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# On the spherical functions with one dimensional K-types and the Paley-Wiener type theorem on some simple Lie groups

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## Introduction.

Let $G$ be a noncompact connected semisimple Lie group with finite center and $K$ a maximal compact subgroup of $G$. We fix a one dimensional unitary representation $\tau$ of $K$. A function $f$ on $G$ is called $\tau$-spherical if

$$f(kxk') = \tau(k)f(x)\tau(k') \qquad (x \in G,\ k,k' \in K).$$

The set $\mathcal{D}_\tau(G)$ of compactly supported $\tau$-spherical $C^\infty$ functions on $G$ is a commutative algebra under convolution. When $\tau_0$ is trivial, R. Gangolli [3] characterized the Fourier transforms of the elements of $\mathcal{D}_{\tau_0}(G)$. Our purpose of this note is to characterize the Fourier transforms of the members of $\mathcal{D}_\tau(G)$ for any simple matrix groups and any one dimensional representations $\tau$.

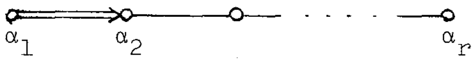
From now on let $G$ be a simple matrix group. If $K$ is semisimple, then $\tau$ must be trivial. We may therefore suppose that $K$ is not semisimple. But it is well known that such a group is one of the following:

$$SO_0(n, 2),\quad Sp(n,R),\quad SO^*(2n) \quad \text{and} \quad SU(p, q).$$

We call $SO_0(n+2,2)$, $Sp(r,R)$, $SO^*(4r)$, $SU(r,r)$ $(n,r \geqq 1)$ the groups of the first kind, and call $SO^*(4r+2)$, $SU(n+r,r)$ $(n,r \geqq 1)$ the groups of the second kind. $r$ denotes the real rank.

$G = KAN$ is an Iwasawa decomposition of $G$, and $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ are Lie algebras of $G$, $K$, $A$, $N$ respectively. $\mathfrak{a}$ extends to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We fix a compatible orderings on the duals of $\mathfrak{a}$ and $\mathfrak{a} + i(\mathfrak{h} \cap \mathfrak{k})$. Let $P$ be the set of positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and put $P_+ = \{\beta \in P : \tilde{\beta} \equiv \beta|\mathfrak{a} \neq 0\}$, $\Delta^+ = \{\tilde{\beta} : \beta \in P_+\}$. The inner product $\langle \cdot, \cdot \rangle$ on the dual $\mathfrak{a}^*$ of $\mathfrak{a}$ defined by the Killing form extends to a bilinear form on $\mathfrak{a}^*_{\mathbb{C}}$. We denote it also by the same notation. For $\alpha \in \Delta^+ \cup (-\Delta^+)$ (resp. $\beta \in P \cup (-P)$) we write $\mathfrak{g}^{\alpha}$ (resp. $\mathfrak{g}_{\beta}$) for the corresponding root space in $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{C}}$).

The simple roots $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ of $\Delta^+$ may be so arranged that the root diagram is

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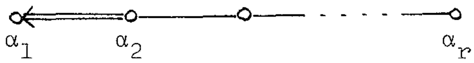


according as $G$ is of the first or second kind. Let

$$e_1 = \alpha_1, \quad e_2 = \alpha_1 + 2\alpha_2, \quad \cdots, \quad e_r = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r$$

or

$$e_1 = \alpha_1, \quad e_2 = \alpha_1 + \alpha_2, \quad \cdots\cdots, \quad e_r = \alpha_1 + \alpha_2 + \cdots + \alpha_r$$

according as $G$ is of the first or second kind. Then $\{e_1, e_2, \cdots, e_r\}$ is an orthogonal basis of $\mathfrak{a}^*_{\mathbb{C}}$ with same norm. $\mathfrak{a}^*_{\mathbb{C}}$ is then identified with $\mathbb{C}^r$ via

$$\mathfrak{a}^*_{\mathbb{C}} \ni \nu = \sum_{j=1}^{r} \nu_j e_j \longleftrightarrow (\nu_1, \nu_2, \cdots, \nu_r) \in \mathbb{C}^r$$

and the Weyl group is identified withe the group of all linear mappings

$$(\nu_1, \nu_2, \cdots, \nu_r) \mapsto (\varepsilon_1 \nu_{j_1}, \varepsilon_2 \nu_{j_2}, \cdots, \varepsilon_r \nu_{j_r}), \ \varepsilon_j = \pm 1, \begin{pmatrix} 1 & 2 & \cdots & r \\ j_1 & j_2 & \cdots & j_r \end{pmatrix} \in \mathfrak{S}_r.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ and $\mathfrak{k} = \mathfrak{k}^a \oplus \mathfrak{k}^s$, where $\mathfrak{k}^a$ and $\mathfrak{k}^s$ are the abelian and semisimple parts of $\mathfrak{k}$ respectively. For $x \in G$, $\kappa(x) \in K$, $H(x) \in \mathfrak{a}$, $n(x) \in N$ are defined by $x = \kappa(x)\exp H(x) n(x)$. Let $\mathfrak{a}^+$ be the positive chamber of $\mathfrak{a}$ and $A^+ = \exp(\mathfrak{a}^+)$. Then $G = K \cdot Cl(A^+) \cdot K$. We write $\omega$ and $\omega_{\mathfrak{m}}$ for the Casimir operators of $G$ and $M$ respectively, where $M$ being the centralizer of $A$ in $K$.

## §1. Elementary τ-spherical functions.

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ the elementary τ-spherical function is defined by

$$\phi(\nu : x) = \int_K \tau(\kappa(xk))\overline{\tau(k)}\ e^{(i\nu - \rho)(H(xk))}\ dk.$$

(1 - 1) $\phi(\nu : x)$ is a W-invariant entire function of $\nu$, and satisfies the differential equation

$$\omega\phi = (\tau(\omega_{\mathfrak{m}}) - \langle\nu,\nu\rangle - \langle\rho,\rho\rangle)\phi.$$

Since $G = K \cdot Cl(A^+) \cdot K$, $\phi$ is determined by its restriction to $A^+$. Let $\Delta$ be the function on $A^+$ defined by

$$\Delta(h) = \prod_{\beta \in P_+} (e^{\beta(H)} - e^{-\beta(H)}) \quad (h = \exp H \in A^+)$$

and $\Upsilon(\omega)$ the radial component of $\omega$. Then we have

(1 - 2) There exist meromorphic functions $\Psi(\nu: h)$ and $C^{\tau}(\nu)$ such that

$$\Delta(h)^{1/2}\phi(\nu: h) = \sum_{s\in W} C^{\tau}(s\nu)\Psi(s\nu: h) \quad (\nu\in \mathfrak{a}_{\mathbb{C}}^{*}, \ h\in A^{+}).$$

Moreover $\Psi$ satisfies the differential equation

$$(\Delta^{1/2}\circ\Upsilon(\omega)\circ\Delta^{-1/2})\Psi = (\tau(\omega_{\mathfrak{m}}) - \langle\nu,\nu\rangle - \langle\rho,\rho\rangle)\Psi.$$

Let $H_1, H_2, \cdots, H_r$ be the basis of $\mathfrak{a}$ dual to $e_1, e_2, \cdots, e_r$. For $\beta\in P_+$ choose $X_{\pm\beta}\in\mathfrak{g}_{\pm\beta}$ such that $\langle X_{\beta}, X_{-\beta}\rangle = 1$. If

$$X_{\pm\beta} = Y_{\pm\beta} + Z_{\pm\beta} \qquad Y_{\pm\beta}\in \mathfrak{k}_{\mathbb{C}}^{a}, \ Z_{\pm\beta}\in \mathfrak{k}_{\mathbb{C}}^{s}\oplus\mathfrak{r}_{\mathbb{C}},$$

then

$$\Delta^{1/2}\circ\Upsilon(\omega)\circ\Delta^{-1/2} = \tau(\omega_{\mathfrak{m}}) + \|e_1\|^{-2}\sum_{j} H_j^2$$

(1 - 3)
$$+ \frac{1}{2}\sum_{\beta\in P_+}\langle\tilde{\beta},\tilde{\beta}\rangle(\mathrm{sh}\beta)^{-2} - \frac{1}{4}\sum_{\beta,\gamma\in P_+}\langle\tilde{\beta},\tilde{\gamma}\rangle(\coth\beta)(\coth\gamma)$$

$$- 4\sum_{\beta\in P_+}(1-\mathrm{ch}\beta)(\mathrm{sh}\beta)^{-2}\tau(Y_{\beta})\tau(Y_{-\beta})$$

Let $L$ denote the semilattice of elements $\sum_j m_j\alpha_j$ $(m_j\in\mathbb{Z}_+)$. For $\lambda = \sum m_j\alpha_j\in L$, we let $m(\lambda) = \sum m_j$. Using (1-2) and (1-3) we see that $\Psi$ has a series expansion

(1 - 4)
$$\Psi(\nu: h) = \sum_{\lambda\in L} a_{\lambda}(\nu)\, e^{(i\nu-\lambda)(H)} \quad (h = \exp H\in A^{+}).$$

Here $a_{\lambda}(\nu)$ $(\lambda\in L)$ are rational functions determined by the recurrence relation $a_0(\nu)\equiv 1$,

$$(\langle\lambda,\lambda\rangle - 2i\langle\nu,\lambda\rangle)a_{\lambda}(\nu) = 2\sum_{\beta}\sum_{n\geq 1}(8\tau(Y_{\beta})\tau(Y_{-\beta}) - \langle\tilde{\beta},\tilde{\beta}\rangle)n\, a_{\lambda-2n\tilde{\beta}}(\nu)$$

$$+ 2 \sum_{\beta} \sum_{n \geqq 1} \langle \rho, \tilde{\beta} \rangle \, \mathfrak{a}_{\lambda-2n\tilde{\beta}}(\nu) + \sum_{\beta,\gamma} \sum_{\substack{m,n \geqq 0 \\ m+n \geqq 1}} \langle \tilde{\beta}, \tilde{\gamma} \rangle \, \mathfrak{a}_{\lambda-2m\tilde{\beta}-2n\tilde{\gamma}}(\nu)$$

$$- 8 \sum_{\beta} \sum_{n \geqq 1} (2n-1) \tau(Y_{\beta}) \tau(Y_{-\beta}) \, \mathfrak{a}_{\lambda-(2n-1)\tilde{\beta}}(\nu) \quad (\lambda \neq 0).$$

We let $\mathfrak{a}^*_+ = \{\nu \in \mathfrak{a}^* : 0 \leqq \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_r\}$, $`\mathfrak{a}^* = \{\nu \in \mathfrak{a}^* : \langle\lambda,\lambda\rangle + 2\langle\nu,\lambda\rangle \neq 0 \ \forall \lambda \in L-\{0\}\}$.

(1 - 5) For $\eta \in {}`\mathfrak{a}^*$ we can find constants $C(\eta)$, $d(\eta) > 0$ such that

$$|\mathfrak{a}_{\lambda}(\nu+i\eta)| \leqq C(\eta) m(\lambda)^{d(\eta)} \quad (\nu \in \mathfrak{a}^*, \ \lambda \in L-\{0\}).$$

There exist constants $C$, $d > 0$ such that

$$|\mathfrak{a}_{\lambda}(\nu+i\eta)| \leqq C \cdot m(\lambda)^{d} \quad (\nu \in \mathfrak{a}^*, \ \eta \in \mathfrak{a}^*_+, \ \lambda \in L-\{0\}).$$

## §2. Harish-Chandra's generalized C-function $C^{\tau}(\nu)$.

The function $C^{\tau}(\nu)$ in (1-2) is meromorphic on $\mathfrak{a}^*_{\mathbb{C}}$ and is given by

$$(2 - 1) \qquad C^{\tau}(\nu) = \int_{\bar{N}} \overline{\tau(\kappa(\bar{n}))} \, e^{-(i\nu+\rho)(H(\bar{n}))} \, d\bar{n} \, .$$

Here we normalize the Haar measure $d\bar{n}$ of $\bar{N}$ so that the integral of $\exp\{-2\rho(H(\bar{n}))\}$ equals one. We shall find an explicit form of $C^{\tau}(\nu)$ for our groups. Since the center of $K$ is one dimensional, $\tau$ is parametrized by an integer $\ell$. We denote this $\tau$ by $\tau_{\ell}$.

Example. If $G = SO_0(n+2,2)$, $K = SO(n+2) \ SO(2)$, then

$$\tau_{\ell}\left(\begin{bmatrix} k' & & \\ & \cos\theta & \sin\theta \\ & -\sin\theta & \cos\theta \end{bmatrix}\right) = e^{i\ell\theta} \qquad (k' \in SO(n+2)).$$

To compute the integral (2-1) we use the reduction theory of G. Schiffmann [7]. Let $s_1, s_2, \cdots, s_r$ be the Weyl reflections defined by $\alpha_1, \alpha_2, \cdots, \alpha_r$ respectively, and denote the length of $s \in W$ by $\ell(s)$. For $s \in W$ let $\bar{N}(s)$ be the analytic subgroup with Lie algebra $\bar{\mathfrak{n}}(s) = \sum_{\substack{\alpha>0 \\ s\alpha<0}} \mathfrak{g}^{-\alpha}$ and let

$$C^{\ell}(\nu : s) = \int_{\bar{N}(s)} \overline{\tau_{\ell}(\kappa(\bar{n}))}\, e^{-(i\nu+\rho_s)(H(\bar{n}))}\, d\bar{n} .$$

(2 - 2)(G.Schiffmann [7]) If $s = s's''$ with $\ell(s) = \ell(s') + \ell(s'')$, then

$$C^{\ell}(\nu : s) = C^{\ell}(s''\nu : s')C^{\ell}(\nu : s'').$$

On the other hand, the element $-1$ of $W$ has the following property.

(2 - 3) $\ell(-1) = r^2$ and

$$-1 = \underbrace{\underbrace{s_r s_{r-1} \cdots s_1}\underbrace{s_r s_{r-1} \cdots s_1} \cdots \underbrace{s_r s_{r-1} \cdots s_1}}_{r}$$

is a reduced expression of $-1$.

Since $\bar{N} = \bar{N}(-1)$, the question of computing $C^{\ell}(\nu)$ is therefore reduced to finding $C^{\ell}(\nu : s_j)$ $(1 \leqq j \leqq r)$. Fix a simple root $\alpha$. Let $\mathfrak{n}_\alpha$ and $\bar{\mathfrak{n}}_\alpha$ be the subalgebras $\mathfrak{g}^{\alpha} + \mathfrak{g}^{2\alpha}$ and $\mathfrak{g}^{-\alpha} + \mathfrak{g}^{-2\alpha}$ respectively, and $\mathfrak{g}(\alpha)$ the semisimple subalgebra generated by $\mathfrak{n}_\alpha + \bar{\mathfrak{n}}_\alpha$. We write $N_\alpha$, $\bar{N}_\alpha$ and $G(\alpha)$ for the analytic subgroups corresponding to the subalgebras $\mathfrak{n}_\alpha$, $\bar{\mathfrak{n}}_\alpha$ and $\mathfrak{g}(\alpha)$. Then $G(\alpha)$ is a real rank one semisimple Lie group with finite center and has Iwasawa decomposition $G(\alpha) = K_\alpha A_\alpha N_\alpha$, where $K_\alpha = K \cap G(\alpha)$ and $A_\alpha = \exp(\mathbb{R}H_\alpha)$. Let $X \in \mathfrak{g}^{-\alpha_j}$ and $Y \in \mathfrak{g}^{-2\alpha_j}$. Finding $K_{\alpha_j}$-component

$\kappa(\bar{n})$ and $A_{\alpha_j}$-component $\exp H(\bar{n})$ of $\bar{n} = \exp(X+Y)$, we can calculate $c^{\ell}(\nu : s_j)$.

(2 - 4) Let $n_j = \dim \mathfrak{g}^{\alpha_j}$ $(1 \leqq j \leqq r)$. Then

$$c^{\ell}(\nu : s_j) = \frac{\Gamma(n_j)}{\Gamma\left(\frac{n_j}{2}\right)} \frac{\Gamma\left(\frac{\langle i\nu, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle}\right)}{\Gamma\left(\frac{\langle i\nu, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} + \frac{n_j}{2}\right)} \qquad (2 \leqq j \leqq r),$$

and $c^{\ell}(\nu : s_1)$ is given by

$$c^{\ell}(\nu : s_1) = \frac{\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\right)\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle} + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle} + \frac{1}{2} + \frac{\ell}{2}\right)\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle} + \frac{1}{2} - \frac{\ell}{2}\right)}$$

or

$$c^{\ell}(\nu : s_1) = \frac{\Gamma(n_1+1)}{\Gamma\left(\frac{n_1+1}{2}\right)} \frac{\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle}\right)}{\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{\langle \alpha_1, \alpha_1\rangle} + \frac{n_1}{2}\right)} \frac{\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{2\langle \alpha_1, \alpha_1\rangle} + \frac{n_1}{4}\right)\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{2\langle \alpha_1, \alpha_1\rangle} + \frac{n_1}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{2\langle \alpha_1, \alpha_1\rangle} + \frac{n_1}{4} + \frac{1}{2} + \frac{\ell}{2}\right)\Gamma\left(\frac{\langle i\nu, \alpha_1\rangle}{2\langle \alpha_1, \alpha_1\rangle} + \frac{n_1}{4} + \frac{1}{2} - \frac{\ell}{2}\right)}$$

according as $G$ is of the first or second kind.

From (2-2), (2-3) and (2-4) we can derive the following result.

<u>THEOREM 1.</u> $c^{\ell}(\nu)$ is expressed by

$$c^{\ell}(\nu) = \frac{\Gamma(2m'+1)^r \Gamma(m)^{r(r-1)}}{\Gamma(m'+\frac{1}{2})^r \Gamma(\frac{m}{2})^{r(r-1)}} \prod_{j<k} \frac{\Gamma(i\nu_j + i\nu_k)\Gamma(i\nu_k - i\nu_j)}{\Gamma(i\nu_j + i\nu_k + \frac{m}{2})\Gamma(i\nu_k - i\nu_j + \frac{m}{2})}$$

$$\times \prod_{j=1}^{r} \frac{\Gamma(2i\nu_j)}{\Gamma(2i\nu_j + m')} \frac{\Gamma(i\nu_j + \frac{m'}{2})\Gamma(i\nu_j + \frac{m'+1}{2})}{\Gamma(i\nu_j + \frac{m'+1}{2} + \frac{\ell}{2})\Gamma(i\nu_j + \frac{m'+1}{2} - \frac{\ell}{2})},$$

where m and m' are given as follows.

| G | $SO_0(n+2,2)$ | $Sp(r,R)$ | $SO^*(4r)$ | $SO^*(4r+2)$ | $SU(n+r,r)$ |
|---|---|---|---|---|---|
| m | n | 1 | 4 | 4 | 2 |
| m' | 0 | 0 | 0 | 2 | n |

§3. Fourier transform on $\mathcal{D}_\ell(G)$.

We define the Fourier transform $\hat{f}$ of $f \in \mathcal{D}_\ell(G) = \mathcal{D}_{\tau_\ell}(G)$ by

$$\hat{f}(\nu) = \int_G f(x)\phi(\nu : x^{-1})dx.$$

For $R > 0$, $\mathcal{D}_\ell(R)$ and $\mathcal{H}_W(R)$ are defined as follows. $\mathcal{D}_\ell(R)$ is the space of those elements in $\mathcal{D}_\ell(G)$ that are supported in $B_R = \{x \in G : \sigma(x) \leqq R\}$, where $\sigma$ is the K-biinvariant continuous function on G defined by $\sigma(\exp H) = \|H\|$ $(H \in \mathfrak{a})$. $\mathcal{H}_W(R)$ is the space of W-invariant entire functions F on $\mathfrak{a}_{\mathbb{C}}^*$ that satisfy

$$\forall M \geqq 0 \ \exists C_M > 0 : |F(\nu)| \leqq C_M(1 + \|\nu\|)^{-M} e^{R\|\mathrm{Im}\nu\|}.$$

We write $\mathcal{H}_W(\mathfrak{a}_c^*)$ for the union of all $\mathcal{H}_W(R)$ $(R > 0)$.

(3 - 1) If $f \in \mathcal{D}_\ell(R)$, then $f \in \mathcal{H}_W(R)$.

Now we let

$$\mu^\ell(\nu) = (c^\ell(\nu)c^\ell(-\nu))^{-1}.$$

This is a meromorphic function on $\mathfrak{a}_c^*$. In fact, Theorem 1 implies that

$$\mu^\ell(\nu) = \begin{cases} X_\ell(\nu)Y(\nu) & (2|m) \\ X_\ell(\nu)Z(\nu) & (2\nmid m), \end{cases}$$

where

$$X_\ell(\nu) = \frac{4^{m'}\Gamma(m'+\frac{1}{2})^{2r}\Gamma(\frac{m}{2})^{2r(r-1)}}{\Gamma(2m'+1)^{2r}\Gamma(m)^{2r(r-1)}}$$

$$\times \prod_{j=1}^{r}\left\{\nu_j \operatorname{th}\pi(\nu_j + \frac{i(\ell+m)}{2})\prod_{p=1}^{m'}(\nu_j^2 + (\frac{|\ell|-m'-1}{2})^2)\right\},$$

$$Y(\nu) = \prod_{j<k}\prod_{p=1}^{m/2}((\nu_j+\nu_k)^2 + (\frac{m}{2}-p)^2)((\nu_j-\nu_k)^2 + (\frac{m}{2}-p)^2)$$

and

$$Z(\nu) = \prod_{j<k}\left\{(\nu_j^2 - \nu_k^2)\operatorname{th}\pi(\nu_j+\nu_k)\operatorname{th}\pi(\nu_j-\nu_k)\right.$$

$$\left.\times \prod_{p=1}^{[\frac{m}{2}]}((\nu_j+\nu_k)^2 + (\frac{m}{2}-p)^2)((\nu_j-\nu_k)^2 + (\frac{m}{2}-p)^2)\right\}.$$

As a function of $\nu_r$, $\mu^\ell(\nu)$ has infinite simple poles on the imaginary axis. Let $\Pi_1 = \Gamma_1$ be the set of those poles that are between 0 and $i(|\ell|-m')/2$. For $a \in \Gamma_1$ we let

$$\mu_a^\ell(\nu^{(a)}) = -2\pi i \cdot \operatorname{Res}[\mu^\ell(\nu) : \nu_r = a],$$

where $\nu^{(a)}$ denoting $(\nu_1, \nu_2, \cdots, \nu_{r-1})$. As a function of $\nu_{r-1}$, $\mu_a^\ell(\nu^{(a)})$ has simple poles on the imaginary axis. Let $\Pi_a$ be the set of those poles that are between 0 and a, and put $\Gamma_2 = \{(a,b): a \in \Gamma_1, b \in \Pi_a\}$. For $\mathfrak{p} = (a, b) \in \Gamma_2$ we let

$$\mu_{\mathfrak{p}}^\ell(\nu^{(\mathfrak{p})}) = -2\pi i \cdot \operatorname{Res}[\mu_a^\ell(\nu^{(a)}) : \nu_{r-1} = b],$$

where $\nu^{(\mathfrak{p})}$ denoting $(\nu_1, \nu_2, \cdots, \nu_{r-2})$. In this manner we define $\Gamma_1, \Gamma_2, \cdots, \Gamma_r, \Pi_{\mathfrak{p}}$ and $\mu_{\mathfrak{p}}^\ell(\nu^{(\mathfrak{p})})$. For simplisity we let $\Gamma_0 = \{o\}$, $\mu_o^\ell(\nu^{(o)}) = \mu^\ell(\nu)$ and $\Gamma = \bigcup(\Gamma_p : 0 \leqq p \leqq r)$. For $\mathfrak{p} = (a_1, a_2, \cdots,$

$a_p) \in \Gamma_p$ we let $\mathfrak{p}' = (a_p, \cdots, a_2, a_1)$, $\mathbb{R}^{(\mathfrak{p})} = \mathbb{R}^{r-p}$ and let $W_{\mathfrak{p}}$ denote the subgroup of $W$ composed of those elements that leave $\nu_j$ $(r-p < j \leqq r)$ fixed.

(3 - 2) For every $\mathfrak{p} \in \Gamma$, $\mu_{\mathfrak{p}}^{\ell}(\nu^{(\mathfrak{p})})$ is a $W_{\mathfrak{p}}$-invariant meromorphic function which is positive-valued on $\mathbb{R}^{(\mathfrak{p})}$.

For $F \in \mathcal{H}_W(\mathfrak{a}_{\mathbb{C}}^*)$ we let

$$\mathcal{F}(F: x) = \sum_{\mathfrak{p}\in\Gamma} \frac{1}{|W_{\mathfrak{p}}|} \int_{\mathbb{R}^{(\mathfrak{p})}} F(\nu^{(\mathfrak{p})}, \mathfrak{p}')\phi(\nu^{(\mathfrak{p})}, \mathfrak{p}': x)\mu_{\mathfrak{p}}^{\ell}(\nu^{(\mathfrak{p})})d\nu^{(\mathfrak{p})}.$$

We say that $G$ has property (S) if the following condition is satisfied:

If $F \in \mathcal{H}_W(R)$, then $\mathcal{F}(F: \cdot)$ is supported in $B_R$.

(3 - 3) If $r = 1$ then $G$ has property (S).

This is proved using the results obtained in the previous sections.

Now we assume property (S) and continue our discussion. Since $\tau$-spherical functions are completely determined by their restrictions to $A$, we can regard a linear functional on $\mathcal{D}_{\ell}(G)$ as a $W$-invariant distribution on $A$. Define the linear functional $T$ on $\mathcal{D}_{\ell}(G)$ by

$$Tf = \mathcal{F}(\hat{f}: 1).$$

Let $F_0$ be a function in $\mathcal{H}_W(1)$ such that $F_0(0) = 1$. Then

$$Tf = \lim_{\varepsilon\downarrow 0} \mathcal{F}(\hat{f}(\cdot)F_0(\varepsilon\cdot): 1) = \lim_{\varepsilon\downarrow 0} \int_G f(x)g_{\varepsilon}(x)dx,$$

where $g_{\varepsilon}(x) = \mathcal{F}(F_0(\varepsilon\cdot): x^{-1})$. Property (S) implies that $g_{\varepsilon}$ is supported in $B_{\varepsilon}$. Since $T = \lim_{\varepsilon\downarrow 0} g_{\varepsilon}$ as a distribution, $T$ must have support $\{1\}$. Moreover we have

(3 - 4) $T$ is a positive measure with support $\{1\}$.

Hence there exists a positive constant $\gamma > 0$ such that

$$Tf = \gamma \cdot f(1) \qquad (f \in \mathcal{D}_\ell(G)).$$

(3 - 6) For $f \in \mathcal{D}_\ell(G)$ we have

$$\gamma \cdot f(x) = \mathcal{F}(f: x) \qquad (x \in G),$$

$$\gamma \cdot \|f\|^2_{L^2(G)} = \mathcal{F}(|\hat{f}|^2: 1).$$

The set $\{\hat{f} : f \in \mathcal{D}_\ell(G)\}$ is dense in the space $C_0(\Omega)$ of continuous functions on $\Omega$ which vanish at infinity. Here $\Omega$ is the support of the Plancherel measure.

<u>THEOREM 2.</u> Assume that $G$ has property (S). Then the map $f \mapsto \hat{f}$ is a linear isomorphism of $\mathcal{D}_\ell(G)$ onto $\mathcal{H}_W(\mathfrak{a}^*_{\mathbb{C}})$. More precisely, for every $R > 0$ $\mathcal{D}_\ell(R)$ is transformed onto $\mathcal{H}_W(R)$.

Proof. Let $F \in \mathcal{H}_W(R)$ and define $f$ by $\gamma \cdot f(x) = \mathcal{F}(F: x)$. Then property (S) implies that $f \in \mathcal{D}_\ell(R)$. Let $F' = F - \hat{f}$. We must prove that $F'$ vanishes identically. But it follows from the definition of $f$ that

$$\mathcal{F}(F': x) = 0 \quad \text{for all} \quad x \in G.$$

Hence

$$\mathcal{F}(F' \cdot \hat{g}: 1) = \int_G \mathcal{F}(F': x) g(x) dx = 0 \quad \text{for all} \quad g \in \mathcal{D}_\ell(G).$$

$\{\hat{g} : g \in \mathcal{D}_\ell(G)\}$ is dense in $C_0(\Omega)$; so $F'$ vanishes on $\Omega$. Since $F'$ is holomorphic, $F'$ vanishes identically.

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