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AN INTEGRAL REPRESENTATION OF THE HARISH-CHANDRA SERIES ON $SO_0(n,1)$

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§0. Introduction

Let G be a connected noncompact real form of a connected complex semisimple Lie group G_c and assume that G is of split rank one. Let K be a maximal compact subgroup of G and G=KAN be an Iwasawa decomposition. Then dimA = 1. Choose an element H_0 of the Lie algebra $\mathfrak{a}$ of A so that eigenvalues of $ad(H_0)$ are $\{0,\pm 1\}$ or $\{0,\pm 1,\pm 2\}$ and put $a_t = \exp(tH_0)$. Then the Iwasawa decomposition of an element $x \in G$ is

$$x=k(x)a_{t(x)}n(x), \quad \text{where } k(x)\in K \text{ and } n(x)\in N.$$

Let M be the centralizer of A in K. Then MAN is a minimal parabolic subgroup of G. Let (τ_1,τ_2) be a double unitary representation of K on a finite dimensional Hilbert space V and denote by V_M the subspace of V comprised of those elements v which have the property that $\tau_1(m)v=v\tau_2(m)$ (all $m\in M$). The Eisenstein integral $E(s,v,x)$ for MAN is defined by the following formula([2],[4]);

(1) $$E(s,v,x)= \int_K e^{(s-\rho)t(xk)}\tau_1(k(xk))v\tau_2(k^{-1})dk,$$

When $G=SU(1,1)$, by the change of variable $z=(\mathrm{th}(\frac{t}{2}))^2$, we can see that the Eisenstein integral $E(s,v,a_t)$ coincide with certain hypergeometric function $F(a,b;c;z)$ (up to a constant factor) and the formula (1) gives a its integral representation of Euler type. Moreover the expansion (2) corresponds to the following formula;

$$F(a,b;c;z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)$$

$$+\ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b;c-a-b+1;1-z).$$

Hence the Harish-Chandra series $E(s,t)$ has an integral representation of Euler type.

In general, Does the series $E(s,t)$ have an integral representation? When G is the general Lorentz group $SO_0(n,1)$, we can fined an integral representation of $E(s,t)$ which is a singular integral over certain noncompact real form H of the complexification K_c of K (Theorem 8.)

Main idea is to consider the analytic continuation of the Iwasawa decomposition and to find a noncompact real form of K_c with certain properties.

The present note is a sketch of results and we shall give proofs in [3].

§1. Analytic continuation of the Iwasawa decomposition

Let G be the general Lorentz group $SO_0(n,1)$ $(n\geq 2)$. Denote by $\mathfrak{g}$, $\mathfrak{g}_c$ the Lie algebra of G and its complexification respectively. We denote by G_c the complex analytic subgroup of $GL(n+1,C)$ with Lie algebra $\mathfrak{g}_c$. Let K,M,A,N be the same as in §0.

for a complex number s, $v \in V_M$ and $x \in G$,

where dk is the normalized Haar measure on K and ρ is a real number obtained from the modular function δ of MAN by $\delta(ma_t n) = e^{-2\rho t}$.

The Eisenstein integral $E(s,v,a_t)$ has the following series expansion ([1],[5]).

Theorem 0.(Harish-Chandra)

There is an open connected dense subset $O(\tau_1,\tau_2)$ of $\mathbb{C}$ which is stable under the action $s \to -s$ and are functions $C_e, C_w, A_k (k=0,1,..)$ with all values in $\mathrm{End}(V_M)$ such that

1) The complement of $O(\tau_1,\tau_2)$ is a discrete set.

2) The functions C_e, C_w (resp. A_k (k=o,1,..)) are meromorphic (resp. rational) on $\mathbb{C}$ and holomorphic on $O(\tau_1,\tau_2)$.

3) Fix any compact subset B of $O(\tau_1,\tau_2)$ and $r>0$. Then, for each integers $i \geq 0$, $j \geq 0$, the series

$$\sum_{k=0}^{\infty} \left(\frac{\partial}{\partial s}\right)^i \left(\frac{\partial}{\partial t}\right)^j \left(A_k(s)e^{-kt}\right)$$

is absolutely and uniformly convergent on $B \times [r,\infty)$.

4) Put $E(s,t) = e^{(s-\rho)t} \sum_{k=0}^{\infty} A_k(s)e^{-kt}$. Then

(2) $E(s,v,a_t) = E(s,t)C_e(s)v + E(-s,t)C_w(s)v$,

for $s \in O(\tau_1,\tau_2)$, $v \in V_M$ and $t>0$.

In this note, we call E(s,t) the Harish- Chandra series of the Eisenstein integral $E(s,v,a_t)$.

For any closed subgroup of G we use the similar notations; for example, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ are the Lie algebras of K,M,A,N respectively. Choose aelement $H_0 \in \mathfrak{a}$ with the same property as in §0 and put

$$'A_c = \{ a_z = \exp(zH_0);\ \ z \text{ is a complex number with } |\mathrm{Im}(z)| < \pi \},$$

$$'G_c = K_c\, 'A_c N_c .$$

Then we obtain from the explicit formula of the Iwasawa decomposition the following results.

Lemma 1.

$'G_c$ is an open connected dense submanifold of G_c and there are holomorphic mappings

$$k : 'G_c \longrightarrow K_c, \quad t : 'G_c \longrightarrow \mathbb{C}, \quad n : 'G_c \longrightarrow N_c$$

such that for each $x \in 'G_c$ the decomposition

(3) $\qquad x = k(x) a_{t(x)} n(x)$

exists and is unique. Moreover if $x \in G$ then this decomposition coincide with the Iwasawa decomposition.

Let θ be the Cartan involution with respect to the pair $(\mathfrak{g}, \mathfrak{k})$ and put

$$\mathfrak{q} = \{ X + \theta X \ ; \quad X \in \mathfrak{n} \}.$$

Then $\mathfrak{k} = \mathfrak{m} + \mathfrak{q}$, $[\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{m}$, $[\mathfrak{m}, \mathfrak{q}] \subset \mathfrak{q}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$. Therefore the real subspace $\mathfrak{h} = \mathfrak{m} + \sqrt{-1}\,\mathfrak{q}$ is a noncompact real form $\mathfrak{k}_c$. Let H be the analytic subgroup of K_c with Lie algebra $\mathfrak{h}$. Then H is isomorphic to $SO_0(n-1,1)$ and M is a maximal compact subgroup of H.

Lemma 2.

For any $t \geq 0$ *and* $h \in H$ $a_t h$ *belongs to* $'G_c$.

Since any finite dimensional representation of K can be extended to a holomorphic representation of K_c, we may regard a double unitary representation of K as a holomorphic double representation of K_c. Hence the function

$$e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h)) v \tau_2(h^{-1})$$

is well-defined.

Now we cosider the following integral;

$$(4) \qquad F(s,t)v = \int_H e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h)) v \tau_2(h^{-1}) dh,$$

for a complex number s, $v \in V_M$ and $t > 0$,

where dh is a Haar measure on H.

Lemma 3.

There is a real number c *depending only on* (τ_1, τ_2) *such that if* $Re(s) < c$ *then the integral* (4) *converges for each* $t > 0$ *and* $v \in V_M$, *and gives a linear endomorphism* $F(s,t)$: $v \longrightarrow F(s,t)v$ *of* V_M. *Moreover there exist meromorphic functions* $F_k (k=0,1,2,\ldots)$ *with values in* $End(V_M)$ *such that*

1) $F_k (k=0,1,2,\ldots)$ *are all holomorphic on the half space* $\{s;\ Re(s) < c\}$.

2) *For any compact subset* B *of* $\{s;\ Re(s) < c\}$ *and* $r > 0$, *the series*

$$\sum_{k=0}^{\infty} F_k(s) e^{-kt}$$

is absolutely and uniformly convergent on $B\times[r,\infty)$ and

$$F(s,t)= e^{(s-\rho)t}\sum_{k=0}^{\infty} F_k(s)e^{-kt},$$

for $t>0$ and $Re(s)<c$.

Corollary 4.

Fix $t>0$. Then the function $s \longrightarrow F(s,t)$ can be extended meromorphiclly onto the whole plane. Moreover if $Re(s)<c$ then the limit $\lim_{t\to\infty} e^{(\rho-s)t}F(s,t)=F_0(s)$ exists.

Most complicated part in proofs of these results is to show that the function $e^{(\rho-s)t}F(s,t)$ is a linear combination of integrals in the followins form;

$$I_{p,q}(s,t)= \int_0^1 r^{2\ -1}(1-r^2)^{-(s+\rho)}(1-x^2r^2)^{s-\rho}\left(\frac{1+xr}{1-xr}\right)^p\left(\frac{1-r}{1+r}\right)^q dr,$$

where $x=e^{-t}$ and p,q are integers.

§2. Differential equations satisfied by $E(s,t)$, $F(s,t)$.

Let $\mathcal{G}$, $\mathcal{K}$ be the universal enveloping algebra of $\mathfrak{g}_c$, $\mathfrak{k}_c$ respectively. Denote by $\Delta(D)$ " the radial part" of $D\in\mathcal{G}$ in the sense of Chap.9 of [5]. Then, in our case, we may regard $\Delta(D)$ as an ordinary differential operator on $(0,\infty)$ whose coefficients are all real analytic functions with values in End(V). Let $\mathfrak{Z}$ be the center of $\mathcal{G}$ and denote by $\Omega(Z,s)$ ($Z\in\mathfrak{Z}$, s is a complex number) an element introduced in [5](p.283). Then

$$\Delta(Z)(E(s,t))=E(s,t)\tau_2(\Omega(Z,s-\rho)),$$

for each $s\in O(\tau_1,\tau_2)$ and $Z\in\mathfrak{Z}$,

and

(5) $\Delta(Z)(E(s,v,a_t))=E(s,v\tau_2(\Omega(Z,s-\rho)),a_t)$

for any complex number s and $Z\in\mathfrak{Z}$.

Since H is a real form of K_c and the function

$$f(x)=e^{(s-\rho)t(x)}\tau_1(k(x)) \quad (x\in{}'G_c)$$

is holomorphic on $'G_c$ which satisfies

$$f(kxman)=e^{(s-\rho)t(a)}\tau_1(k)f(x)\tau_1(m),$$

for $k\in K_c$, $x\in{}'G_c$, $m\in M_c$, $a\in{}'A_c$, $n\in N_c$ with $kxman\in{}'G_c$,

by similar arguments in the proof of (5), we have the next lemma

Lemma 5.

For each $Z\in\mathfrak{Z}$, there is a real number c' such that if $Re(s) < c'$ then

$$\Delta(z)(F(s,t))=F(s,t)\tau_2(\Omega(Z,s-\rho)).$$

Noting that the following properties characterize the function $f_s(t)=E(s,t)$ ($s\in O(\tau_1,\tau_2)$ fixed) (Chap.9 of [5])

a) $f_s(t)= e^{(s-\rho)t}\sum_{k=0}^{\infty}A_k(s)e^{-kt}$,

b) The limit $A_0(s)=\lim_{t\to\infty}e^{(\rho-s)t}f_s(t)$ exists and coincides with the idetity operator of V_M,

c) Let ω be the Casimir operator of G. Then

$$\Delta(\omega)f_s(t)=f_s(t)\tau_2(\Omega(\omega,s-\rho)),$$

we have from Lemma 3. and Lemma 5. the next lemma.

Lemma 6.

For each $t>0$, $F(s,t)=E(s,t)F_0(s)$ *as* $End(V_M)$-*valued meromorphic functions*.

Calulations of $F_0(s)$ is very difficult except special case but the next Lemma is hold.

Lemmma 7.

There is a constant c_0 *depending only on normalizations of a Haar measure of H such that*

$$F_0(s)=c_0((\sin\pi s)/(\sin\pi(s+\rho)))C_e(s),$$

as meromorphic functions.

Thus we have the following theorem.

Theorem 8.

There is a real number c *such that if* $Re(s)<c$ *then the integral*

$$F(s,t)v=\int_H e^{(s-\rho)t(a_t h)}\tau_1(k(a_t h))v\tau_2(h^{-1})dh$$

is absolutely convergent for each $t>0$, $v\in V_M$. *Moreover for fixed* $t>0$ *the function* $s\longrightarrow F(s,t)$ *can be extended to a* $End(V_M)$-*valued meromorphic function on* $\mathbb{C}$ *and*

$$F(s,t)=c_0((\sin\pi s)/(\sin\pi(s+\rho)))E(s,t)C_e(s).$$

Remark.

When G=SU(1,1), the formula (4) gives an integral representation of Euler type. More generally, if the Eisenstein integral $E(s,v,a_t)$ coincide with a hypergeometric function then the formula (4) gives an integral representation of Euler type.

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