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Kyoto University
AN INTEGRAL REPRESENTATION OF THE HARISH-CHANDRA SERIES ON $SO(n,1)$

By

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§0. Introduction

Let $G$ be a connected noncompact real form of a connected complex semisimple Lie group $G_c$ and assume that $G$ is of split rank one. Let $K$ be a maximal compact subgroup of $G$ and $G=KAN$ be an Iwasawa decomposition. Then $\dim A = 1$. Choose an element $H_0$ of the Lie algebra $\mathfrak{a}$ of $A$ so that eigenvalues of $\text{ad}(H_0)$ are $\{0, \pm 1\}$ or $\{0, \pm 1, \pm 2\}$ and put $a_t=\exp(tH_0)$. Then the Iwasawa decomposition of an element $x\in G$ is

$$x=k(x)a_t(x)n(x), \text{ where } k(x)\in K \text{ and } n(x)\in N.$$ 

Let $M$ be the centralizer of $A$ in $K$. Then $MAN$ is a minimal parabolic subgroup of $G$. Let $(\tau_1, \tau_2)$ be a double unitary representation of $K$ on a finite dimensional Hilbert space $V$ and denote by $V_M$ the subspace of $V$ comprised of those elements $v$ which have the property that $\tau_1(m)v=v\tau_2(m)$ (all $m\in M$). The Eisenstein integral $E(s,v,x)$ for $MAN$ is defined by the following formula([2],[4]);

$$E(s,v,x) = \int_M e(s-p)t(xk)\tau_1(k(xk))v\tau_2(k^{-1})dk,$$
When \( G = SU(1,1) \), by the change of variable \( z = (\text{th}(\frac{t}{2}))^2 \), we can see that the Eisenstein integral \( E(s,v,a_t) \) coincide with certain hypergeometric function \( F(a,b;c;z) \) (up to a constant factor) and the formula (1) gives its integral representation of Euler type. Moreover the expansion (2) corresponds to the following formula:

\[
F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(a)\Gamma(c-b)} F(a,b;c-a-b+1;1-z) + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-z).
\]

Hence the Harish-Chandra series \( E(s,t) \) has an integral representation of Euler type.

In general, Does the series \( E(s,t) \) have an integral representation?

When \( G \) is the general Lorentz group \( SO_0(n,1) \), we can find an integral representation of \( E(s,t) \) which is a singular integral over certain noncompact real form \( H \) of the complexification \( K_c \) of \( K \) (Theorem 8.)

Main idea is to consider the analytic continuation of the Iwasawa decomposition and to find a noncompact real form of \( K_c \) with certain properties.

The present note is a sketch of results and we shall give proofs in [3].

\section{1. Analytic continuation of the Iwasawa decomposition}

Let \( G \) be the general Lorentz group \( SO_0(n,1) \) \((n \geq 2)\). Denote by \( g, g_c \) the Lie algebra of \( G \) and its complexification respectively. We denote by \( G_c \) the complex analytic subgroup of \( GL(n+1,\mathbb{C}) \) with Lie algebra \( g_c \). Let \( K, M, A, N \) be the same as in \( \S 0 \).
for a complex number \( s \), \( v \in V_\mathbb{M} \) and \( x \in G \),
where \( d_k \) is the normalized Haar measure on \( K \) and \( \rho \) is a real number
obtained from the modular function \( \delta \) of \( \text{MAN} \) by \( \delta(ma_\mathbb{C}n) = e^{-2\rho t} \).
The Eisenstein integral \( E(s, v, a_t) \) has the following series expansion ([1],[5]).

**Theorem 0. (Harish-Chandra)**

There is an open connected dense subset \( O(\tau_1, \tau_2) \) of \( \mathcal{O} \) which is
stable under the action \( s \rightarrow -s \) and are functions \( C_e, C_w, A_k (k=0,1,\ldots) \)
with all values in \( \text{End}(V_\mathbb{M}) \) such that
1) The complement of \( O(\tau_1, \tau_2) \) is a discrete set.
2) The functions \( C_e, C_w \) (resp. \( A_k (k=0,1,\ldots) \)) are meromorphic (resp. rational) on \( \mathcal{O} \) and holomorphic on \( O(\tau_1, \tau_2) \).
3) Fix any compact subset \( B \) of \( O(\tau_1, \tau_2) \) and \( r>0 \). Then, for each
integers \( j \geq 0 \), \( i \geq 0 \), the series
\[
\sum_{k=0}^{\infty} \left( \frac{2}{s} \right)^i \left( \frac{2}{t} \right)^j A_k(s)e^{-kt}
\]
is absolutely and uniformly convergent on \( B \times [r, \infty) \).
4) Put \( E(s,t) = e(s-\rho t) \sum_{k=0}^{\infty} A_k(s)e^{-kt} \). Then
\[
(2) \quad E(s,v,a_t) = E(s,t)C_e(s)v + E(-s,t)C_w(s)v,
\]
for \( s \in O(\tau_1, \tau_2) \), \( v \in V_\mathbb{M} \) and \( t>0 \).

In this note, we call \( E(s,t) \) the **Harish- Chandra series** of the
Eisenstein integral \( E(s,v,a_t) \).
For any closed subgroup of $G$ we use the similar notations; for example, $\mathfrak{K}, \mathfrak{M}, \mathfrak{A}, \mathfrak{N}$ are the Lie algebras of $K,M,A,N$ respectively. Choose an element $H_0 \in \mathfrak{H}$ with the same property as in §0 and put

$$A_c = \{ a \in \mathfrak{A} : z = \exp(zH_0) \}, \quad z \text{ is a complex number with } |\operatorname{Im}(z)| < \pi,$$

$$G_c = K_c A_c N_c.$$

Then we obtain from the explicit formula of the Iwasawa decomposition the following results.

**Lemma 1.**

$G_c$ is an open connected dense submanifold of $G_c$ and there are holomorphic mappings

$$k : G_c \rightarrow K_c, \quad t : G_c \rightarrow \mathfrak{C}, \quad n : G_c \rightarrow N_c$$

such that for each $x \in G_c$ the decomposition

(3) $x = k(x) a(x) t(x) n(x)$

exists and is unique. Moreover if $x \in G$ then this decomposition coincide with the Iwasawa decomposition.

Let $\Theta$ be the Cartan involution with respect to the pair $(\mathfrak{A}, \mathfrak{K})$ and put

$$\mathfrak{H} = \{ X + \Theta X : X \in \mathfrak{H} \}.$$ Then $\mathfrak{K} = \mathfrak{M} + \mathfrak{H}, \quad \mathfrak{A} \subset \mathfrak{M}, \quad [\mathfrak{A}, \mathfrak{A}] \subset \mathfrak{H}$ and $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{M}$. Therefore the real subspace $\mathfrak{H} = \mathfrak{M} + [-1] \mathfrak{H}$ is a noncompact real form $\mathfrak{H}_c$. Let $H$ be the analytic subgroup of $K_c$ with Lie algebra $\mathfrak{H}_c$. Then $H$ is isomorphic to $SO_0(n-1,1)$ and $M$ is a maximal compact subgroup of $H$.
Lemma 2.

For any $t \geq 0$ and $h \in H$, $a_t h$ belongs to $G$.

Since any finite dimensional representation of $K$ can be extended to a holomorphic representation of $K_c$, we may regard a double unitary representation of $K$ as a holomorphic double representation of $K_c$. Hence the function

$$e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h))v \tau_2(h^{-1})$$

is well-defined.

Now we consider the following integral;

\begin{equation}
F(s, t)v = \int_H e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h))v \tau_2(h^{-1})dh,
\end{equation}

for a complex number $s$, $v \in V_M$ and $t > 0$,
where $dh$ is a Haar measure on $H$.

Lemma 3.

There is a real number $c$ depending only on $(\tau_1, \tau_2)$ such that if $\Re(s) < c$ then the integral (4) converges for each $t > 0$ and $v \in V_M$, and gives a linear endomorphism $F(s, t)$ on $V_M$.

Moreover there exist meromorphic functions $F_k (k=0,1,2,\ldots)$ with values in $\text{End}(V_M)$ such that

1) $F_k (k=0,1,2,\ldots)$ are all holomorphic on the half space \{ $s$; $\Re(s) < c$ \}.

2) For any compact subset $B$ of \{ $s$; $\Re(s) < c$ \} and $r > 0$, the series

$$\sum_{k=0}^{\infty} F_k(s)e^{-kt}$$

 converges uniformly for $s \in B$. 

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is absolutely and uniformly convergent on $B(x, r, \infty)$ and

$$F(s, t) = e^{(s-p)t} \sum_{k=0}^{\infty} F_k(s)e^{-kt},$$

for $t > 0$ and $\text{Re}(s) < c$.

**Corollary 4.**

Fix $t > 0$. Then the function $s \mapsto F(s, t)$ can be extended meromorphically onto the whole plane. Moreover if $\text{Re}(s) < c$ then the limit $\lim_{t \to \infty} e^{(s-p)t}F(s, t) = F_0(s)$ exists.

Most complicated part in proofs of these results is to show that the function $e^{(s-p)t}F(s, t)$ is a linear combination of integrals in the followins form;

$$I_{p, q}(s, t) = \int_0^1 r^2 - (1-r^2) - (s+p)(1-x^2)r^2_s-x^2r^2_s-x^2r^2_s-p(1-xr)^p(1-r)^q dr,$$

where $x = e^{-t}$ and $p, q$ are integers.

§2. Differential equations satisfied by $E(s, t)$, $F(s, t)$.

Let $G$, $G_c$ be the universal enveloping algebra of $\mathfrak{g}$, $\mathfrak{g}_c$ respectively. Denote by $\Delta(D)$ "the radial part" of $D \in G$ in the sense of Chap.9 of [5]. Then, in our case, we may regard $\Delta(D)$ as an ordinary differential operator on $(0, \infty)$ whose coefficients are all real analytic functions with values in $\text{End}(V)$. Let $\mathfrak{g}$ be the center of $\mathfrak{g}$ and denote by $\Omega(Z, s)$ ($Z \in \mathfrak{g}$, $s$ is a complex number) an element introduced in [5](p.283). Then

$$\Delta(Z)(E(s, t)) = E(s, t) \mathcal{C}_Z(\Omega(Z, s-p)).$$

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for each \( s \in \mathbb{O}(\tau_1', \tau_2') \) and \( z \in \mathcal{H} \),

and

\[
(5) \quad \Delta(z)(E(s,v,a_t)) = E(s,v \tau_2'((\Omega(z,s-\rho)),a_t)
\]

for any complex number \( s \) and \( z \in \mathcal{H} \).

Since \( H \) is a real form of \( K_c \) and the function

\[
f(x) = e^{(s-\rho)t(x) \tau_1'(k(x))}(x \in G_c)
\]

is holomorphic on \( G_c \) which satisfies

\[
f(kx = e^{(s-\rho)t(a) \tau_1'(k)f(x) \tau_1'(m)},
\]

for \( k \in K_c, \ x \in G_c, \ m \in M_c, \ a \in A_c, \ m \in N_c \) with \( kx \in G_c \),

by similar arguments in the proof of (5), we have the next lemma

**Lemma 5.**

For each \( z \in \mathcal{H} \), there is a real number \( c' \) such that if \( \text{Re}(s) < c' \), then

\[
\Delta(z)(F(s,t)) = F(s,t) \tau_2'((\Omega(z,s-\rho)).
\]

Noting that the following properties characterize the function

\[
f_s(t) = E(s,t) \ (s \in \mathbb{O}(\tau_1', \tau_2') \text{ fixed}) \ (\text{Chap. 9 of [5]})
\]

a) \( f_s(t) = e^{(s-\rho)t} \sum_{k=0}^{\infty} A_k(s)e^{-kt} \)

b) The limit \( A_0(s) = \lim_{t \to \infty} e^{(s-\rho)t}f_s(t) \) exists and coincides with the identity operator of \( V_M \).

c) Let \( \omega \) be the Casimir operator of \( G \). Then

\[
\Delta(\omega)f_s(t) = f_s(t) \tau_2'((\Omega(\omega,s-\rho)),
\]

we have from Lemma 3. and Lemma 5. the next lemma.

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Lemma 6.

For each $t>0$, $F(s,t) = E(s,t)F_0(s)$ as $\text{End}(V_M)$-valued meromorphic functions.

Calculations of $F_0(s)$ is very difficult except special case but the next Lemma is hold.

Lemma 7.

There is a constant $c_0$ depending only on normalizations of a Haar measure of $H$ such that

$$F_0(s) = c_0 \left( \frac{\sin \pi s}{\sin \pi (s+\rho)} \right) C \epsilon(s),$$

as meromorphic functions.

Thus we have the following theorem.

Theorem 8.

There is a real number $c$ such that if $\Re(s) < c$ then the integral

$$F(s,t)v = \int_H e^{(s-\rho)t(a_t h)} \mathcal{C}_1(k(a_t h)) \mathcal{C}_2(h^{-1}) dh$$

is absolutely convergent for each $t>0$, $v \in V_M$. Moreover for fixed $t>0$ the function $s \mapsto F(s,t)$ can be extended to a $\text{End}(V_M)$-valued meromorphic function on $\mathbb{C}$ and

$$F(s,t) = c_0 \left( \frac{\sin \pi s}{\sin \pi (s+\rho)} \right) E(s,t)C \epsilon(s).$$
Remark.

When $G = \text{SU}(1,1)$, the formula (4) gives an integral representation of Euler type. More generally, if the Eisenstein integral $E(s,v,a_u)$ coincide with a hypergeometric function then the formula (4) gives an integral representation of Euler type.

REFERENCES.