k-k'-2p-j'$. When both r and s denote integers, we assume that

$$(15) \qquad r < s.$$

Denote by $\bar{A}$ the set of integers $\{a_1, a_2, \ldots, a_{K'}\}$, then we assume further that

$$\bar{A} \cup \bar{B} \cup \bar{C} \cup \bar{P} \cup \bar{Q} \cup \{r, s\} = \{1, 2, \ldots, k\}.$$

Put $q = \ell' - p$ and we define ordered sets of different integers, satisfying the following conditions:

$$\begin{cases} D = (d_1, d_2, \ldots, d_q), \\ F = (f_1, f_2, \ldots, f_{\ell-q}), \qquad \bar{D} \cup \bar{F} = \{1, 3, \ldots, 2\ell-1\}. \end{cases}$$

For brevity, we denote the collection (A, B, C, P, Q, r, s, D, F) simply by μ. Denote by $B \circ C$ the following ordered set of integers corresponding to B and C:

$$B \circ C = (b_1, c_1, b_2, c_2, \ldots, b_p, c_p).$$

Moreover we denote by

$$\operatorname{sgn} \mu = \operatorname{sgn}(A, B \circ C, P \circ Q, r, s).$$

We define also ordered sets of different integers satisfying the following conditions:

$$\begin{cases} \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p), \qquad \alpha_1 < \alpha_2 < \ldots < \alpha_p, \\ \beta = (\beta_1, \beta_2, \ldots, \beta_g), \qquad \beta_1 < \beta_2 < \ldots < \beta_g, \end{cases}$$

$$\bar{\alpha} \cup \bar{\beta} = \{1, 2, \ldots, \ell'\};$$

(16) ${}_pW_{j'} = (w_1, w_2, \ldots, w_{K-K'-2p}), \quad w_1 < w_2 < \ldots < w_{K-K'-2p},$

$\{w_1, w_2, \ldots, w_{j'}\} \subset \{k'+2\ell'+1, k'+2\ell'+2, \ldots, k'+2\ell'+j\},$

$\{w_{j'+1}, w_{j'+2}, \ldots, w_{K-K'-2p}\} \subset \{k'+2\ell'+j+1, k'+2\ell'+j+2, \ldots, n\};$

$$(17)\quad \begin{cases} {}_pU_{j'} = (u_1, u_2, \ldots, u_{J+I}), \\ {}_pV_{j'} = (v_1, v_2, \ldots, v_{J+I}), \quad v_1 < v_2 < \cdots < v_{J+I}, \quad u_i < v_i \ (1 \leq i \leq J+I). \end{cases}$$

We prepare two symbols w and x. The former denotes an integer when $2J = j'-1$ and ϕ when $2J = j'$ respectively, and the latter denotes an integer when $2I = k-k'-2p-j'-1$ and ϕ when $2I = k-k'-2p-j'$ respectively. We assume the following conditions:

$$(18)\quad \begin{aligned} &\{u_i,\ v_i\ (1 \leq i \leq J),\ w\} = \{w_1, w_2, \ldots, w_{j'}\}, \\ &\{u_i,\ v_i\ (J+1 \leq i \leq J+I),\ x\} = \{w_{j'+1}, w_{j'+2}, \ldots, w_{k-k'-2p}\}. \end{aligned}$$

We define ordered sets of different integers, satisfying the following conditions:

$$\begin{cases} \text{Б} = (\text{ъ}_1, \text{ъ}_2, \ldots, \text{ъ}_{\ell-q}), \quad \text{ъ}_1 < \text{ъ}_2 < \cdots < \text{ъ}_{\ell-q}, \\ \text{Ы} = (\text{ы}_1, \text{ы}_2, \ldots, \text{ы}_{\ell-q}), \quad \text{ъ}_i < \text{ы}_i \ (1 \leq i \leq \ell-q); \end{cases}$$

$$\{\text{ъ}_i,\ \text{ы}_i\} \subset \{k'+2\ell'+1,\ k'+2\ell'+2, \ldots,\ k'+2\ell'+j\}$$

or

$$\{\text{ъ}_i,\ \text{ы}_i\} \subset \{k'+2\ell'+j+1,\ k'+2\ell'+j+2, \ldots,\ n\} \quad (1 \leq i \leq \ell-q);$$

$$\text{Э} = (\text{э}_1, \text{э}_2, \ldots, \text{э}_m).$$

We assume that

$${}_p\bar{U}_{j'} \cup {}_p\bar{V}_{j'} \cup \{w,\ x\} \cup \bar{\text{Б}} \cup \bar{\text{Ы}} \cup \bar{\text{Э}} = \{k'+2\ell'+1,\ k'+2\ell'+2, \ldots,\ n\}.$$

For brevity, we denote the collection $(\alpha,\ \beta,\ {}_pW_{j'},\ {}_pU_{j'},\ {}_pV_{j'},\ w,\ x,\ \text{Б},\ \text{Ы},\ \text{Э})$ simply by ${}_p\text{Ш}_{j'}$.

After these preparations, we define a function $\tilde{K}_j^{k',\ell'}$ on $F_j^{k',\ell'}$ as follows: for $h \in F_j^{k',\ell'}\ (\subset H^j_{k',\ell'})$,

$$(19)\quad \tilde{K}_j^{k',\ell'}(h) = (-1)^{k'm+k(k+1)/2} \sum_{\text{Ц}} \operatorname{sgn} \text{Ц} \sum_{p=0}^{\mathrm{Min}([\frac{k-k'}{2}],\ \ell')} \ \sum_{j'=\mathrm{Max}(j-m'+a,\,0)}^{\mathrm{Min}(j,\,a)} \ \sum_{{}_p\text{Ш}_{j'}}$$

$$\xi_{k'}(\delta_1, \delta_2, \ldots, \delta_{k'};\ \lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_{k'}}) \prod_{1 \leq i \leq p} \eta_2(\delta_{k'+2\alpha_i-1}, \delta_{k'+2\alpha_i};\ \lambda_{b_i}, \lambda_{c_i}) \times$$

$$\prod_{1 \leq i \leq J+I} \zeta_2(\delta_{u_i}, \delta_{v_i};\ \lambda_{p_i}, \lambda_{q_i})\ \chi(\delta_w, \delta_x;\ \lambda_r, \lambda_s) \prod_{1 \leq i \leq q} H_2(\delta_{k'+2\beta_i-1}, \delta_{k'+2\beta_i};\ \lambda_{k+d_i}, \lambda_{k+d_i+1}) \times$$

$$\prod_{1 \leq i \leq \ell-q} Z_2(\delta_{\text{ъ}_i}, \delta_{\text{ы}_i};\ \lambda_{k+f_i}, \lambda_{k+f_i+1})\ \Xi_m(\delta_{\text{э}_1}, \delta_{\text{э}_2}, \ldots, \delta_{\text{э}_m};\ \lambda_{k+2\ell+1}, \lambda_{k+2\ell+2}, \ldots, \lambda_n) \times$$

$$\prod_{1 \leq i \leq m'} \varepsilon_i^{m'+k+1-i} \prod_{1 \leq i \leq a} \varepsilon_{w_i-k'-2\ell'}^{k-2p+1-i} \prod_{1 \leq i \leq \ell-q} \varepsilon_{\text{ъ}_i-k'-2\ell'},$$

where $p+q = \ell'$,

$$(20)\quad \begin{cases} \delta_i = e^{\sqrt{-1}\,\phi_i} & (1 \leq i \leq k'), \\ \delta_{k'+2i-1} = e^{z_i}, \quad \delta_{k'+2i} = e^{\bar{z}_i} & (1 \leq i \leq \ell'), \\ \delta_{k'+2\ell'+i} = \varepsilon_i\, e^{t_i} & (1 \leq i \leq m'); \end{cases}$$

$$(21)\quad \begin{cases} \lambda_i = \nu_i\, \ell_i, \quad \nu_i = \pm 1 & (1 \leq i \leq k), \\ \lambda_{k+2i-1} = (m_i, \sqrt{-1}\,\xi_i), \quad \lambda_{k+2i} = (m_i, -\sqrt{-1}\,\xi_i) & (1 \leq i \leq \ell), \\ \lambda_{k+2\ell+i} = (\epsilon_i, \sqrt{-1}\,\rho_i) & (1 \leq i \leq m). \end{cases}$$

We put

$$(22)\qquad {}^{+}\tilde{K}_j^{k',\ell'} = \sum_{\nu_1,\nu_2,\cdots,\nu_k} \tilde{K}_j^{k',\ell'}.$$

Because $G' = \bigcup_{x\in G} \bigcup_{k+2\ell+m=n} \bigcup_{0\leq i\leq m} x(F_i^{k,\ell})'\, x^{-1}$, for $g \in G'$ we take $h_g \in (F_i^{k,\ell})'$ which is conjugate to $g \in G'$. We define analytic functions Θ on G' as follows:

$$(23)\quad \Theta(\omega(\nu_1\ell_1, \nu_2\ell_2, \ldots, \nu_k\ell_k; M; \epsilon), \nu(\xi; \rho))\ (g)$$

$$= \begin{cases} \dfrac{\tilde{K}_j^{k',\ell'}(h_g)}{\Delta^{k',\ell'}(h_g)} & h_g \in (F_j^{k',\ell'})', \quad 0 \leq k' \leq k,\ 0 \leq \ell',\ 0 \leq j \leq m',\ k'+2\ell' \leq k+2\ell, \\ 0 & \text{otherwise}, \end{cases}$$

where $M = (m_1, m_2, \ldots, m_\ell)$, $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_m)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_\ell)$, and $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$. And put

$${}^{+}\Theta(\omega(\ell_1, \ell_2, \cdots, \ell_k; *), \nu(*)) = \sum_{\nu_1,\nu_2,\cdots,\nu_k} \Theta(\omega(\nu_1\ell_1, \nu_2\ell_2, \ldots, \nu_k\ell_k; *), \nu(*)).$$

Theorem 1. Take the analytic functions $\Theta(\omega, \nu)$ on G as in (23). Then $\Theta(\omega, \nu)$ are invariant eigendistributions. Especially for $\ell_1 > \ell_2 > \cdots > \ell_k > 0$, $\nu_1, \nu_2, \ldots, \nu_k = \pm 1$, $m_1 > 0$, $m_2 > 0, \ldots, m_\ell > 0$, $\xi_1 > 0$, $\xi_2 > 0, \ldots, \xi_\ell > 0$, $\rho_1 > \rho_2 > \cdots > \rho_m > 0$, $\epsilon_1, \epsilon_2, \ldots, \epsilon_m = 0, 1$, Θ are the characters of irreducible unitary representations of non degenerate principal series. If ℓ, $m = 0$, then Θ are the characters of discrete series. And if k, $\ell = 0$, then Θ are the characters of continuous series.

Proof. Let $Ж_{k,\ell}$ be the centralizer of ${}^{\Theta}H_{k,\ell}$ in G. Put

$$E(Ж_{k,\ell}) = \{ \eta ;\ Ж_{k,\ell} \to \mathbb{R}^*,\ \eta \text{ continuous homomorphism } \} \qquad (\mathbb{R}^* = \mathbb{R} - \{0\}),$$

$$M_{k,\ell} = \bigcap_{\eta \in E(Ж_{k,\ell})} \operatorname{Ker} |\eta| \ .$$

Let $Z_{Ж_{k,\ell}}$ be the center of $Ж_{k,\ell}$, then ${}^{\Theta}H_{k,\ell}$ is the maximal connected split (i.e. Ad(h_) is diagonizable over $\mathbb{R}$) abelian subgroup of $Z_{Ж_{k,\ell}}$. There exists a cuspidal parabolic subgroup $P^{k,\ell}$ whose Langlands decomposition is given as follows:

$$P^{k,\ell} = M_{k,\ell}\, {}^{\Theta}H_{k,\ell}\, N_{k,\ell}\ ,$$

where $N_{k,\ell}$ is the unipotent radical of $P^{k,\ell}$. $M_{k,\ell}$ consist of the elements of the following forms:

$$M_{k,\ell}: \left(\begin{array}{c|c|c|c|c|c}
A_1 & & & B_1 & & \\ \hline
 & C_1\ C_2 \ddots C_\ell & & & D_1\ D_2 \ddots D_\ell & \\ \hline
 & & \varepsilon_1\ \varepsilon_2 \ddots \varepsilon_m & & & \\ \hline
B'_1 & & & A'_1 & & \\ \hline
 & \bar{D}_1\ \bar{D}_2 \ddots \bar{D}_\ell & & & \bar{C}_1\ \bar{C}_2 \ddots \bar{C}_\ell & \\ \hline
 & & & & & \varepsilon_1\ \varepsilon_2 \ddots \varepsilon_m
\end{array}\right),$$

where

$$\begin{pmatrix} A_1 & B_1 \\ B'_1 & A'_1 \end{pmatrix} \in G_k \simeq Sp(k, \mathbb{R}), \quad C_i = \begin{pmatrix} \alpha_i \backepsilon_i^+ & \alpha_i \backepsilon_i^- \\ \bar{\alpha}_i \backepsilon_i^- & \bar{\alpha}_i \backepsilon_i^+ \end{pmatrix}, \quad D_i = \begin{pmatrix} \bar{\beta}_i \backepsilon_i^+ & \bar{\beta}_i \backepsilon_i^- \\ \beta_i \backepsilon_i^- & \beta_i \backepsilon_i^+ \end{pmatrix}$$

$$\begin{pmatrix} \alpha_i & \bar{\beta}_i \\ \beta_i & \bar{\alpha}_i \end{pmatrix} \in SU(1,1), \text{i.e.,}\ |\alpha_i|^2 - |\beta_i|^2 = 1, \quad \backepsilon_i^+ = \frac{1+\backepsilon_i}{2}, \quad \backepsilon_i^- = \frac{1-\backepsilon_i}{2}, \quad \backepsilon_i = \pm 1$$

$$(1 \le i \le \ell).$$

We take non negative integers k', ℓ', m', j, p, q and j' as follows:

$$(24)\quad \begin{array}{l} k'+2\ell' \leq k+2\ell,\ k' \leq k,\ k'+2\ell'+m' = n,\ j \leq m',\ p \leq \mathrm{Min}([\frac{k-k'}{2}], \ell'),\ p+q = \ell', \\ \mathrm{Max}(j-m'+a,\ 0) \leq j' \leq \mathrm{Min}(j,\ a) \quad (a = k-k'-2p). \end{array}$$

Let u, v be non negative integers such that $u+v = 2(\ell-q)$, $j'+u \leq j$.
Let Δ_1, Δ_2 be ordered sets of positive integers as follows:

$$\Delta_1 = (j'+1,\ j'+2, \ldots,\ j'+u),$$
$$\Delta_2 = (j+a-j'+1,\ j+a-j'+2, \ldots,\ j+a-j'+v).$$

Put $c = \mathrm{Min}(u, v)$ and define

$$\Omega(u) = \left\{c,\ c-2, \ldots,\ c-2[\tfrac{c}{2}]+2,\ c-2[\tfrac{c}{2}]\right\}.$$

For $d \in \Omega(u)$, we define positive integers ъ_i, ы_i $(1 \leq i \leq \ell-q)$ as follows:

$$\Delta_1 = (\text{ъ}_1, \text{ы}_1, \text{ъ}_2, \text{ы}_2, \ldots, \text{ъ}_{\frac{u-d}{2}}, \text{ы}_{\frac{u-d}{2}}; \text{ъ}_{\frac{u-d}{2}+1}, \text{ъ}_{\frac{u-d}{2}+2}, \ldots, \text{ъ}_{\frac{u+d}{2}}),$$
$$\Delta_2 = (\text{ы}_{\frac{u-d}{2}+1}, \text{ы}_{\frac{u-d}{2}+2}, \ldots, \text{ы}_{\frac{u+d}{2}}; \text{ъ}_{\frac{u+d}{2}+1}, \text{ы}_{\frac{u+d}{2}+1}, \text{ъ}_{\frac{u+d}{2}+2}, \text{ы}_{\frac{u+d}{2}+2}, \ldots, \text{ъ}_{\frac{u+v}{2}}, \text{ы}_{\frac{u+v}{2}}).$$

Put $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_j = 1$, $\varepsilon_{j+1} = \varepsilon_{j+2} = \ldots = \varepsilon_{m'} = -1$.

Using these notations, we define 2×2 matrices C_i^+ $(1 \leq i \leq \ell)$ as follows:

$$C_i^+ = \begin{pmatrix} e^{\sqrt{-1}\,\theta_{p+i}} & 0 \\ 0 & e^{-\sqrt{-1}\,\theta_{p+i}} \end{pmatrix} \qquad (1 \leq i \leq q),$$

$$C_{q+i}^+ = \begin{pmatrix} \frac{1+\varepsilon_{\text{ъ}_i}\varepsilon_{\text{ы}_i}}{2} & \frac{1-\varepsilon_{\text{ъ}_i}\varepsilon_{\text{ы}_i}}{2} \\ \frac{1-\varepsilon_{\text{ъ}_i}\varepsilon_{\text{ы}_i}}{2} & \frac{1+\varepsilon_{\text{ъ}_i}\varepsilon_{\text{ы}_i}}{2} \end{pmatrix} \qquad (1 \leq i \leq \ell-q).$$

We also define 2×2 matrices C_i^-, D_i^- $(1 \leq i \leq \ell)$ as follows:

$$C_i^- = \mathrm{ch}\,\tau_{p+i}\,1_2 \qquad (1 \leq i \leq q),$$

$$C_{q+i}^- = \begin{pmatrix} \frac{\mathrm{ch}\,t_{\text{ъ}_i}+\mathrm{ch}\,t_{\text{ы}_i}}{2} & \frac{\mathrm{ch}\,t_{\text{ъ}_i}-\mathrm{ch}\,t_{\text{ы}_i}}{2} \\ \frac{\mathrm{ch}\,t_{\text{ъ}_i}-\mathrm{ch}\,t_{\text{ы}_i}}{2} & \frac{\mathrm{ch}\,t_{\text{ъ}_i}+\mathrm{ch}\,t_{\text{ы}_i}}{2} \end{pmatrix} \qquad (1 \leq i \leq \ell-q);$$

$$D_i^- = \operatorname{sh} \tau_{p+i} j_2 \qquad (1 \leq i \leq q),$$

$$D_{q+i}^- = \begin{pmatrix} \frac{\operatorname{sh} t_{b_i} - \operatorname{sh} t_{b_{l_i}}}{2} & \frac{\operatorname{sh} t_{b_i} + \operatorname{sh} t_{b_{l_i}}}{2} \\ \frac{\operatorname{sh} t_{b_i} - \operatorname{sh} t_{b_{l_i}}}{2} & \frac{\operatorname{sh} t_{b_i} + \operatorname{sh} t_{b_{l_i}}}{2} \end{pmatrix} \qquad (1 \leq i \leq \ell - q).$$

Let Ч be an element of S_ℓ satisfying

(25) $$Ч^{-1} = \begin{pmatrix} 1 & 2 & \ldots & q & q+1 & q+2 & \ldots & \ell \\ d_1 & d_2 & \ldots & d_q & f_1 & f_2 & \ldots & f_{\ell-q} \end{pmatrix}, \quad d_1 < d_2 < \ldots < d_q, \quad f_1 < f_2 < \ldots < f_{\ell-q}.$$

Take a permutation Я as follows:

(26) $$Я^{-1} = \begin{pmatrix} j'+u+1 & j'+u+2 & \ldots & j & j+a-j'+v+1 & j+a-j'+v+2 & \ldots & m' \\ э_1 & э_2 & \ldots & э_{j-j'-u} & э_{j-j'-u+1} & э_{j-j'-u+2} & \ldots & э_m \end{pmatrix},$$

$$э_1 < э_2 < \ldots < э_{j-j'-u}, \quad э_{j-j'-u+1} < э_{j-j'-u+2} < \ldots < э_m.$$

The following subsets $H^j_{k',\ell'}(p, j', u, d, ч, я)$ of $Ж_{k,\ell}$ are conjugate to $H^j_{k',\ell'}$ under inner automorphisms of G. They are given $H^j_{k',\ell'}(*) = {}^{\oplus}H^j_{k',\ell'}(*)\,{}^{\ominus}H^j_{k',\ell'}(*)$, where ${}^{\oplus}H^j_{k',\ell'}(*) = H^j_{k',\ell'}(*) \cap K$, ${}^{\ominus}H^j_{k',\ell'}(*) = H^j_{k',\ell'}(*) \cap \exp\mathfrak{P}$ consist of the elements of the following forms respectively.

${}^{\oplus}H^j_{k',\ell'}(p, j', u, d, ч, я)$:

$$\begin{pmatrix} A_1^+ & & & & & \\ & C^+_{ч(1)} \; C^+_{ч(2)} \cdots C^+_{ч(\ell)} & & & & \\ & & \varepsilon_{э_1}\, \varepsilon_{э_2} \cdots \varepsilon_{э_m} & & & \\ & & & \bar{A}_1^+ & & \\ & & & & \bar{C}^+_{ч(1)} \; \bar{C}^+_{ч(2)} \cdots \bar{C}^+_{ч(\ell)} & \\ & & & & & \varepsilon_{э_1}\, \varepsilon_{э_2} \cdots \varepsilon_{э_m} \end{pmatrix}$$

where $A_1^{+} = \mathrm{diag}(e^{\sqrt{-1}\phi_1},\dots,e^{\sqrt{-1}\phi_{k'}}; e^{\sqrt{-1}\theta_1}, e^{-\sqrt{-1}\theta_1},\dots,e^{\sqrt{-1}\theta_p},\ e^{-\sqrt{-1}\theta_p}; \varepsilon_1,\dots,\varepsilon_{j'},\varepsilon_{j+1},\dots,\varepsilon_{j+a-j'})$.

$^{\Theta}H^{j}_{k',\ell'}(p,\ j',\ u,\ d,\ \text{ч},\ \text{я})$:

$$\left(\begin{array}{c|c|c|c|c|c}
A_1^{-} & & & B_1^{-} & & \\ \hline
 & \begin{matrix} C^{-}_{\text{ч}(1)} & & \\ & C^{-}_{\text{ч}(2)} & \\ & & \ddots \\ & & & C^{-}_{\text{ч}(\ell)} \end{matrix} & & & \begin{matrix} D^{-}_{\text{ч}(1)} & & \\ & D^{-}_{\text{ч}(1)} & \\ & & \ddots \\ & & & D^{-}_{\text{ч}(\ell)} \end{matrix} & \\ \hline
 & & \begin{matrix} \mathrm{ch}\,t_{э_1} & & \\ & \mathrm{ch}\,t_{э_2} & \\ & & \ddots \\ & & & \mathrm{ch}\,t_{э_m} \end{matrix} & & & \begin{matrix} \mathrm{sh}\,t_{э_1} & & \\ & \mathrm{sh}\,t_{э_2} & \\ & & \ddots \\ & & & \mathrm{sh}\,t_{э_m} \end{matrix} \\ \hline
B_1^{-} & & & A_1^{-} & & \\ \hline
 & \begin{matrix} D^{-}_{\text{ч}(1)} & & \\ & D^{-}_{\text{ч}(2)} & \\ & & \ddots \\ & & & D^{-}_{\text{ч}(\ell)} \end{matrix} & & & \begin{matrix} C^{-}_{\text{ч}(1)} & & \\ & C^{-}_{\text{ч}(1)} & \\ & & \ddots \\ & & & C^{-}_{\text{ч}(\ell)} \end{matrix} & \\ \hline
 & & \begin{matrix} \mathrm{sh}\,t_{э_1} & & \\ & \mathrm{sh}\,t_{э_2} & \\ & & \ddots \\ & & & \mathrm{sh}\,t_{э_m} \end{matrix} & & & \begin{matrix} \mathrm{ch}\,t_{э_1} & & \\ & \mathrm{ch}\,t_{э_2} & \\ & & \ddots \\ & & & \mathrm{ch}\,t_{э_m} \end{matrix}
\end{array}\right),$$

where

$$A_1^{-} = \begin{pmatrix} 1_{k'} & & & & & & & \\ & \mathrm{ch}\,\tau_1\, 1_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \mathrm{ch}\,\tau_p\, 1_2 & & & & \\ & & & & \mathrm{ch}\,t_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \mathrm{ch}\,t_{j'} & \\ & & & & & & & \mathrm{ch}\,t_{j+1} \\ & & & & & & & & \ddots \\ & & & & & & & & & \mathrm{ch}\,t_{j+a-j'} \end{pmatrix}, \quad B_1^{-} = \begin{pmatrix} 0_{k'} & & & & & & & \\ & \mathrm{sh}\,\tau_1\, J_2 & & & & & & \\ & & \ddots & & & & & \\ & & & \mathrm{sh}\,\tau_p\, J_2 & & & & \\ & & & & \mathrm{sh}\,t_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \mathrm{sh}\,t_{j'} & \\ & & & & & & & \mathrm{sh}\,t_{j+1} \\ & & & & & & & & \ddots \\ & & & & & & & & & \mathrm{sh}\,t_{j+a-j'} \end{pmatrix}.$$

Then

$$(27)\quad \left\{\begin{array}{ll} H^{j}_{k',\ell'}(p,\ j',\ u,\ d,\ \text{ч},\ \text{я}); & j,\ k',\ \ell',\ p,\ j' \ \text{satisfying (24)} \\ & 1 \le u \le \mathrm{Min}(2(\ell+p-\ell'),\ j-j') \\ & d \in \Omega(u) \\ & \text{ч} \ \text{satisfying (25)} \\ & \text{я} \ \text{satisfying (26)} \end{array}\right\}$$

form a maximal system of representative which are not conjugate to each other under inner automorphism of $Ж_{k,\ell}$.

We take A, B, C, P, Q, r and s as in (13), (14) and (15) respectively. We denote the collection (A, B, C, P, Q, r, s) simply by $\tilde{\text{Д}}$ and put sgn $\tilde{\text{Д}}$ = sgn(A, B·C, P·Q, r, s). Let $(\delta_1, \delta_2, \ldots, \delta_n)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be as in (20) and (21) respectively. We take ${}_pW_{j'}$, ${}_pU_{j'}$, ${}_pV_{j'}$, w and x as in (16), (17) and (18) respectively. We denote the collection $({}_pW_{j'}, {}_pU_{j'}, {}_pV_{j'}, w, x)$ simply by ${}_p\tilde{\text{Ш}}_{j'}$. Let ω be an irreducible square integrable representation of $M_{k,\ell}$ and ν be a purely imaginary valued regular linear form ${}^{\Theta}\mathfrak{g}_{k,\ell}$ (the Lie algebra of ${}^{\Theta}H_{k,\ell}$). The character of this representation ω has been calculated in T. Hirai [2(d)]. Using this formula, we can express the character $\gamma(\omega, \nu)$ of the representation $\omega\otimes\nu$ as follows: for $h \in (H^{j}_{k',\ell'}(p, j', u, d, \text{ч}, \text{я}))'$,

(28) $\gamma(\omega, \nu)(h)$

$$
= \begin{cases}
\dfrac{(-1)^{k(k+1)/2} \sum_{\tilde{\text{Д}}} \operatorname{sgn}\tilde{\text{Д}}\ \xi_{k'}(e^{\sqrt{-1}\phi_1}, \ldots, e^{\sqrt{-1}\phi_{k'}}; \lambda_{a_1}, \ldots, \lambda_{a_{k'}}) \prod_{1\le i\le p} \eta_2(\delta_{k'+2d_i-1}, \delta_{k'+2d_i}; \lambda_{b_i}, \lambda_{c_i})}{{}_k\Delta^{k',\ell'}(e^{\sqrt{-1}\phi_1}, \ldots, e^{\sqrt{-1}\phi_{k'}}, e^{\sqrt{-1}\theta_1}, e^{-\sqrt{-1}\theta_1}, \ldots, e^{\sqrt{-1}\theta_\ell}, e^{-\sqrt{-1}\theta_\ell}, \varepsilon_1 e^{t_1}, \ldots, \varepsilon_{j'} e^{t_{j'}}, \varepsilon_{j+1} e^{t_{j+1}}, \ldots, \varepsilon_{j+a_{j'}} e^{t_{j+a-j'}})} \times \\[2ex]
\dfrac{\sum_{{}_p\tilde{\text{Ш}}_{j'}} \prod_{1\le i\le J+I} \zeta_2(\delta_{u_i}, \delta_{v_i}; \lambda_{p_i}, \lambda_{q_i})\ \chi(\delta_w, \delta_x; \lambda_r, \lambda_s)}{} \times \\[2ex]
\prod_{i=1}^{q} \dfrac{e^{-\sqrt{-1}m_{d_i}\theta_{p+i}} - e^{\sqrt{-1}m_{d_i}\theta_{p+i}}}{e^{\sqrt{-1}\theta_{p+i}} - e^{-\sqrt{-1}\theta_{p+i}}} e^{\sqrt{-1}\xi_{d_i}\tau_{p+i}} \prod_{i=1}^{\ell-q} \dfrac{2e^{-m_{f_i}|t_{\text{ъ}_i} - t_{\text{ы}_i}|/2}}{|e^{(t_{\text{ъ}_i} - t_{\text{ы}_i})/2} - e^{-(t_{\text{ъ}_i} - t_{\text{ы}_i})/2}|} e^{\sqrt{-1}\xi_{f_i}(t_{\text{ъ}_i} + t_{\text{ы}_i})/2}\ \varepsilon_{\text{ъ}_i}^{m_{f_i}+1} \times \\[2ex]
\prod_{i=1}^{m} \varepsilon_{\text{э}_i}^{\epsilon_i} e^{\sqrt{-1}\rho_i t_{\text{э}_i}}, \qquad \text{if } \varepsilon_{\text{ъ}_1} = \varepsilon_{\text{ы}_1} \cdots \varepsilon_{\text{ъ}_{\ell-q}} = \varepsilon_{\text{ы}_{\ell-q}}; \\[2ex]
0 \qquad \text{otherwise.}
\end{cases}
$$

We extend $\omega\otimes\nu$ as the representation $p^{k,\ell}$ naturally and get the induced representation $\pi(\omega, \nu) = \operatorname{Ind}_{p^{k,\ell}\uparrow G} \omega\otimes\nu$ of G. Its character $\theta(\omega, \nu)$ can be obtained by T. Hirai [2(a)]. This is exactly (22). The irreducibility of $\pi(\omega, \nu)$ is given by Harish-Chandra [1(d)]. Even if a parameter $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is not regular, $\gamma(\omega, \nu)$ is an invariant eigendistribution on $\mathfrak{X}_{k,\ell}$. Then $\Theta(\omega,\nu)$ is an invariant eigendistribution on G.

§2. Fourier transforms

Notations 2. We define functions corresponding to those in Notations 1. Each function $(*-2)$ has the same conditions of the corresponding function $(*-1)$.

(a-2) $$ {}^{+}\xi'_p(\delta_1, \delta_2, \ldots, \delta_p; \ell_1, \ell_2, \ldots, \ell_p) = \sum_{\sigma\in S_p} (\delta_{\sigma(1)}^{\ell_1} + \delta_{\sigma(1)}^{-\ell_1})(\delta_{\sigma(2)}^{\ell_2} + \delta_{\sigma(2)}^{-\ell_2}) \ldots (\delta_{\sigma(p)}^{\ell_p} + \delta_{\sigma(p)}^{-\ell_p}) ; $$

(b-2) $$ {}^{+}\eta'_2(\delta_1, \delta_2; \ell_1, \ell_2) = 2(\delta_1^{\ell_2} + \delta_1^{-\ell_2})\delta_2^{-\ell_1} + 2(\delta_2^{\ell_2} + \delta_2^{-\ell_2})\delta_1^{-\ell_1} ; $$

(c-2) $$ {}^{+}\zeta'_2(\delta_1, \delta_2; \ell_1, \ell_2) = 4(\delta_1^{-\ell_1}\delta_2^{\ell_2} + \delta_1^{-\ell_2}\delta_2^{-\ell_1}) ; $$

(d-2) $$ {}^{+}\chi'(\delta_1, \delta_2; \ell_1, \ell_2) = 4(\delta_1^{-\ell_1}\delta_2^{-\ell_2} + \delta_1^{-\ell_2}\delta_2^{-\ell_1}) ; $$

(d′-2) $$ {}^{+}\chi'(\delta_1; \ell_1) = 2\delta_1^{-\ell_1} ; $$

(e-2) $$ H'_2(\delta_1, \delta_2; \lambda, \bar{\lambda}) = (e^{\sqrt{-1}m\theta} + e^{-\sqrt{-1}m\theta})(e^{\sqrt{-1}\xi\tau} - e^{-\sqrt{-1}\xi\tau}) ; $$

(f-2) $$ Z'_2(\delta_1, \delta_2; \lambda, \bar{\lambda}) = 2e^{-m(t_1-t_2)/2}(e^{\sqrt{-1}\xi(t_1+t_2)/2} - e^{-\sqrt{-1}\xi(t_1+t_2)/2})\varepsilon^{m} - 2e^{-m(t_1+t_2)/2}(e^{\sqrt{-1}\xi(t_1-t_2)/2} - e^{-\sqrt{-1}\xi(t_1-t_2)/2})\varepsilon^{m} ; $$

(g-2) $$ \Xi'_p(\delta_1, \delta_2, \ldots, \delta_p; \lambda_1, \lambda_2, \ldots, \lambda_p) = \sum_{\sigma\in S_p} \operatorname{sgn}\sigma \prod_{j=1}^{p} (e^{\sqrt{-1}\xi_{\sigma(j)}t_j} - e^{-\sqrt{-1}\xi_{\sigma(j)}t_j})\varepsilon^{m_{\sigma(j)}} . $$

For $f\in C_0^\infty(G)$, we put

(29) $$ F_f^{k,\ell}(h) = L^{k,\ell} K_f^{k,\ell}(h) \qquad (h\in H'_{k,\ell}(I)) . $$

Every $F_f^{k,\ell}$ $(0\le k,\ell,\ k+2\ell\le m)$ has the following properties.

(i) $F_f^{k,\ell}$ is zero outside some relatively compact subset of $H'_{k,\ell}(I)$ and can be extended to a continuous function on the whole $H_{k,\ell}$. ([1(a)] , Lemma 40)

(ii) $F_f^{k,\ell}(wh) = \varepsilon'(w) F_f^{k,\ell}(h) \qquad (h\in H'_{k,\ell}(I),\ w\in W^{k,\ell})$.

(iii) The restriction of $F_f^{k,\ell}$ on every connected component of $H'_{k,\ell}(I)$ may be considered as an infinitely differentiable function on the closure of the component.

Using the notions of §2, we define functions $({}^{+}\tilde{K}_j^{k',\ell'})'$ on $F_j^{k',\ell'}$ as follows:

$$({}^{+}\tilde{K}_j^{k',\ell'})'(h) = (-1)^{k'm+k(k+1)/2} \sum_{Ц} \sum_{p=0}^{\mathrm{Min}([\frac{k-k'}{2}],\ell')} \sum_{j'=\mathrm{Max}(j-m'+a,0)}^{\mathrm{Min}(j,a)} \sum_{p\,Ш j'}$$

$${}^{+}\xi'_{k'}(\delta_1, \delta_2, \ldots, \delta_{k'}; \lambda_{a_1}, \lambda_{a_2}, \ldots, \lambda_{a_{k'}}) \prod_{1\le i\le p} {}^{+}\eta'_2(\delta_{k'+2\alpha_i-1}, \delta_{k'+2\alpha_i}; \lambda_{b_i}, \lambda_{c_i}) \times$$

$$\prod_{1\le i\le J+I} {}^{+}\zeta'_2(\delta_{u_i}, \delta_{v_i}; \lambda_{p_i}, \lambda_{q_i}) \; {}^{+}\chi'(\delta_w, \delta_x; \lambda_r, \lambda_s) \prod_{1\le i\le q} H'_2(\delta_{k'+2\beta_i-1}, \delta_{k'+2\beta_i}; \lambda_{d_i}, \lambda_{d_{i+1}}) \times$$

$$\prod_{1\le i\le \ell-q} Z'_2(\delta_{Ъ_i}, \delta_{Ы_i}; \lambda_{f_i}, \lambda_{f_{i+1}}) \; \Xi'_m(\delta_{Э_1}, \delta_{Э_2}, \ldots, \delta_{Э_m}; \lambda_{k+2\ell+1}, \lambda_{k+2\ell+2}, \ldots, \lambda_n) \times$$

$$\prod_{1\le i\le m} \varepsilon_i^{m'+k'+1-i} \prod_{1\le i\le a} \varepsilon_{w_i-k'-2\ell}^{k-2p+1-i} \prod_{1\le i\le \ell-q} \varepsilon_{Ъ_i-k'-2\ell'} .$$

<u>Lemma</u> 2.1. Define the functions ${}^{+}\tilde{K}_j^{k',\ell'}$ and $({}^{+}\tilde{K}_j^{k',\ell'})'$ on $F_j^{k',\ell'}$ by (22) and (30) respectively. Then we get

$$(31) \quad L^{k',\ell'}\,{}^{+}\tilde{K}^{k',\ell'} = L\left(\ell_1, \ell_2, \ldots, \ell_k, \frac{m_1+\sqrt{-1}\xi_1}{2}, \frac{m_1-\sqrt{-1}\xi_1}{2}, \ldots, \frac{m_\ell+\sqrt{-1}\xi_\ell}{2}, \frac{m_\ell-\sqrt{-1}\xi_\ell}{2}, \sqrt{-1}\rho_1, \sqrt{-1}\rho_2, \ldots, \sqrt{-1}\rho_m\right) ({}^{+}\tilde{K}^{k',\ell'})'.$$

We define functions $({}^{+}K_j^{k',\ell'})'$ on $(H_{k',\ell'}^j(R))' = H_{k',\ell'}^j(R) \cap H'_{k',\ell'}$ as follows:

$$(32) \quad ({}^{+}K_j^{k',\ell'})'(h) = \varepsilon'(w)\,({}^{+}\tilde{K}_j^{k',\ell'})'(h_0) \quad (h = wh_0 \in (H_{k',\ell'}^j(R))',\ h_0 \in F_j^{k',\ell'},\ w \in W^{k',\ell'}).$$

Let W_p be the permutation group on $\mathbb{Z}^p = \{(\ell_1, \ell_2, \ldots, \ell_p); \ell_1, \ell_2, \ldots, \ell_p$ integers$\}$ generated by the following permutations:

(33)
any permutation of $\ell_1, \ell_2, \ldots, \ell_p$;
change of the sign of ℓ_i, i.e., $\ell_i \longrightarrow -\ell_i$ $(1 \le i \le p)$.

Let f be a function with parameters $(\ell_1, \ell_2, \ldots, \ell_p) \in \mathbb{Z}^p$. Let A be a subset of $\mathbb{Z}^p$. We define the sum on A as follows:

$$(34) \quad \sum_{(\ell_1,\ell_2,\ldots,\ell_p)\in S(A)} f_{(\ell_1,\ell_2,\ldots,\ell_p)} = \sum_{(\ell_1,\ell_2,\ldots,\ell_p)\in A} \frac{1}{|W_p(\ell_1,\ell_2,\ldots,\ell_p)|} f_{(\ell_1,\ell_2,\ldots,\ell_p)},$$

where $W_p(\ell_1, \ell_2, \ldots, \ell_p) = \{w \in W_p;\ w(\ell_1, \ell_2, \ldots, \ell_p) = (\ell_1, \ell_2, \ldots, \ell_p)\}$. For $A = \{(\ell_1, \ell_2, \ldots, \ell_p) \in \mathbb{Z}^p; \ell_1 \ge \ell_2 \ge \ldots \ge \ell_p \ge 0\}$, we simply denote S(A) by $S(\ell_1 \ge \ell_2 \ge \cdots \ge \ell_p \ge 0)$. When A is infinite, the sum on the right hand side of (34) defined by the following limit,

$$\sum_{(\ell_1,\ell_2,\dots,\ell_p)\in A} f_{(\ell_1,\ell_2,\dots,\ell_p)} = \lim_{\substack{M_1,N_1,M_2,N_2,\dots,M_p,N_p\to+\infty \\ (\ell_1,\ell_2,\dots,\ell_p)\in A}} \sum_{\ell_1=-N_1}^{M_1}\sum_{\ell_2=-N_2}^{M_2}\cdots\sum_{\ell_p=-N_p}^{M_p} f_{(\ell_1,\ell_2,\dots,\ell_p)} .$$

Lemma 2.2. Define the functions $({}^{+}K_j^{k',\ell'})'$ on $(H^j_{k',\ell'}(R))'$ by (32). For $f\in C_0^\infty(G)$, we put the functions $F_f^{k',\ell'}(h)$ on $H_{k',\ell'}$ as in (29). Then we get

$$(35)\quad \sum_{\substack{S(\ell_1\ge\ell_2\ge\cdots\ge\ell_p\ge0)\\ S(m_1\ge0)\cdots S(m_\ell\ge0)}} \int_{(H^j_{k',\ell'})'} F_f^{k',\ell'}(h)\ ({}^{+}K_j^{k',\ell'})'(h)\ dh$$

$$= (-1)^{k'm+k(k+1)/2} \sum_{p=0}^{\mathrm{Min}([\frac{k-k'}{2}],\ell')} \sum_{j'=\mathrm{Max}(j-m'+a,0)}^{\mathrm{Min}(j,a)} \sum_{\substack{u+v\\=\ell-g}} \frac{m'!\ 2^{\ell'}\ \ell'!\ \ell!}{J!\ I!\ u!\ v!} \sum_{\circledast}$$

$$\int_{\circledast\!\!\!\!\!\bigcirc} F_f^{k',\ell'}(\delta_1,\delta_2,\dots,\delta_n)\ {}^{+}\xi'_{k'}(\delta_1,\delta_2,\dots,\delta_{k'};\lambda_1,\lambda_2,\dots,\lambda_{k'})\times$$

$$\prod_{1\le i\le p} {}^{+}\eta'_2(\delta_{k'+2i-1},\delta_{k'+2i};\lambda_{k'+2i-1},\lambda_{k'+2i}) \prod_{1\le i\le J} {}^{+}\zeta'_2(\varepsilon_{2i-1}e^{t_{2i-1}},\varepsilon_{2i}e^{t_{2i}};\lambda_{k'+2p+2i-1},\lambda_{k'+2p+2i})\times$$

$$\prod_{1\le i\le I} {}^{+}\zeta'_2(\varepsilon_{j'+2i-1}e^{t_{j'+2i-1}},\varepsilon_{j'+2i}e^{t_{j'+2i}};\lambda_{k'+2p+2J+2i-1},\lambda_{k'+2p+2J+2i})\quad {}^{+}\chi(\delta_{w_0},\delta_{x_0};\lambda_{r_0},\lambda_{s_0})\times$$

$$\prod_{1\le i\le g} H'_2(\delta_{k'+2p+2i-1},\delta_{k'+2p+2i};\lambda_{k+2i-1},\lambda_{k+2i})\times$$

$$\prod_{1\le i\le u} Z'_2(\varepsilon_{j'+2i-1}e^{t_{j'+2i-1}},\varepsilon_{j'+2i}e^{t_{j'+2i}};\lambda_{k+2g+2i-1},\lambda_{k+2g+2i})\times$$

$$\prod_{1\le i\le v} Z'_2(\varepsilon_{j+a-j'+2i-1}e^{t_{j+a-j'+2i-1}},\varepsilon_{j+a-j'+2i}e^{t_{j+a-j'+2i}};\lambda_{k+2g+2u+2i-1},\lambda_{k+2g+2u+2i})\times$$

$$\Xi'_m(\varepsilon_{j'+2u+1}e^{t_{j'+2u+1}},\dots,\varepsilon_j e^{t_j},\varepsilon_{j+a-j'+2v+1}e^{t_{j+a-j'+2v+1}},\dots,\varepsilon_{m'}e^{t_{m'}};\lambda_{k+2\ell+1},\dots,\lambda_n)\times$$

$$\prod_{1\le i\le m}\varepsilon_i^{m'+k'+1-i}\prod_{1\le i\le j'}\varepsilon_i^{k-2p+1-i}\prod_{1\le i\le a-j'}\varepsilon_{j+2i-1}^{k-2p+1-(j'+i)},\ \prod_{1\le i\le u}\varepsilon_{j'+2i-1}\prod_{1\le i\le v}\varepsilon_{j+a-j'+2i-1}$$

$$d\phi_1\dots d\phi_{k'}\ d\theta_1\dots d\theta_{\ell'}\ d\tau_1\dots d\tau_{\ell'}\ dt_1\dots dt_{m'},$$

where $\circledast$ means

$S(\ell_1\ge\ell_2\ge\dots\ge\ell_{k'}\ge0)$, $S(\ell_{k'+1}\ge\ell_{k'+2}\ge0),\dots,$

$S(\ell_{k'+2p-1}\ge\ell_{k'+2p}\ge0)$, $S(\ell_{k'+2p+1}\ge\ell_{k'+2p+2}\ge0),\dots,$

$S(\ell_{k'+2p+J+I-1}\ge\ell_{k'+2p+J+I}\ge0)$, $S(\ell_{r_0}\ge\ell_{s_0}\ge0)$, $S(m_1\ge0),\dots,$ $S(m_\ell\ge0)$;

$\circledast\!\!\!\!\!\bigcirc$ is the following domain,

$t_1>t_2>0,\dots,$ $t_{2J-1}>t_{2J}>0$, $t_{j+1}>t_{j+2}>0,\dots,$ $t_{j+2J-1}>t_{j+2J}>0$,

$t_{w_0}>t_{x_0}>0$,

$t_{j'+1} > t_{j'+2} > 0, \ldots, \quad t_{j'+2u-1} > t_{j'+2u} > 0,$

$t_{j+a-j'+1} > t_{j+a-j'+2} > 0, \ldots, \quad t_{j+a-j'+2v-1} > t_{j+a-j'+2v} > 0, \quad t_{j'+2u+1} > 0, \ldots, \quad t_j > 0,$

$t_{j+a-j'+2v+1} > 0, \ldots, \quad t_{m'} > 0, \quad \tau_1 > 0, \ldots, \quad \tau_{\ell'} > 0;$

w_0 , x_0 , r_0 and s_0 are the following symbols,

$$w_0 = \begin{cases} j' & \text{if} \quad 2J = j'-1 \\ \phi & \text{if} \quad 2J = j' \end{cases}, \quad x_0 = \begin{cases} k-k'-2p-j' & \text{if} \quad 2I = k-k'-2p-j'-1 \\ \phi & \text{if} \quad 2I = k-k'-2p-j' \end{cases}$$

$$r_0 = \begin{cases} \text{integer } r_0 & \text{if} \quad 2J = j'-1 \\ \phi & \text{if} \quad 2J = j' \end{cases},$$

$$s_0 = \begin{cases} \text{integer } s_0 & \text{if} \quad 2I = k-k'-2p-j'-1 \\ \phi & \text{if} \quad 2I = k-k'-2p-j' \end{cases},$$

satisfying

$$\{1, 2, \ldots, k'+2p+J+I, r_0, s_0\} = \{1, 2, \ldots, k\},$$

$$r_0 < s_0 \quad , \quad \text{if } r_0 \text{ and } s_0 \text{ are integral.}$$

Notations 3. We define functions corresponding to those in Notations 1, 2. Following notations express the histories of functions.

(b-3) $$^{+}\hat{\eta}'_2(m, \xi) = \begin{cases} -\frac{\sqrt{-1}}{4} \coth\frac{\pi\xi}{2} & m \in \mathbb{Z}_0 \quad (\mathbb{Z}_0 = 2\mathbb{Z}), \\ -\frac{\sqrt{-1}}{4} \tanh\frac{\pi\xi}{2} & m \in \mathbb{Z}_1 \quad (\mathbb{Z}_1 = 2\mathbb{Z}+1), \end{cases}$$

$$^{+}\hat{\eta}'_2(\xi; \mathbb{Z}_p) = {}^{+}\hat{\eta}'_2(m, \xi) \qquad m \in \mathbb{Z}_p \qquad (p = 0, 1);$$

(c-3) $$^{+}\hat{\zeta}'_2(\rho_1, \varepsilon; \rho_2, \varepsilon) = \begin{cases} \frac{1}{4} \coth\pi\rho_2 \ \coth\pi(\rho_1+\rho_2) & \varepsilon = 1, \\ \frac{1}{4} \coth\pi\rho_2 \ \mathrm{cosech}\,\pi(\rho_1+\rho_2) & \varepsilon = -1; \end{cases}$$

(d-3) $$^{+}\hat{\chi}'(\rho_1, \varepsilon; \rho_2, \varepsilon) = \begin{cases} -\frac{1}{4} \coth\pi\rho_1 \ \mathrm{cosech}\,\pi\rho_2 & \varepsilon = 1, \\ -\frac{1}{4} \mathrm{cosech}\,\pi\rho_1 \ \coth\pi\rho_2 & \varepsilon = -1; \end{cases}$$

(d'-3) $$^{+}\hat{\chi}'(\rho_1, \varepsilon_1) = \begin{cases} -\frac{\sqrt{-1}}{2} \coth\pi\rho_1 & \varepsilon_1 = 1, \\ -\frac{\sqrt{-1}}{2} \mathrm{cosech}\,\pi\rho_1 & \varepsilon_1 = -1; \end{cases}$$

$$(f\text{-}3)\qquad \hat{Z}_2'(\eta;\varepsilon;\mathbb{Z}_p) = \begin{cases} -\frac{\sqrt{-1}}{4}\coth\frac{\pi\eta}{2} & p = 0, \\ -\frac{\sqrt{-1}}{4}\tanh\frac{\pi\eta}{2}\,\varepsilon & p = 1. \end{cases}$$

Lemma 2.3. Let F be a continuous function of bounded variation on the p-dimensional torus $\mathbb{T}^p$ and satisfy

$$F(e^{\sqrt{-1}\phi_{\sigma(1)}}, e^{\sqrt{-1}\phi_{\sigma(2)}}, \ldots, e^{\sqrt{-1}\phi_{\sigma(p)}}) = F(e^{\sqrt{-1}\phi_1}, e^{\sqrt{-1}\phi_2}, \ldots, e^{\sqrt{-1}\phi_p}) \qquad (\sigma\in S_p).$$

Then we get

$$(a\text{-}4)\qquad \sum_{S(\ell_1\geq\ell_2\geq\cdots\geq\ell_p\geq 0)} \int_{\phi_1,\phi_2,\ldots,\phi_p\in[-\pi,\pi]} F(e^{\sqrt{-1}\phi_1}, e^{\sqrt{-1}\phi_2}, \ldots, e^{\sqrt{-1}\phi_p})\ {}^{+}\xi_p'(e^{\sqrt{-1}\phi_1}, e^{\sqrt{-1}\phi_2}, \ldots, e^{\sqrt{-1}\phi_p}; \ell_1, \ell_2, \ldots, \ell_p)\, d\phi_1\, d\phi_2 \ldots d\phi_p$$

$$= \sum_{\ell_1,\ell_2,\ldots,\ell_p\in\mathbb{Z}} \int_{\phi_1,\phi_2,\ldots,\phi_p\in[-\pi,\pi]} F(e^{\sqrt{-1}\phi_1}, e^{\sqrt{-1}\phi_2}, \ldots, e^{\sqrt{-1}\phi_p})\ e(\sqrt{-1}(\ell_1\phi_1+\ell_2\phi_2+\ldots+\ell_p\phi_p))\, d\phi_1 d\phi_2 \ldots d\phi_p$$

$$= F(1,1,\ldots,1)$$

where $e(x) = e^x$.

Proof. From the definition of the sum, every $(\ell_1, \ell_2, \ldots, \ell_p)\in\mathbb{Z}^p$ is counted exactly once. And from Jordan's theorem we get the second assertion.

We put $I(*)$ = the left hand side of $(*\text{-}4)$.

Lemma 2.4. Let F be a continuous function on $\mathbb{C}^* = \mathbb{C}-\{0\}$ which vanishes outside some compact set and continuously differentiable on $\mathbb{C}-\mathbb{R}^+$ ($\mathbb{R}^+ = \{t\in\mathbb{R}; t\geq 0\}$). Suppose F satisfies $F(e^z) = F(e^{\bar{z}}) = -F(e^{-\bar{z}})$, then we get

$$(b\text{-}4)\qquad \sum_{S(\ell_1\geq\ell_2\geq 0)} \int_{\substack{\theta\in[-\pi,\pi]\\ \tau>0}} F(e^z)\ {}^{+}\eta_2'(e^z, e^{\bar{z}}; \ell_1, \ell_2)\, d\theta\, d\tau$$

$$= \sum_{m\in\mathbb{Z}} \int_{\xi\in\mathbb{R}} \hat{F}(m,\xi)\,{}^{+}\hat{\eta}_2'(m,\xi)\,d\xi = \sum_{p=0,1} \sum_{m\in\mathbb{Z}_p} \int_{\xi\in\mathbb{R}} \hat{F}(m,\xi)\,{}^{+}\hat{\eta}_2'(\xi,\mathbb{Z}_p)\,d\xi,$$

where $\hat{F}(m,\xi) = \displaystyle\int_{\substack{\theta\in[-\pi,\pi]\\ \tau\in\mathbb{R}}} F(e^z)\,e^{\sqrt{-1}\,(m\theta+\xi\tau)}\,d\theta d\tau.$

<u>Proof</u>. From the symmetry of F, we have

$$I(b) = \sum_{\ell_1,\ell_2\in\mathbb{Z}} \int_{\substack{\theta\in[-\pi,\pi]\\ \tau>0}} F(e^z)\,e^{-|\ell_1-\ell_2|\tau+\sqrt{-1}(\ell_1+\ell_2)\theta}\,d\theta d\tau.$$

We define

$$a_\ell(\tau) = e^{-|\ell|\tau}, \qquad b_m(\tau) = \int_{\theta\in[-\pi,\pi]} F(e^z)\,e^{\sqrt{-1}m\theta}\,d\theta \quad (\tau>0).$$

We sometimes denote $a_\ell(\tau)$, $b_m(\tau)$ by a_ℓ, b_m respectively.

Let $\alpha, \alpha_0, \beta, \beta_0$ be non negative integers.

$$\sum_{\ell_1=-\alpha_0}^{\alpha} \sum_{\ell_2=-\beta_0}^{\beta} a_{\ell_1-\ell_2}\, b_{\ell_1+\ell_2} = \sum_{\ell=-\alpha_0-\beta}^{\alpha+\beta_0} \sum_{\substack{m=u(\ell,\alpha_0,\beta_0)\\ m\equiv\ell\,(\mathrm{mod}\,2)}}^{v(\ell,\alpha,\beta)} a_\ell\, b_m \quad ,$$

where $u(\ell,\alpha_0,\beta_0) = \mathrm{Max}(-\ell-2\alpha_0, \ell-2\beta_0)$, $v(\ell,\alpha,\beta) = \mathrm{Min}(2\alpha-\ell, \ell+2\beta)$.

We put the limit of the sum of b_m as follows:

$$\nu(\mathbb{Z}_p) = \sum_{m\in\mathbb{Z}_p} b_m \qquad (p = 0,1),$$

$$\nu(k) = \begin{cases} \nu(\mathbb{Z}_0) & k \equiv 0 \ (\mathrm{mod}2), \\ \nu(\mathbb{Z}_1) & k \equiv 1 \ (\mathrm{mod}2). \end{cases}$$

Then there exist positive numbers M,K satisfying

$$\left| \sum_{\substack{m=u\\ m\equiv\ell\,(\mathrm{mod}2)}}^{v} b_m - \nu(\ell) \right| < M \ , \quad \text{for all } u < v,\ u,v\in\mathbb{Z};$$

$$|a_\ell| < K\ , \quad \text{for all } \ell\in\mathbb{Z}.$$

Because $\sum_{\ell\in\mathbb{Z}} a_\ell$ converges absolutely, for an arbitrary positive number ε, there exists a positive integer r satisfying

$$\sum_{\ell=-k}^{-r-1}\ \sum_{\ell=r+1}^{m} |a_\ell(\tau)| < \frac{\varepsilon}{2M} \qquad (k, m > r,\ \tau > 0).$$

For this ε and r, we can take a positive number s satisfying:

if $u(\ell,\alpha_0,\beta_0),\ v(\ell,\alpha,\beta) \geq s \quad (-r \leq \ell \leq r)$,

$$\left| \sum_{\substack{m=u(\ell,\alpha_0,\beta_0)\\ m\equiv \ell \pmod 2}}^{v(\ell,\alpha,\beta)} b_m - \nu(\ell) \right| < \frac{\varepsilon}{2K} \qquad (-r \leq \ell \leq r).$$

Then if we take sufficiently large positive integers $\alpha, \alpha_0, \beta, \beta_0$, we get

$$\left| \sum_{\ell_1=-\alpha_0}^{\alpha} \sum_{\ell_2=-\beta_0}^{\beta} a_{\ell_1-\ell_2} b_{\ell_1+\ell_2} - \sum_{\ell=-\alpha_0-\beta}^{\alpha+\beta_0} a_\ell \nu(\ell) \right|$$

$$\leq \sum_{\ell=-\alpha_0-\beta}^{\alpha+\beta_0} |a_\ell| \left| \sum_{\substack{m=u(\ell,\alpha_0,\beta_0)\\ m\equiv \ell \pmod 2}}^{v(\ell,\alpha,\beta)} b_m - \nu(\ell) \right|$$

$$= \left(\sum_{\ell=-r}^{r} + \sum_{\ell=-\alpha_0-\beta}^{-r-1}\ \sum_{\ell=r+1}^{\alpha+\beta_0} \right) |a_\ell| \left| \sum_{\substack{m=u(\ell,\alpha_0,\beta_0)\\ m\equiv \ell \pmod 2}}^{v(\ell,\alpha,\beta)} b_m - \nu(\ell) \right| < \frac{\varepsilon}{2K} K + \frac{\varepsilon}{2M} M = \varepsilon .$$

Hence we have

$$\sum_{\ell_1,\ell_2\in\mathbb{Z}} a_{\ell_1-\ell_2} b_{\ell_1+\ell_2} = \sum_{p=0,1}\ \sum_{\ell,m\in\mathbb{Z}_p} a_\ell b_m .$$

On the other hand,

$$\sum_{\ell,m\in\mathbb{Z}_p} a_\ell(\tau) b_m(\tau) = \begin{cases} \coth\tau \sum_{m\in\mathbb{Z}_0} b_m(\tau) & p = 0 \\ \operatorname{cosech}\tau \sum_{m\in\mathbb{Z}_1} b_m(\tau) & p = 1 \end{cases}$$

are integrable functions and there exist positive numbers N, τ_0 satisfying

$$\begin{cases} |a_\ell(\tau) b_m(\tau)| < N & (\ell, m \in \mathbb{Z},\ \tau > 0), \\ a_\ell(\tau) b_m(\tau) = 0 & (\ell, m \in \mathbb{Z},\ \tau > \tau_0). \end{cases}$$

Then from Lebesque's theorem and Fubini's theorem, we get

$$I(b) = \sum_{p=0,1} \sum_{m \in \mathbb{Z}_p} \int_{\theta \in [-\pi,\pi]} \left\{ \int_{\tau>0} F(e^z)\, c(m)\, d\tau \right\} e^{\sqrt{-1} m\theta} d\theta ,$$

where

$$c(m) = \begin{cases} \coth\tau & m \in \mathbb{Z}_0 \\ \operatorname{cosech}\tau & m \in \mathbb{Z}_1 . \end{cases}$$

we know

$$\int_{\tau>0} F(e^z) \coth\tau \, d\tau = \frac{1}{2} \int_{\tau \in \mathbb{R}} F(e^z) \coth\tau \, d\tau$$

$$= \frac{1}{4\sqrt{-1}} \int_{\xi \in \mathbb{R}} \hat{F}(\xi) \coth\pi\xi/2 \, d\xi ,$$

$$\int_{\tau>0} F(e^z) \operatorname{cosech}\tau \, d\tau = \frac{1}{2} \int_{\tau \in \mathbb{R}} F(e^z) \operatorname{cosech}\tau \, d\tau$$

$$= \frac{1}{4\sqrt{-1}} \int_{\xi \in \mathbb{R}} \hat{F}(\xi) \tanh\pi\xi/2 \, d\xi ,$$

where $\hat{F}(\xi) = \int_{\tau \in \mathbb{R}} F(e^z) e^{\sqrt{-1}\xi\tau} d\tau$. Applying these formulas and Fubini's theorem, we obtain the assertion of Lemma 2.4.

<u>Lemma</u> 2.5. For $F \in C_0^\infty(\mathbb{R}^* \times \mathbb{R}^*)$ ($\mathbb{R}^* = \mathbb{R} - \{0\}$) (the space of all infinitely differentiable functions which vanish outside of some compact set) satisfying $F(\varepsilon e^{t_1}, \varepsilon e^{t_2}) = -F(\varepsilon e^{-t_1}, \varepsilon e^{t_2}) = -F(\varepsilon e^{t_2}, \varepsilon e^{t_1})$, we get

(c-4) $$\sum_{S(\ell_1 \geq \ell_2 \geq 0)} \int_{t_1 > t_2 > 0} F(\varepsilon e^{t_1}, \varepsilon e^{t_2})\, {}^{+}\zeta'_2(\varepsilon e^{t_1}, \varepsilon e^{t_2}; \ell_1, \ell_2)\, dt_1\, dt_2$$

$$= \int_{\rho_1, \rho_2 \in \mathbb{R}} \hat{F}(\rho_1, \varepsilon; \rho_2, \varepsilon)\, {}^{+}\hat{\zeta}'_2(\rho_1, \varepsilon; \rho_2, \varepsilon)\, d\rho_1\, d\rho_2 ,$$

where

$$\hat{F}(\rho_1, \varepsilon; \rho_2, \varepsilon) = \int_{t_1, t_2 \in \mathbb{R}} F(\varepsilon e^{t_1}, \varepsilon e^{t_2})\, e^{\sqrt{-1}(\rho_1 t_1 + \rho_2 t_2)} dt_1\, dt_2 .$$

Proof. From the symmetry of F, we have

$$I(c) = 4 \sum_{(\ell_1, \ell_2) \in S(A)} \int_{t_1 > t_2 > 0} F(\varepsilon e^{t_1}, \varepsilon e^{t_2})\, e^{\ell_1 t_1 + \ell_2 t_2} \varepsilon^{\ell_1 + \ell_2}\, dt_1\, dt_2$$

where $A = \{(\ell_1, \ell_2) \in \mathbb{Z}^2 ; \ell_2 \geq 0 \geq \ell_1 + \ell_2 \text{ , or } 0 \geq \ell_1 \geq \ell_2 \}$.

Because F is infinitely differentiable and with compact support from Lebesque's theorem and Parseval's equality, we obtain

$$I(c) = 4 \int_{t_1 > t_2 > 0} F(\varepsilon e^{t_1}, \varepsilon e^{t_2}) \sum_{(\ell_1, \ell_2) \in S(A)} e^{\ell_1 t_1 + \ell_2 t_2} \varepsilon^{\ell_1 + \ell_2}\, dt_1\, dt_2$$

$$= \frac{4}{(2\pi)^2} \int_{\rho_1, \rho_2 \in \mathbb{R}} \hat{F}(\rho_1, \varepsilon; \rho_2, \varepsilon) \sum_{(\ell_1, \ell_2) \in S(A)} K(\rho_1, \rho_2; \ell_1, \ell_2; \varepsilon)\, d\rho_1\, d\rho_2 .$$

where

$$K(\rho_1, \rho_2; \ell_1, \ell_2; \varepsilon) = \int_{t_1 > t_2 > 0} \varepsilon^{\ell_1 + \ell_2}\, e(\ell_1 t_1 + \ell_2 t_2 - \sqrt{-1}(\rho_1 t_1 + \rho_2 t_2))\, dt_1\, dt_2$$

$$= \frac{\varepsilon^{\ell_1 + \ell_2}}{(\ell_1 - \sqrt{-1}\rho_1)(\ell_1 + \ell_2 - \sqrt{-1}(\rho_1 + \rho_2))} .$$

The Weyl group W of type C_2 is realized as the set of the following transformations on $\mathbb{R}^2$: $w_1(a,b) = (a,b)$, $w_2(a,b) = (-a,b)$, $w_3(a,b) = (a,-b)$, $w_4(a,b) = (-a,-b)$, $w_5(a,b) = (b,a)$, $w_6(a,b) = (-b,a)$, $w_7(a,b) = (b,-a)$, $w_8(a,b) = (-b,-a)$. For $w \in W$, we define $S(w)$, as follows:

$$\frac{\varepsilon^{\ell_1 + \ell_2}}{(\ell_1 - \sqrt{-1}w\rho_1)(\ell_2 - \sqrt{-1}w\rho_2)} = S(w) \frac{\varepsilon^{w^{-1}(\ell_1 + \ell_2)}}{(w^{-1}\ell_1 - \sqrt{-1}\rho_1)(w^{-1}\ell_2 - \sqrt{-1}\rho_2)}$$

Then we have

$$S(w_j) = \begin{cases} 1 & j = 1,4,5,8, \\ -1 & j = 2,3,6,7 . \end{cases}$$

$$\sum_{(\ell_1,\ell_2)\in S(A)} \sum_{w\in W} \frac{\varepsilon^{\ell_1+\ell_2}\,\mathrm{sgn}w}{(\ell_1-\sqrt{-1}w\rho_1)(\ell_2-\sqrt{-1}w\rho_2)} = \sum_{(\ell_1,\ell_2)\in S(A)} \sum_{w\in W} \frac{\varepsilon^{w^{-1}(\ell_1+\ell_2)}\,\mathrm{sgn}w\ S(w)}{(w^{-1}\ell_1-\sqrt{-1}\rho_1)(w^{-1}\ell_2-\sqrt{-1}\rho_2)}$$

$$= \sum_{(\ell_1,\ell_2)\in S(A)} \left\{ \sum_{j=1,2,3,4} \frac{\varepsilon^{w_j^{-1}(\ell_1+\ell_2)}}{(w_j^{-1}\ell_1-\sqrt{-1}\rho_1)(w_j^{-1}\ell_2-\sqrt{-1}\rho_2)} - \sum_{j=5,6,7,8} \frac{\varepsilon^{w_j^{-1}(\ell_1+\ell_2)}}{(w_j^{-1}\ell_1-\sqrt{-1}\rho_1)(w_j^{-1}\ell_2-\sqrt{-1}\rho_2)} \right\}$$

$$= \sum_{(\ell_1,\ell_2)\in \mathbb{Z}^2} \frac{\varepsilon^{\ell_1+\ell_2}}{(\ell_1-\sqrt{-1}\rho_1)(\ell_2-\sqrt{-1}\rho_2)} \sum_{(\ell_1,\ell_2)\in \mathbb{Z}^2} \frac{\varepsilon^{\ell_1+\ell_2}}{(\ell_1-\sqrt{-1}\rho_1)(\ell_2-\sqrt{-1}\rho_2)} = 0 .$$

From this equality, we have

$$\sum_{(\ell_1,\ell_2)\in S(A)} \sum_{j=1,4,5,8} \sum_{w\in W} \frac{\mathrm{sgn}w_j w\ \varepsilon^{\ell_1+\ell_2}}{(\ell_1-\sqrt{-1}ww_j\rho_1)(\ell_1+\ell_2-\sqrt{-1}ww_j(\rho_1+\rho_2))}$$

$$= \sum_{(\ell_1,\ell_2)\in S(A)} \Bigg[\sum_{j=5,8} \sum_{w\in W} \frac{-\mathrm{sgn}w\ \varepsilon^{\ell_1+\ell_2}}{(w_j^{-1}\ell_2-\sqrt{-1}w\rho_2)(w_j^{-1}(\ell_1+\ell_2)-\sqrt{-1}w(\rho_1+\rho_2))}$$

$$+ \sum_{j=1,4} \sum_{w\in W} \left\{ \frac{\mathrm{sgn}\ w_j w\ \varepsilon^{\ell_1+\ell_2}}{(\ell_1-\sqrt{-1}ww_j\rho_1)(\ell_2-\sqrt{-1}ww_j\rho_2)} - \frac{\mathrm{sgn}\ w_j w\ \varepsilon^{\ell_1+\ell_2}}{(\ell_2-\sqrt{-1}ww_j\rho_2)(\ell_1+\ell_2-\sqrt{-1}ww_j(\rho_1+\rho_2))} \right\} \Bigg]$$

$$= \sum_{(\ell_1,\ell_2)\in S(A)} \sum_{j=1,4,5,8} \sum_{w\in W} \frac{-\mathrm{sgn}\ w\ \varepsilon^{\ell_1+\ell_2}}{(w_j^{-1}\ell_2-\sqrt{-1}w\rho_2)(w_j^{-1}(\ell_1+\ell_2)-\sqrt{-1}w(\rho_1+\rho_2))}$$

$$= \sum_{(\ell_1,\ell_2)\in \mathbb{Z}^2} \sum_{w\in W} \frac{-\mathrm{sgn}\ w\ \varepsilon^{\ell_1+\ell_2}}{(\ell_2-\sqrt{-1}w\rho_2)(\ell_1+\ell_2-\sqrt{-1}w(\rho_1+\rho_2))} .$$

Then we obtain

$$I(c)$$

$$= \frac{4}{(2\pi)^2} \int_{\rho_1,\rho_2\in\mathbb{R}} \hat{F}(\rho_1,\varepsilon;\rho_2,\varepsilon)\ \frac{1}{4|W|} \sum_{\ell_1,\ell_2\in S(A)} \sum_{j=1,4,5,8} \sum_{w\in W} \frac{\mathrm{sgn}\ w_j w\ \varepsilon^{\ell_1+\ell_2}}{(\ell_1-\sqrt{-1}ww_j\rho_1)(\ell_1+\ell_2-\sqrt{-1}ww_j(\rho_1+\rho_2))} d\rho_1 d\rho_2$$

$$= \frac{4}{(2\pi)^2} \int_{\rho_1,\rho_2\in\mathbb{R}} \hat{F}(\rho_1,\varepsilon;\rho_2,\varepsilon)\ \frac{1}{4|W|} \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \sum_{w\in W} \frac{-\mathrm{sgn}\ w\ \varepsilon^{\ell_1+\ell_2}}{(\ell_2-\sqrt{-1}w\rho_2)(\ell_1+\ell_2-\sqrt{-1}w(\rho_1+\rho_2))}\ d\rho_1 d\rho_2$$

$$= \frac{1}{(2\pi)^2} \int_{\rho_1,\rho_2\in\mathbb{R}} \hat{F}(\rho_1,\varepsilon;\rho_2,\varepsilon) \sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \frac{\varepsilon^{\ell_1+\ell_2}}{(\rho_2+\sqrt{-1}\ell_2)(\rho_1+\rho_2+\sqrt{-1}(\ell_1+\ell_2))}\ d\rho_1 d\rho_2 .$$

$$\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \frac{\varepsilon^{\ell_1+\ell_2}}{(\rho_2+\sqrt{-1}\ell_1)(\rho_1+\rho_2+\sqrt{-1}(\ell_1+\ell_2))}$$

$$= \sum_{\ell_1\in\mathbb{Z}}\sum_{\ell_2\in\mathbb{Z}} \frac{\varepsilon^{\ell_1+\ell_2}}{(\rho_1+\sqrt{-1}\ell_1)(\rho_1+\rho_2+\sqrt{-1}(\ell_1+\ell_2))}$$

$$= \begin{cases} \sum_{\ell_1\in\mathbb{Z}} \frac{1}{\rho_2+\sqrt{-1}\ell_1}\,\pi\ \coth\pi(\rho_1+\rho_2) & \varepsilon = 1 \\ \sum_{\ell_1\in\mathbb{Z}} \frac{1}{\rho_2+\sqrt{-1}\ell_1}\,\pi\ \operatorname{cosech}\pi(\rho_1+\rho_2) & \varepsilon = -1 \end{cases}$$

$$= \begin{cases} \pi^2 \coth\pi\rho_2\ \coth\pi(\rho_1+\rho_2) & \varepsilon = 1 \\ \pi^2 \coth\pi\rho_2\ \operatorname{cosech}\pi(\rho_1+\rho_2) & \varepsilon = -1. \end{cases}$$

Hence, we obtain the assertion of Lemma 2.5.

<u>Lemma</u> 2.6. (1) For $F\in C_0^\infty(\mathbb{R}^*\times\mathbb{R}^*)$ satisfying $F(\varepsilon e^{t_1},-\varepsilon e^{t_2}) = -F(\varepsilon e^{-t_1},-\varepsilon e^{t_2}) = -F(-\varepsilon e^{t_2},\varepsilon e^{t_1})$, we get

$$\text{(d-4)}\qquad \sum_{S(\ell_1\geq\ell_2\geq 0)} \int_{t_1,t_2>0} F(\varepsilon e^{t_1},-\varepsilon e^{t_2})\,{}^{+}\chi_2'(\varepsilon e^{t_1},-\varepsilon e^{t_2};\ell_1,\ell_2)\ dt_1 dt_2$$

$$= \int_{\rho_1,\rho_2\in\mathbb{R}} \hat{F}(\rho_1,\varepsilon;\rho_2,-\varepsilon)\,{}^{+}\hat{\chi}'(\rho_1,\varepsilon;\rho_2,-\varepsilon)\ d\rho_1 d\rho_2 .$$

(2) For $F\in C_0^\infty(\mathbb{R}^*)$ satisfying $F(\varepsilon e^{t_1}) = -F(\varepsilon e^{-t_1})$, we get

$$\text{(d}'\text{-4)}\qquad \sum_{S(\ell_1\geq 0)} \int_{t_1>0} F(\varepsilon_1 e^{t_1})\ {}^{+}\chi'(\varepsilon_1 e^{t_1};\ell_1)\ dt_1$$

$$= \int_{\rho_1\in\mathbb{R}} \hat{F}(\rho_1,\varepsilon_1)\,{}^{+}\hat{\chi}'(\rho_1,\varepsilon_1)\ d\rho_1 .$$

<u>Proof</u>. From the symmetry of F, we have

$$I(d) = 4\sum_{(\ell_1,\ell_2)\in S(A)} \int_{t_1,t_2>0} F(\varepsilon e^{t_1},-\varepsilon e^{t_2})\ e^{\ell_1 t_1+\ell_2 t_2}\ \varepsilon^{\ell_1}\ (-\varepsilon)^{\ell_2}\ dt_1\, dt_2$$

where

$$A = \{ (\ell_1,\ell_2)\in\mathbb{Z}^2 ;\ \ell_1,\ell_2\leq 0 \} .$$

Because F is infinitely differentiable and with compact support, from Lebesque's theorem and Parseval's equality, we get

I(d)

$$= 4 \int_{t_1, t_2 > 0} F(\varepsilon e^{t_1}, -\varepsilon e^{t_2}) \sum_{(\ell_1, \ell_2) \in S(A)} e^{\ell_1 t_1 + \ell_2 t_2} \varepsilon^{\ell_1} (-\varepsilon)^{\ell_2} \, dt_1 \, dt_2$$

$$= \frac{4}{(2\pi)^2} \int_{\rho_1, \rho_2 \in \mathbb{R}} \hat{F}(\rho_1, \varepsilon; \rho_2, -\varepsilon) \sum_{(\ell_1, \ell_2) \in S(A)} K(\rho_1, \rho_2; \ell_1, \ell_2; \varepsilon) \, d\rho_1 d\rho_2 ,$$

where

$$K(\rho_1, \rho_2; \ell_1, \ell_2; \varepsilon) = \int_{t_1, t_2 > 0} \varepsilon^{\ell_1} (-\varepsilon)^{\ell_2} \, e(\ell_1 t_1 + \ell_2 t_2 - \sqrt{-1}(\rho_1 t_1 + \rho_2 t_2)) \, dt_1 \, dt_2$$

$$= \frac{\varepsilon^{\ell_1} (-\varepsilon)^{\ell_2}}{(\ell_1 - \sqrt{-1}\rho_1)(\ell_2 - \sqrt{-1}\rho_2)} .$$

Let W have the same meaning as in the proof of Lemma 2.5. Put $W_0 = \{w_1, w_2, w_3, w_4\}$, then W_0 is a subgroup of W.

$$\sum_{(\ell_1, \ell_2) \in S(A)} \sum_{w, w' \in W_0} \frac{\operatorname{sgn} ww' \, \varepsilon^{\ell_1} (-\varepsilon)^{\ell_2}}{(\ell_1 - \sqrt{-1}\, ww'\rho_1)(\ell_2 - \sqrt{-1}\, ww'\rho_2)}$$

$$= \sum_{(\ell_1, \ell_2) \in S(A)} \sum_{w' \in W_0} \sum_{w \in W_0} \frac{\operatorname{sgn} w \, \varepsilon^{w'\ell_1} (-\varepsilon)^{w'\ell_2}}{(w'\ell_1 - \sqrt{-1}\, w\rho_1)(w'\ell_2 - \sqrt{-1}\, w\rho_2)}$$

$$= \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \sum_{w \in W_0} \frac{\operatorname{sgn} w \, \varepsilon^{\ell_1} (-\varepsilon)^{\ell_2}}{(\ell_1 - \sqrt{-1}\, w\rho_1)(\ell_2 - \sqrt{-1}\, w\rho_2)} .$$

Using this equality, we have

I(d)

$$= \frac{4}{(2\pi)^2} \int_{\rho_1, \rho_2 \in \mathbb{R}} \hat{F}(\rho_1, \varepsilon; \rho_2, -\varepsilon) \frac{1}{|W_0|^2} \sum_{(\ell_1, \ell_2) \in S(A)} \sum_{w, w' \in W_0} \frac{\operatorname{sgn} ww' \, \varepsilon^{\ell_1} (-\varepsilon)^{\ell_2}}{(\ell_1 - \sqrt{-1}\, ww'\rho_1)(\ell_2 - \sqrt{-1}\, ww'\rho_2)} \, d\rho_1 d\rho_2$$

$$= \frac{4}{(2\pi)^2} \int_{\rho_1, \rho_2 \in \mathbb{R}} \hat{F}(\rho_1, \varepsilon; \rho_2, -\varepsilon) \frac{1}{|W_0|^2} \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \sum_{w \in W_0} \frac{\operatorname{sgn} w \, \varepsilon^{\ell_1} (-\varepsilon)^{\ell_2}}{(\ell_1 - \sqrt{-1}\, w\rho_1)(\ell_2 - \sqrt{-1}\, w\rho_2)} \, d\rho_1 d\rho_2$$

$$= -\frac{1}{(2\pi)^2} \int_{\rho_1, \rho_2 \in \mathbb{R}} \hat{F}(\rho_1, \varepsilon; \rho_2, -\varepsilon) \sum_{(\ell_1, \ell_2) \in \mathbb{Z}^2} \frac{\varepsilon^{\ell_1} (-\varepsilon)^{\ell_2}}{(\rho_1 + \sqrt{-1}\,\ell_1)(\rho_2 + \sqrt{-1}\,\ell_2)} \, d\rho_1 d\rho_2 .$$

$$\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2} \frac{\varepsilon^{\ell_1}(-\varepsilon)^{\ell_2}}{(\rho_1+\sqrt{-1}\ell_1)(\rho_2+\sqrt{-1}\ell_2)} = \begin{cases} \pi^2 \coth\pi\rho_1 \operatorname{cosech}\pi\rho_2 & \varepsilon = 1 \\ \pi^2 \operatorname{cosech}\pi\rho_1 \coth\pi\rho_2 & \varepsilon = -1 . \end{cases}$$

Hence, we obtain the assertion (1) of Lemma 2.6. We obtain easily the assertion (2) of Lemma 2.6.

Lemma 2.7. Let $F \in C_0(\mathbb{C}^*)$ be as in Lemma 2.4. We get

(e-4) $$\sum_{S(m\geq 0)} \int_{\substack{\theta\in[-\pi,\pi] \\ \tau>0}} F(e^{z})\, H_2'(e^{z}, e^{\bar{z}}; \lambda_1, \lambda_2)\, d\theta d\tau = \hat{F}(m, \xi).$$

Proof. From the symmetry of F, we obtain the assertion.

Lemma 2.8. For $F \in C_0^\infty(\mathbb{R}^*\times\mathbb{R}^*)$ satisfying the symmetry of Lemma 2.5, we get

(f-4) $$\sum_{\substack{S(m\geq 0) \\ m\in\mathbb{Z}_p}} \int_{t_1>t_2>0} F(\varepsilon e^{t_1}, \varepsilon e^{t_2})\, Z_2'(\varepsilon e^{t_1}, \varepsilon e^{t_2}; \lambda_1, \lambda_2)\, dt_1 dt_2 = \int_{\eta\in\mathbb{R}} \hat{F}\left(\frac{\eta+\xi}{2}, \varepsilon; \frac{-\eta+\xi}{2}, \varepsilon\right) \hat{Z}_2'(\eta; \varepsilon; \mathbb{Z}_p)\, d\eta \qquad (p = 0,1),$$

where

$$\hat{F}\left(\frac{\eta+\xi}{2}, \varepsilon; \frac{-\eta+\xi}{2}, \varepsilon\right) = \int_{t_1,t_2\in\mathbb{R}} F(\varepsilon e^{t_1}, \varepsilon e^{t_2})\, e\left(\frac{\sqrt{-1}(\eta+\xi)t_1}{2} + \frac{\sqrt{-1}(-\eta+\xi)t_2}{2}\right) dt_1 dt_2 .$$

Proof. Put $\sigma = \frac{t_1-t_2}{2}$, $\tau = \frac{t_1+t_2}{2}$. Then from the symmetry of F, we have

$$I(f) = 2\sum_{m\in\mathbb{Z}_p} \int_{\sigma>0,\tau\in\mathbb{R}} F(\varepsilon e^{\tau+\sigma}, \varepsilon e^{\tau-\sigma})\, e^{-m\sigma+\sqrt{-1}\xi\tau} \varepsilon^m\, d\sigma\, d\tau .$$

$$\sum_{m\in\mathbb{Z}_p} e^{-m\sigma}\varepsilon^m = \begin{cases} \coth\sigma & p = 0 \\ \varepsilon \operatorname{cosech}\sigma & p = 1 . \end{cases}$$

Because F is infinitely differentiable and with compact support, from Lebesque's theorem and Parseval's equality, we obtain

$$I(f) = \begin{cases} 2 \displaystyle\int_{\sigma>0,\tau\in\mathbb{R}} F(\varepsilon e^{\tau+\sigma}, \varepsilon e^{\tau-\sigma}) \coth\sigma \, e^{\sqrt{-1}\xi\tau} \, d\sigma \, d\tau & p = 0 , \\ 2 \displaystyle\int_{\sigma>0,\tau\in\mathbb{R}} F(\varepsilon e^{\tau+\sigma}, \varepsilon e^{\tau-\sigma}) \, \varepsilon \, \mathrm{cosech}\sigma \, e^{\sqrt{-1}\xi\tau} \, d\sigma \, d\tau & p = 1 , \end{cases}$$

$$= \begin{cases} -\dfrac{\sqrt{-1}}{2} \displaystyle\int_{\eta,\xi\in\mathbb{R}} \int_{\sigma,\tau\in\mathbb{R}} F(\varepsilon e^{\tau+\sigma}, \varepsilon e^{\tau-\sigma}) e^{\sqrt{-1}(\eta\sigma+\xi\tau)} d\sigma \, d\tau \, \coth\dfrac{\pi\eta}{2} \, d\eta \, d\xi & p = 0, \\ -\dfrac{\sqrt{-1}}{2} \displaystyle\int_{\eta,\xi\in\mathbb{R}} \int_{\sigma,\tau\in\mathbb{R}} F(\varepsilon e^{\tau+\sigma}, \varepsilon e^{\tau-\sigma}) e^{\sqrt{-1}(\eta\sigma+\xi\tau)} d\sigma \, d\tau \, \varepsilon \tanh\dfrac{\pi\eta}{2} \, d\eta \, d\xi & p = 1. \end{cases}$$

By the change of the variables $t_1 = \tau+\sigma$, $t_2 = \tau-\sigma$, we obtain the assertion of Lemma 2.8.

Lemma 2.9. Let $F \in C_0^\infty((\mathbb{R}^*)^p)$ satisfy the following symmetry:

$$\varepsilon(*) \quad \begin{cases} F(\ldots \varepsilon_i e^{t_i} \ldots) = -F(\ldots \varepsilon_i e^{-t_i} \ldots) & (1 \le i \le p), \\ F(\ldots \varepsilon_i e^{t_i} \ldots \varepsilon_j e^{t_j} \ldots) = -(\varepsilon_i \varepsilon_j)^{j-i} F(\ldots \varepsilon_j e^{t_j} \ldots \varepsilon_i e^{t_i} \ldots) & (1 \le i \le j \le p). \end{cases}$$

For $0 \le q \le p$, we put $\delta_i = \varepsilon_i e^{t_i}$ $(1 \le i \le p)$, where $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_q = 1$, $\varepsilon_{q+1} = \varepsilon_{q+2} = \ldots = \varepsilon_p = -1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be integers satisfying $\{\alpha_1, \alpha_2, \ldots, \alpha_p\} = \{1, 2, \ldots, p\}$. Then we get

$$\text{(g-4)} \quad \int_{t_1,t_2,\ldots,t_p>0} F(\delta_{\alpha_1}, \delta_{\alpha_2}, \ldots, \delta_{\alpha_p}) \, \Xi_p'(\delta_{\alpha_1}, \delta_{\alpha_2}, \ldots, \delta_{\alpha_p} : \lambda_1, \lambda_2, \ldots, \lambda_p) \prod_{i=1}^{p} \varepsilon_i^{p-i+1} \, dt_1 \, dt_2 \cdots dt_p$$

$$= \int_{t_1,t_2,\ldots,t_p>0} F(\delta_1, \delta_2, \ldots, \delta_p) \, \Xi_p'(\delta_1, \delta_2, \ldots, \delta_p : \lambda_1, \lambda_2, \ldots, \lambda_p) \prod_{i=1}^{p} \varepsilon_i^{p-i+1} \, dt_1 \, dt_2 \cdots dt_p$$

$$= p! \int_{t_1,t_2,\ldots,t_p\in\mathbb{R}} F(\delta_1, \delta_2, \ldots, \delta_p) \, e(\sqrt{-1}(\rho_1 t_1 + \rho_2 t_2 + \cdots + \rho_p t_p)) \prod_{i=1}^{p} \varepsilon_i^{\varepsilon_i+p-i+1} dt_1 \, dt_2 \cdots dt_p .$$

Proof. Because $\Xi_p' \prod_{i=1}^{p} \varepsilon_i^{p-i+1}$ satisfies the symmetry $\varepsilon(*)$, we have the first equality of (g-4). Using the symmetry $\varepsilon(*)$ of F. We obtain the second equality.

For $f \in C_0^\infty(G)$, we define the Fourier transformation of $F_f^{k,\ell}$ as follows:

$$(36) \quad \hat{F}_f^{k,\ell}(\ell_1,\ell_2,\ldots,\ell_k;m_1,\xi_1,\ldots,m_\ell,\xi_\ell;\rho_1,\varepsilon_1,\rho_2,\varepsilon_2,\ldots,\rho_m,\varepsilon_m)$$

$$= \int_{\substack{\phi_1,\phi_2,\ldots,\phi_k\in[-\pi,\pi]\\ \theta_1,\ldots,\theta_\ell\in[-\pi,\pi]\\ \tau_1,\ldots,\tau_\ell,t_1,t_2,\ldots,t_m\in\mathbb{R}}} F_f^{k,\ell}(e^{\sqrt{-1}\phi_1},e^{\sqrt{-1}\phi_2},\ldots,e^{\sqrt{-1}\phi_k},e^{z_1},e^{\bar{z}_1},\ldots,e^{z_\ell},e^{\bar{z}_\ell},\varepsilon_1 e^{t_1},\varepsilon_2 e^{t_2},\ldots,\varepsilon_m e^{t_m})\times$$
$$e^{(\sqrt{-1}(\ell_1\phi_1+\ell_2\phi_2+\cdots+\ell_k\phi_k+m_1\theta_1+\cdots+m_\ell\theta_\ell+\xi_1\tau_1+\cdots+\xi_\ell\tau_\ell+\rho_1 t_1+\rho_2 t_2+\cdots+\rho_m t_m)} \, d\phi_1\, d\phi_2\ldots d\phi_k\, d\theta_1\ldots d\theta_\ell\, d\tau_1\ldots d\tau_\ell\, dt_1\, dt_2\ldots dt_m.$$

<u>Lemma</u> 2.10. For $f\in C_0^\infty(G)$, we define $F_f^{k',\ell'}$ and $\hat{F}_f^{k',\ell'}$ as in (29), (36) respectively. And we put $({}^+K_j^{k',\ell'})'$ as in (32). Then we obtain

$$(37) \quad \sum_{\substack{S(\ell_1\geq\ell_2\geq\cdots\geq\ell_k\geq 0)\\ S(m_1\geq 0)\cdots S(m_\ell\geq 0)}} \int_{(H^j_{k',\ell'})'} F_f^{k',\ell'}(h)\,({}^+K_j^{k',\ell'})'(h)\,dh$$

$$= (-1)^{k'm+k(k+1)/2} \sum_{p=0}^{\mathrm{Min}([\frac{k-k'}{2}],\ell')} \sum_{j'=\mathrm{Max}(j-m'+a,0)}^{\mathrm{Min}(j,a)} \sum_{u+v=\ell-q} \sum_{\substack{r_{q+1},\ldots,r_\ell=0,1\\ r_{\ell+1},\ldots,r_{\ell+p}=0,1}} \sum_{\substack{m_1,\ldots,m_q\in\mathbb{Z}\\ m_{\ell+1}\in\mathbb{Z}_{r_{\ell+1}},\ldots,m_{\ell+p}\in\mathbb{Z}_{r_{\ell+p}}}} \frac{m'!\,m!\,2^{\ell'}\,\ell'!\,\ell!}{J!\,I!\,u!\,v!}$$
$$\int_{\circledast} \hat{F}_f^{k',\ell'}(\ell_1,\ell_2,\ldots,\ell_{k'};m_1,\xi_1,\ldots,m_q,\xi_q,m_{\ell+1},\xi_{\ell+1},\ldots,m_{\ell+p},\xi_{\ell+p};\circledast)\times$$
$$\prod_{1\leq i\leq p} {}^+\hat{\eta}_2'(\xi_{\ell+i};\mathbb{Z}_{r_{\ell+i}}) \prod_{1\leq i\leq J} {}^+\hat{\xi}_2'(\rho_{2i-1},\varepsilon_{2i-1};\rho_{2i},\varepsilon_{2i})\times$$
$$\prod_{1\leq i\leq I} {}^+\hat{\zeta}_2'(\rho_{j+2i-1},\varepsilon_{j+2i-1};\rho_{j+2i},\varepsilon_{j+2i})\ {}^+\hat{\chi}'(\rho_{w_0},\varepsilon_{w_0};\rho_{x_0},\varepsilon_{x_0})\times$$
$$\prod_{1\leq i\leq u} \hat{Z}_2'(\eta_{q+i},\varepsilon_{j'+2i-1};\mathbb{Z}_{r_{q+i}}) \prod_{1\leq i\leq v} \hat{Z}_2'(\eta_{q+u+i};\varepsilon_{j+a-j'+2i-1};\mathbb{Z}_{r_{q+u+i}})\times$$
$$\prod_{1\leq i\leq m} \varepsilon_i^{\epsilon_i+m'+k'+1-i} \prod_{1\leq i\leq j'} \varepsilon_i^{k-2p+1-i} \prod_{1\leq i\leq a-j'} \varepsilon_{j+2i-1}^{k-2p+1-(j'+i)} \prod_{1\leq i\leq u} \varepsilon_{j'+2i-1} \prod_{1\leq i\leq v} \varepsilon_{j+a-j'+2i-1}$$
$$d\rho_1\ldots d\rho_a \quad d\xi_{\ell+1}\ldots d\xi_{\ell+p}\ d\eta_{q+1}\ldots d\eta_\ell,$$

where

$$\circledast = (\rho_1,\varepsilon_1,\ldots,\rho_{j'},\varepsilon_{j'};\frac{\eta_{q+1}+\xi_{q+1}}{2},\varepsilon_{j'+1},\frac{-\eta_{q+1}+\xi_{q+1}}{2},\varepsilon_{j'+2},\ldots,\frac{\eta_{q+u}+\xi_{q+u}}{2},\varepsilon_{j'+2u-1},$$
$$\frac{-\eta_{q+u}+\xi_{q+u}}{2},\varepsilon_{j'+2u};\rho_{j'+2u+1},\varepsilon_{j'+2u+1},\ldots,\rho_j,\varepsilon_j,\rho_{j+1},\varepsilon_{j+1},\ldots,\rho_{j+a-j'},\varepsilon_{j+a-j'};$$
$$\frac{\eta_{q+u+1}+\xi_{q+u+1}}{2},\varepsilon_{j+a-j'+1},\frac{-\eta_{q+u+1}+\xi_{q+u+1}}{2},\varepsilon_{j+a-j'+2},\ldots,\frac{\eta_\ell+\xi_\ell}{2},\varepsilon_{j+a-j'+2v-1},\frac{-\eta_\ell+\xi_\ell}{2},$$
$$\varepsilon_{j+a-j'+2v};\rho_{j+a-j'+2v+1},\varepsilon_{j+a-j'+2v+1},\ldots,\rho_{m'},\varepsilon_{m'});$$

$\circledast$ is the following domain,

$$(\rho_1,\ldots,\rho_{k-k'-2p},\xi_{\ell+1},\ldots,\xi_{\ell+p},\eta_{q+1},\ldots,\eta_\ell)\in\mathbb{R}^{\ell+m'-(\ell+m)};$$

$$w_0 = \begin{cases} j' & 2J = j'-1 \\ \phi & 2J = j' \end{cases}, \qquad x_0 = \begin{cases} k-k'-2p-j' & 2I = k-k'-2p-j'-1 \\ \phi & 2I = k-k'-2p-j' \end{cases},$$

$a = k-k'-2p.$

Proof. We apply Lemma 2.3,..., Lemma 2.9 to Lemma 2.2. Then we obtain the assertion.

§3. Plancherel formula

Let dg be the normalized Haar measure on G ([1(c)], p.115). Let dh be the Haar measure on $H_{k,\ell}$ defined by (10). Then put the left invariant measure $d_{k,\ell}\bar{g}$ on $G/H_{k,\ell}$ as follows:

$$dg = d_{k,\ell}\bar{g}\, dh.$$

Using this measure, we define the Harish-Chandra transform on $H'_{k,\ell}$ as in (7). Recall, for $f \in C_0^\infty(G)$,

$$(38) \qquad K_f^{k,\ell}(h) = \varepsilon_R^{k,\ell}(h) \operatorname{conj} \Delta^{k,\ell}(h) \int_{G/H_{k,\ell}} f(ghg^{-1})\, d_{k,\ell}\bar{g} \qquad (h \in H'_{k,\ell}).$$

Then we have

$$(39) \qquad \int_{G} f(g)\, dg = \sum_{k+2\ell \le n} \alpha^{k,\ell} \int_{H'_{k,\ell}} K_f^{k,\ell}(h)\, \varepsilon_R^{k,\ell}(h)\, \Delta^{k,\ell}(h)\, dh,$$

where $\alpha^{k,\ell} = 1/|W^{k,\ell}|$.

Lemma 3.1. For $f \in C_0^\infty(G)$, we put $K_f^{n,0}$ as in (38). Then we get

$$(40) \qquad L^{n,0} K_f^{n,0}(e) = (-1)^q c f(e),$$

where $q = n(n+3)/2$, $c = 2^{3n/2} \pi^{n(n+1)/2} \prod_{1 \le p \le n} p!\, (n+1)^{-n(n-1)/2}$ and e is the identity element of G.

Proof. Since $\operatorname{conj} \Delta^{n,0}(h) = (-1)^n \Delta^{n,0}(h)$, we apply ([1(c)], Lemma 17.5) to G. We get the assertion of Lemma 3.1.

We call (ω,ν) of type (k,ℓ) if (ω,ν) is parameterized by (21). For (ω,ν) of type (k,ℓ), we take the invariant eigendistribution $\Theta_{k,\ell}(\omega,\nu)$ defined by (23). We sometimes denote $\Theta_{k,\ell}(\omega,\nu)$ by $\Theta(\omega,\nu)$.

Put the analytic function $K_{k,\ell}^{k',\ell'}(\omega,\nu)$ on $H'_{k',\ell'}$ as follows:

$$(41) \qquad K_{k,\ell}^{k',\ell'}(\omega,\nu)(h) = \varepsilon_R^{k',\ell'}(h)\ \Delta^{k',\ell'}(h)\ \Theta_{k,\ell}^{k',\ell'}(\omega,\nu)(h) \qquad (h \in H'_{k',\ell'}).$$

Then we have from (39),

$$(42) \qquad (\Theta_{k,\ell}(\omega,\nu),\ f) = \int_{G'} f(g)\ \Theta_{k,\ell}(\omega,\nu)\ dg$$

$$= \sum_{k+2\ell\leq n} \alpha^{k',\ell'} \int_{H'_{k\ell}} K_f^{k',\ell'}(h)\ K_{k,\ell}^{k',\ell'}(\omega,\nu)\ dh.$$

There exists a constant α such that $\left|K_{k,\ell}^{k',\ell'}(\omega,\nu)(h)\right| \leq \alpha$ for any k',ℓ' and (ω,ν). Then for any (ω,ν), we obtain

$$(43) \qquad |(\Theta(\omega,\nu),f)| < \alpha M_f \qquad (f \in C_0^\infty(G)),$$

where

$$M_f = \sum_{k+2\ell\leq n} \alpha^{k',\ell'} \int_{H'_{k',\ell'}} \left|K_f^{k',\ell'}(h)\right|\ dh.$$

Let W_n be the Weyl group of type C_n realized as the permutation group on $\mathbb{R}^n$. Let $\mathfrak{S}^n$ be the set of all polynomials of n indeterminates. And put the subset $\mathfrak{S}_0^n$ of $\mathfrak{S}^n$ as $\{ p \in \mathfrak{S}^n;\ wp = p\ (\forall w \in W_n) \}$. When $X = (X_1, X_2, \ldots, X_n)$ are differential operators on $H_{k,\ell}$ defined by (8), we consider $P(X)$ $(P \in \mathfrak{S}_0^n)$ are differential operators on $H_{k,\ell}$. For any Laplace operator D on G, there exists a unique $P_D \in \mathfrak{S}_0^n$ such that for any $f \in C_0^\infty(G)$ and (ω,ν),

$$(44) \qquad K_{Df}^{k',\ell'} = P_D(X)\ K_f^{k',\ell'},\quad D\ \Theta_{k,\ell}(\omega,\nu) = P_D(\omega,\nu)\Theta_{k,\ell}(\omega,\nu),$$

where

$$(45)\quad P_D(\omega,\nu) = P_D\left(\ell_1,\ell_2,\ldots,\ell_k,\frac{m_1+\sqrt{-1}\,\xi_1}{2},\frac{m_1-\sqrt{-1}\,\xi_1}{2},\ldots,\frac{m_\ell+\sqrt{-1}\,\xi_\ell}{2},\frac{m_\ell-\sqrt{-1}\,\xi_\ell}{2},\sqrt{-1}\rho_1,\sqrt{-1}\rho_2,\ldots,\sqrt{-1}\rho_m\right)$$

(46) $X = (X_1, X_2, \ldots, X_n)$ are differential operators on $H_{k',\ell'}$ ([2(c)], §6). Hence from (43), (44), we have that for any $f\in C_o^\infty(G)$ and $P\in \mathfrak{S}_o^n$, there exists a constant $M_{D,f}$ such that for any (ω,ν),

$$(47)\quad |P(\omega,\nu)(\Theta(\omega,\nu),f)| \le M_{D,f}.$$

<u>Lemma</u> 3.2. Let $f\in C_o^\infty(G)$. For any positive integer N, there exists a positive constant $M_{N,f}$ such that for any (ω,ν),

$$(48)\quad (1+|(\omega,\nu)|^2)^N\,|(\Theta(\omega,\nu),f)| \le M_{N,f}$$

<u>Proof</u>. From (47), we get the assertion as in ([2(b)], Lemma 3.2) corollary. Let (ω,ν) be type (k,ℓ). The series

$$\sum_{\ell_1,\ldots,\ell_k\in\mathbb{Z}}\ \sum_{m_1,\ldots,m_\ell\in\mathbb{Z}}\ \sum_{\epsilon_1,\ldots,\epsilon_m} (\Theta(\omega(\ell_1,\ldots,\ell_k,m_1,\ldots,m_\ell,\epsilon_1,\ldots,\epsilon_m),\nu(\xi_1,\ldots,\xi_\ell,\rho_1,\ldots,\rho_m)),\ f)$$

is absolutely convergent and the convergence is uniform with respect to $(\xi_1,\ldots,\xi_\ell,\rho_1,\ldots,\rho_m)\in\mathbb{R}^{\ell+m}$.

The following lemmas are due to T. Hirai.

<u>Lemma</u> 3.3 ([2(c)], Cor. p55). Let $K_{k,\ell}^{k',\ell'}(\omega,\nu)$ be defined by (41) Then for $f\in C_o^\infty(G)$

$$(49)\quad \sum_{k'+2\ell'\le n}\alpha^{k',\ell'}\int_{H'_{k',\ell'}} L^{k',\ell'}K_f^{k',\ell'}(h)\ L^{k',\ell'}K_{k,\ell}^{k',\ell'}(\omega,\nu)(h)\,dh = (-1)^{n^2}L(\omega,\nu)^2(\Theta_{k,\ell}(\omega,\nu),f)\qquad (k+2\ell\le n),$$

where L is the polynomial defined in §1, 1.2 and L (ω,ν) is defined as in (45).

Lemma 3.4 ([2(c)], Remark 6.3). Let (ω,ν) be type (k,ℓ). Let $({}^{+}K_j^{k',\ell'})'$ be defined by (32). Then we get for $f\in C_0^\infty(G)$,

$$(50)\quad \sum_{k'+2\ell'\leq n}\ \sum_{0\leq j\leq m'} \binom{m'}{j} \alpha^{k'\ell'} \int_{(H^j_{k',\ell'})'} F_f^{k',\ell'}(h)\,({}^{+}K_j^{k',\ell'})'(h)\,dh = (-1)^{n^2} L_{(\omega,\nu)}({}^{+}\Theta(\omega,\nu),f),$$

where $\binom{m'}{j} = \dfrac{m'!}{(m'-j)!\,j!}$.

Theorem 2. For $f\in C_0^\infty(G)$, we get

$$f(e) = \sum_{k+2\ell\leq n} C(k,\ell) \sum_{\substack{\ell_1>\ell_2>\cdots>\ell_k>0\\ m_1>0,\dots,m_\ell>0\\ \epsilon_1,\epsilon_2,\dots,\epsilon_m=0,1}} \int_{\substack{\xi_1>0,\dots,\xi_\ell>0\\ \rho_1>\rho_2>\cdots>\rho_m>0}} \gamma(m,\epsilon;\xi,\rho)\quad L(\ell,m;\xi,\rho)$$

$$({}^{+}\Theta(\omega(\ell_1,\ell_2,\dots,\ell_k;m_1,\dots,m_\ell;\epsilon_1,\epsilon_2,\dots,\epsilon_m),\nu(\xi_1,\dots,\xi_\ell,\rho_1,\rho_2,\dots,\rho_m)),f)$$
$$d\xi_1\dots d\xi_\ell\, d\rho_1\, d\rho_2 \dots d\rho_m,$$

Where

$$\gamma(m,\epsilon;\xi,\rho) = \prod_{1\leq p\leq\ell}\gamma(\xi;m_p-1)\prod_{1\leq p<q\leq m}\gamma(\rho_p+\rho_q;\epsilon_p+\epsilon_q)\ \gamma(\rho_p-\rho_q;\epsilon_p-\epsilon_q)$$
$$\prod_{1\leq p\leq m}\gamma(\rho_p;k+\epsilon_p);$$

$$\gamma(\xi;p) = \begin{cases}\tanh\pi\xi/2 & (p\equiv 0 \bmod 2),\\ \coth\pi\xi/2 & (p\equiv 1 \bmod 2);\end{cases}$$

$L(\ell,m;\xi,\rho)$

$$= L\left(\ell_1,\ell_2,\dots,\ell_k,\frac{m_1+\sqrt{-1}\,\xi_1}{2},\frac{m_1-\sqrt{-1}\,\xi_1}{2},\dots,\frac{m_\ell+\sqrt{-1}\,\xi_\ell}{2},\frac{m_\ell-\sqrt{-1}\,\xi_\ell}{2},\sqrt{-1}\rho_1,\sqrt{-1}\rho_2,\dots,\sqrt{-1}\rho_m\right);$$

$c(k,\ell)$ positive constants.

Outline of the proof. Using the notation in Lemma 3.4, we put

$$(51)\quad b_{k,\ell}^{k',\ell'} = \sum_{0\leq j\leq m'}\binom{m'}{j} \sum_{\substack{S(\ell_1\geq\ell_2\geq\cdots>\ell_k\geq 0)\\ S(m_1\geq 0)\cdots S(m_\ell\geq 0)\\ \epsilon_1,\epsilon_2,\dots,\epsilon_m=0,1}} \int_{\substack{\xi_1>0,\dots,\xi_\ell>0\\ \rho_1>\rho_2>\cdots>\rho_m>0}} \int_{(H^j_{k',\ell'})'} F_f^{k',\ell'}(h)\,({}^{+}K_j^{k',\ell'})'(h)\,dh$$
$$\gamma(m,\epsilon;\xi,\rho)\ d\xi_1\dots d\xi_\ell\, d\rho_1 d\rho_2\dots d\rho_m.$$

Then from Lemma 2.3 and Lemma 3.1, we have

(52) $$b^{n,o}_{n,o} = F^{n,o}_{f}(e) = (-1)^{q} c\ f(e).$$

Put

(53) $$a_{k,\ell} = (-1)^{n^2} \sum_{\substack{\ell_1 > \ell_2 > \cdots > \ell_k > 0 \\ m_1 > 0, \ldots, m_\ell > 0 \\ \epsilon_1, \epsilon_2, \ldots, \epsilon_m = 0,1}} \int_{\substack{\xi_1 > 0 \cdots \xi_\ell > 0 \\ \rho_1 > \rho_2 > \cdots > \rho_m > 0}} L(\ell, m; \xi, \rho) \quad \gamma(m, \epsilon; \xi, \rho)$$

$$({}^{+}\Theta(\omega(\ell_1, \ell_2, \cdots, \ell_k, m_1, \cdots, m_\ell, \epsilon_1, \epsilon_2, \cdots, \epsilon_m), \nu(\xi_1, \cdots, \xi_\ell, \rho_1, \rho_2, \cdots, \rho_m)), f)$$
$$d\xi_1 \ldots d\xi_\ell d\rho_1 d\rho_2 \ldots d\rho_m .$$

Then from Lemma 3.4, we have

(54) $$\sum_{\substack{k'+2\ell' \leq k+2\ell \\ k' \leq k}} \alpha^{k',\ell'}\ b^{k',\ell'}_{k,\ell} = a_{k,\ell} \qquad (k+2\ell \leq n).$$

Using Lemma 2.10, we solve this equation with respect to $b^{n,o}_{n,o}$.
Hence from (52), we obtain the assertion of Theorem 2.

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