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AN INTEGRAL REPRESENTATION OF THE HARISH-CHANDRA SERIES ON $SO_0(n,1)$

By

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§0. Introduction

Let $G$ be a connected noncompact real form of a connected complex semisimple Lie group $G_c$ and assume that $G$ is of split rank one. Let $K$ be a maximal compact subgroup of $G$ and $G=KAN$ be an Iwasawa decomposition. Then $\dim A = 1$. Choose an element $H_0$ of the Lie algebra $\mathfrak{a}$ of $A$ so that eigenvalues of $\text{ad}(H_0)$ are $\{0,1\}$ or $\{0,1,2\}$ and put $a_t=\exp(tH_0)$. Then the Iwasawa decomposition of an element $x \in G$ is

$$x=k(x)a_t(x)n(x), \text{ where } k(x) \in K \text{ and } n(x) \in N.$$ 

Let $M$ be the centralizer of $A$ in $K$. Then $MAN$ is a minimal parabolic subgroup of $G$. Let $(\tau_1, \tau_2)$ be a double unitary representation of $K$ on a finite dimensional Hilbert space $V$ and denote by $V_M$ the subspace of $V$ comprised of those elements $v$ which have the property that $\tau_1(m)v=v\tau_2(m)$ (all $m \in M$). The Eisenstein integral $E(s,v,x)$ for $MAN$ is defined by the following formula([2],[4]);

$$(1) \quad E(s,v,x) = \int_K e(s-p)t(xk)\tau_1(k(xk))v\tau_2(k^{-1})dk,$$
When $G = SU(1,1)$, by the change of variable $z = (\text{th}^{2}(\frac{t}{2}))^{2}$, we can see that the Eisenstein integral $E(s,v,a_{t})$ coincide with certain hypergeometric function $F(a,b;c;z)$ (up to a constant factor) and the formula (1) gives its integral representation of Euler type. Moreover the expansion (2) corresponds to the following formula:

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b; a+b-c+1; 1-z)$$

$$+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1; 1-z).$$

Hence the Harish-Chandra series $E(s,t)$ has an integral representation of Euler type.

In general, Does the series $E(s,t)$ have an integral representation? When $G$ is the general Lorentz group $SO_{0}(n,1)$, we can find an integral representation of $E(s,t)$ which is a singular integral over certain noncompact real form $H$ of the complexification $K_{c}$ of $K$ (Theorem 8.)

Main idea is to consider the analytic continuation of the Iwasawa decomposition and to find a noncompact real form of $K_{c}$ with certain properties.

The present note is a sketch of results and we shall give proofs in [3].

§ 1. Analytic continuation of the Iwasawa decomposition

Let $G$ be the general Lorentz group $SO_{0}(n,1)$ $(n \geq 2)$. Denote by $\mathfrak{g}$, $\mathfrak{g}_{c}$ the Lie algebra of $G$ and its complexification respectively. We denote by $G_{c}$ the complex analytic subgroup of $GL(n+1, \mathbb{C})$ with Lie algebra $\mathfrak{g}_{c}$. Let $K,M,A,N$ be the same as in §0.
for a complex number $s$, $v \in V_M$ and $x \in \mathbb{C}$,
where $d_k$ is the normalized Haar measure on $K$ and $\rho$ is a real number
obtained from the modular function $\delta$ of MAN by $\delta(ma_n) = e^{-2\rho t}$.
The Eisenstein integral $E(s,v,a_t)$ has the following series expansion ([1],[5]).

**Theorem 0. (Harish-Chandra)**

There is an open connected dense subset $O(\tau_1, \tau_2)$ of $\Phi$ which is
stable under the action $s \rightarrow -s$ and are functions $C_e, C_w, A_k (k=0,1,\ldots)$
with all values in $\text{End}(V_M)$ such that

1) The complement of $O(\tau_1, \tau_2)$ is a discrete set.
2) The functions $C_e, C_w$ (resp. $A_k (k=0,1,\ldots)$) are meromorphic (resp. rational)
on $\Phi$ and holomorphic on $O(\tau_1, \tau_2)$.
3) Fix any compact subset $B$ of $O(\tau_1, \tau_2)$ and $r>0$. Then, for each integers $i, j \geq 0$, the series

$$
\sum_{k=0}^{\infty} \left( \frac{2}{3s} \right)^i \left( \frac{2}{3t} \right)^j A_k(s)e^{-kt}
$$

is absolutely and uniformly convergent on $B \times [r, \infty)$.
4) Put $E(s,t) = e^{(s-\rho)t} \sum_{k=0}^{\infty} A_k(s)e^{-kt}$. Then

$$(2) \quad E(s,v,a_t) = E(s,t)C_e(s)v + E(-s,t)C_w(s)v,$$

for $s \in O(\tau_1, \tau_2)$, $v \in V_M$ and $t\geq 0$.

In this note, we call $E(s,t)$ the Harish-Chandra series of the
Eisenstein integral $E(s,v,a_t)$. 

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For any closed subgroup of $G$ we use the similar notations; for example, $\mathfrak{K}, \mathfrak{M}, \mathfrak{A}, \mathfrak{N}$ are the Lie algebras of $K, M, A, N$ respectively. Choose an element $H_0 \in \mathfrak{A}$ with the same property as in 5.0 and put

$$A_c = \left\{ a \in \mathbb{C} \right| \exp(zH_0) \right\}; \quad z \text{ is a complex number with } |\text{Im}(z)| < \pi,$$

$$G_c = K_c \cdot A_c \cdot N_c.$$

Then we obtain from the explicit formula of the Iwasawa decomposition the following results.

**Lemma 1.**

$G_c$ is an open connected dense submanifold of $G_c$ and there are holomorphic mappings

$$k : G_c \to K_c, \quad t : G_c \to \mathbb{C}, \quad n : G_c \to N_c$$

such that for each $x \in G_c$ the decomposition

$$(3) \quad x = k(x) a_c(t(x)) n(x)$$

exists and is unique. Moreover if $x \in G$ then this decomposition coincide with the Iwasawa decomposition.

Let $\Theta$ be the Cartan involution with respect to the pair $(\mathfrak{g}, \mathfrak{k})$ and put

$$\mathfrak{g}_\Theta = \left\{ x + \Theta x \right| x \in \mathfrak{g} \right\}.$$

Then $\mathfrak{K} = \mathfrak{m} + \mathfrak{g}_\Theta, \quad [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{g}] \subseteq \mathfrak{g}_\Theta$ and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}$. Therefore the real subspace $\mathfrak{h}^\Theta = \mathfrak{m} + i \mathfrak{g}_\Theta$ is a noncompact real form $\mathfrak{h}_c^\Theta$. Let $H$ be the analytic subgroup of $K_c$ with Lie algebra $\mathfrak{h}_c^\Theta$. Then $H$ is isomorphic to $SO_0(n-1,1)$ and $M$ is a maximal compact subgroup of $H$. 

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Lemma 2.

For any $t \geq 0$ and $h \in H$, $a_t h$ belongs to $G$.

Since any finite dimensional representation of $K$ can be extended to a holomorphic representation of $K_c$, we may regard a double unitary representation of $K$ as a holomorphic double representation of $K_c$. Hence the function

$$e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h)) \tau_2(h^{-1})$$

is well-defined.

Now we consider the following integral;

$$I(s,t)v = \int_H e^{(s-\rho)t(a_t h)} \tau_1(k(a_t h)) \tau_2(h^{-1}) dh,$$

for a complex number $s$, $v \in V_M$ and $t \geq 0$,

where $dh$ is a Haar measure on $H$.

Lemma 3.

There is a real number $c$ depending only on $(\tau_1, \tau_2)$ such that if $\Re(s) < c$ then the integral (4) converges for each $t \geq 0$ and $v \in V_M$, and gives a linear endomorphism $F(s,t): v \mapsto F(s,t)v$ of $V_M$.

Moreover there exist meromorphic functions $F_k(k=0,1,2,\ldots)$ with values in $\text{End}(V_M)$ such that

1) $F_k(k=0,1,2,\ldots)$ are all holomorphic on the half space \{ $s$; $\Re(s) < c$ \}.

2) For any compact subset $B$ of \{ $s$; $\Re(s) < c$ \} and $r > 0$, the series

$$\sum_{k=0}^{\infty} F_k(s)e^{-kt}$$

is well-defined.
is absolutely and uniformly convergent on $B_{x}[r, \infty)$ and

$$F(s,t) = e^{(s-F)t} \sum_{k=0}^{\infty} F_k(s)e^{-kt},$$

for $t > 0$ and $\text{Re}(s) < c$.

**Corollary 4.**

Fix $t > 0$. Then the function $s \mapsto F(s,t)$ can be extended meromorphically onto the whole plane. Moreover, if $\text{Re}(s) < c$ then

the limit $\lim_{t \to \infty} e^{(p-s)t}F(s,t) = F_0(s)$ exists.

Most complicated part in proofs of these results is to show that the function $e^{(p-s)t}F(s,t)$ is a linear combination of integrals in the following form:

$$I_{p,q}(s,t) = \int_{0}^{1} r^2 \frac{1}{1-r^2} (s+p)(1-x^2) \frac{1-xr}{1-x} \frac{1-(1-x)^p}{1+(1-x)^q} dr,$$

where $x = e^{-t}$ and $p, q$ are integers.

§2. Differential equations satisfied by $E(s,t), F(s,t)$.

Let $\mathcal{G}, \mathcal{A}$ be the universal enveloping algebra of $\mathcal{L}_c, \mathcal{K}_c$ respectively. Denote by $\Delta(D)$ "the radial part" of $D \in \mathcal{G}$ in the sense of Chap. 9 of [5]. Then, in our case, we may regard $\Delta(D)$ as an ordinary differential operator on $(0, \infty)$ whose coefficients are all real analytic functions with values in $\text{End}(V)$. Let $\mathcal{G}_c$ be the center of $\mathcal{G}$ and denote by $\Omega(Z,s)$ $(Z \in \mathcal{G}_c$, $s$ is a complex number) an element introduced in [5] (p. 283). Then

$$\Delta(Z)(E(s,t)) = E(s,t) \tilde{T}_Z(\Omega(Z,s-p)),$$
for each $s \in \mathcal{O}(\tau_1, \tau_2)$ and $Z \in \mathcal{F}$,

and

$$(5) \quad \Delta(Z)(E(s,v,a_t)) = E(s,v_2 \tau_2(\Omega(Z,s-\rho)),a_t)$$

for any complex number $s$ and $Z \in \mathcal{F}$.

Since $H$ is a real form of $K_c$ and the function

$$f(x) = e(s-\rho)t(x) \tau_1(k(x)) \quad (x \in G_c)$$

is holomorphic on $G_c$ which satisfies

$$f(kx) = e(s-\rho)t(a) \tau_1(k)f(x) \tau_1(m),$$

for $k \in K_c, \, x \in G_c, \, m \in M_c, \, a \in A_c, \, n \in N_c$ with $kx \in G_c$,

by similar arguments in the proof of (5), we have the next lemma

**Lemma 5.**

For each $Z \in \mathcal{F}$, there is a real number $c'$ such that if $Re(s) < c'$ then

$$\Delta(Z)(F(s,t)) = F(s,t) \tau_2(\Omega(Z,s-\rho)).$$

Noting that the following properties characterize the function

$$f_s(t) = E(s,t) \quad (s \in \mathcal{O}(\tau_1, \tau_2) \text{ fixed}) \quad (\text{Chap. 9 of [5]})$$

a) \quad $f_s(t) = e(s-\rho)t \sum_{k=0}^{\infty} A_k(s)e^{-kt}$,

b) \quad The limit $A_0(s) = \lim_{t \to \infty} e^{(\rho-s)t}f_s(t)$ exists and coincides with the identity operator of $V_M$,

c) \quad Let $\omega$ be the Casimir operator of $G$. Then

$$\Delta(\omega)f_s(t) = f_s(t) \tau_2(\Omega(\omega,s-\rho)),$$

we have from Lemma 3. and Lemma 5. the next lemma.
Lemma 6.

For each \( t > 0 \), \( F(s,t) = E(s,t)F_0(s) \) as \( \text{End}(V_M) \)-valued meromorphic functions.

Calculations of \( F_0(s) \) is very difficult except special case but the next Lemma is hold.

Lemma 7.

There is a constant \( c_0 \) depending only on normalizations of a Haar measure of \( H \) such that

\[
F_0(s) = c_0(\frac{(\sin \pi s)}{(\sin \pi (s+\rho))}) C_e(s),
\]

as meromorphic functions.

Thus we have the following theorem.

Theorem 8.

There is a real number \( c \) such that if \( \text{Re}(s) < c \) then the integral

\[
F(s,t)v = \int_{H} e^{(s-\rho)t(a_{\lambda}h)} \gamma_1(k(a_{\lambda}h)) \gamma_2(h^{-1}) dh
\]

is absolutely convergent for each \( t > 0 \), \( v \in V_M \). Moreover for fixed \( t > 0 \) the function \( s \mapsto F(s,t) \) can be extended to a \( \text{End}(V_M) \)-valued meromorphic function on \( \mathbb{C} \) and

\[
F(s,t) = c_0(\frac{(\sin \pi s)}{(\sin \pi (s+\rho))}) E(s,t)C_e(s).
\]
Remark.

When $G=SU(1,1)$, the formula (4) gives an integral representation of Euler type. More generally, if the Eisenstein integral $E(s,v,a_t)$ coincide with a hypergeometric function then the formula (4) gives an integral representation of Euler type.

REFERENCES.