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AN INTEGRAL REPRESENTATION OF THE HARISH-CHANDRA SERIES ON $SO_0(n,1)$

By

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§0. Introduction

Let $G$ be a connected noncompact real form of a connected complex semisimple Lie group $G$ and assume that $G$ is of split rank one. Let $K$ be a maximal compact subgroup of $G$ and $G=KAN$ be an Iwasawa decomposition. Then $\dim A = 1$. Choose an element $H_0$ of the Lie algebra $\mathfrak{a}$ of $A$ so that eigenvalues of $\text{ad}(H_0)$ are $\{0, \pm 1\}$ or $\{0, \pm 1, \pm 2\}$ and put $a_t=\exp(tH_0)$. Then the Iwasawa decomposition of an element $x \in G$ is

$$x=k(x)a_t(x)n(x), \text{ where } k(x) \in K \text{ and } n(x) \in N.$$ 

Let $M$ be the centralizer of $A$ in $K$. Then $MAN$ is a minimal parabolic subgroup of $G$. Let $(\mathcal{T}_1, \mathcal{T}_2)$ be a double unitary representation of $K$ on a finite dimensional Hilbert space $V$ and denote by $V_M$ the subspace of $V$ comprised of those elements $v$ which have the property that $\mathcal{T}_1(m)v=v\mathcal{T}_2(m)$ (all $m \in M$). The Eisenstein integral $E(s,v,x)$ for $MAN$ is defined by the following formula([2],[4]);

(1) $E(s,v,x) = \int_K e(s-P)t(xk)\mathcal{T}_1(k(xk))v\mathcal{T}_2(k^{-1})dk,$
When $G = \text{SU}(1,1)$, by the change of variable $z = (\text{th}(\frac{t}{2}))^2$, we can see that the Eisenstein integral $E(s,v,a_v)$ coincide with certain hypergeometric function $F(a,b;c;z)$ (up to a constant factor) and the formula (1) gives its integral representation of Euler type.  
Moreover the expansion (2) corresponds to the following formula;

\[
F(a,b;c;z) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)} F(a,b; a+b-c+1; 1-z) \\
+ (1-z)^{c-a-b} \frac{\Gamma(c) \Gamma(a+b-c)}{\Gamma(a) \Gamma(b)} F(c-a,c-b; c-a-b+1; 1-z).
\]

Hence the Harish-Chandra series $E(s,t)$ has an integral representation of Euler type.

In general, Does the series $E(s,t)$ have an integral representation?

When $G$ is the general Lorentz group $\text{SO}_0(n,1)$, we can find an integral representation of $E(s,t)$ which is a singular integral over certain noncompact real form $H$ of the complexification $K_c$ of $K$ (Theorem 8).  

Main idea is to consider the analytic continuation of the Iwasawa decomposition and to find a noncompact real form of $K_c$ with certain properties.

The present note is a sketch of results and we shall give proofs in [3].

§ 1. Analytic continuation of the Iwasawa decomposition

Let $G$ be the general Lorentz group $\text{SO}_0(n,1)$ ($n \geq 2$). Denote by $\mathfrak{g}_c$ the Lie algebra of $G$ and its complexification respectively. We denote by $\mathfrak{g}_c$ the complex analytic subgroup of $\text{GL}(n+1,C)$ with Lie algebra $\mathfrak{g}_c$. Let $K,M,A,N$ be the same as in § 0.
for a complex number $s$, $v \in V_M$ and $x \in G$, where $\delta_k$ is the normalized Haar measure on $K$ and $\rho$ is a real number obtained from the modular function $\delta$ of MAN by $\delta(ma_n) = e^{-2\rho t}$.

The Eisenstein integral $E(s,v,a_t)$ has the following series expansion ([1],[5]).

**Theorem 0. (Harish-Chandra)**

There is an open connected dense subset $O(\tau_1, \tau_2)$ of $G$ which is stable under the action $s \mapsto -s$ and are functions $C_e, C_w, A_k (k=0,1,\ldots)$ with all values in $\text{End}(V_M)$ such that

1) The complement of $O(\tau_1, \tau_2)$ is a discrete set.

2) The functions $C_e, C_w$ (resp. $A_k (k=0,1,\ldots)$) are meromorphic (resp. rational) on $G$ and holomorphic on $O(\tau_1, \tau_2)$.

3) Fix any compact subset $B$ of $O(\tau_1, \tau_2)$ and $r > 0$. Then, for each integers $i \geq 0$, $j \geq 0$, the series

$$
\sum_{k=0}^{\infty} \left( \frac{2}{s} \right)^i \left( \frac{2}{t} \right)^j (A_k(s)e^{-kt})
$$

is absolutely and uniformly convergent on $B \times [r, \infty)$.

4) Put $E(s,t) = e(s-\rho)t \sum_{k=0}^{\infty} A_k(s)e^{-kt}$. Then

$$
E(s,t) = E(s,v,a_t) = E(s,t)C_e(s)v + E(-s,t)C_w(s)v,
$$

for $s \in O(\tau_1, \tau_2)$, $v \in V_M$ and $t > 0$.

In this note, we call $E(s,t)$ the Harish-Chandra series of the Eisenstein integral $E(s,v,a_t)$. 
For any closed subgroup of $G$ we use the similar notations; for example, $\mathfrak{K}, \mathfrak{W}, \mathfrak{M}, \mathfrak{N}$ are the Lie algebras of $K, M, A, N$ respectively. Choose a element $H_0 \in \mathfrak{H}$ with the same property as in §0 and put

$$A_c = \{ a_z = \exp(zH_0) ; z \text{ is a complex number with } |\text{Im}(z)| < \pi \},$$

$$G_c = K_c ^A N_c.$$

Then we obtain from the explicit formula of the Iwasawa decomposition the following results.

Lemma 1.

$G_c$ is an open connected dense submanifold of $G_c$ and there are holomorphic mappings

$$k : G_c \longrightarrow K_c, \quad t : G_c \longrightarrow C, \quad n : G_c \longrightarrow N_c$$

such that for each $x \in G_c$ the decomposition

$$(3) \quad x = k(x)a_t(x)n(x)$$

exists and is unique. Moreover if $x \in G$ then this decomposition coincide with the Iwasawa decomposition.

Let $\Theta$ be the Cartan involution with respect to the pair $(\mathfrak{K}, \mathfrak{M})$ and put

$$\Theta = \frac{1}{2} \left\{ x + \Theta x \right\}, \quad x \in \mathfrak{N}.$$  

Then $\mathfrak{K} = \mathfrak{M} + \Theta \mathfrak{M}$, $[\mathfrak{K}, \Theta \mathfrak{M}] \subseteq \mathfrak{M}$, $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{K}$ and $[\mathfrak{M}, \mathfrak{M}] \subseteq \mathfrak{M}$. Therefore the real subspace $\mathfrak{H} = \mathfrak{M} + \mathfrak{M} \mathfrak{I}$ is a noncompact real form $\mathfrak{K}_c$. Let $H$ be the analytic subgroup of $K_c$ with Lie algebra $\mathfrak{H}$. Then $H$ is isomorphic to $SO_0(n-1,1)$ and $M$ is a maximal compact subgroup of $H$. 

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Lemma 2.

For any \( t \geq 0 \) and \( h \in H \), \( \sigma_t h \) belongs to \( G \).

Since any finite dimensional representation of \( K \) can be extended to a holomorphic representation of \( K_0 \), we may regard a double unitary representation of \( K \) as a holomorphic double representation of \( K_0 \). Hence the function

\[
\exp((-s-\rho)t(a_t h)) \varphi_1(k(a_t h)) \varphi_2(h^{-1})
\]

is well-defined.

Now we consider the following integral:

\[
I(s,t)v = \int_H \exp((-s-\rho)t(a_t h)) \varphi_1(k(a_t h)) \varphi_2(h^{-1})dh,
\]

for a complex number \( s \), \( v \in V_M \) and \( t > 0 \),

where \( dh \) is a Haar measure on \( H \).

Lemma 3.

There is a real number \( c \) depending only on \((\varphi_1, \varphi_2)\) such that if \( \text{Re}(s) < c \) then the integral (4) converges for each \( t > 0 \) and \( v \in V_M \), and gives a linear endomorphism \( F(s,t)v \) of \( V_M \).

Moreover there exist meromorphic functions \( F_k(k=0,1,2,...) \) with values in \( \text{End}(V_M) \) such that

1) \( F_k(k=0,1,2,...) \) are all holomorphic on the half space \( \{ s ; \text{Re}(s) < c \} \).

2) For any compact subset \( B \) of \( \{ s ; \text{Re}(s) < c \} \) and \( r > 0 \), the series

\[
\sum_{k=0}^{\infty} F_k(s)e^{-kt}
\]

converges.
is absolutely and uniformly convergent on $B(x, r)$ and

$$
P(s, t) = e^{(s - P)x} \sum_{k=0}^{\infty} F_k(s) e^{-xt},
$$

for $t > 0$ and $\text{Re}(s) < c$.

**Corollary 4.**

Fix $t > 0$. Then the function $s \mapsto P(s, t)$ can be extended meromorphically onto the whole plane. Moreover if $\text{Re}(s) < c$ then the limit $\lim_{t \to \infty} e^{(s - P)x} P(s, t) = P_0(s)$ exists.

Most complicated part in proofs of these results is to show that the function $e^{(s - P)x} P(s, t)$ is a linear combination of integrals in the following form:

$$
I_{p, q}(s, t) = \int_0^1 r^2 \frac{1}{1 - (1 - r^2)(s + P)(1 - x^2 r^2)} s - P \frac{1 + x}{1 - x} \frac{1 - r}{1 + r} q \, dr,
$$

where $x = e^{-t}$ and $p, q$ are integers.

§ 2. Differential equations satisfied by $E(s, t), F(s, t)$.

Let $\mathcal{G}, \mathcal{L}$ be the universal enveloping algebra of $\mathcal{G}_c, \mathcal{L}_c$ respectively. Denote by $\Delta(D)$ "the radial part" of $D \in \mathcal{G}$ in the sense of Chap. 9 of [5]. Then, in our case, we may regard $\Delta(D)$ as an ordinary differential operator on $(0, \infty)$ whose coefficients are all real analytic functions with values in $\text{End}(V)$. Let $\mathcal{G}$ be the center of $\mathcal{G}$ and denote by $\Omega(z, s) (z \in \mathcal{G}, s$ is a complex number) an element introduced in [5](p.283). Then

$$
\Delta(z)(E(s, t)) = E(s, t) \mathcal{T}_2(\Omega(z, s - P)),
$$

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for each \( s \in \Omega(\tau_1, \tau_2) \) and \( Z \in \mathcal{F} \),

and

\[(5) \quad \Delta(Z)(E(s,v,a_t)) = E(s,v \tau_2(\Omega(Z,s-\rho)), a_t) \]

for any complex number \( s \) and \( Z \in \mathcal{F} \).

Since \( H \) is a real form of \( K_c \) and the function

\[ f(x) = e^{(s-\rho)t(x) \tau_1(k(x))} \quad (x \in G_c) \]

is holomorphic on \( G_c \) which satisfies

\[ f(kx, m, n) = e^{(s-\rho)t(a) \tau_1(k)f(x) \tau_1(m)}, \]

for \( k \in K_c, x \in G_c, m \in M_c, a \in A_c, n \in N_c \) with \( kx, m, n \in G_c \),

by similar arguments in the proof of (5), we have the next lemma.

**Lemma 5.**

For each \( Z \in \mathcal{F} \), there is a real number \( c' \) such that if \( \text{Re}(s) < c' \) then

\[ \Delta(Z)(F(s,t)) = F(s,t) \tau_2(\Omega(Z,s-\rho)). \]

Noting that the following properties characterize the function \( f_s(t) = E(s,t) \) (\( s \in \Omega(\tau_1, \tau_2) \) fixed) (Chap. 9 of [5]):

a) \( f_s(t) = e^{(s-\rho)t \sum_{k=0}^{\infty} A_k(s)e^{-kt}} \)

b) The limit \( A_0(s) = \lim_{t \to \infty} e^{(s-\rho)t} f_s(t) \) exists and coincides with the identity operator of \( \mathcal{V}_M \).

c) Let \( \omega \) be the Casimir operator of \( G \). Then

\[ \Delta(\omega)f_s(t) = f_s(t) \tau_2(\Omega(\omega,s-\rho)), \]

we have from Lemma 3 and Lemma 5 the next lemma.
Lemma 6.

For each $t > 0$, $F(s, t) = E(s, t)F_0(s)$ as $\text{End}(V_m)$-valued meromorphic functions.

Calculations of $F_0(s)$ is very difficult except special case but the next Lemma is hold.

Lemma 7.

There is a constant $c_0$ depending only on normalizations of a Haar measure of $H$ such that

$$F_0(s) = c_0 \left( \frac{\sin\pi s}{\sin\pi(s+\rho)} \right) C_e(s),$$

as meromorphic functions.

Thus we have the following theorem.

Theorem 8.

There is a real number $c$ such that if $\Re(s) < c$ then the integral

$$F(s, t)v = \int_H e^{(s-\rho)t(a, h)} \tau_1(k(a, h)) \nu \tau_2(h^{-1}) dh$$

is absolutely convergent for each $t > 0$, $v \in V_m$. Moreover for fixed $t > 0$ the function $s \mapsto F(s, t)$ can be extended to a $\text{End}(V_m)$-valued meromorphic function on $\mathbb{C}$ and

$$F(s, t) = c_0 \left( \frac{\sin\pi s}{\sin\pi(s+\rho)} \right) E(s, t) C_e(s).$$
Remark.

When $G=SU(1,1)$, the formula (4) gives an integral representation of Euler type. More generally, if the Eisenstein integral $E(s,v,a_t)$ coincide with a hypergeometric function then the formula (4) gives an integral representation of Euler type.

REFERENCES.


