Title
Lectures on harmonic analysis on Lie groups and related topics (An integral representation of the Harish-Chandra series on SO_0(n,1). / M. Mamiuda)

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Citation
Lectures in Mathematics (1982), 14

Issue Date
1982

URL
http://hdl.handle.net/2433/84919

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Type
Book

Textversion
publisher Kyoto University
AN INTEGRAL REPRESENTATION OF THE HARISH-CHANDRA SERIES ON $SO_0(n,1)$

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§0. Introduction

Let $G$ be a connected noncompact real form of a connected complex semisimple Lie group $G_c$ and assume that $G$ is of split rank one. Let $K$ be a maximal compact subgroup of $G$ and $G=KAN$ be an Iwasawa decomposition. Then $\text{dim} A = 1$. Choose an element $H_0$ of the Lie algebra $\mathfrak{a}$ of $A$ so that eigenvalues of $\text{ad}(H_0)$ are $\{0,1\}$ or $\{0,\pm 1,\pm 2\}$ and put $a_t=\exp(tH_0)$. Then the Iwasawa decomposition of an element $x \in G$ is

$$x=k(x)a_t(x)n(x), \text{ where } k(x) \in K \text{ and } n(x) \in N.$$  

Let $M$ be the centralizer of $A$ in $K$. Then $MAN$ is a minimal parabolic subgroup of $G$. Let $(\pi_1, \pi_2)$ be a double unitary representation of $K$ on a finite dimensional Hilbert space $V$ and denote by $V_M$ the subspace of $V$ comprised of those elements $v$ which have the property that $\pi_1(m)v=v\pi_2(m)$ (all $m \in M$). The Eisenstein integral $E(s,v,x)$ for $MAN$ is defined by the following formula([2],[4]);

(1)  
$$E(s,v,x)=\int_K e(s-p)t(xk)\pi_1(k(xk))v\pi_2(k^{-1})dk,$$
When $G = SU(1,1)$, by the change of variable $z = (\theta(\frac{t}{2}))^2$, we can see that the Eisenstein integral $E(s,v,a_t)$ coincide with certain hypergeometric function $F(a,b;c;z)$ (up to a constant factor) and the formula (1) gives an integral representation of Euler type. Moreover, the expansion (2) corresponds to the following formula:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b; a+b-c+1; 1-z)$$

$$+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a, c-b; c-a-b+1; 1-z).$$

Hence the Harish-Chandra series $E(s,t)$ has an integral representation of Euler type.

In general, does the series $E(s,t)$ have an integral representation? When $G$ is the general Lorentz group $SO_0(n,1)$, we can find an integral representation of $E(s,t)$ which is a singular integral over certain noncompact real form $H$ of the complexification $K_c$ of $K$ (Theorem 8.)

Main idea is to consider the analytic continuation of the Iwasawa decomposition and to find a noncompact real form of $K_c$ with certain properties.

The present note is a sketch of results and we shall give proofs in [3].

§ 1. Analytic continuation of the Iwasawa decomposition

Let $G$ be the general Lorentz group $SO_0(n,1)$ $(n \geq 2)$. Denote by $g$, $g_c$ the Lie algebra of $G$ and its complexification respectively. We denote by $G_c$ the complex analytic subgroup of $GL(n+1,\mathbb{C})$ with Lie algebra $g_c$. Let $X,M,A,N$ be the same as in §0.
for a complex number $s$, $v \in V_M$ and $x \in G$, where $d_k$ is the normalized Haar measure on $K$ and $\rho$ is a real number obtained from the modular function $\delta$ of $\text{MAN}$ by $\delta(ma_n) = e^{-2\rho t}$.

The Eisenstein integral $E(s, v, a_t)$ has the following series expansion ([1],[5]).

**Theorem 0. (Harish-Chandra)**

There is an open connected dense subset $\mathcal{O}(\tau_1, \tau_2)$ of $\mathcal{O}$ which is stable under the action $s \rightarrow -s$ and are functions $C_e, C_w, A_k (k=0,1,\ldots)$ with all values in $\text{End}(V_M)$ such that

1) The complement of $\mathcal{O}(\tau_1, \tau_2)$ is a discrete set.

2) The functions $C_e, C_w$ (resp. $A_k (k=0,1,\ldots)$) are meromorphic (resp. rational) on $\mathcal{O}$ and holomorphic on $\mathcal{O}(\tau_1, \tau_2)$.

3) Fix any compact subset $B$ of $\mathcal{O}(\tau_1, \tau_2)$ and $r > 0$. Then, for each integers $i \geq 0$, $j \geq 0$, the series

$$\sum_{k=0}^{\infty} \left( \frac{2}{\tau_1} \right)^i \left( \frac{2}{\tau_2} \right)^j (A_k(s)e^{-kt})$$

is absolutely and uniformly convergent on $B \times [r, \infty)$.

4) Put $E(s, t) = e^{(s-\rho)t} \sum_{k=0}^{\infty} A_k(s)e^{-kt}$. Then

$$(2) \quad E(s, v, a_t) = E(s, t)C_e(s)v + E(-s, t)C_w(s)v,$$

for $s \in \mathcal{O}(\tau_1, \tau_2)$, $v \in V_M$ and $t > 0$.

In this note, we call $E(s, t)$ the Harish-Chandra series of the Eisenstein integral $E(s, v, a_t)$.
For any closed subgroup of $G$ we use the similar notations; for example, $\mathfrak{K}, \mathfrak{M}, \mathfrak{A}, \mathfrak{N}$ are the Lie algebras of $K, M, A, N$ respectively. Choose an element $H_0 \in \mathfrak{N}$ with the same property as in 8.0 and put

$$A_c = \left\{ \exp(\mathfrak{z}H_0) : \mathfrak{z} \text{ is a complex number with } |\text{Im}(\mathfrak{z})| < \pi \right\},$$

$$G_c = K_c A_c N_c.$$  

Then we obtain from the explicit formula of the Iwasawa decomposition the following results.

**Lemma 1.**

$G_c$ is an open connected dense submanifold of $G_c$ and there are holomorphic mappings

$$k : 'G_c \rightarrow K_c, \quad t : 'G_c \rightarrow C, \quad n : 'G_c \rightarrow N_c$$

such that for each $x \in G_c$ the decomposition

$$x = k(x)a_c(x)n(x)$$

exists and is unique. Moreover if $x \in G$ then this decomposition coincide with the Iwasawa decomposition.

Let $\Theta$ be the Cartan involution with respect to the pair $(\mathfrak{M}, \mathfrak{N})$ and put

$$\mathfrak{C} = \frac{1}{2} \mathfrak{M} + \mathfrak{N}; \quad \mathfrak{R} = \mathfrak{M} + \mathfrak{I} \mathfrak{N}. $$

Then $\mathfrak{R} = \mathfrak{M} + \mathfrak{C}^\perp$, $[\mathfrak{C}, \mathfrak{C}] \subset \mathfrak{M}$, $[\mathfrak{M}, \mathfrak{C}] \subset \mathfrak{C}$ and $[\mathfrak{M}, \mathfrak{N}] \subset \mathfrak{N}$. Therefore the real subspace $\mathfrak{K}_c = \mathfrak{M} + \mathfrak{I} \mathfrak{N}$ is a noncompact real form $\mathfrak{K}_c$. Let $H$ be the analytic subgroup of $K_c$ with Lie algebra $\mathfrak{K}$. Then $H$ is isomorphic to $SO_0(n-1,1)$ and $M$ is a maximal compact subgroup of $H$. 

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Lemma 2.

For any $t \geq 0$ and $h \in H$, $a_t h$ belongs to $G$.

Since any finite dimensional representation of $K$ can be extended to a holomorphic representation of $K_c$, we may regard a double unitary representation of $K$ as a holomorphic double representation of $K_c$. Hence the function

$$e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h))v \tau_2(h^{-1})$$

is well-defined.

Now we consider the following integral;

$$I(s,t)v = \int_H e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h))v \tau_2(h^{-1})dh,$$

for a complex number $s$, $v \in V_M$ and $t \geq 0$, where $dh$ is a Haar measure on $H$.

Lemma 3.

There is a real number $c$ depending only on $(\tau_1, \tau_2)$ such that if $\text{Re}(s) < c$, then the integral $(4)$ converges for each $t > 0$ and $v \in V_M$, and gives a linear endomorphism $F(s,t): v \rightarrow F(s,t)v$ of $V_M$.

Moreover there exist meromorphic functions $F_k(k=0,1,2,...)$ with values in $\text{End}(V_M)$ such that

1) $F_k(k=0,1,2,...)$ are all holomorphic on the half space \{ $s; \text{Re}(s) < c$ \}.

2) For any compact subset $B$ of \{ $s; \text{Re}(s) < c$ \} and $r > 0$, the series

$$\sum_{k=0}^{\infty} F_k(s)e^{-kt}$$

converges uniformly on $B$. 

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is absolutely and uniformly convergent on $Bx[r, \infty)$ and

$$F(s,t) = e^{(s-\rho)t} \sum_{k=0}^{\infty} F_k(s)e^{-k\rho t},$$

for $t \geq 0$ and $\Re(s) < c$.

**Corollary 4.**
Fix $t \geq 0$. Then the function $s \mapsto F(s,t)$ can be extended meromorphically onto the whole plane. Moreover if $\Re(s) < c$ then the limit

$$\lim_{t \to \infty} e^{(s-\rho)t}F(s,t) = F_0(s)$$

exists.

Most complicated part in proofs of these results is to show that the function $e^{(s-\rho)t}F(s,t)$ is a linear combination of integrals in the following form;

$$I_{p,q}(s,t) = \int_0^1 r^2 -1(1-r^2)^{-s}(s+p)(1-x^2r^2,s-p)(1+xr)^p(1-r)^q dr,$$

where $x = e^{-t}$ and $p,q$ are integers.

§2. Differential equations satisfied by $E(s,t)$, $F(s,t)$.

Let $G$, $\mathfrak{g}$ be the universal enveloping algebra of $G_c$, $\mathfrak{g}_c$ respectively. Denote by $\Delta(D)$ "the radial part" of $D \in G_c$ in the sense of Chap.9 of [5]. Then, in our case, we may regard $\Delta(D)$ as an ordinary differential operator on $(0, \infty)$ whose coefficients are all real analytic functions with values in $\text{End}(V)$. Let $\mathfrak{g}$ be the center of $G$ and denote by $\Omega(z,s)$ ($z \in \mathfrak{g}$, $s$ is a complex number) an element introduced in [5](p.283). Then

$$\Delta(z)(E(s,t)) = E(s,t)\overline{T}_z(\Omega(z,s-\rho)),$$
for each $s \in \mathcal{O}(\tau_1, \tau_2)$ and $Z \in \mathcal{F}$, and

(5) $\Delta(Z)(E(s,v,a_t)) = E(s,v,\tau_2(\mathcal{O}(Z,s-\rho)),a_t)$

for any complex number $s$ and $Z \in \mathcal{F}$.

Since $H$ is a real form of $K_c$ and the function

$$f(x) = e^{(s-\rho)t} \tau_1(k(x)) \quad (x \in G_c)$$

is holomorphic on $G_c$ which satisfies

$$f(kx^m) = e^{(s-\rho)t(a)} \tau_1(k)f(x) \tau_1(m),$$

for $k \in K_c$, $x \in G_c$, $m \in M_c$, $a \in A_c$, $n \in N_c$ with $kx^m \in G_c$,

by similar arguments in the proof of (5), we have the next lemma

**Lemma 5.**

For each $Z \in \mathcal{F}$, there is a real number $c'$ such that if $\Re(s) < c'$ then

$$\Delta(Z)(F(s,t)) = F(s,t) \tau_2(\mathcal{O}(Z,s-\rho)).$$

Noting that the following properties characterize the function $f_s(t) = E(s,t)$ ($s \in \mathcal{O}(\tau_1, \tau_2)$ fixed) (Chap. 9 of [5])

- (a) $f_s(t) = e^{(s-\rho)t} \sum_{k=0}^{\infty} A_k(s)e^{-kt}$,
- (b) The limit $A_0(s) = \lim_{t \to \infty} e^{(s-\rho)t}f_s(t)$ exists and coincides with the identity operator of $V_M$,
- (c) Let $\omega$ be the Casimir operator of $G$. Then

$$\Delta(\omega)f_s(t) = f_s(t)\tau_2(\mathcal{O}(\omega,s-\rho)),$$

we have from Lemma 3. and Lemma 5. the next lemma.
Lemma 6.  
For each \( t > 0 \), \( F(s,t) = E(s,t)F_0(s) \) as \( \text{End}(V_M) \)-valued meromorphic functions.

Calculations of \( F_0(s) \) is very difficult except special case but the next Lemma is hold.

Lemma 7.  
There is a constant \( c_0 \) depending only on normalizations of a Haar measure of \( H \) such that 
\[ F_0(s) = c_0 \left( \frac{\sin \pi s}{\sin \pi (s+\rho)} \right) C_e(s), \]
as meromorphic functions.

Thus we have the following theorem.

Theorem 8.  
There is a real number \( c \) such that if \( \text{Re}(s) < c \) then the integral 
\[ F(s,t)v = \int_H e^{(s-\rho)t} \Gamma_1(k(a_t h))v \Gamma_2(h^{-1}) dh \]
is absolutely convergent for each \( t > 0 \), \( v \in V_M \). Moreover for fixed \( t > 0 \) the function \( s \mapsto F(s,t) \) can be extended to a \( \text{End}(V_M) \)-valued meromorphic function on \( \mathcal{C} \) and 
\[ F(s,t) = c_0 \left( \frac{\sin \pi s}{\sin \pi (s+\rho)} \right) E(s,t)C_e(s). \]
Remark.

When $G=SU(1,1)$, the formula (4) gives an integral representation of Euler type. More generally, if the Eisenstein integral $E(s,v,a_t)$ coincide with a hypergeometric function then the formula (4) gives an integral representation of Euler type.

REFERENCES.


