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Structure of unipotent orbits and Fourier transform of unipotent orbital integrals for semisimple Lie groups

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Introduction

Let G be a connected real semisimple Lie group with finite center, and consider the action of G on itself through inner automorphisms. An orbit under this action is nothing but a conjugate class of G. We know [8] that an orbit O has on it a G-invariant measure, and it can be considered as a tempered measure on G. We denote it by μ_O and call it an orbital integral on O. A Fourier transform of μ_O is by definition an expression of μ_O as a superposition of irreducible characters of G (i.e., of characters of quasi-simple irreducible representations of G on Hilbert spaces).

When O consists of regular elements, this Fourier transform was given for real rank one groups by P. Sally, Jr. and G. Warner[9], and in general by B. Herb[3] for "almost all" O by means of integro-summation, not necessarily absolutely convergent, of irreducible characters appearing in the Plancherel formula for G.

An orbit is called unipotent if it consists of unipotent elements. We know that G has only a finite number of unipotent orbits. For this type of orbits, the case of real rank one is treated by D. Barbasch [1a].

The purpose of this paper is threefold and concerned with the Fourier transform of μ_O for a unipotent O. Firstly we give in §1 a method of inducing invariant distributions from a certain reductive subgroup of G, and study how we can apply it to the Fourier transform. Secondly we investigate in Part I the structure of unipotent orbits for $SL(n, F)$ for a local field F (i.e., a locally compact, non-discrete, commutative field), and determine the closure relation between them, and then apply it to the case of symplectic or orthogonal groups. Thirdly we give explicitly in Part II the Fourier transform of unipotent orbital integrals for $SL(n, \mathbb{R})$ (cf. [1b]).

Let us explain the contents of this paper in more detail. In §1, analogously as for representations, we give a method of inducing invariant distributions from a reductive subgroup given as a Levi subgroup of a parabolic subgroup of G (Theorem 1.1), and also a criterion for a unipotent orbit O to be "almost" equal to a certain standard subset. This enables us to reduce in a certain extent the problem of obtaining the Fourier transform of μ_O to a similar problem or to the Plancherel formula for certain reductive subgroups (Theorems 1.3 and 1.4). In §2, we give an expression of a unipotent orbit in $SL(n, F)$ with a local field F by means of the unipotent radical of a parabolic subgroup (Theorem 2.3). Using this expression and with elementary discussions, we determine in §3 the closure relation for unipotent orbits in $GL(n, F)$ (Theorem 3.3). Here we define the closure relation as follows: let O, O' be unipotent orbits, then $O \succcurlyeq O'$ if and only if $Cl(O) \supset O'$. This result is applied in §4 for $SL(n, F)$, and in §5 for classical groups over $\mathbb{C}$ to study further the relation between unipotent orbits and unipotent radicals of parabolic subgroups (Theorems 4.1 and 5.3). Concerning the results in Part I, the author expresses his thanks to Prof. N. Iwahori for his kind suggestions.

In §6, Part II, we apply the results in §1 to $SL(n, \mathbb{R})$, and reduce the problem of Fourier transform to a simple case of special unipotent orbits $O_{\pm}$ for $G = SL(N, \mathbb{R})$ with even N. In §7, we follow the method of D. Barbasch[1a] and give a formula expressing $\mu_{O_+}(f)$ for $f \in C_0^{\infty}(G)$ by means of the Harish-Chandra's invariant integral F_f defined on a fundamental Cartan subgroup B (Theorem 7.1). In §8, we prove that, modulo the Plancherel formula for $SL(N/2, \mathbb{R})$, the Fourier transform of μ_{O_+} is obtained by studying the Fourier transform of a C^{∞}-functions on B coming from F_f. Then the explicit form of the Fourier transform of $\mu_{O_{\pm}}$ is given in Theorem 8.5 modulo the known Plancherel formula for $SL(N/2, \mathbb{R})$.

Remark. The results in this paper have some overlappings with those of D. Barbasch in [1b], though they were worked out independently. See also Acknowledgements at the end of the paper.

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§1. Method of inducing invariant distributions

Let G be a connected semisimple Lie group with finite center, K a maximal compact subgroup of G. Take a parabolic subgroup P, then $G = KP$. Let N_P be the unipotent radical of P and S_P a Levi subgroup of P, that is, a reductive subgroup such that $P = N_P S_P$ is a semidirect product decomposition of P. We may assume that S_P is so chosen that $P \cap K = S_P \cap K$. We define a method of inducing an invariant distribution on S_P to such a one on G, analogously as for representations of S_P.

Let τ be an invariant distribution on S_P. We define a distribution π on G from τ as follows. Denote by $C_o^\infty(S_P)$ or $C_o^\infty(G)$ the space of all C^∞-functions with compact supports on S_P or G respectively. For $s \in S_P$, put $\beta(s) = |\det \mathrm{Ad}_{\underline{n}_P}(s)|$, where $\mathrm{Ad}_{\underline{n}_P}(s)$ denotes the restriction of $\mathrm{Ad}(s)$ on the Lie algebra $\underline{n}_P$ of N_P. For $f \in C_o^\infty(G)$, put

$$(1.1) \qquad f^P(s) = \beta^{-1/2}(s) \int_K \int_{N_P} f(knsk^{-1})\, dn\, dk,$$

where dn and dk denote Haar measures on N_P and K respectively. Then $f^P \in C_o^\infty(S_P)$ and it is invariant under $K \cap S_P$ through inner automorphisms. We put

$$(1.2) \qquad \pi(f) = \tau(f^P).$$

This π is called the distribution induced from τ and is denoted by $\mathrm{Ind}_{S_P}^G \tau$. We know [5a, p.345] that, when τ is the character of an irreducible unitary representation of S_P, π is also the character of the induced representation of it.

For $f \in C_o^\infty(G)$, put

$$(1.3) \qquad f^N(k, s) = \beta^{-1/2}(s) \int_{N_P} f(knsk^{-1})\, dn,$$

then $f^N \in C_o^\infty(K \times S_P)$, and $f^P(s) = \int_K f^N(k, s) dk$. If a change of order of "integration" is possible in the right hand side of (1.2), π is also expressed as

$$(1.4) \qquad \pi(f) = \int_K \tau(f^N(k, \cdot))\, dk.$$

<u>Theorem</u> 1.1. Assume that the expression (1.4) holds for any $f \in C_o^\infty(G)$. Then π is invariant under G.

<u>Proof</u>. For $g_o \in G$, put $f_{g_o}(g) = f(g_o g g_o^{-1})$. We prove that $\pi(f_{g_o}) = \pi(f)$. Fix a (Borel-)measurable section V of $K \cap S_P \backslash S_P$ in S_P. Let for $k \in K$, $g_o k = k'n's'$ with $k' \in K$, $n' \in N_P$, $s' \in V$. Then k', n' and s' are uniquely determined by k, and the map $k \longrightarrow k'$ defines a measurable bijection from K onto itself. We know that the Haar measure on K is transformed as

$$(1.5) \qquad dk' = \beta^{-1}(s')\, dk.$$

Moreover

$$g_o knsk^{-1} g_o^{-1} = k'n's'ns(k'n's')^{-1} = k'n''s''$$

with $n'' = n'(s'ns'^{-1})(s'ss'^{-1}n'^{-1}s's^{-1}s'^{-1}) \in N_P$, and $s'' =$

$s'ss'^{-1} \in S_P$. The map $n \longrightarrow n''$ is bijective on N_P for every fixed k, and $dn'' = \beta(s')dn$. Therefore

$$(f_{g_o})^N(k, s) = \beta^{-1/2}(s) \int_{N_P} f_{g_o}(knsk^{-1})dn$$

$$= \beta^{-1/2}(s) \int_{N_P} f(k'n's'ns(k's'n')^{-1})\beta^{-1}(s')dn''$$

$$= \beta^{-1}(s')\, f^N(k', s'ss'^{-1}).$$

Then by (1.4),

$$\pi(f_{g_o}) = \int_K \tau(f_{g_o}{}^N(k, \cdot))dk = \int_K \tau(f^N(k', \cdot))\beta^{-1}(s')dk$$

$$= \pi(f) \qquad \text{(by (1.5))}. \qquad \text{Q.E.D.}$$

For a subset A of G, put

$$K(A) = \{ kak^{-1};\ k \in K,\ a \in A \}.$$

Then, since K is compact, $K(A)$ is closed if so is A. Moreover $Cl(K(A)) = K(Cl(A))$, where $Cl(A)$ denotes the closure of A. Let ω be an S_P-orbit in S_P. Then $K(N_P\omega)$ is G-invariant because $G = KP = KN_PS_P$. When ω is unipotent, it is a finite union of unipotent orbits in G.

<u>Corollary</u>. Let ω be an orbit in S_P. Then an invariant measure on $K(N_P\omega)$ is given by $\mu_1 = \mathrm{Ind}_{S_P}^G \mu_\omega$:

$$(1.6) \quad \mu_1(f) = \int_K \int_{N_P} \int_\omega f(knsk^{-1})\, d\mu_\omega(s)\, dn\, dk \qquad (f \in C_o^\infty(G)),$$

where μ_ω denotes an S_P-invariant measure on ω. In particular an invariant measure on $K(N_P)$ is given by

$$(1.7) \qquad \mu_0(f) = \int_K \int_{N_P} f(knk^{-1})\, dn\, dk.$$

Further, as a corollary of the proof of the theorem, we get:

<u>Lemma</u> 1.2. Let ω be a unipotent S_P-orbit in S_P. Assume that there exists an S_P-invariant measurable subset Ω in N_P such that $\int_\Omega dn > 0$, and $\Omega\omega$ is N_P-invariant. Then $K(\Omega\omega)$ is G-invariant and an invariant measure on it is given by

$$(1.8) \qquad \mu(f) = \int_K \int_\Omega \int_\omega f(knsk^{-1})\, d\mu_\omega(s)\, dn\, dk \qquad (f \in C_0^\infty(G)).$$

In particular, if $\omega = \{e\}$, an invariant measure on $K(\Omega)$ is given by

$$(1.9) \qquad \mu(f) = \int_K \int_\Omega f(knk^{-1})\, dn\, dk.$$

For application to orbital integrals, let us characterize that an orbit $\mathcal{O}$ is "almost" equal to $K(\Omega\omega)$. For example, assume that there exists a measurable subset A of $N_P\omega$ such that (1) $\mathcal{O} = K(A)$, (2) for every $x \in \omega$, the section A_x of A at x (i.e., $A = \bigcup_{x\in\omega} A_x x$, $A_x \subset N_P$) is equal to Ω modulo null sets with respect to dn. Then, by Fubini's theorem applied to (1.8), we see that μ in (1.8) is supported by $\mathcal{O}$, i.e., $\mu(E) = \mu(E \cap \mathcal{O})$ for any measurable subset E. Hence μ gives an invariant measure on $\mathcal{O}$.

For later applications, we take here a little different formulation. Let $P_o \subset P$ be another parabolic subgroup, then $N_P \subset N_{P_o}$. Put $G' = S_P$, $K' = G' \cap K$, $P' = G' \cap P_o$, and $N' = G' \cap N_{P_o}$. Then, G' is reductive and not necessarily connected, and P' is a parabolic subgroup of G' with unipotent radical N'.

Definition 1.1. Let $\omega \subset S_P$ and $\Omega \subset N_P$ be as in Lemma 1.2. We say that a G-orbit $\mathcal{O}$ saturates $K(\Omega\omega)$ if the following condition holds: for a parabolic subgroup $P_o \subset P$, there exist measurable subsets σ of N' and A of $N_P\sigma$ such that (1) $\mathcal{O} = K(A)$, (2) $\omega = K'(\sigma)$, and an invariant measure μ_ω on ω is given as

$$\mu_\omega(\varphi) = \int_\sigma \int_{K'} \varphi(k'n'k'^{-1})\, d\mu'(n')\, dk' \qquad (\varphi \in C_o^\infty(S_P)),$$

where μ' is a measure on σ and dk' is the normalized Haar measure on K', (3) for any $x \in \sigma$, the section $A_x \subset N_P$ of A at x coincides with Ω modulo null sets with respect to dn.

Theorem 1.3. Let $\omega \subset S_P$ be an S_P-orbit and $\Omega \subset N_P$ an S_P-invariant measurable subset such that $\int_\Omega dn > 0$, and $\Omega\omega$ is N_P-invariant (hence P-invariant). If a G-orbit $\mathcal{O}$ saturates $K(\Omega\omega)$, then an invariant measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$ is given by (1.8). In paricular, if $\Omega = N_P$, then $\mu_{\mathcal{O}} = \mathrm{Ind}_{S_P}^G \mu_\omega$.

Proof. Inserting the above expression for μ_ω in (1.8), we get

$$\mu(f) = \int_K \int_\Omega \int_{K'} \int_\sigma f(knk'n'k'^{-1}k^{-1})\, dk'\, d\mu'(n')\, dk\, dn$$

$$= \int_K \int_{K'} \iint_{\Omega \times \sigma} f(kk'nn'(kk')^{-1})\, d\mu'(n')\, dn\, dk'\, dk$$

$$= \int_K \iint_A f(knn'k^{-1})\, d\mu'(n')\, dn\, dk.$$

This proves that μ is supported by $\mathcal{O} = K(A)$, and so gives an invariant measure on $\mathcal{O}$. Q.E.D.

This theorem may be used to deduce the Fourier transform of $\mu_{\mathcal{O}}$ to that of μ_ω by studying the structure of $\mathcal{O}$ as in Definition 1.1. This works very well for $SL(n, F)$, F a local field (cf. §2), and especially for $F = \mathbb{R}$, we shall work out for μ_ω and then for $\mu_{\mathcal{O}}$ in Part II.

Let us explain how it works. Assume that the Fourier transform of an S_P-orbital integral μ_ω is given in such a form that for a signed measure ν on the unitary dual $\widehat{S}_P$ of S_P,

$$(1.10) \qquad \mu_\omega = \int_{\widehat{S}_P} \chi_\delta \, d\nu(\delta).$$

Here the unitary dual of S_P is by definition the set of all equivalent classes of irreducible unitary representations of S_P, and χ_δ denotes the character of representations of class $\delta \in \widehat{S}_P$. Insert this into the right hand side of (1.6). Then, if a change of order of "integration" is possible, the invariant measure $\mu_1 = \mathrm{Ind}_{S_P}^G \mu_\omega$ is expressed as

$$(1.11) \qquad \mu_1 = \int_{\widehat{S}_P} \mathrm{Ind}_{S_P}^G \chi_\delta \, d\nu(\delta).$$

Note that $\mathrm{Ind}_{S_P}^G \chi_\delta$ is the character of the induced representation of an element of class δ. This representation is irreducible for almost all $\delta \in \hat{S}_P$ with respect to the Plancherel measure ν_0 for S_P, and the equivalence between them corresponds to the coincidence of their characters.

Moreover assume that there exist unipotent orbits O_i ($1 \leqslant i \leqslant q$) in G such that every O_i saturates $K(\Omega_i\omega)$ for an S_P-invariant $\Omega_i \subset N_P$ with positive measure, where $\Omega_i\omega$ is N_P-invariant and $N_P - \bigcup_i \Omega_i$ is of measure zero. Then, by Theorem 1.3, μ_1 is expressed as

$$\mu_1 = \mu_{O_1} + \mu_{O_2} + \cdots + \mu_{O_q} ,$$

and therefore the formula (1.11) gives almost the Fourier transform of this sum of orbital integrals. In particular, when we consider ν_0 in (1.7), we get the following theorem.

<u>Theorem</u> 1.4. Assume that there exist unipotent orbits O_i ($1 \leqslant i \leqslant q$) such that every O_i saturates $K(\Omega_i)$ for a P-invariant $\Omega_i \subset N_P$ with positive measure, and $N_P - \bigcup_i \Omega_i$ is of measure zero. Then the Fourier transform of the sum $\mu_{O_1} + \mu_{O_2} + \cdots + \mu_{O_q}$ is given by

$$\mu_{O_1} + \mu_{O_2} + \cdots + \mu_{O_q} = \int_{\hat{S}_P} \mathrm{Ind}_{S_P}^{G} \chi_\delta \, d\nu_0(\delta),$$

where ν_0 denotes the Plancherel measure for S_P.

<u>Note</u>. Let F be a non-archimedean local field and $G = SL(n, F)$, $K = SL(n, \underline{O})$, where $\underline{O}$ denotes the maximal compact subring of F. Then the results in this section can be translated for this case appropriately.

Part I. Unipotent orbits, their structure and closure relation

In Part I, we put $G = SL(n, F)$, $\tilde{G} = GL(n, F)$, with F a local field except for Lemmas 3.1, 3.2 and §5. For $g \in \tilde{G}$, denote by $i(g)$ the automorphism of G given by $i(g)h = ghg^{-1}$ $(h \in G)$. Put $i(\tilde{G}) = \{ i(g);\ g \in \tilde{G} \}$, $i(G) = \{ i(g);\ g \in G \}$. Then $[i(\tilde{G}): i(G)] = \#(F^\times/(F^\times)^n)$, where $(F^\times)^n = \{ a^n;\ a \in F^\times \}$. Moreover put for $a \in F^\times$, a diagonal matrix $g_a \in \tilde{G}$ as

$$g_a = \begin{pmatrix} a & 0 \\ 0 & 1_{n-1} \end{pmatrix},$$

where 1_p denotes the unit matrix of degree p. Then every class of $i(\tilde{G})/i(G)$ is represented by a certain $i(g_a)$. Put $d = [i(\tilde{G}): i(G)]$, then, $d = 1$ for $F = \mathbb{C}$, $d = 1$ or 2 according as n is odd or even for $F = \mathbb{R}$, and $d > 1$ for F non-archimedean and $n > 1$. Put $K = SU(n)$, $SO(n)$ or $SL(n, \underline{O})$ according as $F = \mathbb{C}$, $\mathbb{R}$ or non-archimedean.

§2. Structure of unipotent orbits

Every unipotent element in G is expressed as $1_n + X$ with a nilpotent matrix X. Therefore it is sufficient for us to study the conjugate class of X. We denote again by $i(g)$ the transformation on X given by $i(g)(1_n + X) = 1_n + i(g)X$, and similarly for $i(\tilde{G})$ and $i(G)$. We know that any nilpotent matrix X is conjugate under $\tilde{G}$ to one of the following Jordan matrices: for a partition $\alpha = (p_1, p_2, \ldots, p_s)$ of n such that

(2.1) $$p_1 \geqslant p_2 \geqslant \cdots \geqslant p_s \geqslant 1.$$

Put

(2.2) $$J(\alpha) = J(p_1) \oplus J(p_2) \oplus \cdots \oplus J(p_s),$$

where $J(p)$ is a matrix of degree p given by

$$J(p) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \quad \text{and} \quad A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let m_p be the multiplicity of $J(p)$ in $J(\alpha)$. Assume $m_r > 0$ and $m_p = 0$ for $p > r$, and put

(2.3) $$n_j = m_j + m_{j+1} + \cdots + m_r \quad \text{for} \quad 1 \leqslant j \leqslant r, \text{ and}$$

(2.4) $$\beta = (n_1, n_2, \ldots, n_r).$$

Then β is a partition of n such that

(2.5) $$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r \geqslant 1.$$

Let $X(\beta)$ be an $n \times n$ matrix given as follows by a blockwise expression (with respect to the partition β of n):

(2.6) $$X(\beta) = \begin{pmatrix} 0_{n_1} & I_{n_1 n_2} & & & \\ & 0_{n_2} & I_{n_2 n_3} & & \\ & & \ddots & \ddots & \\ & & & 0_{n_{r-1}} & I_{n_{r-1} n_r} \\ & & & & 0_{n_r} \end{pmatrix} \quad \text{with} \quad I_{pq} = \begin{pmatrix} 0 \\ 1_q \end{pmatrix},$$

where 0_p denotes the zero matrix of degree p, and for $p \geqslant q$, I_{pq} is a $p \times q$ matrix of the above form.

<u>Lemma</u> 2.1. The matrix $J(\alpha)$ is conjugate to $X(\beta)$ under $\tilde{G}$.

<u>Proof</u>. By a permutation matrix, $J(\alpha)$ is conjugate to $X(\beta)$. Q.E.D.

We call α Jordan type and β parabolic type of the conjugate class of $J(\alpha)$ and $X(\beta)$ under $\tilde{G}$ or of its element. Here we get the following.

<u>Lemma</u> 2.2. Any unipotent element $1_n + X$ in G is conjugate under $\tilde{G}$ to $1_n + X(\beta)$ for some β with the condition (2.5). Further it is conjugate under G to $1_n + i(g_a)X(\beta)$ for some $a \in F^{\times}$.

For g or X, we denote by $O(g)$ or $O(X)$ the G-orbit of g or X respectively. Let us determine $O(i(g_a)X(\beta))$. Let $S(\beta)$ and $N(\beta)$ be subgroups of G consisting of all matrices in G expressed blockwisely as follows:

(2.7) $S(\beta)$: $\mathrm{diag}(c_1, c_2, \ldots, c_r)$ with $c_j \in GL(n_j, F)$,

where $\mathrm{diag}(c_1, c_2, \ldots, c_r)$ denotes a blockwise diagonal matrix with diagonal elements $c_1, c_2, \ldots, c_r$, and

(2.8) $N(\beta)$: $1_n + X$ with $X = \begin{pmatrix} 0_{n_1} & & & * \\ & 0_{n_2} & & \\ & & \ddots & \\ 0 & & & 0_{n_r} \end{pmatrix}$ (upper triangular).

Then $P(\beta) = S(\beta)N(\beta)$ is a parabolic subgroup of G, and $N(\beta)$ its unipotent radical, and $S(\beta)$ a Levi subgroup of it. Let $\underline{n}(\beta)$ be the set of all nilpotent matrices X appearing in (2.8) as $1_n + X$. Then it is a nilpotent Lie algebra under the natural bracket operation, and is stable under $i(P(\beta))$. For a subset A of $\underline{n}(\beta)$, put

$$(2.9) \qquad K(A) = i(K)A = \{ i(k)X;\ X \in A,\ k \in K \}.$$

Then, since $X(\beta) \in \underline{n}(\beta)$ and $G = KP(\beta)$, we have for $a \in F^\times$, $O(i(g_a)X(\beta)) = K(i(g_a)\Omega'(\beta))$ with $\Omega'(\beta) = i(P(\beta))X(\beta) \subset \underline{n}(\beta)$, and $O(1_n + i(g_a)X(\beta)) = K(i(g_a)\Omega(\beta))$ with $\Omega(\beta) = 1_n + \Omega'(\beta) \subset N(\beta)$. Thus we wish to determine $i(g_a)\Omega'(\beta) \subset \underline{n}(\beta)$ and establish a close relation between the orbit and the unipotent radical $N(\beta)$. The result is given as follows.

Theorem 2.3. Let $\beta = (n_1, n_2, \ldots, n_r)$ be a partition of n satisfying (2.5). Let $t \geqslant 1$ be the maximal of divisors of r such that for $q = r/t$,

$$n_{jt+1} = n_{jt+2} = \cdots = n_{jt+t-1} = n_{(j+1)t} \qquad (0 \leqslant j \leqslant q-1).$$

Then the G-orbit of $i(g_a)X(\beta)$, $a \in F^\times$, is given by $O(i(g_a)X(\beta)) = K(i(g_a)\Omega'(\beta))$, where $i(g_a)\Omega'(\beta)$ is an open subset of $\underline{n}(\beta)$ consisting of elements expressed blockwisely (with respect to the partition β) as follows: let $X = (x_{ij})$, x_{ij} is of type $n_i \times n_j$, then

$$\mathrm{rank}(x_{i,i+1}) = \max = n_{i+1} \quad (1 \leqslant i \leqslant r), \quad \text{and}$$

$$(2.10) \qquad \prod_{0 \leqslant j < q} \ \prod_{1 \leqslant i < t} \det(x_{jt+i,jt+i+1})^i \in a(F^\times)^t.$$

Note that $x_{jt+i,jt+i+1}$ are square matrices for $1 \leqslant i < t$. For $F = \mathbb{R}$, $(\mathbb{R}^\times)^t = \mathbb{R}^\times$ or $\mathbb{R}_+^\times = \{ a \in \mathbb{R};\ a > 0 \}$, according as t is odd or even. Hence, when t is odd, (2.10) is trivially satisfied. When t is even or equivalently $n_{2j-1} = n_{2j}$ for any j, (2.10) is rewritten as follows according as $a > 0$ or $a < 0$,

$$\text{(2.10')} \qquad \prod_{1 \leqslant j \leqslant r/2} \det(x_{2j-1,2j}) > 0 \quad \text{or} \quad < 0.$$

<u>Proof</u>. Since $O(i(g_a)X(\beta)) = i(g_a)O(X(\beta))$, it is sufficient for us to prove the assertion for $X(\beta)$, i.e., for $a = 1$. For $1 \leqslant m < r$, let $\underline{h}_m$ be a subspace of $\underline{n}(\beta)$ consisting of $X = (x_{ij})$ such that $x_{ij} = 0$ for $j - i \neq m$, and put $\underline{h}(m) = \underline{h}_m + \underline{h}_{m+1} + \dots + \underline{h}_{r-1}$. Then $\underline{h}(1) = \underline{n}(\beta)$ and $[\underline{h}(m), \underline{h}(m')] = \underline{h}(m + m')$. First we assert

$$\text{(2.11)} \qquad i(N(\beta))X(\beta) = X(\beta) + \underline{h}(2).$$

In fact, by an explicit calculation, we have $[\underline{h}_m, X(\beta)] = \underline{h}_{m+1}$ for $m \geqslant 1$, because of (2.5), (2.6) and (2.8). Fix $m \geqslant 2$ and an element $X_o \in \underline{h}(2)$. Then for $g = 1_n + X \in N(\beta)$ with $X \in \underline{h}_m$, we have

$$i(g)(X(\beta) + X_o) \equiv X(\beta) + X_o + [X, X(\beta)] \quad \text{modulo} \quad \underline{h}(m+2),$$

and $[X, X(\beta)] \in \underline{h}_{m+1}$. Since $[\underline{h}_m, X(\beta)] = \underline{h}_{m+1}$, this gives us $i(N(\beta))X(\beta) \equiv X(\beta) + \underline{h}_2 + \underline{h}_3 + \dots + \underline{h}_{m+1}$ modulo $\underline{h}_{m+2}$ by induction on m, whence (2.11).

By (2.11), it rests for us to prove that $i(S(\beta))X(\beta)$ is the subset of $\underline{h}_1$ consisting of $X = (x_{ij})$ for which the conditions in the theorem hold for $a = 1$. Let $g \in S(\beta)$ be as in (2.7), then $X = (x_{ij}) = i(g)X(\beta) \in \underline{h}_1$ is given as

$$(2.12) \qquad x_{i,i+1} = c_i I_{n_i n_{i+1}} c_{i+1}^{-1}.$$

Therefore $\mathrm{rank}(x_{i,i+1}) = \max$, and the product of determinants in (2.10) is equal to

$$(2.13) \qquad \prod_{1 \leqslant i \leqslant r} \det(c_i) \prod_{1 \leqslant j \leqslant q} \det(c_{jt})^{-t}.$$

Thus (2.10) holds for $X \in i(S(\beta))X(\beta)$.

Conversely assume that we are given $X = (x_{ij}) \in \underline{h}_1$ satisfying the conditions in the theorem. Then there exists a set of matrices $c_i \in GL(n_i, F)$ satisfying (2.12), i.e., $i(g)X(\beta) = X$ for $g = \mathrm{diag}(c_1, c_2, \ldots, c_r)$. By (2.13), the condition (2.10) means that $\det(g) \in (F^\times)^t$. We can replace g by gh with $h = \mathrm{diag}(d_1, d_2, \ldots, d_r)$ such that $i(h)X(\beta) = X(\beta)$. Since $\det(gh) = \det(g)\det(h)$, it is sufficient to see that $\det(h)$ can take any value in $(F^\times)^t$. Put $m_p = n_p - n_{p+1} \geqslant 0$. If $m_p > 0$, put $e(m_p, a_p) = \mathrm{diag}(a_p, 1_{m_p - 1}) \in GL(m_p, F)$, $a_p \in F^\times$. Since $n_p = m_p + m_{p+1} + \ldots + m_r$, we get an $n_p \times n_p$ matrix d_p by

$$d_p = e(m_p, a_p) \oplus e(m_{p+1}, a_{p+1}) \oplus \ldots \oplus e(m_r, a_r),$$

omitting $e(m_i, a_i)$ for $m_i = 0$. Then $i(h)X(\beta) = X(\beta)$, because $d_p I_{n_p n_{p+1}} d_{p+1}^{-1} = I_{n_p n_{p+1}}$. On the other hand, $\det(h)$ is a product of $(a_p)^p$ over all p such that $m_p \neq 0$. Note that the set of $a^p b^{p'}$ $(a, b \in F^\times)$ is equal to $(F^\times)^m$ for m the greatest common divisor (GCD) of p and p'. Then we see that $\det(h)$ runs over $(F^\times)^t$ with $t = \mathrm{GCD}\{p;\ m_p \neq 0\} = \mathrm{GCD}\{r,\ p;\ n_p > n_{p+1}\}$. Q.E.D.

Corollary 1. Two elements $i(g_a)X(\beta)$ and $i(g_b)X(\beta)$ are conjugate under G if and only if $a^{-1}b \in (F^\times)^t$, where t is given in Theorem 2.1.

Corollary 2. The closure of $O(i(g_a)X(\beta))$ is given by

$$(2.14) \qquad \mathrm{Cl}(O(i(g_a)X(\beta))) = K(i(g_a)\mathrm{Cl}(\Omega'(\beta))),$$

where $i(g_a)\mathrm{Cl}(\Omega'(\beta))$ consists of elements $X = (x_{ij})$ such that

$$(2.15) \qquad \prod_{0 \leqslant j < q} \prod_{1 \leqslant i < t} \det(x_{jt+i,\,jt+i+1})^i \in a(F^\times)^t \cup \{0\} .$$

For the case $F = \mathbb{R}$ and t even, (2.15) is rewritten as

$$(2.15') \qquad \prod_{1 \leqslant j \leqslant r/2} \det(x_{2j-1,\,2j}) \geqslant 0 \quad \text{or} \quad \leqslant 0.$$

Put r_j, N_j as follows: in case $F = \mathbb{R}$ and t even,

$$(2.16) \qquad r_j = n_{2j-1}, \quad N_j = n_{2j-1} + n_{2j} = 2r_j \quad (1 \leqslant j \leqslant r/2),$$

and in case F non-archimedean, for $0 \leqslant j < q$,

$$(2.17) \qquad r_{j+1} = n_{jt+1}, \quad N_{j+1} = n_{jt+1} + n_{jt+2} + \dots + n_{(j+1)t} = tr_{j+1},$$

and in both cases, $\beta' = (N_1, N_2, \dots, N_Q)$ with $Q = r/2$ or $= q$ respectively. Then, β is a subpartition of β', and $P(\beta') \supset P(\beta)$, $S(\beta') \supset S(\beta)$, $N(\beta') \subset N(\beta)$. Every $X \in \underline{n}(\beta)$ is decomposed uniquely as $X = X_1 + X_2$ such that $1_n + X_1 \in S(\beta')$,

$X_2 \in \underline{n}(\beta')$. Then, for $X = X(\beta)$,

$$X(\beta)_1 = \mathrm{diag}(x_1, x_2, \ldots, x_Q), \tag{2.18}$$

where $x_j = X(r_j, r_j, \ldots, r_j)$, the standard upper triangular matrix of degree N_j corresponding to the partition $(r_j, r_j, \ldots, r_j)$ of N_j. Let ω be the $S(\beta')$-orbit of $1_n + i(g_a)X(\beta)_1$. Then we have the following.

Corollary 3. The orbit $\mathcal{O} = O(1_n + i(g_a)X(\beta))$ saturates $K(N(\beta')\omega)$, and an invariant measure on $\mathcal{O}$ is given by $\mu_{\mathcal{O}} = \mathrm{Ind}_{S(\beta')}^{G} \mu_\omega$, where μ_ω denotes an invariant measure on ω.

Proof. We apply Theorem 1.3 to $\mathcal{O}$, $P = P(\beta')$, $S_P = S(\beta')$, $P_o = P(\beta)$ and ω. We may assume $a = 1$, i.e., $\mathcal{O} = O(1_n + X(\beta))$. Let σ', ρ' be the sets of X_1 and of X_2 for $X \in \Omega'(\beta) \subset \underline{n}(\beta)$ respectively. Then by Theorem 2.3, $\underline{n}(\beta') - \rho'$ is of measure zero and $\Omega'(\beta) = \sigma' + \rho'$. Put $\Omega^o = 1_n + \rho'$, $\Omega = 1_n + \underline{n}(\beta') = N_P$, $\sigma = 1_n + \sigma'$. Then $\omega = i(K \cap S_P)(\sigma)$ and $K(\Omega^o\sigma) = \mathcal{O} \subset K(\Omega\omega)$. Since $\Omega - \Omega^o$ is of measure zero, $\mathcal{O}$ saturates $K(\Omega\omega) = K(N_P\omega)$. Hence Theorem 1.3 gives the desired result. Q.E.D.

Remark 2.1. As a consequence of Corollary 3, the Fourier transform of $\mu_{\mathcal{O}}$ is reduced to a much simpler case of $S(\beta')$ and ω, by means of (1.10), (1.11). Note that $S(\beta')$ is nearly a direct product of $GL(N_j, F)$ for $1 \leqslant j \leqslant Q$, and the orbit ω corresponds to the simple subpartition β of $\beta' = (N_1, N_2, \ldots, N_Q)$ (cf. (2.16), (2.17)).

Remark 2.2. In case $F = \mathbb{R}$, t even, put $\Omega_{\pm} = i(g_{\pm 1})\Omega(\beta)$, $O_{\pm} = K(\Omega_{\pm})$. Then, since $Cl(O_+ \cup O_-) = N(\beta)$, the Fourier transform of $\mu_{O_+} + \mu_{O_-}$ is given by Theorem 1.4 by means of the Plancherel formula for $S(\beta)$.

§3. Closure relation between unipotent orbits

Let O and O' (resp. $\tilde{O}$ and $\tilde{O}'$) be unipotent G-orbits (resp. $\tilde{G}$-orbits) in G. We denote by $O \underset{G}{\geqslant} O'$ (resp. $\tilde{O} \geqslant \tilde{O}'$) the relation $Cl(O) \supset O'$ (resp. $Cl(\tilde{O}) \supset \tilde{O}'$). Similar notations are used for orbits of nilpotent matrices. In this case, if $\tilde{O} \geqslant \tilde{O}'$, we denote $X \geqslant X'$ for any $X \in \tilde{O}$, $X' \in \tilde{O}'$, and further if $\tilde{O} \neq \tilde{O}'$, we denote this by $\tilde{O} \succ \tilde{O}'$ and similarly for $X \succ X'$. Let us describe these relations by means of the parameters of unipotent orbits introduced in §2. This is equivalent to doing it for nilpotent matrices. By Corollary 2 to Theorem 2.3, we have $Cl(O(i(g_a)X(\beta)) = K(i(g_a)Cl(\Omega'(\beta))) = i(g_a)K(Cl(\Omega'(\beta)))$, where $\Omega'(\beta)$ is given by (2.15) with $a = 1$. Therefore it is sufficient for us to see which orbits intersect with $Cl(\Omega'(\beta)) \subset \underline{n}(\beta)$. The $\tilde{G}$-orbit $\tilde{O}(X(\beta))$ of $X(\beta)$ is given by $\tilde{O}(X(\beta)) = K(\tilde{\Omega}'(\beta))$, where $\tilde{\Omega}'(\beta)$ is the set of $X = (x_{ij}) \in \underline{n}(\beta)$ satisfying $\operatorname{rank}(x_{i,i+1}) = \max$ $(1 \leqslant i \leqslant r)$ (cf. Theorem 2.3). Therefore $Cl(\tilde{O}(X(\beta))) = K(\underline{n}(\beta))$. Here we are mainly concerned with $\tilde{G}$-orbits. The result for G-orbits can be obtained from it.

For the next two lemmas, F is an arbitrary field. Let $\beta = (n_1, n_2, \ldots, n_r)$ be a partition of n, not necessarily satisfying $n_1 \geqslant n_2 \geqslant \dots \geqslant n_r$. We can define $P(\beta) = S(\beta)N(\beta)$ and $\underline{n}(\beta)$ analogously, and put $\tilde{P}(\beta) = \tilde{S}(\beta)N(\beta)$ with $\tilde{S}(\beta) = \bigcup_{a \in F^\times} g_a S(\beta)$. An element X in $\underline{n}(\beta)$ is called proper if it satisfies the following:

(P) in any row and in any column of X, there exists at most one non-zero component which is equal to 1.

Lemma 3.1. Put $\beta_o = (n_1, n_2 + n_3 + \dots + n_r)$. Then any element in $\underline{n}(\beta)$ is conjugate undet $\tilde{P}(\beta_o)$ to a proper element in $\underline{n}(\beta)$.

Proof. We prove this by induction on r. The assertion is true for $r = 2$. Assume that it is true for $r - 1$. Put $\beta_1 = (n_2, n_3, \dots, n_r)$, $m = n_2 + n_3 + \dots + n_r$. Take an $X \in \underline{n}(\beta)$ and express it as

$$X = \begin{pmatrix} 0_{n_1} & * \\ 0 & X_1 \end{pmatrix} \qquad \text{with} \quad X_1 \in \underline{n}(\beta_1).$$

Then by assumption, there exists a $g_1 \in GL(m, F)$ such that $Y_1 = i(g_1)X_1$ is a proper element in $\underline{n}(\beta_1)$. Moreover we see from Lemma 2.1 that there exist a permutation matrix g_2 of degree m and a partition $\beta' = (n_1', n_2', \dots, n_s')$ of m such that $n_1' \geqslant n_2' \geqslant \dots \geqslant n_s'$ and $i(g_2)Y_1 = X(\beta')$. Put $\bar{g}_i = \mathrm{diag}(1_{n_1}, g_i) \in GL(n, F)$, then

$$i(\bar{g}_2\bar{g}_1)X = \begin{pmatrix} 0_{n_1} & x \\ 0 & X(\beta') \end{pmatrix},$$

where x is an $n_1 \times m$ matrix. Let

$$h_1 = \begin{bmatrix} 1_{n_1} & x' \\ 0 & 1_m \end{bmatrix}, \text{ then } Y = i(h_1)i(\overline{g}_2\overline{g}_1)X = \begin{bmatrix} 0_{n_1} & y \\ 0 & X(\beta') \end{bmatrix},$$

where $y = x + x'X(\beta')$. Put $m_1' = n_1' - n_2'$, $m' = m - m_1'$. We see from (2.6) that there exists an x' for which $y = (z, 0_{n_1 m'})$, where z is an $n_1 \times m_1'$ matrix and $0_{n_1 m'}$ denotes the $n_1 \times m'$ zero matrix. Take an $h_2 = \mathrm{diag}(a, b) \in S(\beta_0)$ with $a \in GL(n_1, F)$, $b = \mathrm{diag}(c, 1_{m'})$ with $c \in GL(m_1', F)$. Then $i(b)X(\beta') = X(\beta')$, and so

$$i(h_2)Y = \begin{bmatrix} 1_{n_1} & y' \\ 0 & X(\beta') \end{bmatrix} \quad \text{with} \quad y' = ayb^{-1} = (z', 0_{n_1 m'}),$$

$z' = azc^{-1}$. We see that for some a and c, z' has analogous property as (P), whence $i(h_2)Y$ has the property (P). Put $g = \overline{g}_2{}^{-1}h_2h_1\overline{g}_2\overline{g}_1$, then

$$i(g)X = i(\overline{g}_2)^{-1}i(h_2)Y = \begin{bmatrix} 0_{n_1} & y'g_2{}^{-1} \\ 0 & Y_1 \end{bmatrix}$$

is a proper element in $\underline{n}(\beta)$, because g_2 is a permutation matrix and Y_1 is proper in $\underline{n}(\beta_1)$. Q.E.D.

The next lemma determines which conjugate classes intersect with $\underline{n}(\beta)$. Here again F is an arbitrary field.

Lemma 3.2. Let $\beta = (n_1, n_2, \ldots, n_r)$ be a partition of n such that $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 1$, and $\alpha = (p_1, p_2, \ldots, p_s)$ be the Jordan type of $X(\beta)$, i.e., the partition of n defined from β by (2.1), (2.3). Then the set of Jordan types $\alpha' = (p_1', p_2', \ldots, p_t')$ of elements in $\underline{n}(\beta)$ are characterized as follows: the set $\{p_1', p_2', \ldots, p_t'\}$ corresponding to α' is obtained from $\{p_1, p_2, \ldots, p_s\}$ by a repetition of replacements

(i) an element p by $\{p - a, a\}$ with $1 \leqslant a \leqslant p$,

(ii) two elements $\{p, q\}$, $p > q$, by $\{p - a, q + a\}$ with $1 \leqslant a < p - q$.

Befor proving this lemma, we give the closure relation as its direct consequence.

Theorem 3.3. Let $\widetilde{G} = GL(n, F)$ with F a local field. For two partitions $\alpha = (p_1, p_2, \ldots, p_s)$, $\alpha' = (p_1', p_2', \ldots, p_t')$ of n, let $J(\alpha)$, $J(\alpha')$ be the corresponding Jordan matrices in (2.2). Then $Cl(\widetilde{O}(J(\alpha))) \supset \widetilde{O}(J(\alpha'))$ or $J(\alpha) \succcurlyeq J(\alpha')$ if and only if $\{p_1', p_2', \ldots, p_t'\}$ is obtained from $\{p_1, p_2, \ldots, p_s\}$ by the process in Lemma 3.2.

Further $J(\alpha) \succcurlyeq J(\alpha')$ can be expressed also in the form

$$(3.1) \qquad p_1 + p_2 + \cdots + p_i \geqslant p_1' + p_2' + \cdots + p_i' \quad \text{for } i \geqslant 1,$$

where we put $p_i = 0$ for $i > s$, $p_j' = 0$ for $j > t$.

Remark 3.1. After I obtained Lemma 3.2 and the first expression for $\succcurlyeq$ in Theorem 3.3, Prof. N. Iwahori informed me that he obtained Theorem 3.3 in case $F = \mathbb{C}$. The second expression for $\succcurlyeq$ is given to me by him.

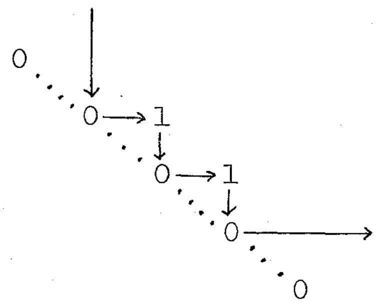
Proof of Lemma 3.2. We see easily that the replacements (i) and (ii) are possible in $\underline{n}(\beta)$.

Conversely let us prove the following. Starting from an arbitrary proper element X in $\underline{n}(\beta)$, we replace it by

$X' \in \underline{n}(\beta)$, where X' is conjugate to X, or $X \to X'$ corresponds to an inverse of (i) or (ii). Repeating these replacements appropriately, we come to an element in $\underline{n}(\beta)$ conjugate to $X(\beta)$.

First remark that the Jordan type of X can be determined by drawing zigzag lines in the matrix X as follows. We start along a column in X downward on which no numeral 1 exists. When we come to numeral 0 on the diagonal, we turn to the left along the row. When we encounter 1 on the row, we turn downward. Continuing this process, we get a zigzag line as shown below. If there exist (p-1) numerals 1 on the line, it represents $J(p)$ in the Jordan normal form of X.

(3.2) X:



(The other numerals 0 or 1 are not shown explicitly. The zigzag line represents $J(3)$ in X.)

We call a column, a row or a position (i, j) of a matrix in $\underline{n}(\beta)$ admissible (with respect to β) if the components of $Y \in \underline{n}(\beta)$ are not identically zero on it. Now assume that m-th column of X coincides with that of $X(\beta)$ for $1 \leqslant m \leqslant j-1$, and not for $m = j$. We apply an induction on j. Let the numeral 1 on the j-th column of $X(\beta)$ be on the position (i, j). Let us discuss a replacement of X in three cases.

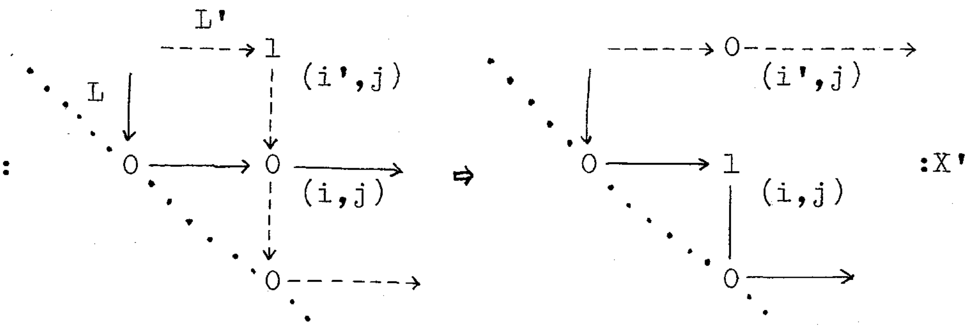
(I) Suppose there is no numeral 1 in X both on j-th column and on i-th row. Then, putting 1 on the position (i, j) of X, we get another proper element X' in $\underline{n}(\beta)$. This replacement corresponds to an inverse of (i). In fact, consider two zigzag lines in X, the one ending on i-th row and the other starting on j-th column which represent respectively $J(p)$ and $J(q)$ in the Jordan form of X. They are connected into one at the position (i, j) in X', and the resulting line represents $J(p+q)$ in X'. Thus $J(p) \oplus J(q) \longrightarrow J(p+q)$.

(II) Suppose there exists numeral 1 on j-th column, at the position (i', j). Let i and i' belong to a-th and a'-th blocks of rows (with respect to β), then $a' \leqslant a$. When $a' = a$, we can find a permutation matrix g_0 in $\widetilde{S}(\beta)$ such that the numeral 1 at (i', j) in X is removed to the position (i, j) in $X' = i(g_0)X$ and the m-th columns of X for $1 \leqslant m < j$ are left unchanged by $X \rightarrow X'$. Then X' coincides with $X(\beta)$ even on the j-th column.

We assume now that $a' < a$, whence $i' < i$. Let L and L' be the zigzag lines in X passing (i, j) horizontally and vertically respectively. The numbers of 1 on L and L' before the intersecting point (i, j) are $a - 1$ and $a' - 1$ respectively because of the assumption and (2.5), (2.6). Let the similar numbers of 1 after (i, j) be b and b', then L and L' represent $J(p)$ with $p = a + b$ and $J(p')$ with $p' = a' + b'$ respectively.

(A) When $b = 0$, we connect at (i, j) L and the second part of L' by removing 1 from (i', j) to (i, j) as shown below.

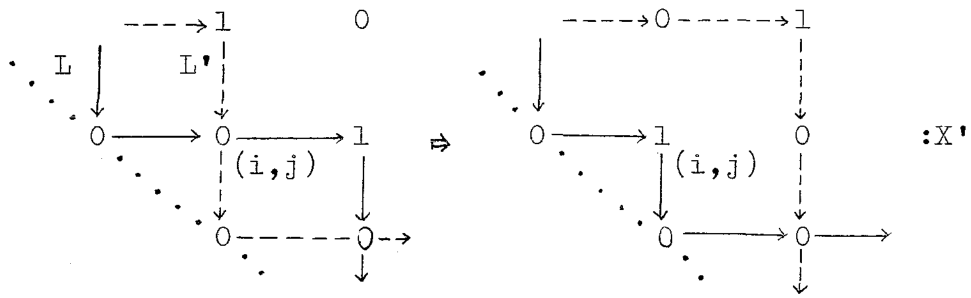
(3.3) X:



Then we get from X an $X' \in \underline{n}(\beta)$. This yields a replacement of $J(p) \oplus J(q)$ in X by $J(p') \oplus J(q')$ with $p' = a + 1 + b' > \max(p, q)$, $q' = a' - 1$. This is an inverse of (ii).

(B) When $b \geqslant 1$, we switch L and L' at (i, j) by removing two numerals 1 as shown below.

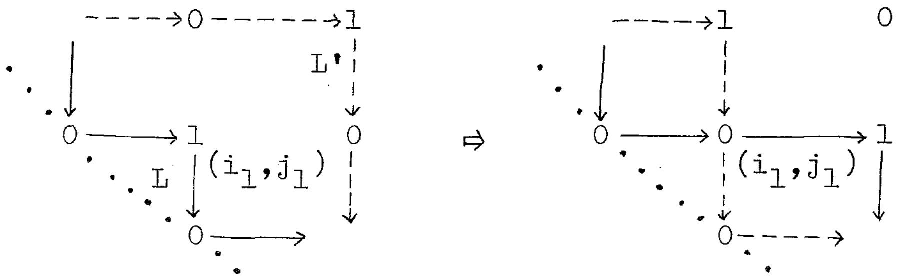
(3.4) X:



This gives a replacement of $J(p) \oplus J(q)$ in X by $J(p') \oplus J(q')$ with $p' = a + b'$, $q' = a' + b$. If $p' > \max(p, q)$, this is an inverse of (ii), and if $p' = \max(p, q)$, X and X' are conjugate to each other.

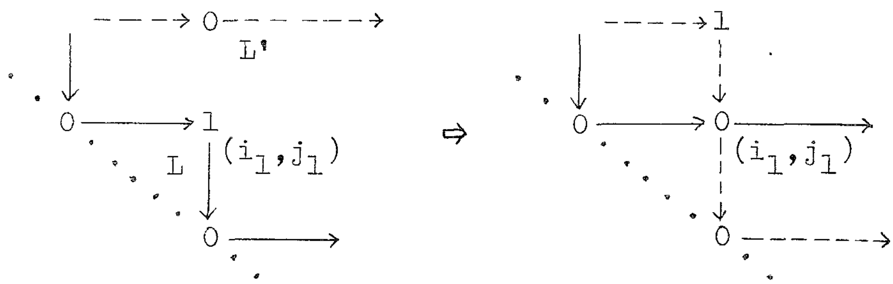
Thus it rests only to consider the case $b' < b$. In this case, we can find 1 on L after (i, j), at (i_1, j_1), such that it is not removed by $X \longrightarrow X'$, and L does not intersect with L' on its strait segments starting and ending at (i_1, j_1). Choose the first such position (i_1, j_1). We switch again in X' as shown in (3.5) or (3.5') the two lines obtained from L and L' by the previous switching, thus getting $X'' \in \underline{n}(\beta)$.

(3.5) X':



$:X''$

(3.5') X':



$:X''$

Let c, c' be the numbers of 1 on L, L' between those 1's replaced by $X \to X'$ and those 1's replaced by $X' \to X''$, not containing both extremities. Then $c \leqslant c'$, and $X' \to X''$ gives rise to a replacement of $J(p') \oplus J(q')$ in X' by $J(p'') \oplus J(q'')$ in X'' with

$$p'' = a + c' + (b - c) = a + b + (c' - c),$$

$$q'' = a' + c + (b' - c') = a' + b' - (c' - c).$$

Hence, if $c' - c \geqslant 1$, $X \to X''$ gives rise to an inverse of (ii), and if $c' - c = 0$, X and X'' are conjugate.

In any case, the new matrix and $X(\beta)$ coincide with each other on j-th column.

(III) Assume that there exists a numeral 1 on i-th row of X. We can treat this case similarly as (II), changing the rolls of columns and rows.

Thus the proof of Lemma 3.2 is now complete.

§4. Unipotent radicals and unipotent orbits

Let $\beta = (n_1, n_2, \ldots, n_r)$ be a partition of n, and $P(\beta) = S(\beta)N(\beta)$ the corresponding parabolic subgroup of G. Then we see in §2 that, if $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 1$, $K(N(\beta)) = Cl(\tilde{O}(1_n + X(\beta)))$. In this section, we study what happens when the above condition on β is not satisfied. We get the following result.

<u>Theorem</u> 4.1. Let $\beta = (n_1, n_2, \ldots, n_r)$ be a partition of n. Then $K(N(\beta)) = Cl(\tilde{O}(1_n + X(\beta')))$, where $\beta' = (n_1', n_2', \ldots, n_r')$ is a rearrangement of β such that $n_1' \geqslant n_2' \geqslant \ldots \geqslant n_r' \geqslant 1$, $\{n_1', n_2', \ldots, n_r'\} = \{n_1, n_2, \ldots, n_r\}$, and $\tilde{O}(1_n + X(\beta'))$ denotes the $\tilde{G}$-orbit of $1_n + X(\beta')$. Moreover any element in $N(\beta)$ is conjugate under K to an element in $N(\beta')$.

<u>Proof</u>. It is sufficient for us to prove that there exists only one maximal element in $\underline{n}(\beta)$ with respect to the order $\geqslant$, modulo conjugacy, and it is conjugate to $X(\beta')$. In fact, this gives us $\tilde{O}(X(\beta')) \subset K(\underline{n}(\beta)) \subset Cl(\tilde{O}(X(\beta')))$, whence $K(\underline{n}(\beta)) = Cl(\tilde{O}(X(\beta'))$ and so $K(N(\beta)) = Cl(\tilde{O}(1_n + X(\beta')))$. Thus finally $K(N(\beta)) = K(N(\beta'))$.

Let us first prove that there exists in $\underline{n}(\beta)$ a maximal element conjugate to $X(\beta')$. We apply the characterization (3.1) of the order $\succcurlyeq$. Let X be a proper element in $\underline{n}(\beta)$ and $\alpha'' = (p''_1, p''_2, \ldots, p''_s)$, $p''_1 \geqslant p''_2 \geqslant \cdots \geqslant p''_s$, be its Jordan type. Consider a solution $X = X_0$ of the following maximum problem on X:

firstly make p''_1 maximum, then,

secondly make p''_2 maximum, and then,

thirdly make p''_3 maximum, and so on.

Let $\alpha = (p_1, p_2, \ldots, p_t)$ be the Jordan type of X_0. Take a zigzag line in X representing a Jordan matrix $J(p)$ for X (cf. (3.2)). It touches the diagonal at most once in any (k, k)-block (with respect to β), an $n_k \times n_k$ matrix. Hence we have always $p''_1 \leqslant r$. Conversely $p''_1 = r$ is attained for instance by an X which has 1 at the last row of the last column in every (k, k+1)-block:

$$X = \begin{bmatrix} 0_{n_1} & x_{12} & & & \bigcirc \\ & 0_{n_2} & x_{23} & & \\ & & \ddots & \ddots & \\ & & & 0_{n_{r-1}} & x_{r-1,r} \\ \bigcirc & & & & 0_{n_r} \end{bmatrix} \quad \text{with} \quad x_{k,k+1} = \begin{bmatrix} & & & 0 \\ & * & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Thus we get $p_1 = r$.

Take any proper X for which $p''_1 = p_1$ $(= r)$, and take out from X all columns and rows on which some segments of the zigzag line for $J(p_1)$ pass. Then we get a matrix X_1 in $\underline{n}(\beta_1)$, where $\beta_1 = (n_1 - 1, n_2 - 1, \ldots, n_r - 1)$, a partirion of $n - r$. Thus the second maximum problem for $X \in \underline{n}(\beta)$ is nothing but the first one for $X_1 \in \underline{n}(\beta_1)$. Inductively we see that the multiplicity m_p of $J(p)$ in the Jordan form of X_o is $n'_p - n'_{p+1}$. Thus we see that X_o is conjugate to $X(\beta')$ under G. By (3.1), X_o is maximal in $\underline{n}(\beta)$.

Let us now prove that any maximal element in $\underline{n}(\beta)$ is $\widetilde{G}$-conjugate to X_o. Suppose $X \in \underline{n}(\beta)$ be maximal and proper, and let $(p''_1, p''_2, \ldots, p''_s)$ be its Jordan type. Suppose $p''_1 = p_1$ $(= r)$. Then replacing X by $i(g)X$ with an appropriate permutation matrix $g \in \widetilde{S}(\beta)$, we can make the zigzag line L in X representing $J(p''_1)$ of X coincides with the line M for $J(p_1)$ of X_o. Taking out all columns and rows on which some segments of $L = M$ pass, we get X_1, $X_{o,1}$ in $\underline{n}(\beta_1)$. Thus by induction on n, we may assume that $p''_1 < p_1$.

When $p''_1 < p_1$ $(= r)$, there must exist k, $1 \leqslant k < r$, such that L has numeral 1 in $(m, m+1)$-block of X for $1 \leqslant m < k$ and not for $m = k$. Take a position (i, j) on L in $(k, k+1)$-block, and let L' be a zigzag line in X passing (i, j) vertically. Note that L passes (i, j) horizontally. Thus we come to the analogous situation as in the proof of Lemma 3.2. Then, by the same argument as in (I)$\sim$(III) there, we get an $X' \in \underline{n}(\beta)$ such that $X' \not\succ X$ having numeral 1 in $(k, k+1)$-block. By induction on k if necessary, we get an $X'' \in \underline{n}(\beta)$ such that $X'' \succ X$. This contradicts that X is maximal. Q.E.D.

§5. Unipotent orbits in symplectic or orthogonal groups

We saw until now that for $SL(n, F)$ there exists a close relation between unipotent orbits and unipotent radicals of parabolic subgroups. For groups of other types, even for classical groups over $\mathbb{C}$, the relation between them is not so direct in general. Here we study it for symplectic or orthogonal groups over $\mathbb{C}$. To do so, we apply a theorem giving the closure relation for unipotent orbits for these groups from that for general linear groups. This theorem is due to Prof. N. Iwahori who explained it to the author at the same time as for Theorem 3.3 (for $F = \mathbb{C}$), to whom the author expresses his hearty thanks.

Let L_n be an $n \times n$ matrix such that

$$L_n = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & 0 \end{pmatrix}, \text{ and put } \quad M_{2n} = \begin{pmatrix} 0_n & L_n \\ -L_n & 0_n \end{pmatrix}.$$

We define

$$Sp(N, \mathbb{C}) = \{ g \in GL(N, \mathbb{C});\ {}^t g M_N g = M_N \} \quad \text{with} \quad N = 2n,$$
$$O(N, \mathbb{C}) = \{ g \in GL(N, \mathbb{C});\ {}^t g L_N g = L_N \}.$$

Let G be one of these groups and put $G_A = GL(N, \mathbb{C})$. Let $\underline{g}$ and $\underline{g}_A$ be Lie algebras of G and G_A respectively, given in the form of matrices.

Theorem 5.1 (Iwahori). Let $x, y \in \underline{g}$. Then,

(1) $\quad \mathrm{Ad}(G_A)x \cap \underline{g} = \mathrm{Ad}(G)x,$

(2) $\mathrm{Cl}(\mathrm{Ad}(G)x) \supset \mathrm{Ad}(G)y$ if and only if $\mathrm{Cl}(\mathrm{Ad}(G_A)x) \supset \mathrm{Ad}(G_A)y$.

<u>Theorem</u> 5.2 (cf. [11]). Let $x \in \mathfrak{g}_A$ be nilpotent. Then $\mathrm{Ad}(G_A)x \cap \mathfrak{g} \neq \emptyset$ if and only if the Jordan type $(p_1, p_2, \ldots, p_s)$ of x satisfies the following condition (C1) or (BD1) according as G is symplectic or orthogonal.

(C1) The multiplicity of any odd integer in p_i's is even.

(BD1) The multiplicity of any even integer in p_i's is even.

Assume that $1_N + X \in G$ is unipotent. Then $x = \log(1_N + X) \in \mathfrak{g}$ is nilpotent, and the correspondence $X \to x$ is bijective and G-homomorphic: $i(g)X \to \mathrm{Ad}(g)x$ $(g \in G)$. Moreover the Jordan types of X and x coincide with each other. Therefore, for the nilpotent case, Theorems 5.1 and 5.2 can be stated for X (instead of x) in the same way.

Put $K_A = U(N)$, and for a partition β of N, let $P_A(\beta)$ be the parabolic subgroup of G_A corresponding to β and $N_A(\beta)$ its unipotent radical (see §2). Put $K = G \cap K_A$, and let P be a parabolic subgroup containing a Borel subgroup of all upper triangular matrices in G. Then $G = KP$, and there exists a partition $\beta = (n_1, n_2, \ldots, n_r)$ of N satisfying

(5.1) $$n_i = n_{r-i+1} \quad (1 \leqslant i \leqslant r/2)$$

such that $P = G \cap P_A(\beta)$. We denote P by $P(\beta)$ and its unipotent radical $G \cap N_A(\beta)$ by $N(\beta)$.

For simplicity, we give our result only for symlectic case.

<u>Theorem</u> 5.3. Let $G = Sp(N, \mathbb{C})$, $N = 2n$. The set $K(N(\beta)) =$

$i(K)N(\beta)$ is equal to the closure of a unipotent G-orbit in G. Moreover the closure of G-orbit of a unipotent element $1_N + X$ in G can be expressed as $K(N(\beta))$ for some β if and only if the Jordan type $\alpha_C = (p_1, p_2, \dots, p_s)$, $p_1 \geqslant p_2 \geqslant \dots \geqslant p_s \geqslant 1$, of X satisfies in addition to (C1) the following condition.

(C2) Let p_t be odd and p_j for $j > t$ be all even. Then, for p_j with $j \leqslant t$, (i) the multiplicity of any even integer in p_j's is at most 2, and (ii) if $p_i, p_{i+1}, \dots, p_{j-1}$ are of multiplicity 1, and $i = 1$ or p_{i-1} is of multiplicity $\geqslant 2$, and so is p_j, then $j - i$ is even.

The correspondence of β with (5.1) to α_C with (C1), (C2) is not necessarily 1-1. In the way of proving the theorem we show how α_C is determined explicitly from β.

For the proof, we prepare three lemmas.

Lemma 5.4. Let $1_N + X$ be a unipotent element in G_A. In order that it is maximal in $N_A(\beta)$ with respect to the order $\succ$ for some $\beta = (n_1, n_2, \dots, n_r)$ with (5.1), it is necessary and sufficient that the Jordan type $\alpha = (q_1, q_2, \dots, q_s)$, $q_1 \geqslant q_2 \geqslant \dots \geqslant q_s$, of X satisfies the following condition:

(OE) if q_i is odd and q_j is even, then $i < j$.

Proof. Let $\beta' = (n'_1, n'_2, \dots, n'_r)$ be a rearrangement of β such that $n'_1 \geqslant n'_2 \geqslant \dots \geqslant n'_r$, $\{n'_1, n'_2, \dots, n'_r\} = \{n_1, n_2, \dots, n_r\}$. Then, by Theorem 4.1, $1_N + X(\beta')$ is maximal in $N_A(\beta)$. For the Jordan type $\alpha = (q_1, q_2, \dots, q_s)$ of $X(\beta')$, the multiplicity of p in it is $n'_p - n'_{p+1}$. Then (5.1) gives the condition (OE). Q.E.D.

For two partitions α, α' of N representing Jordan types, we define $\alpha \geqslant \alpha'$ by $J(\alpha) \geqslant J(\alpha')$ (see Theorem 3.3). For an α, we define a set of Jordan types

$$A_C(\alpha) = \{\alpha';\ \alpha' \leqslant \alpha, \text{ and } \alpha' \text{ satisfies (C1)}\}.$$

Lemma 5.5. Assume that $\alpha = (q_1, q_2, \ldots, q_s)$ satisfies (OE). Then there exists in $A_C(\alpha)$ a unique maximal element α_C, and it is obtained from α as follows:

(C3) if q_{2i-1}, q_{2i} in α are different odd integers, then replace (q_{2i-1}, q_{2i}) by $(q_{2i-1}-1, q_{2i}+1)$, for $1 \leqslant i \leqslant s/2$.

Proof. Let $\alpha_o = (p_1, p_2, \ldots, p_s)$ be a partition of N obtained from α by (C3). Then

$$(5.2) \qquad p_1 + p_2 + \ldots + p_j = q_1 + q_2 + \ldots + q_j$$

except for $j = 2i - 1$ such that $q_{2i-1} > q_{2i}$ are odd, and in that case,

$$(5.2') \qquad p_1 + p_2 + \ldots + p_{2i-1} = q_1 + q_2 + \ldots + q_{2i-1} - 1.$$

First, applying (C1) and the characterization (3.1) of $\geqslant$, we see from (5.2), (5.2') that α_o is maximal in $A_C(\alpha)$. Next we prove the uniqueness. Suppose $\alpha' = (p_1', p_2', \ldots, p_t')$ be maximal in $A_C(\alpha)$ and different from α_o. Then, by (5.2), (5.2'), there exists $j = 2i - 1$ such that $q_{2i-1} > q_{2i}$ are odd and

$$p_1' + p_2' + \ldots + p_{2i-1}' = q_1 + q_2 + \ldots + q_{2i-1}.$$

Put $p_1' + p_2' + \dots + p_{2i-2}' = q_1 + q_2 + \dots + q_{2i-2} - m$, $m \geqslant 0$, then $p_{2i-1}' = q_{2i-1} + m$. Note that $q_1 + q_2 + \dots + q_{2i-2}$ is even, then we have

$$p_1' + p_2' + \dots + p_{2i-2}' \equiv m, \quad p_{2i-1}' \equiv m + 1 \pmod{2}.$$

Since any odd integer has even multiplicity in α, we see that m is even and p_{2i-1}' is odd. Hence by (C1), $p_{2i}' = p_{2i-1}' = q_{2i-1} + m$, and so

$$\begin{aligned} p_1' + p_2' + \dots + p_{2i}' &= q_1 + q_2 + \dots + q_{2i-1} + (q_{2i-1} + m) \\ &> q_1 + q_2 + \dots + q_{2i-1} + q_{2i}. \end{aligned}$$

This contradicts that $\alpha' \leqslant \alpha$. Q.E.D.

<u>Lemma</u> 5.6. Assume that $\beta = (n_1, n_2, \dots, n_r)$ satisfies (5.1). Let α be the Jordan type corresponding to β as in Lemma 5.4, and α_C the unique maximal element in $A_C(\alpha)$. Then there exists an element $1_N + X$, unique modulo conjugacy under G, in $N(\beta) = G \cap N_A(\beta)$ such that the Jordan type of X is α_C.

<u>Note</u> 5.1. For symplectic or orthogonal groups, the analogy of Lemma 3.1 does not hold in general.

<u>Proof</u>. For the uniqueness of X, we refer Theorem 5.1(1).

To prove the existence, we recall that α_C is obtained from α by (C3). According to the process in (C3), we first study the case where $\alpha = (q_1, q_2)$. The corresponding β is given by

$$(5.3)\qquad \beta = (\underbrace{1, 1, \ldots, 1}_{(q_1-q_2)/2}, \underbrace{2, 2, \ldots, 2}_{q_2}, \underbrace{1, 1, \ldots, 1}_{(q_1-q_2)/2})$$

or a rearrangement of it. We discuss for β in (5.3), and the other cases are quite similar. Put $x = \log(1_N + X)$, then ${}^t x M_N + M_N x = 0_N$. Therefore x has the following form:

$$(5.4)\qquad x = \begin{pmatrix} y & z \\ 0_n & y' \end{pmatrix}, \quad y' = -L_n {}^t y L_n, \quad {}^t z L_n = L_n z,$$

where y, z are $n \times n$ matrices. We denote here by $\underline{n}_A(\beta)$ the set of matrices $\underline{n}(\beta)$ in §§2-4. Put

$$A(p,q) = \underbrace{\begin{pmatrix} 0 & 1 & & & & & & & \\ & 0 & 1 & & & & & 0 & \\ & & \ddots & \ddots & & & & & \\ & & & 0 & 1 & & & & \\ & & & & 0 & I_{12} & & & \\ & & & & & 0_2 & 1_2 & & \\ & & & & & & \ddots & \ddots & \\ & 0 & & & & & & 0_2 & 1_2 \\ & & & & & & & & 0_2 \end{pmatrix}}_{p-q \quad\quad q\text{-times}}, \quad \text{of type } (p+q)\times(p+q)$$

$$I_{12} = (0, 1),$$

$$A'(p,q) = \begin{pmatrix} & & & & 0 \\ & & & & \vdots \\ & A(p,q) & & & 0 \\ & & & & 1 \\ & & & & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & & 0_{n-2} \\ 0 & 1 & \\ 1 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0_{n-2} \\ 1_2 & 0 \end{pmatrix}.$$

(I) Let $\alpha = (2p, 2q)$, $p \geqslant q$. Then $\alpha_C = \alpha$. Put in (5.4) $y = A(p, q)$, $z = B$. Then, by drawing zigzag lines in x as in (3.2), we see that x is conjugate under G_A to $J(\alpha_C) = J(2p) \oplus J(2q)$, and so is $X = \exp x - 1_N$. Since $x \in \underline{g} \cap \underline{n}_A(\beta)$, we have $1_N + X \in N(\beta)$.

(II) Let $\alpha = (2p+1, 2p+1)$. Then $\alpha_C = \alpha$. Put in (5.4),

$y = A'(p, p)$, $z = C$. Then x is conjugate to $J(\alpha_C)$, and $x \in \underline{g} \cap \underline{n}_A(\beta)$, whence $\exp x = 1_N + X \in N(\beta)$.

(III) Let $\alpha = (2p+1, 2q-1)$, $p \geqslant q$. Then $\alpha_C = (2p, 2q)$. Let β_C correspond to α_C as in (5.3). Put $y = A'(p, q-1)$, $z = B$, then x is conjugate to $J(\alpha_C)$ and $x \in \underline{g} \cap \underline{n}_A(\beta_C)$, but not $x \in \underline{n}_A(\beta)$. To get an $x' = \mathrm{Ad}(g)x$ in $\underline{g} \cap \underline{n}_A(\beta)$, we choose as $g \in G$ a blockwise diagonal matrix with respect to β_C as

$$g = \mathrm{diag}(1_{p-q}, \underbrace{u, u, \ldots, u}_{q}, \underbrace{u', u', \ldots, u'}_{q}, 1_{p-q})$$

where $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ with $i = \sqrt{-1}$, and $u' = L_2 {}^t u^{-1} L_2$.

Now let us reduce the general case to the above special case. Take an $\alpha = (q_1, q_2, \ldots, q_s)$ satisfying (OE). Put $\alpha' = (q_1, q_2)$, $N' = q_1 + q_2$. Apply (I)-(III) to α' and $Sp(N', \mathbb{C})$, we get the element x or x' above. We imbed this $Sp(N', \mathbb{C})$ into G appropriately. Imitating some discussions in the proof of Theorem 4.1, we see that there exists a subset S' of $\{1, 2, \ldots, N\}$ consisting of N'-elements such that (1) if $j \in S'$, then $N - j + 1 \in S'$, and (2) when we imbed $Sp(N', \mathbb{C})$ into $G = Sp(N, \mathbb{C})$ by using j-th rows and columns with $j \in S'$, the above element x or x' is imbedded in $\underline{g} \cap \underline{n}_A(\beta)$. Now taking out these rows and columns, we come to the similar situation for $\alpha'' = (q_3, q_4, \ldots, q_s)$ and $N'' = N - N'$. By induction on s, the assertion of the lemma is proved. Q.E.D.

<u>Proof of Theorem</u> 5.3. Note first that

(5.5) $$G \cap K_A(N_A(\beta)) \supset K(N(\beta)).$$

Let $\alpha = (q_1, q_2, \ldots, q_s)$ correspond to β as in Lemma 5.4. Put $\tilde{O}(\alpha) = \tilde{O}(1_N + J(\alpha))$, then by Theorem 3.3, $K_A(N_A(\beta)) = Cl(\tilde{O}(\alpha))$ is a union of $\tilde{O}(\alpha')$ over $\alpha' \leqslant \alpha$. By Theorems 5.1(1) and 5.2, $G \cap \tilde{O}(\alpha')$ is a G-orbit for $\alpha' \in A_C(\alpha)$, and empty otherwise. Further, using Theorem 5.1(2) and Lemma 5.5, we get

(5.6) $$G \cap K_A(N_A(\beta)) = \bigcup_{\alpha' \in A_C(\alpha)} G \cap \tilde{O}(\alpha') = Cl(G \cap \tilde{O}(\alpha_C)).$$

On the other hand, we have from Lemma 5.6

(5.7) $$G \cap \tilde{O}(\alpha_C) \subset K(N(\beta)).$$

The assertion of Theorem 5.3 follows from (5.5)-(5.7). Q.E.D.

Corollary. Assume that β satisfies (5.1). Then

$$G \cap K_A(N_A(\beta)) = K(N(\beta)),$$

where $K = G \cap K_A$, $N(\beta) = G \cap N_A(\beta)$.

Remark 5.2. Suppose α satisfies (C1), and $\alpha \neq (1, 1, \ldots, 1), (2, 1, 1, \ldots, 1)$. Then the G-orbit $G \cap O(1_N + J(\alpha))$ contains a non-trivial set of the form $K(N(\beta))$ for some β with (5.1).

Part II. Fourier transform of unipotent orbital integrals for $SL(n, \mathbb{R})$

§6. Reduction to groups of lower ranks

6.1. Put $G = SL(n, \mathbb{R})$. Let $\beta = (n_1, n_2, \ldots, n_r)$, $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_r \geqslant 1$, be a partition of n, and let the notations be as in §2.

Assume that $n_{2i-1} - n_{2i} \neq 0$ for some i (we put $n_{r+1} = 0$). Then we see in §2 that for the G-orbit $\mathcal{O} = O(1_n + X(\beta))$, $Cl(\mathcal{O}) = K(N(\beta))$, and then, by Theorem 1.4, the Fourier transform of $\mu_{\mathcal{O}}$ is obtained directly from the Plancherel formula for $S(\beta)$.

Assume now that $n_{2i-1} - n_{2i} = 0$ for any i. Then necessarily n is even. The two orbits $O_{\pm} = O(1_n+i(g_{\pm 1})X(\beta))$ are given as $O_{\pm} = K(\Omega_{\pm})$ with $\Omega_{\pm} = i(g_{\pm 1})\Omega(\beta)$, and invariant measures on them are given respectively as

$$(6.1) \qquad \mu_{O_{\pm}}(f) = \int_K \int_{\Omega'_{\pm}} f(k(1_n+X)k^{-1})\, dk\, dX,$$

where dk and dX denote the normalized Haar measure on K and the usual Lebesgue measure on $\underline{n}(\beta)$ respectively, and $\Omega'_{\pm} = i(g_{\pm 1})\Omega'(\beta)$. Since $Cl(O_+ \cup O_-) = K(N(\beta))$, the Fourier transform of $\mu_{O_+} + \mu_{O_-}$ is given by Theorem 1.4 form the Plancherel formula for $S(\beta)$. Therefore it rest for us to obtain the Fourier transform of $\mu = \mu_{O_+} - \mu_{O_-}$. For $g_o \in GL(n, \mathbb{R})$ and $f \in C_o^{\infty}(G)$, put

$$(6.2)\qquad (i(g_o)f)(g) = f(i(g_o)^{-1}g) = f(g_o^{-1}gg_o) \qquad (g \in G).$$

Then, since $\Omega_- = i(g_{-1})\Omega_+$, we get from (6.1) that $\mu_{0_-}(f) = \mu_{0_+}(i(g_{-1})f)$ and

$$(6.3)\qquad \mu(f) = \mu_{0_+}(\psi) \qquad \text{with} \qquad \psi = f - i(g_{-1})f.$$

6.2. We have also another way of reduction. Put

$$(6.4)\qquad \begin{aligned} &r_j = n_{2j-1},\quad N_j = n_{2j-1} + n_{2j} = 2r_j \quad (1 \leqslant j \leqslant Q = r/2), \\ &\beta' = (N_1, N_2, \ldots, N_Q). \end{aligned}$$

Then $P(\beta') \supset P(\beta)$, $S(\beta') \supset S(\beta)$ and $N(\beta') \subset N(\beta)$. Put $G' = S(\beta')$, $K' = G' \cap K$, $P' = G' \cap P(\beta)$, and $N' = G' \cap N(\beta)$. Then P' is a parabolic subgroup of G', and the latter is given as follows with respect to the partition β':

$$(6.5)\qquad G' = \Big\{ \operatorname{diag}(g_1, g_2, \ldots, g_Q);\ g_j \in GL(N_j, \mathbb{R}),\ \prod_{1 \leqslant j \leqslant Q} \det(g_j) = 1 \Big\},$$

and N' consists of elements in G' of the form

$$(6.6)\qquad g_j = \begin{pmatrix} 1_{r_j} & X_j \\ 0_{r_j} & 1_{r_j} \end{pmatrix}, \quad X_j \in \underline{gl}(r_j, \mathbb{R}) \quad (1 \leqslant j \leqslant Q).$$

Let $\sigma_\pm$ be the subsets of N' consisting respectively of elements such that in (6.6)

$$(6.7)\qquad \prod_{1 \leqslant j \leqslant Q} \det(X_j) > 0 \quad \text{or} \quad < 0,$$

and put $\omega_{\pm} = K'(\sigma_{\pm})$, then $\omega_{\pm}$ are G'-orbits. By Corollary 3 to Theorem 2.3, $O_{\pm}$ saturate $K(N(\beta')\omega_{\pm}) = K(N(\beta')\sigma_{\pm})$, and then inducing $\mu_{\omega_{\pm}}$ by means of $P(\beta') = S(\beta')N(\beta')$ and $S(\beta') = G'$, we get

$$\mu_{O_{\pm}} = \mathrm{Ind}_{S(\beta')}^{G} \mu_{\omega_{\pm}}. \tag{6.8}$$

Therefore, because of (1.10), (1.11), it will be sufficient for us to get the Fourier transform of $\mu_{\omega_{\pm}}$ on G'.

6.3. Further we can reduce the problem from G' to its connected semisimple part $G'' = [G', G']$ as follows. Let G'_o (resp. Z_o) be the connected component of e in G' (resp. in the center of G'). Then $G'_o = G''Z_o$ is a direct product, and G'', Z_o consist of elements in (6.5) satisfying respectively

$$\begin{aligned} &G'': \quad g_j \in SL(N_j, \mathbb{R}) \quad (1 \leqslant j \leqslant Q), \\ &Z_o: \quad g_j = t_j 1_{N_j}, \quad t_j > 0, \quad \prod_{1 \leqslant j \leqslant Q} t_j^{N_j} = 1. \end{aligned} \tag{6.9}$$

Since G'_o is normal in G' and $[G' : G'_o] = 2^{Q-1} < \infty$, we can induce an invariant distribution ρ on G'_o to such a one $\mathrm{Ind}_{G'_o}^{G'} \rho$ on G' as follows: take a complete system of representatives $\{a_1, a_2, \ldots, a_q\}$, $q = 2^{Q-1}$, of G'/G'_o, and put for $f \in C_o^{\infty}(G')$

$$(\mathrm{Ind}_{G'_o}^{G'} \rho)(f) = \sum_{1 \leqslant j \leqslant q} \rho(i(a_j)f|G'_o). \tag{6.10}$$

If ρ is the character of an irreducible representation T of G'_o, $\mathrm{Ind}_{G'_o}^{G'} \rho$ is the character of the induced representation $\mathrm{Ind}_{G'_o}^{G'} T$ of T. On the other hand, let $\sigma_{\pm,o}$ be the connected components of $\sigma_{\pm}$ consisting of elements in (6.6) satisfying respectively

(6.11) $\det(X_j) > 0$ for $1 \leqslant j \leqslant Q$, or

(6.11') $\det(X_1) < 0$, and $\det(X_j) > 0$ for $2 \leqslant j \leqslant Q$,

and put $\omega_{\pm,o} = K'_o(\sigma_{\pm,o})$ with $K'_o = G'_o \cap K$. Then they are G'_o-orbits in G'_o. Define $\mu_{\omega_{\pm,o}}$ similarly as in (6.1), then, since $[K' : K'_o] = 2^{Q-1}$, we have

(6.12) $$\mu_{\omega_{\pm}} = 2^{Q-1} \operatorname{Ind}_{G'_o}^{G'} \mu_{\omega_{\pm,o}} .$$

Consider now the reduction from $G'_o = G''Z_o$ to G''. Note that we have $\widehat{G'_o} = \widehat{G''} \times \widehat{Z}_o$, or more exactly, an irreducible character of G'_o is of the form $\pi_\gamma \times \chi$, where π_γ is the character of class $\gamma \in \widehat{G''}$, and $\chi \in \widehat{Z}_o$. We will get in the sequel an expression of $\mu_{\omega_{\pm,o}}$ on G'' by means of π_γ ($\gamma \in G''$) of the form

(6.13) $$\mu_{\omega_{\pm,o}} = \int_{\widehat{G''}} \pi_\gamma \, d\nu(\gamma),$$

where ν is a signed measure on G''. Then we get on $G'_o = G''Z_o$

(6.14) $$\mu_{\omega_{\pm,o}} = \int_{\widehat{G''}} \int_{\widehat{Z}_o} (\pi_\gamma \times \chi) \, d\nu(\gamma) \, d\nu_{Z_o}(\chi),$$

where ν_{Z_o} is a Haar measure on Z_o normalized in such a way that

(6.15) $$\int_{\widehat{Z}_o} \int_{Z_o} \varphi(z) \, \chi(z) \, dz \, d\nu_{Z_o}(\chi) = \varphi(e) \qquad (\varphi \in C_o^\infty(Z_o)),$$

where dz is a Haar mesure on Z_o fixed to give a Haar measure on $G'_o = G''Z_o$. Note that the decomposition of $\mathrm{Ind}_{G'_o}^{G'}(\pi_\gamma \times \chi)$ into irreducible characters of G' is easy for γ appearing in (6.13) (cf. §8). Then (6.14) gives immediately the Fourier transform of $\mu_{\omega_\pm}$ by (6.12). By a similar reason, the reduction by (6.8) from G' to G is easy.

Since G'' is a direct product of $SL(N_j, \mathbb{R})$, $1 \leqslant j \leqslant Q$, we are now reduced to the following case: $G = SL(N, \mathbb{R})$, $N = 2n$, $\beta = (n, n)$, $O_\pm = K(\Omega_\pm)$ with $\Omega_\pm = 1_N + \Omega'_\pm$,

$$(6.16) \qquad \Omega'_\pm = \left\{ \begin{pmatrix} 0_n & X \\ 0_n & 0_n \end{pmatrix} : X \in \underline{gl}(n, \mathbb{R}),\ \det X > 0 \text{ or } < 0 \right\}.$$

§7. A limit expression for μ_{O_+}

7.1. Let $X_o, Y_o, H_o \in \underline{g}$ be

$$(7.1) \qquad X_o = \begin{pmatrix} 0_n & 1_n \\ 0_n & 0_n \end{pmatrix}, \quad Y_o = \begin{pmatrix} 0_n & 0_n \\ 1_n & 0_n \end{pmatrix}, \quad H_o = \frac{1}{2}\begin{pmatrix} -1_n & 0_n \\ 0_n & 1_n \end{pmatrix}.$$

and put $Z = X_o - Y_o$. Then $O_+ = i(G)(\exp X_o)$, $O_- = i(g_{-1})O_+$ with $g_{-1} = \mathrm{diag}(-1, 1_{N-1})$, and $\{X_o, Y_o, H_o\}$ is a Lie triplet such that $[X_o, Y_o] = -2H_o$. Let σ be a Cartan involution of $\underline{g}$ defined by $X \to -{}^tX$ $(X \in \underline{g})$, and $\underline{g} = \underline{k} + \underline{p}$ the corresponding Cartan decomposition. Put $z(\theta) = \exp \theta Z \in K$. We give a σ-invariant Cartan subgroup B of G containing the one-parameter subgroup $z(\theta)$ as follows: $b \in B$ is expressed as $b = b_K b_p$, where

$$(7.2)\qquad \begin{aligned} b_K &= \exp X, \quad X = \begin{pmatrix} 0_n & x \\ -x & 0_n \end{pmatrix}, \; x = \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_n), \\ b_p &= \mathrm{diag}(e^{t_1}, e^{t_2}, \ldots, e^{t_n}, e^{t_1}, e^{t_2}, \ldots, e^{t_n}), \; t_j \in \mathbb{R}. \end{aligned}$$

Let G_Z be the centralizer of the one-parameter subgroup $z(\theta)$ in G. Then there exists an invariant measure $d\bar{g}$ ($\bar{g} = gG_Z$) on G/G_Z. Put for $f \in C_0^\infty(G)$,

$$(7.3)\qquad I_f(\theta) = \int_{G/G_Z} f(gz(\theta)g^{-1})\, d\bar{g}.$$

<u>Theorem</u> 7.1. For $f \in C_0^\infty(G)$,

$$\lim_{\theta \to +0} \theta^{n^2} I_f(\theta) = c_1 \mu_{0_+}(f),$$

where c_1 is a constant depending only on the normalization of the invariant measure $d\bar{g}$ on G/G_Z.

Before proving this theorem, we give a result of Harish-Chandra giving an expression of $I_f(\theta)$ as a limit of his function F_f on B. Let us recall the definition of F_f in our case. For a moment, let H be any Cartan subgroup of G, and $\underline{h}_c$ the complexification of the Lie algebra $\underline{h}$ of H. We define for a linear form δ on $\underline{h}_c$, if possible, a character ξ_δ on H by $\xi_\delta(h) = e^{\delta(\log h)}$ $(h \in H)$, where $\log h$ denotes an inverse image of h under $\exp: \underline{h}_c \to G_c = SL(N, \mathbb{C})$. Introduce in the root system of $(\underline{g}_c, \underline{h}_c)$ an order, and let S_R denote the set of all positive real roots in it. Let ρ be half the sum of all positive roots, and put

$$(7.4)\qquad \begin{aligned} \Delta^H(h) &= \xi_\rho(h) \prod_{\alpha>0} (1 - \xi_\alpha(h)^{-1}), \\ \varepsilon_R^H(h) &= \operatorname{sgn}\Big(\prod_{\alpha \in S_R} (1 - \xi_\alpha(h)^{-1}\Big). \end{aligned}$$

Let H' be the set of regular elements in H. For $h \in H'$, we put

$$(7.5)\qquad F_f^H(h) = (\varepsilon_R^H \Delta^H)(h) \int_{G/H} f(ghg^{-1})\, d\tilde{g} ,$$

where $d\tilde{g}$ $(\tilde{g} = gH)$ denotes an invariant measure on G/H. For $H = B$, we denote F_f^B simply by F_f. Note that B is fundamental in G in the sense of Harish-Chandra, and $S_R = \phi$ for B.

For a $\gamma \in B$, let P_γ be the set of positive roots α of $(\underline{g}_c, \underline{h}_c)$ such that $\xi_\alpha(\gamma) = 1$, and G_γ the centralizer of γ in G. Denote by H_α the element of $\underline{b}_c$ such that $\operatorname{Tr}(\operatorname{ad} H_\alpha \ \operatorname{ad} X) = \alpha(X)$ $(X \in \underline{b}_c)$, and by $\partial(H_\alpha)$ the differential operator on B corresponding naturally to H_α. Let D_γ be the product of $\partial(H_\alpha)$ over $\alpha \in P_\gamma$, and $d\dot{g}$ $(\dot{g} = gG_\gamma)$ an invariant measure on G/G_γ. Then Lemma 23 in [2a] says that

$$\lim_{\substack{b \to \gamma \\ b \in B'}} (D_\gamma F_f)(b) = c_2\ \xi_\rho(\gamma) \prod_{\alpha \in P_\gamma} (1 - \xi_\alpha(\gamma)^{-1}) \int_{G/G_\gamma} f(g\gamma g^{-1}) d\dot{g},$$

where c_2 is a positive constant depending only on the normalization of invariant measures.

Now let $\gamma = z(\theta)$, $\theta \neq 0$ sufficiently small. Then $G_\gamma = G_Z$. Put for $b = b_K b_p$ in (7.2), $a_j(b) = \exp(t_j + i\theta_j)$, $a_{j+n}(b) =$

$\exp(t_j - i\theta_j)$ $(i = \sqrt{-1})$. Denote by α_{jk} the root α for which $\xi_\alpha(b) = a_j(b)a_k(b)^{-1}$. We introduce an order such that $\alpha > 0$ if $\alpha = \alpha_{jk}$ for some $j < k$. Then $P_\gamma = P_Z$, where

$$(7.5) \qquad P_Z = \{ \alpha_{jk} ; 1 \leqslant j < k \leqslant n \text{ or } n+1 \leqslant j < k \leqslant 2n \}.$$

We summalize the obtained formula in the form of a theorem.

Theorem 7.2. Let $\theta \neq 0$ be sufficiently small. Put

$$(7.6) \qquad D_Z = \prod_{\alpha \in P_Z} \partial(H_\alpha).$$

Then

$$\lim_{\substack{b \to z(\theta) \\ b \in B'}} D_Z F_f(b) = c_2 (e^{i\theta} - e^{-i\theta})^{n^2} \int_{G/G_Z} f(gz(\theta)g^{-1}) d\bar{g}.$$

Combining Theorems 7.1 and 7.2, we get the following.

Theorem 7.3. For $f \in C_0^\infty(G)$,

$$(7.7) \qquad \lim_{\theta \to +0} \lim_{\substack{b \to z(\theta) \\ b \in B'}} D_Z F_f(b) = c_1 c_2 (2i)^{n^2} \mu_{O_+}(f).$$

7.2. Proof of Theorem 7.1. First we give a decomposition of G such that $G = K \exp(\underline{w}) G_Z$, where $\underline{w}$ is an appropriate subspace of $\underline{p}$, and write down an invariant measure on G/G_Z by means of K and $\underline{w}$. Let $\underline{g}_Z$ be the Lie algebra of G_Z, then $\underline{g}_Z = \underline{k}_Z + \underline{p}_Z$ with $\underline{k}_Z = \underline{k} \cap \underline{g}_Z$, $\underline{p}_Z = \underline{p} \cap \underline{g}_Z$. Denote by $\underline{p}_Z{}^\perp$ the orthogonal complement of $\underline{p}_Z$ in $\underline{p}$ with respect to the Killing form. Then, using a result of G.D.Mostow [6, Th.3], we have the following.

Lemma 7.4[1a, Prop.4.4]. As an analytic manifold, G is expressed as a direct product as $G = K \exp(\underline{p}_Z{}^{\perp})\exp(\underline{p}_Z)$.

Put $K_Z = G_Z \cap K$, then $G_Z = K_Z\exp(\underline{p}_Z)$. Therefore, to get the desired decomposition of G, we look for a subspace $\underline{w}$ of $\underline{p}_Z{}^{\perp}$ such that $\mathrm{Ad}(K_Z)\underline{w} = \underline{p}_Z{}^{\perp}$. The space $\underline{p}_Z{}^{\perp}$ consists of elements of the form

$$(7.8)\quad \underline{p}_Z{}^{\perp}: \quad X = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \quad {}^t x = x,\ {}^t y = y, \quad x, y \in \underline{gl}(n, \mathbb{R}).$$

We take as $\underline{w}$ the space consisting of $W \in \underline{p}_Z{}^{\perp}$ such that

$$(7.9)\quad W = \begin{pmatrix} T & 0_n \\ 0_n & -T \end{pmatrix}, \quad T = \mathrm{diag}(t_1, t_2, \ldots, t_n),$$

and put

$$(7.10)\quad \begin{aligned} \underline{w}' &= \left\{ W \in \underline{w} ;\ \prod_{1 \leq j < k \leq n} (t_j{}^2 - t_k{}^2) \neq 0 \right\}, \\ \underline{w}'_+ &= \left\{ W \in \underline{w} ;\ t_1 > t_2 > \cdots > t_n > 0 \right\}. \end{aligned}$$

Lemma 7.5. Let φ be a mapping from $K_Z \times \underline{w}$ to $\underline{p}_Z{}^{\perp}$ given by $\varphi(k, W) = \mathrm{Ad}(k)W$. Then φ is differentiable, and everywhere regular on $K_Z \times \underline{w}'$. It is surjective and $\varphi(K_Z \times \underline{w}')$ is open and dense in $\underline{p}_Z{}^{\perp}$. Moreover $\varphi(K_Z \times \underline{w}') = \varphi(K_Z \times \underline{w}'_+)$, and $\varphi(k_1, W_1) = \varphi(k_2, W_2)$ for $(k_i, W_i) \in K_Z \times \underline{w}'_+$ if and only if $W_1 = W_2$, $k_1{}^{-1}k_2 \in D_K$, the group of diagonal elements in K_Z.

Proof. For $Q \in \underline{k}_Z$, $R \in \underline{w}$, we get

$$d\varphi_{(k,W)}(Q, R) = \frac{d}{dt}\varphi(k\exp(tQ), W + tR)\Big|_{t=0} =$$

$$= \mathrm{Ad}(k)(\mathrm{ad}(Q)W + R).$$

After simple calculations, this gives us the regularity of φ on $K_Z \times \underline{w}'$.

Let us prove that φ is onto, Let $X \in \underline{p}$, then $X \in \underline{p}_Z{}^{\perp}$ if and only if $XZ = -ZX$, i.e., $ZXZ^{-1} = -X$. Therefore, for $X \in \underline{p}_Z{}^{\perp}$, we can find a $k \in K$ and $W \in \underline{w}$ such that

$$\mathrm{Ad}(k)X = W = \begin{pmatrix} T & 0_n \\ 0_n & -T \end{pmatrix}, \quad T = \mathrm{diag}(t_1, t_2, \ldots, t_n).$$

Here $\{t_1, t_2, \ldots, t_n, -t_1, -t_2, \ldots, -t_n\}$ is the set of all eigenvalues of X. Putting $g^Z = ZgZ^{-1}$ for $g \in G$, we have $\mathrm{Ad}(k^Z)X = W$, whence for $\ell = k^Z k^{-1}$, $\mathrm{Ad}(\ell)W = W$. Therefore, for $W \in \underline{w}'$, ℓ is diagonal. Since $\ell^Z = \ell^{-1}$, we have $\ell = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\varepsilon_j = \pm 1$. We can find $m \in K$ such that $\ell = (m^Z)^{-1}m$ and $\mathrm{Ad}(m)W \in \underline{w}'$. Thus $(mk)^Z = mk$, i.e., $mk \in K_Z$, and $\mathrm{Ad}(mk)X = \mathrm{Ad}(m)W \in \underline{w}'$. This proves that $\varphi(K_Z \times \underline{w}')$ is the set of all regular elements in $\underline{p}_Z{}^{\perp}$. On the other hand, Image(φ) is closed, because K_Z is compact. Hence Image(φ) $= \underline{p}_Z^{\perp}$, that is, φ is onto.

The rest of the lemma is easy to prove. Q.E.D.

<u>Lemma</u> 7.6. The mapping $\varphi_o\colon (k, W, g) \longrightarrow k\exp(W)g$ from $K \times \underline{w} \times G_Z$ to G is differentiable and surjective. It is regular on $K \times \underline{w}' \times G_Z$ and $\varphi_o(K \times \underline{w}' \times G_Z) = \varphi_o(K \times \underline{w}'_+ \times G_Z)$ is open and dense in G. For $(k_i, W_i, g_i) \in K \times \underline{w}'_+ \times G_Z$ $(i = 1, 2)$, $\varphi_o(k_1, W_1, g_1) = \varphi_o(k_2, W_2, g_2)$ if and only if $W_1 = W_2$, $k_1 = k_2 z$, $g_1 = z^{-1}g_2$ for some $z \in D_K$.

Proof. Surjectivity follows directly from Lemmas 7.4, 7.5. Regularity is easy to prove. Now assume that $k_1\exp(W_1)g_1 = k_2\exp(W_2)g_2$. Put $k = k_2^{-1}k_1$, $g = g_2g_1^{-1}$, then $k\exp(W_1) = \exp(W_2)g = h$ (put), and so

$$hZh^{-1} = \mathrm{Ad}(k)\begin{pmatrix} 0_n & D_1^{\,2} \\ D_1^{-2} & 0_n \end{pmatrix} = \begin{pmatrix} 0_n & D_2^{\,2} \\ D_2^{-2} & 0_n \end{pmatrix},$$

where D_1, D_2 are diagonal matrices of degree n such that $\exp W_i = \mathrm{diag}(D_i, D_i^{-1})$. Since $W_i \in \underline{w}_+'$, we see that k must be diagonal and in D_K. Hence $W_1 = W_2$ and $k = g \in D_K$.

Q.E.D.

By this lemma, an open dense subset of G/G_Z is naturally diffeomorphic to $K/D_K \times \underline{w}_+'$. To get an invariant measure on it, we use the following lemma. Put $t = (t_1, t_2, \ldots, t_n)$, $a(t) = \exp W$ for W in (7.9), and

$$(7.11)\qquad D_t^+ = \{\, t = (t_1, t_2, \ldots, t_n);\ t_1 > t_2 > \cdots > t_n > 0 \,\}.$$

Lemma 7.7. A Haar measure dg on G is given as follows: for $f \in C_0^\infty(G)$,

$$\int_G f(g)dg = c_3 \int_{G_Z}\int_K\int_{D_t^+} f(ka(t)g_Z)\rho_W(t)dt_1dt_2\ldots dt_n\, dk\, dg_Z,$$

where c_3 is a positive constant, dg_Z denotes a Haar measure on G_Z, and, with $\mathrm{sh}\, x = (e^x - e^{-x})/2$,

$$\rho_W(t) = 2^{n^2+n-1}\prod_{1\le i<j\le n}\mathrm{sh}(2t_i-2t_j)\mathrm{sh}(2t_i+2t_j)\prod_{1\le \ell\le n}\mathrm{sh}\, 2t_\ell,$$

Proof. For $g \in G$, let $\delta g = (dg_{ij})$ be $N\times N$ matrix whose (i, j)-component is the differential of (i,j)-component

g_{ij} of g. Put $\delta_\ell g = g^{-1}\delta g$. Then every component $\delta_\ell g_{ij}$ of $\delta_\ell g$ is a left invariant 1-form on G. The exterior product $\bigwedge \delta_\ell g_{ij}$ over all $(i, j) \neq (2n, 2n)$ is a non-zero left invariant form on G of degree $\dim G$, whence it determines a Haar measure on G. Similarly the exterior product $\bigwedge \delta_\ell (g_Z)_{ij}$ (resp. $\bigwedge \delta_\ell k_{ij}$) over (i, j) such that $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant 2n$, $(i, j) \neq (n, n)$ (resp. $1 \leqslant i < j \leqslant 2n$) determines a Haar measure on G_Z(resp. on K). For $g = ka(t)g_Z$, we get at $k = g_Z = e$,

$$\delta_\ell g\Big|_{k=g_Z=e} = a^{-1}\delta_\ell k\Big|_e a + \delta_\ell a + \delta_\ell g_Z\Big|_e ,$$

where $a = a(t)$. Note that

$$\delta_\ell a = \mathrm{diag}(dt_1, dt_2, \ldots, dt_n, -dt_1, -dt_2, \ldots, -dt_n)$$

$${}^t(\delta_\ell k) = -\delta_\ell k, \quad \delta_\ell g_Z \cdot Z = Z \cdot \delta_\ell g_Z .$$

Then we can calculate the Jacobian at $k = g_Z = e$, which is equal to $\rho_W(t)$. This proves the assertion of the lemma. Q.E.D.

Note 7.1. When dg, dg_Z are given as indicated in the above proof, the constant c_3 is given by $c_3 = |D_K| v_{2n} = 2^n v_{2n}$, where v_{2n} is the volume of $K = SO(2n)$ with respect to the measure on K indicated above.

Corollary. An invariant measure $d\bar{g}$ ($\bar{g} = gG_Z$) on G/G_Z is given as follows: for $\varphi \in C_0^\infty(G/G_Z)$,

$$\int_{G/G_Z} \varphi(\bar{g})\, d\bar{g} = \frac{1}{|D_K|}\int_{D_t^+} \varphi(\overline{ka(t)})\rho_W(t)dt_1dt_2\ldots dt_n dk.$$

Using this result, we calculate the limit of $\theta^{n^2} I_f(\theta)$.

Lemma 7.8. For $f \in C_o^\infty(G)$, put $f^K(g) = \int_K f(kgk^{-1})dk$. Then

$$\lim_{\theta \to +0} \theta^{n^2} I_f(\theta) = \frac{1}{2^{n+1}} \int_{D_y^+} f^K\left(\begin{bmatrix} 1_n & Y \\ 0_n & 1_n \end{bmatrix}\right) \rho_o(y)dy_1dy_2\cdots dy_n,$$

where $Y = \mathrm{diag}(y)$ with $y = (y_1, y_2, \ldots, y_n)$, and D_y^+ is similar to (7.11) for y, and further

$$(7.12) \qquad \rho_o(y) = \prod_{1 \leqslant i < j \leqslant n} (y_i^2 - y_j^2).$$

Proof. In the definition (7.3) of $I_f(\theta)$, insert the above expression of $d\bar{g}$. Then we get

$$I_f(\theta) = \frac{1}{|D_K|} \int_{D_t^+} f^K(a(t)z(\theta)a(t)^{-1}) \rho_w(t)\, dt_1dt_2\cdots dt_n.$$

Put $s_j = \exp(2t_j)$, $S = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, and $y_j = \theta s_j$, $Y = \mathrm{diag}(y_1, y_2, \ldots, y_n)$. Then

$$a(t)z(\theta)a(t)^{-1} = \begin{pmatrix} \cos(\theta)1_n & \sin(\theta)S \\ -\sin(\theta)S^{-1} & \cos(\theta)1_n \end{pmatrix} \longrightarrow \begin{pmatrix} 1_n & Y \\ 0_n & 1_n \end{pmatrix},$$

as $\theta \to +0$ for any fixed $y = (y_1, y_2, \ldots, y_n)$. Moreover $dy_j = 2y_jdt_j$ and

$$\theta^{n^2} \rho_w(t) \longrightarrow 2^{n-1}\rho_o(y)y_1y_2\cdots y_n.$$

This gives us the desired result by a simple argument. Q.E.D.

Now we rewrite μ_{O_+} with $O_+ = K(\Omega_+)$. Recall that for $f \in C_o^\infty(G)$,

$$(7.13) \qquad \mu_{O_+}(f) = \int_{D_X^+} f^K(n(X))\, dX, \qquad n(X) = \begin{pmatrix} 1_n & X \\ 0_n & 1_n \end{pmatrix},$$

where $D_X^+ = \{X = (x_{ij}) \in \underline{gl}(n, \mathbb{R});\ \det X > 0\}$, and $dX = \prod_{i,j} dx_{ij}$. Let $K' = G \cap P(\beta)$ with $\beta = (n, n)$, then $k \in K'$ is given as $k = \mathrm{diag}(u, v)$, $u, v \in O(n)$, $\det(uv) = 1$, and $kn(X)k^{-1} = n(uXv^{-1})$.

<u>Lemma</u> 7.9. Let φ_1 be a mapping from $K' \times D_y^+$ to D_X^+ given by $\varphi_1(k, y) = uYv^{-1}$ with $Y = \mathrm{diag}(y)$. Then it is everywhere regular and the image $\varphi_1(K' \times D_y^+)$ is open and dense in D_X^+. For $(k, y), (k', y') \in K' \times D_y^+$, $\varphi_1(k, y) = \varphi_1(k', y')$ if and only if $y = y'$, $k^{-1}k' \in D_K$.

The proof is easy and so omitted.

<u>Lemma</u> 7.10. The measure dX on D_X^+ is expressed as follows: put $k' = \mathrm{diag}(u, v^{-1}) \in K'$, and denote by dk' the normalized Haar measure on K', then for $\psi \in C_o^\infty(D_X^+)$,

$$\int_{D_X^+} \psi(X)\, dX = c_4 \int_{K'} \int_{D_y^+} \psi(uY\mathbf{v})\ \rho_o(y)\ dy_1 dy_2 \ldots dy_n\ dk',$$

where $c_4 = 2\,|D_K|\, v_n{}^2 = 2^{n+1} v_n{}^2$ with v_n similar as v_{2n} in Note 7.1.

<u>Proof</u>. Let $X = uYv$, then we have at $u = v = e$

$$\delta X\big|_{u=v=e} = \delta u\big|_e\ Y + \delta Y + Y\ \delta v\big|_e\ .$$

Noting that $\delta u\big|_e = \delta_\ell u\big|_e$, we can calculate as in the proof of Lemma 7.8 the Jacobian at $u = v = e$, equal to $\rho_o(y)$. Q.E.D.

Applying the lemma to (7.13), we get the following.

<u>Corollary</u>. Let $Y = \mathrm{diag}(y)$ for $y = (y_1, y_2, \ldots, y_n)$.

Then for $f \in C_0^\infty(G)$,

$$(7.14) \qquad \mu_{O_+}(f) = c_4 \int_{D_y^+} f^K(n(Y))\rho_o(y)dy_1dy_2\cdots dy_n.$$

Proof of Theorem 7.1. The formula in the theorem follows from Lemma 7.8 and (7.14) with the constant $c_1 = (c_4 2^{n+1})^{-1} = (2^{n+1}v_n)^{-2}$. Q.E.D.

§8. Fourier transform of $\mu_{O_\pm}$

To get the Fourier transform of $\mu_{O_\pm}$, we apply the expression (7.7) of $\mu_{O_\pm}(f)$ by means of $D_Z F_f$.

8.1. First of all, we study the symmetry of the function F_f on B'. Put $W = N_G(B)/Z_G(B)$ and $\widetilde{W} = N_{\tilde{G}}(B)/Z_{\tilde{G}}(B)$, where $N_G(B)$ and $Z_G(B)$ be the normalizer and the centralizer of B in G respectively, and similarly for $N_{\tilde{G}}(B)$ and $Z_{\tilde{G}}(B)$. Let $a_j(b)$ for $b \in B$ be as in §7.1. Then for $w \in \widetilde{W}$, there exists a permutation σ of $\{1, 2, \ldots, 2n\}$ such that $a_j(wb) = a_{\sigma(j)}(b)$. We denote w by w_σ. We consider subgroups W_o, W_1 and W_2 of W consisting of elements w_σ for which

(8.1) for W_o: $\quad \sigma(n+j) = n + \sigma(j) \qquad (1 \leqslant j \leqslant n)$,

(8.2) for W_1 (resp. W_2): σ is a product of even (resp. any) number of permutations $\sigma_j = (j, n+j)$, $1 \leqslant j \leqslant n$.

Put $w_1 = w_{\sigma_1}$ with $\sigma_1 = (1, n+1)$, then

$$(8.3) \qquad i(g_{-1})b = w_1 b \quad (b \in B).$$

Lemma 8.1. The group W is generated by W_o and W_1, and $\widetilde{W} = W \cup w_1 W$.

Let $\mathrm{sgn}(w)$ be the usual sign of $w \in \widetilde{W}$ as an element in the Weyl group of $(\underline{g}_c, \underline{b}_c)$, then $\mathrm{sgn}(w) = 1$ on W and $\mathrm{sgn}(w_1) = -1$. The symmetry of F_f is given in the following.

Lemma 8.2. Let $f \in C_o^\infty(G)$ and $b \in B'$. Then

(1) $$F_f(wb) = \mathrm{sgn}(w) F_f(b) = F_f(b) \quad \text{for} \quad w \in W,$$

(2) $$F_{i(g_{-1})f}(b) = - F_f(w_1 b),$$

(3) let $\psi = f - i(g_{-1})f$, then $F_\psi(wb) = F_\psi(b)$ for $w \in \widetilde{W}$.

Proof. Recall that $S_R = \phi$ for B, then (1) is easy to see. For (2), we apply (8.3). Finally (3) follows from (1) and (2). Q.E.D.

8.2. Let H be a Cartan subgroup of G with Lie algebra $\underline{h}$. Here we recall some properties of F_f^H given by Harish-Chandra([2a, Th.2] and [2b, Lem.40]). Denote by S_I the set of all positive singular imaginary roots of $(\underline{g}_c, \underline{h}_c)$, and put $H'(I) = \{ h \in H;\ \xi_\alpha(h) \neq 1 \text{ for any } \alpha \in S_I \}$.

Lemma 8.3. Let $f \in C_o^\infty(G)$. Then the function F_f^H on H' vanishes outside a bounded subset, and can be extended to a function on $H'(I)$ which is, on every connected component of $H'(I)$, equal to the restriction of a C^∞-function on its closure. Moreover, for an $a \in H$ and a polynomial P of $\partial(X)$, $X \in \underline{h}_c$, assume that $s_\alpha P = - P$ for any $\alpha \in S_I$ such that $\xi_\alpha(a) = 1$, where s_α denotes the reflexion corresponding to α. Then

PF_f^H can be extended to a continuous function arround a.

Consider F_f on B'. Then the set S_I for B is given by $\{ \alpha_{j,n+j};\ 1 \leqslant j \leqslant n \}$ and $\xi_\alpha(b) = \exp(2i\theta_j)$ for b in (7.2) and $\alpha = \alpha_{j,n+j}$. Hence by Lemma 8.3, F_f can be considered as a C^∞-function on $B'(I) = \{ b;\ \theta_j \neq 0 \ (\mathrm{mod.}\ \pi)$ for $1 \leqslant j \leqslant n \}$. Further we have the following.

<u>Lemma</u> 8.4. Let $\phi = f - i(g_{-1})f$. Then $F_\phi = F_\phi^B$ can be extended to a C^∞-function on B, and $F_\psi^H = 0$ for any Cartan subgroup H not conjugate to B under G.

<u>Proof</u>. For the first assertion, we prove here that F_f can be extended to a C^∞-function arround the unit element e. Arround other non-regular lements, the proof is similar. Remark that F_ϕ is even in every θ_j by Lemma 8.2. Then it follows from this that F_ϕ can be extended to a continuous function arround e. Note further that $\partial/\partial\theta_j$ is a constant multiple of $\partial(H_\alpha)$ for $\alpha = \alpha_{j,n+j}$. Then, by Lemma 8.4, $(\partial/\partial\theta_j)F_\phi$ can be extended to a continuous function for $\theta_2\theta_3\cdots\theta_n \neq 0$, $\theta_1, \theta_2, \ldots, \theta_n$ sufficiently small, and again by the above remark, so does it arround e. In general, let P be a monomial of $\partial/\partial\theta_j$, $\partial/\partial t_j$ $(1 \leqslant j \leqslant n)$, then the extendability of PF_f arround e follows from Lemma 8.3 and the above remark similarly.

For the second assertion, it is sufficient to remark that for $h \in H$, $i(g_{-1})h$ is again conjugate to h under G. This in turn can be seen for instance from the explicit form of H given in [5c, Exemple 3.3]. Q.E.D.

$\underline{8.3}$. Here we study the Fourier transform of the C^∞-function $D_Z F_\psi$ on B with $\psi = f - i(g_{-1})f$. Denote by $\mathbb{Z}$ the set of all integers, by $\mathbb{T}$ the quotient $\mathbb{R}/2\pi\mathbb{Z}$, and by $\mathbb{R}^n_o$ the hyperplane of $\mathbb{R}^n$ defined by $\rho_1 + \rho_2 + \ldots + \rho_n = 0$ for $\rho = (\rho_1, \rho_2, \ldots, \rho_n) \in \mathbb{R}^n$. Let $m = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n$ and $\rho \in \mathbb{R}^n_o$. We put for $b \in B$ in (7.2)

$$(8.4) \qquad e(m, \rho; b) = \exp(i \sum_{1 \leqslant j \leqslant n} (m_j\theta_j + \rho_j t_j)), \quad i = \sqrt{-1}.$$

Note that $t_1 + t_2 + \ldots + t_n = 0$ for $b \in B$, and so B is isomorphic to $\mathbb{T}^n \times \mathbb{R}^n_o$. Then $\mathbb{Z}^n \times \mathbb{R}^n_o$ can be identified with the dual group of B, and the action of $w \in \widetilde{W}$ on B induces the dual action on $\mathbb{Z}^n \times \mathbb{R}^n_o$: $e(m, \rho; wb) = e(w^{-1}(m, \rho); b)$. Now put

$$(8.5) \qquad d(m, \rho) = \int_B D_Z F_\psi(b)\, e(m, \rho; b)\, db,$$

where $db = d\theta_1 d\theta_2 \ldots d\theta_n dt_1 dt_2 \ldots dt_{n-1}$. Then, since F_ψ is in $C_o^\infty(B)$ by Lemmas 8.3 and 8.4, we have

$$(8.6) \quad D_Z F_\psi(e) = n(2\pi)^{-2n+1} \sum_{m \in \mathbb{Z}^n} \int_{\mathbb{R}^n_o} d(m, \rho) d\rho_1 d\rho_2 \ldots d\rho_{n-1},$$

where the right hand side converges absolutely. For $n = 1$, $\mathbb{R}^n_o = \{0\}$ and the integration disappears. By Theorem 7.3, this gives an expression of $\mu_{0_+}(\psi) = \mu_{0_+}(f) - \mu_{0_-}(f)$, which will be rewritten in the following.

Since F_ψ is in $C_o^\infty(B)$, we have by integration by parts,

$$(8.7) \quad d(m, \rho) = \int_B F_\psi(b) D_Z e(m, \rho; b) db = p_Z(m, \rho) d_\psi(m, \rho),$$

where

$$(8.8)\qquad p_Z(m,\ \rho) = (8n)^{-n^2+n} \prod_{1 \leqslant j < k \leqslant n} ((m_j - m_k)^2 + (\rho_j - \rho_k)^2),$$

and for $f \in C_o^\infty(G)$,

$$(8.9)\qquad d_f(m,\ \rho) = \int_B F_f(b) e(m,\ \rho;\ b) db.$$

Note that $F_f(wb) = F_f(b)$ for $w \in W$ by Lemma 8.2, then we have $d_f(w(m,\ \rho)) = d_f(m,\ \rho)$, whence

$$(8.10)\qquad d_f(m,\ \rho) = \frac{1}{|W|} \sum_{w \in W} d_f(w(m,\ \rho))$$

$$= \frac{1}{|W|} \int_B F_f(b) \sum_{w \in W} e(w(m,\ \rho);\ b)\, db.$$

Hence for $\psi = f - i(g_{-1})f$,

$$(8.11)\qquad d_\psi(m,\ \rho) = \frac{1}{|W|} \int_B F_f(b) \sum_{w \in \widetilde{W}} e(w(m,\ \rho);\ b)\, db.$$

The meaning of $d_f(m,\ \rho)$, $d_\psi(m,\ \rho)$ will be seen later. Note that $d_\psi(w(m,\ \rho)) = d_\psi(m,\ \rho)$ for $w \in \widetilde{W}$, and $p_Z(w(m,\ \rho)) = p_Z(m,\ \rho)$ for $w \in W_o$, then

$$\sum_{w \in \widetilde{W}} d(w(m,\ \rho)) = |W_o|\ p(m,\ \rho)\ d_\psi(m,\ \rho) \qquad \text{with}$$

$$(8.12)\qquad p(m,\ \rho) = \sum_{w \in W_2} p_Z(w(m,\ \rho)).$$

Note that any element $w \in \widetilde{W}$ is expressed uniquely as $w = w_2 w_o$ with $w_2 \in W_2$, $w_o \in W_o$, and that $w\rho = w_o\rho$. Then we get from the above equality

$$|W_o| \sum_{w_o' \in W_o} p(w_o'm,\ \rho) d_\psi(w_o'm,\ \rho) =$$

$$= \sum_{w_2 \in W_2} \sum_{w_o, w_o' \in W_o} d(w_2 w_o w_o' m,\ w_o \rho)$$

$$= \sum_{w_2 \in W_2} \sum_{w_o, w_o' \in W_o} d(w_2 w_o' m,\ w_o \rho).$$

For $m \in \mathbb{Z}^n$, put $W_o(m) = \{w \in W_o;\ wm = m\}$, $W_2(m) = \{w \in W_2;\ wm = m\}$. Then $W_2(m) = 2^{d(m)}$ with $d(m) = \#\{j;\ m_j = 0\}$. Let $m_1 \geqslant m_2 \geqslant \dots \geqslant m_n \geqslant 0$ and integrate the both sides above with respect to $d\rho = d\rho_1 d\rho_2 \dots d\rho_{n-1}$ over $D_\rho^+ \subset \mathbb{R}_o^n$ defined by $\rho_1 > \rho_2 > \dots > \rho_n$. Then we have

$$|W_o|\ |W_o(m)| \sum_{m' \in W_o m} \int_{D_\rho^+} p(m', \rho) d_\psi(m', \rho) d\rho$$

$$= |W_o(m)| \sum_{w_2 \in W_2} \sum_{m' \in W_o m} \int_{\mathbb{R}_o^n} d(w_2 m', \rho) d\rho$$

$$= |W_o(m)|\ |W_2(m)| \sum_{m'' \in \widetilde{W} m} \int_{\mathbb{R}_o^n} d(m'', \rho) d\rho.$$

Therefore, using Theorem 7.3, we get from (8.6)

$$(8.13) \qquad \mu_{O_+}(\psi) = c_5 \sum_{\substack{m \in \mathbb{Z}^n \\ m_j \geqslant 0}} 2^{-d(m)} \int_{D_\rho^+} p(m, \rho) d_\psi(m, \rho) d\rho.$$

<u>8.4</u>. Let $G_2 = SL(2, \mathbb{R})$, $B_2 = \left\{ b(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}$. For the character π of an irreducible unitary representation of G_2, put $k(b(\theta)) = \Delta(b(\theta))\pi(b(\theta))$ with $\Delta(b(\theta)) = e^{i\theta} - e^{-i\theta}$. Then we know (see for exemple [5c, p.51]) that for any non-negative integer c, there exist two equivalence classes D_{c+1}^+ and D_{c+1}^- of such representations such that

for D_{c+1}^+, $\quad k(b(\theta)) = -e^{ic\theta}$; $\quad$ for D_{c+1}^-, $\quad k(b(\theta)) = e^{-ic\theta}$.

Now, for $G = SL(N, \mathbb{R})$, $N = 2n$, we consider a parabolic subgroup $P(\beta_o)$ corresponding to the partition $\beta_o = (2, 2, \ldots, 2)$ of N. Then $S(\beta_o)$ consists of elements of the form

$$(8.14) \qquad s = \operatorname{diag}(s_1, s_2, \ldots, s_n); \; s_j \in GL(2, \mathbb{R}), \quad \det(s_1 s_2 \ldots s_n) = 1.$$

Let S_o be the connected component of e in $S(\beta_o)$, then for $s \in S_o$, we have $s_j = e^{t_j} s_j'$ with $t_j \in \mathbb{R}$, $s_j' \in G_2$. Denote again by $D^{\pm}_{c+1}$ certain elements in the classes $D^{\pm}_{c+1}$ respectively. We consider a representation U of S_o given by

$$U(s) = e^{i\rho_1 t_1} D^{+}_{m_1+1}(s_1') \otimes e^{i\rho_2 t_2} D^{+}_{m_2+1}(s'_2) \otimes \cdots \otimes e^{i\rho_n t_n} D^{+}_{m_n+1}(s_n'),$$

where $m_j \in \mathbb{Z}, \geqslant 0$, and $\rho_j \in \mathbb{R}$ with $\rho_1 + \rho_2 + \ldots \rho_n = 0$. Let $U' = \operatorname{Ind}_{S_o}^{S(\beta_o)} U$, and extend it to $P(\beta_o)$ by putting $U'(g) =$ identity for $g \in N(\beta_o)$. Then, inducing it to G, we get a unitary representation $T(m, \rho)$ of G. We define another representation $T'(m, \rho)$ by $T'(m, \rho; g) = T(m, \rho; i(g_{-1})g)$. Then they are always irreducible and their characters can be calculated by Theorem 2 in [5a, p.358]. Note that an s in (8.14) with $s_j = e^{t_j} b(\theta_j)$ is conjugate under G to b or $w_1 b = i(g_{-1})b$ for $b \in B$ in (7.2), according as $[n/2]$ is even or odd. By this reason, we denote $T(m, \rho)$ and $T'(m, \rho)$ by $T^+(m, \rho)$ and $T^-(m, \rho)$ when $[n/2]$ is even, and by $T^-(m, \rho)$ and $T^+(m, \rho)$ when $[n/2]$ is odd respectively. Let $\pi^{\pm}(m, \rho)$ be the characters of $T^{\pm}(m, \rho)$ respectively and put $k^{\pm}(m, \rho; b) = \Delta^B(b)\pi^{\pm}(m, \rho; b)$ for $b \in B'$. Then,

$$(8.15)\qquad \begin{aligned} k^+(m,\ \rho;\ b) &= \sum_{w\in W} e(w(m,\ \rho);\ b), \\ k^-(m,\ \rho;\ b) &= -\sum_{w\in w_1 W} e(w(m,\ \rho);\ b). \end{aligned}$$

We say that this series of representations is associated to B.

On the other hand, there exists a positive constant c_B such that for $f \in C_0^\infty(G)$

$$(8.16)\qquad \int_{i(G)B'} f(g)dg = c_B \int_B \int_{G/B} f(gbg^{-1})\,|\Delta^B(b)|^2\,db\,d\tilde{g},$$

where $\tilde{g} = gB$. Note that the complex conjugate of $\Delta^B(b)$ is equal to $(-1)^n\Delta^B(b)$. Then, by the second assertion of Lemma 8.4, we get from (8.15), (8.16) the following: put

$$\pi^\pm(m,\ \rho;\ f) = \int_G f(g)\pi^\pm(m,\ \rho;\ g)dg = \mathrm{Tr}\Big(\int_G f(g)T^\pm(m,\ \rho;\ g)dg\ \Big),$$

and $\psi = f - i(g_{-1})f$, then

$$\begin{aligned}(8.17)\qquad \pi^+(m,\ \rho;\ f) - \pi^-(m,\ \rho;\ f) &= \\ &= c_B(-1)^n \int_B F_f(b) \sum_{w\in\tilde{W}} e(w(m,\ \rho;\ b)\,db \\ &= c_B(-1)^n\,|W|\,d_\psi(m,\ \rho) \qquad \text{(by (8.11))}.\end{aligned}$$

Thus, by (8.13) and (8.17), we get the Fourier transform of $\mu_{O_+} - \mu_{O_-}$ as follows.

<u>Theorem</u> 8.5. Let $G = SL(N,\ \mathbb{R})$, $N = 2n$, and let $\mu_{O_\pm}$ be the invariant measures on the unipotent orbits $O_\pm = K(\Omega_\pm)$ given by (6.1). Then the Fourier transform of $\mu_{O_+} - \mu_{O_-}$ is given by

$$\mu_{0_+} - \mu_{0_-} = c \sum_{\substack{m \in \mathbb{Z}^n \\ m_j \geqslant 0}} 2^{-d(m)} \int_{D_\rho^+} (\pi^+(m, \rho) - \pi^-(m, \rho)) \times$$
$$\times p(m, \rho) d\rho_1 d\rho_2 \ldots d\rho_{n-1},$$

where c is a constant depending only on the normalization of the Haar measure on G, $d(m) = \#\{j;\ m_j = 0\}$ for $m = (m_1, m_2, \ldots, m_n)$, D_ρ^+ is a subdomain of $\mathbb{R}_0^n = \{\rho = (\rho_1, \rho_2, \ldots, \rho_n);\ \rho_1 + \rho_2 + \ldots + \rho_n = 0\}$ defined by $\rho_1 > \rho_2 > \ldots > \rho_n$, and $\pi^\pm(m, \rho)$ are the characters of irreducible unitary representations $T^\pm(m, \rho)$ of G, and $p(m, \rho)$ is given by (8.8), (8.12). In particular, for $n = 1$, the above formula should be read as

$$\mu_{0_+} - \mu_{0_-} = c \sum_{m \in \mathbb{Z}, \geqslant 0} 2^{-d(m)} (\pi^+(m) - \pi^-(m)),$$

where $\pi^\pm(m)$ denote the characters of $D_{m+1}^\pm$ respectively.

Remark 8.1. The representations $T^\pm(m, \rho)$ with $m_j = 0$ for some j (resp. $D_1^\pm$ if $n = 1$) do not appear in the Plancherel formula for G, but they appear here in the Fourier transform of $\mu_{0_+} - \mu_{0_-}$.

Remark 8.2. As is remarked before, $\mu_{0_+} + \mu_{0_-} = \mathrm{Ind}_{P(\beta)}^G \delta_e$, where δ_e denotes the Dirac's distribution at e on $S(\beta)$, and its Fourier transform is obtained from the Plancherel formula for $S(\beta) \cong \{(g_1, g_2);\ g_1, g_2 \in GL(n, \mathbb{R}), \det(g_1 g_2) = 1\}$. The contribution to it from the characters $\pi^\pm(m, \rho)$ of representations $T^\pm(m, \rho)$ of the series associated to B is zero if n is odd, and is given as follows if $n = 2\ell$ is even:

$$c' \sum_{\substack{m \in \mathbb{Z}^n \\ m_j > 0}} \int_{D_\rho^+} (\pi^+(m, \rho) + \pi^-(m, \rho)) q(m, \rho) d\rho_1 d\rho_2 \ldots d\rho_{n-1},$$

where c' is a constant depending on the normalization of the Haar measure on G, and

$$(8.19) \qquad q(m, \rho) = \sum_{w \in W_0} w q_0(m, \rho) \qquad \text{with}$$

$$q_0(m, \rho) = \prod_{1 \leqslant j \leqslant n} m_j \quad \times$$

$$\times \prod_{\substack{1 \leqslant j < k \leqslant \ell \\ \text{or } \ell+1 \leqslant j < k \leqslant n}} ((m_j - m_k)^2 + (\rho_j - \rho_k)^2)((m_j + m_k)^2 + (\rho_j + \rho_k)^2).$$

Acknowledgements. When I sent to Prof. N. Kawanaka a copy of this paper as a preprint, he kindly informed me that the closure relation in §3 has been already known and a proof can be found in [12]. Then I found that the proof here is elementary (but longer) in the sense that we do not use any notion from algebraic geometry. Therefore I left the section as it was except only a minor change. Later Prof. R. Hotta also gave me the reference [13] and pointed out the similarity between the inducing invariant measures on unipotent orbits in §1 and the inducing unipotent orbits in [13]. I learned the following. Let G be a connected reductive algebraic group defined over an algebraically closed field. A nilpotent element A in the Lie algebra of G is called in [14] "of parabolic type" if there is a parabolic subgroup P of G such that the G-orbit of A intersects densely (with respect to the Zarisky topology) the Lie algebra $\underline{u}_P$ of the unipotent radical U_P of P.

A special meaning of this type of nilpotent elements in Springer's theory of representations of the Weyl group of G, can be seen in [14, Proposition 1.4]. A unipotent class in G is called in [13] "Richardson class" if it has the analogous property for U_P of some P. We note here that the necessary and sufficient condition, given in §5 for type C, that a unipotent class should be a Richardson class, can not be expressed in a simple manner by means of its weighted Dynkin diagram (cf. [11, p.263]).

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