Lectures on harmonic analysis on Lie groups and related topics (On the spherical functions with one-dimensional K-type and the Paley-Wiener type theorem on some simple Lie groups.)

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On the spherical functions with one dimensional K-types and the Paley-Wiener type theorem on some simple Lie groups

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Introduction.

Let $G$ be a noncompact connected semisimple Lie group with finite center and $K$ a maximal compact subgroup of $G$. We fix a one dimensional unitary representation $\tau$ of $K$. A function $f$ on $G$ is called $\tau$-spherical if

$$f(kxk') = \tau(k)f(x)\tau(k') \quad (x \in G, k, k' \in K).$$

The set $\mathcal{S}_{\tau}(G)$ of compactly supported $\tau$-spherical $C^\infty$ functions on $G$ is a commutative algebra under convolution. When $\tau_0$ is trivial, R. Gangolli [3] characterized the Fourier transforms of the elements of $\mathcal{S}_{\tau_0}(G)$. Our purpose of this note is to characterize the Fourier transforms of the members of $\mathcal{S}_{\tau}(G)$ for any simple matrix groups and any one dimensional representations $\tau$.

From now on let $G$ be a simple matrix group. If $K$ is semisimple, then $\tau$ must be trivial. We may therefore suppose that $K$ is not semisimple. But it is well known that such a group is one of the following:

$\text{SO}_0(n, 2), \text{Sp}(n, \mathbb{R}), \text{SO}^*(2n)$ and $\text{SU}(p, q)$.

We call $\text{SO}_0(n+2, 2), \text{Sp}(r, \mathbb{R}), \text{SO}^*(4r)$, $\text{SU}(r, r)$ ($n, r \geq 1$) the groups of the first kind, and call $\text{SO}^*(4r+2), \text{SU}(n+r, r)$ ($n, r \geq 1$) the groups of the second kind. $r$ denotes the real rank.
$G = \text{KAN}$ is an Iwasawa decomposition of $G$, and $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ are Lie algebras of $G$, $K$, $A$, $N$ respectively. $\alpha$ extends to a Cartan subalgebra $\mathfrak{g}^\ast$ of $\mathfrak{g}$. We fix a compatible orderings on the duals of $\mathfrak{g}$ and $\mathfrak{g}^\ast + i(\mathfrak{k}^\ast)$. Let $P$ be the set of positive roots of $(\mathfrak{g}_c, \mathfrak{f}_c)$ and put $P_+ = \{\beta \in P : \beta \equiv \beta | \alpha \neq 0\}$, $\Delta^+ = \{\beta : \beta \in P_+\}$. The inner product $\langle \cdot, \cdot \rangle$ on the dual $\alpha^\ast$ of $\alpha$ defined by the Killing form extends to a bilinear form on $\alpha^\ast_c$. We denote it also by the same notation. For $\alpha \in \Delta^+ \cup (-\Delta^+)$ (resp. $\beta \in P \cup (-P)$) we write $\mathfrak{g}_\alpha$ (resp. $\mathfrak{g}_\beta$) for the corresponding root space in $\mathfrak{g}$ (resp. $\mathfrak{g}_c$).

The simple roots $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ of $\Delta^+$ may be so arranged that the root diagram is

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \cdots & & \alpha_r \\
\end{array}
\]

or

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \cdots & & \alpha_r \\
\end{array}
\]

according as $G$ is of the first or second kind. Let

\[
e_1 = \alpha_1, \quad e_2 = \alpha_1 + 2\alpha_2, \quad \cdots, \quad e_r = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_r
\]

or

\[
e_1 = \alpha_1, \quad e_2 = \alpha_1 + \alpha_2, \quad \cdots, \quad e_r = \alpha_1 + \alpha_2 + \cdots + \alpha_r
\]

according as $G$ is of the first or second kind. Then $\{e_1, e_2, \cdots, e_r\}$ is an orthogonal basis of $\alpha^\ast_c$ with same norm. $\alpha^\ast_c$ is then identified with $\mathbb{C}^r$ via

\[
\alpha^\ast_c \ni \mathfrak{v} = \sum_{j=1}^{r} \mathfrak{v}_j \mathfrak{e}_j \leftrightarrow (\mathfrak{v}_1, \mathfrak{v}_2, \cdots, \mathfrak{v}_r) \in \mathbb{C}^r
\]

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and the Weyl group is identified with the group of all linear mappings

\[(v_1, v_2, \cdots, v_r) \mapsto (\epsilon_1 v_{j_1}, \epsilon_2 v_{j_2}, \cdots, \epsilon_r v_{j_r}), \quad \epsilon_j = \pm 1, \quad (j_1, j_2, \cdots, j_r) \in \mathbb{Z}^r.\]

Let \( g = k \oplus \mathfrak{t} \) be the Cartan decomposition of \( g \) and 
\( k = k^a \oplus k^s \), where \( k^a \) and \( k^s \) are the abelian and semisimple parts of \( k \) respectively. For \( x \in G \), \( \kappa(x) \in K \), \( H(x) \in \mathfrak{a}_1 \), \( n(x) \in N \) are defined by \( x = \kappa(x) \exp H(x) n(x) \). Let \( \alpha^+ \) be the positive chamber of \( \alpha \) and \( A^+ = \exp(\alpha^+) \). Then \( G = K \cdot \text{Cl}(A^+) \cdot K \). We write \( \omega \) and \( \omega_m \) for the Casimir operators of \( G \) and \( M \) respectively, where \( M \) being the centralizer of \( A \) in \( K \).

§1. Elementary \( \tau \)-spherical functions.

For \( v \in \mathfrak{a}_1^* \) the elementary \( \tau \)-spherical function is defined by

\[\phi(v: x) = \int_K \tau(\kappa(xk)) \overline{\tau(k)} \ e^{i\nu(H(xk))} \ dk.\]

\((1 - 1) \phi(v: x)\) is a \( W \)-invariant entire function of \( v \), and satisfies the differential equation

\[\omega \phi = (\tau(\omega_m) - \langle v, v \rangle - \langle \rho, \rho \rangle) \phi.\]

Since \( G = K \cdot \text{Cl}(A^+) \cdot K \), \( \phi \) is determined by its restriction to \( A^+ \). Let \( \Delta \) be the function on \( A^+ \) defined by

\[\Delta(h) = \prod_{\beta \in \Delta_+} (e^\beta(H) - e^{-\beta(H)}) \quad (h = \exp H \in A^+)\]

and \( \Phi(\omega) \) the radial component of \( \omega \). Then we have
There exist meromorphic functions $\Psi(v; h)$ and $C^T(v)$ such that

$$\Delta(h)\phi(v; h) = \sum_{s \in \mathbb{F}} C^T(sv)\Psi(sv; h) \quad (v \in \mathbb{A}_c^\circ, h \in A^+).$$

Moreover, $\Psi$ satisfies the differential equation

$$(\Delta^2 + \lambda)\Psi = (\tau(w_m) - \langle v, v \rangle - \langle p, p \rangle)\Psi.$$

Let $H_1, H_2, \ldots, H_r$ be the basis of $\mathfrak{a}$ dual to $e_1, e_2, \ldots, e_r$. For $\beta \in P_+$ choose $X_{\pm \beta} \in \mathfrak{g}$ such that $\langle X_{\beta}, X_{-\beta} \rangle = 1$. If

$$X_{\pm \beta} = Y_{\pm \beta} + Z_{\pm \beta}, \quad Y_{\pm \beta} \in \mathfrak{h}, \quad Z_{\pm \beta} \in \mathfrak{K}_c \oplus \mathfrak{V}_c,$$

then

$$\Delta^2 \Phi(w) \Delta^{-1/2} = \tau(w_m) + \sum_{j=1}^{r} H_j^2$$

and

$$(1 - 3) \quad + \frac{1}{\xi} \sum_{\beta \in P_+} \langle \beta, \beta \rangle (sh)^{-2} - \frac{1}{\xi} \sum_{\beta, \gamma \in P_+} \langle \beta, \gamma \rangle (coth \beta)(coth \gamma)$$

$$- 4 \sum_{\beta \in P_+} (1 - ch \beta)(sh \beta)^{-2} \tau(\beta) \tau(-\beta)$$

Let $L$ denote the semilattice of elements $\sum_j a_j$ ($m_j \in \mathbb{Z}_+$). For $\lambda = \sum_j a_j \in L$, we let $m(\lambda) = \sum_j m_j$. Using (1-2) and (1-3) we see that $\Psi$ has a series expansion

$$(1 - 4) \quad \Psi(v; h) = \sum_{\lambda \in L} A_\lambda(v) e^{(iv - \lambda)(H)} \quad (h = \exp H \in A^+).$$

Here $A_\lambda(v)$ ($\lambda \in L$) are rational functions determined by the recurrence relation $A_{\lambda}(v) \equiv 1$,

$$\langle \lambda, \lambda \rangle - 21 v, \lambda \rangle \Psi(v) = 2 \sum_{\beta} \sum_{n \geq 1} (8 \tau(\beta) \tau(-\beta) - \langle \beta, \beta \rangle) e^{(iv - \lambda)(H)}$$

Here $A_\lambda(v)$ ($\lambda \in L$) are rational functions determined by the recurrence relation $A_{\lambda}(v) \equiv 1$,

$$\langle \lambda, \lambda \rangle - 21 v, \lambda \rangle \Psi(v) = 2 \sum_{\beta} \sum_{n \geq 1} (8 \tau(\beta) \tau(-\beta) - \langle \beta, \beta \rangle) e^{(iv - \lambda)(H)}$$
We let $\mathcal{A}^* = \{ \nu \in \mathcal{A}^* : 0 \leq \nu_1 \leq \nu_2 \leq \cdots \leq \nu_r \}$, $\mathcal{A} = \{ \nu \in \mathcal{A}^* : <\lambda, \nu> + 2<\nu, \lambda> \neq 0 \ \forall \lambda \in L \setminus \{0\} \}$.

(1 - 5) For $\eta \in \mathcal{A}^*$ we can find constants $C(\eta), d(\eta) > 0$ such that

$$\| \mathcal{A}_\nu (\nu + \eta) \| \leq C(\eta) m(\lambda)^d(\eta) \ (\nu \in \mathcal{A}^*, \lambda \in L \setminus \{0\}).$$

There exist constants $C, d > 0$ such that

$$\| \mathcal{A}_\nu (\nu + \eta) \| \leq C \cdot m(\lambda)^d \ (\nu \in \mathcal{A}^*, \eta \in \mathcal{A}^*, \lambda \in L \setminus \{0\}).$$

§2. Harish-Chandra's generalized C-function $C^T(\nu)$.

The function $C^T(\nu)$ in (1-2) is meromorphic on $\mathcal{A}_c^*$ and is given by

$$C^T(\nu) = \int_{\mathcal{N}} \tau(\kappa(\mathcal{N})) \ e^{-(1+\rho)(H(\mathcal{N}))} \ d\mathcal{N}.$$

Here we normalize the Haar measure $d\mathcal{N}$ of $\mathcal{N}$ so that the integral of $\exp(-2\rho(H(\mathcal{N})))$ equals one. We shall find an explicit form of $C^T(\nu)$ for our groups. Since the center of $K$ is one dimensional, $\tau$ is parametrized by an integer $k$. We denote this $\tau$ by $\tau_k$.

Example. If $G = SO_0(n+2,2)$, $K = SO(n+2) \text{ SO}(2)$, then

$$\tau_k\left(\begin{bmatrix} k' & \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = e^{i k\theta} \ (k' \in SO(n+2)).$$
To compute the integral (2-1) we use the reduction theory of G. Schiffmann [7]. Let $s_1, s_2, \ldots, s_r$ be the Weyl reflections defined by $\alpha_1, \alpha_2, \ldots, \alpha_r$ respectively, and denote the length of $s \in W$ by $\ell(s)$. For $s \in W$ let $\mathcal{N}(s)$ be the analytic subgroup with Lie algebra $\mathfrak{m}(s) = \sum_{\alpha > 0} q^{-\alpha}$ and let

$$C^\ell(v; s) = \int_{\mathcal{N}(s)} \frac{\tau_\ell(\kappa(\mathbb{R}))}{e^{-(iv+\rho_\mathbb{R})(H(\mathbb{R}))}} d\mathbb{R}.$$

(2 - 2)(G.Schiffmann [7]) If $s = s's''$ with $\ell(s) = \ell(s') + \ell(s'')$, then

$$C^\ell(v; s) = C^\ell(s''v; s')C^\ell(v; s').$$

On the other hand, the element $-1$ of $W$ has the following property.

$$\ell(-1) = r^2$$

and

$$-1 = s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}\cdots s_1s_2^{-1}$$

is a reduced expression of $-1$.

Since $\mathcal{N} = \mathcal{N}(-1)$, the question of computing $C^\ell(v)$ is therefore reduced to finding $C^\ell(v; s_j)$ ($1 \leq j \leq r$). Fix a simple root $\alpha$. Let $\mathfrak{n}_\alpha$ and $\overline{\mathfrak{n}}_\alpha$ be the subalgebras $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$ respectively, and $\mathfrak{g}(\alpha)$ the semisimple subalgebra generated by $\mathfrak{n}_\alpha + \overline{\mathfrak{n}}_\alpha$. We write $\mathcal{N}_\alpha$, $\mathcal{N}_\overline{\alpha}$ and $G(\alpha)$ for the analytic subgroups corresponding to the subalgebras $\mathfrak{n}_\alpha$, $\overline{\mathfrak{n}}_\alpha$ and $\mathfrak{g}(\alpha)$. Then $G(\alpha)$ is a real rank one semisimple Lie group with finite center and has Iwasawa decomposition $G(\alpha) = K_\alpha A_\alpha N_\alpha$, where $K_\alpha = K \cap G(\alpha)$ and $A_\alpha = \exp(\mathfrak{R}H_\alpha)$. Let $X \in \mathfrak{g}_{-\alpha}^\perp$ and $Y \in \mathfrak{g}_{-2\alpha}^\perp$. Finding $K_\alpha$-component
\( \kappa(\mathbf{m}) \) and \( A_{\alpha_j} \)-component \( \exp H(\mathbf{m}) \) of \( \mathbf{m} = \exp(X+Y) \), we can calculate \( C^\ell(\nu; s_j) \).

(2 - 4) Let \( n_j = \dim \mathcal{Q}_j^\ell \) \((1 \leq j \leq r)\). Then

\[
C^\ell(\nu; s_j) = \frac{\Gamma(n_j)}{\Gamma(\frac{n_j}{2})} \exp\left(\frac{\nu}{2} \frac{\alpha_j}{\alpha_j^2 + \frac{n_j}{2}}\right) \frac{\Gamma\left(\frac{\nu}{\alpha_j} + \frac{n_j}{2}\right)}{\Gamma\left(\frac{\nu}{\alpha_j} + \frac{n_j}{2} + \frac{1}{2}\right)} \quad \text{for} \quad 2 \leq j \leq r,
\]

and \( C^\ell(\nu; s_1) \) is given by

\[
C^\ell(\nu; s_1) = \frac{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{\nu}{\alpha_1} + \frac{1}{2} + \frac{1}{2}\right)} \frac{\frac{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2}\right)}{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2} + \frac{1}{2}\right)}}{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2} \frac{1}{2}\right)}
\]
or

\[
C^\ell(\nu; s_1) = \frac{\Gamma(n_1+1)}{\Gamma\left(\frac{n_1+1}{2}\right)} \frac{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2}\right)}{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2} + \frac{1}{2}\right)} \frac{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2} \frac{1}{2}\right)}{\Gamma\left(\frac{\nu}{\alpha_1} + \frac{n_1}{2} \frac{1}{2}\right)}
\]

according as \( G \) is of the first or second kind.

From (2-2), (2-3) and (2-4) we can derive the following result.

**Theorem 1.** \( C^\ell(\nu) \) is expressed by

\[
C^\ell(\nu) = \frac{\Gamma(2m'+1)\Gamma(m)\Gamma(m')}{\Gamma(m'+\frac{1}{2})\Gamma(m')^2} \prod_{j<k} \frac{\Gamma(\nu_j+\nu_k)\Gamma(\nu_k-\nu_j)}{\Gamma(\nu_j+\nu_k+m')\Gamma(\nu_k-\nu_j+m')} \exp\left(\frac{\nu_j+\nu_k}{2}\frac{\nu_j+\nu_k}{\nu_j+\nu_k+m'}\right)
\]

\[
\times \prod_{j=1}^{r} \frac{\Gamma(2\nu_j)}{\Gamma(2\nu_j+m')} \frac{\Gamma(\nu_j+m')\Gamma(\nu_j+m'+1)}{\Gamma(\nu_j+m'+\frac{1}{2})\Gamma(\nu_j+m'+\frac{1}{2}+\frac{1}{2})},
\]

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where \( m \) and \( m' \) are given as follows.

<table>
<thead>
<tr>
<th>( G )</th>
<th>( \text{SO}_0(n+2,2) )</th>
<th>( \text{Sp}(r,R) )</th>
<th>( \text{SO}^*(4r) )</th>
<th>( \text{SO}^*(4r+2) )</th>
<th>( \text{SU}(n+r,r) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>( n )</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>( m' )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>( n )</td>
</tr>
</tbody>
</table>

§3. Fourier transform on \( \mathcal{D}_\mathcal{X}(G) \).

We define the Fourier transform \( \hat{f} \) of \( f \in \mathcal{D}_\mathcal{X}(G) = \mathcal{D}_\mathcal{X}(G) \) by

\[
\hat{f}(\nu) = \int_G f(x)\phi(x^{-1})dx.
\]

For \( R > 0 \), \( \mathcal{D}_\mathcal{X}(R) \) and \( \mathcal{H}_W(R) \) are defined as follows. \( \mathcal{D}_\mathcal{X}(R) \) is the space of those elements in \( \mathcal{D}_\mathcal{X}(G) \) that are supported in \( B_R = \{ x \in G : \sigma(x) \leq R \} \), where \( \sigma \) is the K-bi-invariant continuous function on \( G \) defined by \( \sigma(\exp H) = \| H \| (H \in \mathcal{K}) \). \( \mathcal{H}_W(R) \) is the space of \( W \)-invariant entire functions \( F \) on \( \sigma^*_c \) that satisfy

\[
\forall M \geq 0 \exists C_M > 0 : |F(\nu)| \leq C_M (1 + \| \nu \|^2)^{-M} e^{-R \| \Im \nu \|}.
\]

We write \( \mathcal{H}_W(\sigma^*_c) \) for the union of all \( \mathcal{H}_W(R) \) (\( R > 0 \)).

(3 - 1) If \( f \in \mathcal{D}_\mathcal{X}(R) \), then \( f \in \mathcal{H}_W(R) \).

Now we let

\[
\mu^k(\nu) = (C^k(\nu)C^k(-\nu))^{-1}.
\]

This is a meromorphic function on \( \sigma^*_c \). In fact, Theorem 1 implies that

\[
\mu^k(\nu) = \begin{cases} 
X_k(\nu)Y(\nu) & (2|m) \\
X_k(\nu)Z(\nu) & (2|m')
\end{cases},
\]

where
\[ X_k(v) = \frac{4^m r(m+\frac{1}{2})^2 r(m)^2 r(r-1)}{\Gamma(2m+1)^2 \Gamma(m)^2 r(r-1)} \times \prod_{j=1}^{r} \left\{ v_j^2 \pi(v_j + \frac{1}{2}) \prod_{p=1}^{m} (v_j^2 + \frac{1}{2} - m) \right\}, \]

\[ Y(v) = \prod_{j<k} \prod_{p=1}^{\frac{m}{2}} ((v_j + v_k)^2 + (\frac{m}{2} - p)^2)((v_j - v_k)^2 + (\frac{m}{2} - p)^2) \]

and

\[ Z(v) = \prod_{j<k} \left\{ (v_j^2 - v_k^2) \pi(v_j + v_k) \pi(v_j - v_k) \right\} \times \prod_{p=1}^{\frac{m}{2}} ((v_j + v_k)^2 + (\frac{m}{2} - p)^2)((v_j - v_k)^2 + (\frac{m}{2} - p)^2) \].

As a function of \( v_r \), \( \mu^k(v) \) has infinite simple poles on the imaginary axis. Let \( \Pi_1 = \Gamma_1 \) be the set of those poles that are between 0 and \( \frac{1}{2}(m'-m)/2 \). For \( a \in \Gamma_1 \) we let

\[ \mu^k_a(v(a)) = -2\pi i \text{Res}[\mu^k(v) : v_r = a], \]

where \( v(a) \) denoting \( (v_1, v_2, \cdots, v_{r-1}) \). As a function of \( v_{r-1} \), \( \mu^k_a(v(a)) \) has simple poles on the imaginary axis. Let \( \Pi_a \) be the set of those poles that are between 0 and \( a \), and put \( \Gamma_2 = \{(a, b) : a \in \Pi_1, b \in \Pi_a \} \). For \( \mathfrak{p} = (a, b) \in \Gamma_2 \) we let

\[ \mu^k_{\mathfrak{p}}(v(\mathfrak{p})) = -2\pi i \text{Res}[\mu^k_a(v(a)) : v_{r-1} = b], \]

where \( v(\mathfrak{p}) \) denoting \( (v_1, v_2, \cdots, v_{r-2}) \). In this manner we define \( \Gamma_1, \Gamma_2, \cdots, \Gamma_p, \Pi_{\mathfrak{p}} \) and \( \mu^k_{\mathfrak{p}}(v(\mathfrak{p})) \). For simplicity we let \( \Gamma_0 = \{0\} \), \( \mu^k_0(v(0)) = \mu^k(v) \) and \( \Gamma = \bigcup(\Gamma_p : 0 \leq p \leq r) \). For \( \mathfrak{p} = (a_1, a_2, \cdots, a_p) \),
\( a_p \in \Gamma_p \) we let \( \mathcal{P} = (a_p, \ldots, a_r) \), \( R(\mathcal{P}) = \mathbb{R}^{r-p} \) and let \( \mathcal{W} \) denote the subgroup of \( \mathcal{W} \) composed of those elements that leave \( \nu_j \) 
\((r-p < j \leq r)\) fixed.

(3 - 2) For every \( f \in \Gamma \), \( \mu^g_f(\nu(\mathcal{P})) \) is a \( \mathcal{W} \)-invariant meromorphic function which is positive-valued on \( R(\mathcal{P}) \).

For \( F \in \mathcal{N}_W(\mathcal{P}_c) \) we let

\[
\mathcal{F}(F; x) = \sum_{\mathcal{P} \in \Gamma} \int_{R(\mathcal{P})} F(\nu(\mathcal{P}), \mathcal{P}, \mathcal{P}'; x) \mu^g_f(\nu(\mathcal{P})) d\nu(\mathcal{P}).
\]

We say that \( G \) has property (S) if the following condition is satisfied:

If \( F \in \mathcal{N}_W(\mathcal{P}_c) \), then \( \mathcal{F}(F; \cdot) \) is supported in \( B_R \).

(3 - 3) If \( r = 1 \) then \( G \) has property (S).

This is proved using the results obtained in the previous sections.

Now we assume property (S) and continue our discussion. Since \( \tau \)-spherical functions are completely determined by their restrictions to \( A \), we can regard a linear functional on \( \mathcal{D}(G) \) as a \( \mathcal{W} \)-invariant distribution on \( A \). Define the linear functional \( T \) on \( \mathcal{D}(G) \) by

\[
T_f = \mathcal{F}(\mathcal{P}; 1).
\]

Let \( F_0 \) be a function in \( \mathcal{N}_W(1) \) such that \( F_0(0) = 1 \). Then

\[
T_f = \lim_{\varepsilon \to 0} \mathcal{F}(\mathcal{P}; F_0(\varepsilon \cdot); 1) = \lim_{\varepsilon \to 0} \int_G f(x) g_\varepsilon(x) dx,
\]

where \( g_\varepsilon(x) = \mathcal{F}(F_0(\varepsilon \cdot); x^{-1}) \). Property (S) implies that \( g_\varepsilon \) is supported in \( B_\varepsilon \). Since \( T = \lim_{\varepsilon \to 0} \) as a distribution, \( T \) must have support \( \{1\} \). Moreover we have
(3 - 4) \( T \) is a positive measure with support \( \{1\} \).

Hence there exists a positive constant \( \gamma > 0 \) such that
\[
Tf = \gamma \cdot f(1) \quad (f \in L^2(G)).
\]

(3 - 6) For \( f \in L^2(G) \) we have
\[
\gamma \cdot f(x) = \mathcal{F}(f: x) \quad (x \in G),
\]
\[
\gamma \cdot \|f\|_{L^2(G)}^2 = \mathcal{F}(|\hat{f}|)^2 : 1.
\]

The set \( \{\hat{f} : f \in L^2(G)\} \) is dense in the space \( C_0(\Omega) \) of continuous functions on \( \Omega \) which vanish at infinity. Here \( \Omega \) is the support of the Plancherel measure.

**Theorem 2.** Assume that \( G \) has property \((S)\). Then the map \( f \mapsto \hat{f} \) is a linear isomorphism of \( L^2(G) \) onto \( \mathcal{H}_W(\pi_c) \). More precisely, for every \( R > 0 \) \( L^2(G) \) is transformed onto \( \mathcal{H}_W(R) \).

Proof. Let \( F \in \mathcal{H}_W(R) \) and define \( f \) by \( \gamma \cdot f(x) = \mathcal{F}(F: x) \). Then property \((S)\) implies that \( f \in L^2(G) \). Let \( F' = F - \hat{f} \). We must prove that \( F' \) vanishes identically. But it follows from the definition of \( f \) that
\[
\mathcal{F}(F': x) = 0 \quad \text{for all} \quad x \in G.
\]

Hence
\[
\mathcal{F}(F' : \hat{g}; 1) = \int_G \mathcal{F}(F': x)g(x)dx = 0 \quad \text{for all} \quad g \in L^2(G).
\]

\( \{\hat{g} : g \in L^2(G)\} \) is dense in \( C_0(\Omega) \); so \( F' \) vanishes on \( \Omega \). Since \( F' \) is holomorphic, \( F' \) vanishes identically.
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