PAPER Special Section on Nonlinear Theory and its Applications

An Algebraic Approach to Guarantee Harmonic Balance Method Using Gröbner Base

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SUMMARY Harmonic balance (HB) method is well known principle for analyzing periodic oscillations on nonlinear networks and systems. Because the HB method has a truncation error, approximated solutions have been guaranteed by error bounds. However, its numerical computation is very time-consuming compared with solving the HB equation. This paper proposes an algebraic representation of the error bound using Gröbner base. The algebraic representation enables to decrease the computational cost of the error bound considerably. Moreover, using singular points of the algebraic representation, we can obtain accurate break points of the error bound by collisions.

key words: *harmonic balance method, error bound, Gröbner base, algebraic representation, quadratic approximation, singular point*

1. Introduction

Harmonic balance (HB) method is well known principle for analyzing periodic oscillations of nonlinear networks and systems [1], [2]. Using this method, we express circuit equations as simultaneous algebraic equations called HB equation due to an approximation by truncated Fourier series of variables. Although the HB method ignores high frequency components, the HB method enables to clarify essential relations among the system parameters. Recently, some techniques by the HB method for bifurcation analysis have been proposed [3]–[6].

Because the HB method has a truncation error, approximated solutions of the HB equation have been guaranteed by bounded regions, called error bounds, within which the solution must reside [7]–[9]. In particular, Swern presented a method to obtain the error bound for a feedback system with a polynomial-type nonlinear element [9]. However, the numerical computation of the error bound is very time-consuming because we have to express the high dimensional error bound using a set of the numerical values.

In order to overcome the difficulty, we propose an algebraic representation of the error bound using Gröbner base [10], [11]. Several reports have proposed methods to apply Gröbner base to nonlinear circuit systems [12]–[15]. Gröbner base enables to eliminate variables from polynomial simultaneous equations containing parameters [16], [17]. By Gröbner base, the algebraic representation of the error bound is described as only one algebraic equation with system parameters. The proposed method does not depend on the number of specific frequency components. Further, linear elements of the system are contained as parameters. Thus, when we fix the nonlinear elements, the algebraic representation can be uniquely obtained.

In order to visualize the error bound, we project the error bound to a complex plane of a target frequency component using the algebraic representation. However, the computation of its projection is time-consuming. Thus, we propose an approximated error bound using the algebraic representation. Although the proposed error bound approximately guarantees solutions, the approximated error bound reduces the computational cost of the projection considerably [11].

When we set the system parameters close to bifurcation parameters, there exist two error bounds in a neighborhood. In such cases, the error bounds are broken by a collision of each other. Hence, we cannot guarantee the solutions near the bifurcation point. The collision point is a break point which generates a singular point of the error bound. Since the singular point can be calculated by the algebraic representation, we propose a method to obtain the accurate break point of the error bound.

2. Harmonic Balance (HB) Method and Error Bound

2.1 HB Method

We consider the nonlinear feedback system shown in Fig. 1. The system equation is described by

$$u(\tau) = G_1(s;\boldsymbol{\mu})\left\{v(\tau) - G_2(s;\boldsymbol{\mu})N[u(\tau)]\right\}, \tag{1}$$

$$N[u] = \sum_{i=0}^{p} c_{2i+1}u^{2i+1}, \quad c_{2i+1} > 0, \quad i = 0, \ldots, p, \tag{2}$$

where $s = \mathrm{d}/\mathrm{d}\tau$, $v(\tau)$ is an input function with a period 2π,

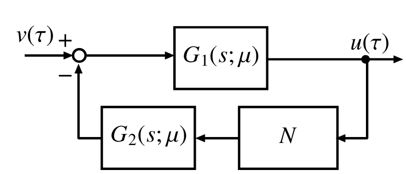

Fig. 1 Nonlinear feedback system.

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$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ is a set of system parameters, transfer functions $G_1(s;\boldsymbol{\mu})$, $G_2(s;\boldsymbol{\mu})$ have a low-pass characteristics and a polynomial-type nonlinear element $N[u]$ is a monotone increasing function of u.

We apply the HB method (cf. Appendix A) to the system Eq. (1). Thus, the HB equation is described by

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) \equiv (f_{0\mathrm{r}}, f_{1\mathrm{r}}, f_{1\mathrm{s}}, \ldots, f_{n\mathrm{r}}, f_{n\mathrm{s}})^{\mathrm{T}} = \mathbf{0} \in \mathbb{R}^m. \quad (3)$$

where $\boldsymbol{x} \equiv (x_{0\mathrm{r}}, x_{1\mathrm{r}}, x_{1\mathrm{s}}, \ldots, x_{n\mathrm{r}}, x_{n\mathrm{s}})^{\mathrm{T}} \in \mathbb{R}^m$ is a vector of variables, $\boldsymbol{e} \equiv (e_{0\mathrm{r}}, e_{1\mathrm{r}}, e_{1\mathrm{s}}, \ldots, e_{n\mathrm{r}}, e_{n\mathrm{s}})^{\mathrm{T}} \in \mathbb{R}^m$ is a vector of frequency components of the input function, $(\cdot)^{\mathrm{T}}$ denote the transposition, n is the number of specific frequency components, and $m = 2n + 1$ denotes the number of unknowns.

2.2 Error Bound for HB Method

Because the HB method has a truncation error, we calculate guaranteed solutions evaluated by an error bound. The error bound is $m - 1$ dimensional surface in m dimensional space. In order to obtain the error bound, we extend the method reported in [9] to the periodically forced system (cf. Appendix B). The error bound is calculated by removing the variable λ from Eqs. (A· 16) and (A· 24).

However, the numerical computation of the error bound is very time-consuming compared with solving the HB equation because we need to express the high dimensional error bound using a set of numerical values.

3. Algebraic Representation of Error Bound

3.1 Error Bound by Gröbner Base

To overcome the difficulty of the numerical method, we try to represent the error bound algebraically using Gröbner base. In order to apply Gröbner techniques [16], [17], we transform Eqs. (A· 16) and (A· 24) into polynomial equations. Multiplying the both sides of Eq. (A· 16) by $(1 - \lambda H)^{2p}$, we rewrite Eq. (A· 16) to the following equation;

$$f_{\mathrm{EB1}}(\lambda, \boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) \equiv \lambda(1 - \lambda H)^{2p} - \sum_{i=0}^{p}(2i+1)c_{2i+1}(1-\lambda H)^{2(p-i)}\left(\sum_{k=0}^{n}\sqrt{x_{k\mathrm{r}}^2 + x_{k\mathrm{s}}^2}\right)^{2i} = 0. \quad (4)$$

Moreover, multiplying the both sides of Eq. (A· 24) by $1 - \lambda H$ and squaring it, we obtain

$$f_{\mathrm{EB2}}(\lambda, \boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) \equiv \lambda^4 H^2 \sum_{k=0}^{n}\left(x_{k\mathrm{r}}^2 + x_{k\mathrm{s}}^2\right) - (1-\lambda H)^2 \sum_{k=0}^{n}\left(f_{k\mathrm{r}}^2 + f_{k\mathrm{s}}^2\right) = 0. \quad (5)$$

Equations (4) and (5) are polynomial equations with respect to λ.

Because Gröbner base of lexicographic order $\lambda \succ x_{k\mathrm{r}}$ (or $\lambda \succ x_{k\mathrm{s}}$) enables to eliminate λ from Eqs. (4) and (5). we can obtain the following algebraic representation of the error bound [10];

$$g_{\mathrm{EB}}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) = 0. \quad (6)$$

However, the computational cost of Gröbner base is highly dependent on the complexity of Eqs. (4) and (5). In particular, the computational cost increases exponentially according to the expansion of the number n. Thus, the algebraic representation (6) can not be calculated by the naive method in [10] if we consider more than 2 frequency components.

3.2 Efficient Method to Obtain Algebraic Representation

In order to resolve the problem of Gröbner base, we propose an efficient method to obtain the algebraic representation of the error bound using transformations of variables [11]. Because the number n of the specific frequency components complicates only the norms in Eqs. (4) and (5), we transform the norms into new variables;

$$\alpha(\boldsymbol{x}) \equiv \|u_{\mathrm{L}}(\tau)\|_1 = \sum_{k=0}^{n}\sqrt{x_{k\mathrm{r}}^2 + x_{k\mathrm{s}}^2}, \quad (7)$$

$$\beta(\boldsymbol{x}) \equiv \|u_{\mathrm{L}}(\tau)\|_2{}^2 = \sum_{k=0}^{n}\left(x_{k\mathrm{r}}^2 + x_{k\mathrm{r}}^2\right), \quad (8)$$

$$\gamma(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) \equiv \|\mathrm{FH}(u_{\mathrm{L}})\|_2{}^2 = \sum_{k=0}^{n}\left(f_{k\mathrm{r}}^2(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) + f_{k\mathrm{s}}^2(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e})\right). \quad (9)$$

Thus, using the transformation of the variable $\boldsymbol{x}$ into α, β, γ, we rewrite Eqs. (4) and (5) by

$$f_{\mathrm{EB1}}(\lambda, \alpha; H) = \lambda(1-\lambda H)^{2p} - \sum_{i=0}^{p}(2i+1)c_{2i+1}(1-\lambda H)^{2(p-i)}\alpha^{2i} = 0, \quad (10)$$

$$f_{\mathrm{EB2}}(\lambda, \beta, \gamma; H) = \lambda^4 H^2 \beta - (1-\lambda H)^2 \gamma = 0. \quad (11)$$

Because the representations (10) and (11) have only 4 variables $\lambda, \alpha, \beta, \gamma$ instead of $m + 1$ variables λ, $x_{0\mathrm{r}}$, $x_{1\mathrm{r}}$, $x_{1\mathrm{s}}, \ldots, x_{n\mathrm{s}}$ in Eqs. (4) and (5), the computational cost of Gröbner base can be reduced.

Then, $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ is obtained by the elimination of λ using Gröbner base from Eqs. (10) and (11). Since the expressions of Eqs. (10) and (11) are far simpler than those of Eqs. (4) and (5), the computational cost of obtaining $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ is remarkably less than the cost of Eq. (6) by the naive method. After we calculate $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$, the algebraic representation (6) is obtained by the substitutions of α, β, γ, H into $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$. Thus, the algorithm is given by

S1. We give the polynomial equations $f_{\mathrm{EB1}}(\lambda, \alpha; H) = 0$ and $f_{\mathrm{EB2}}(\lambda, \beta, \gamma; H) = 0$.
S2. We obtain $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ by the elimination of λ using Gröbner base of order $\lambda \succ (\alpha, \beta, \gamma)$ from f_{EB1} and f_{EB2}.

S3. We obtain the algebraic representation (6) of the error bound by the substitution of α, β, γ, H into $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$.

As is easily seen from this algorithm, we can obtain the algebraic representation (6) of the error bound even if we consider many frequency components. Moreover, the representation $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ is uniquely determined only by the nonlinear element $N[u]$ because the transfer functions G_1 and G_2 are contained only in the variable γ and the parameter H in $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ symbolically.

3.3 Example

We apply the proposed method to Duffing equation;

$$\frac{\mathrm{d}^2 u(\tau)}{\mathrm{d}\tau^2} + \mu \frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} + u^3 = E \cos \tau. \tag{12}$$

This equation can be rewritten as Eq. (13);

$$\begin{aligned} &u(\tau) = G_1(s;\mu) \{ v(\tau) - G_2(s;\mu) N[u(\tau)] \}, \qquad (13) \\ &v(\tau) = E \cos \tau, \quad N[u(\tau)] = u^3, \\ &G_1(s;\mu) = \frac{1}{s^2 + \mu s}, \quad G_2 = 1, \quad G = G_1. \end{aligned}$$

The equations f_{EB1} and f_{EB2} is written by

$$f_{\mathrm{EB1}}(\lambda, \alpha; H) = \lambda(1 - \lambda H)^2 - 3\alpha^2 = 0, \tag{14}$$

$$f_{\mathrm{EB2}}(\lambda, \beta, \gamma; H) = \lambda^4 H^2 \beta - (1 - \lambda H)^2 \gamma = 0. \tag{15}$$

Thus, the following algebraic representation $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ of the error bound is obtained by the elimination of λ using Gröbner base of order $\lambda \succ (\alpha, \beta, \gamma)$;

$$\begin{aligned} g_{\mathrm{EB}}(\alpha, \beta, \gamma; H) = \; & 9\alpha^4 H^6 \gamma^3 - 135\alpha^4 \beta H^4 \gamma^2 \\ & -6\alpha^2 \beta H^3 \gamma^2 - 270\alpha^6 \beta^2 H^3 \gamma + 225\alpha^4 \beta^2 H^2 \gamma \\ & -30\alpha^2 \beta^2 H \gamma + \beta^2 \gamma - 81\alpha^8 \beta^3 H^2 = 0. \end{aligned} \tag{16}$$

Let us compare the proposed method using Gröbner base of order $\lambda \succ \alpha \succ \beta \succ \gamma$ with the naive method using the order $\lambda \succ x_{1\mathrm{r}} \succ x_{1\mathrm{s}}$. The computational cost of both methods is shown in Table 1 where $n = 1$. From this table, we can confirm the efficiency of the proposed method.

Further, Eq. (16) does not contain the transfer function $G(s;\mu)$ and the number of the specific frequency components explicitly. Thus, when we fix the nonlinear element $N[u]$, we can obtain the algebraic representation (6) from $g_{\mathrm{EB}}(\alpha, \beta, \gamma; H)$ even if we consider many frequency components.

Table 1 Comparison of computational cost between proposed method and naive method ($n = 1$).

Method	Order of variables	Computation time [s]	Required memory [MB]
Naive method	$\lambda \succ x_{1\mathrm{r}} \succ x_{1\mathrm{s}}$	7425	956
Proposed method	$\lambda \succ \alpha \succ \beta \succ \gamma$	0.007	1.09

Calculated by a PC with Xeon 3.06 GHz CPU.

4. Fast Computation of Approximated Error Bound

4.1 Quadratic Approximation of Error Bound

In order to visualize the high dimensional error bound, we project the error bound to a complex plane of a target frequency component. However, the computation of the projection is very time-consuming.

We propose a fast computational method to obtain an approximated projection of the error bound. The error bound is in a neighborhood of the solution of the HB equation, and resembles an ellipsoidal body. Thus, we approximate the error bound to the quadratic form using variations of the solution.

Let us consider a projection of the error bound to a complex plane $(x_{l\mathrm{r}}, x_{l\mathrm{s}})$ for $l \in \{1, \dots, n\}$. Then we rewrite the vector of the variable

$$\begin{aligned} \boldsymbol{x} &= (x_{l\mathrm{r}}, x_{l\mathrm{s}}, x_{0\mathrm{r}}, \dots, x_{l-1\mathrm{r}}, x_{l-1\mathrm{s}} x_{l+1\mathrm{r}}, x_{l+1\mathrm{s}} \dots, x_{n\mathrm{r}}, x_{n\mathrm{s}})^{\mathrm{T}} \\ &\equiv (x_1, x_2, \dots, x_m)^{\mathrm{T}}. \end{aligned} \tag{17}$$

Further, we consider that the variable $\boldsymbol{x}$ is described by

$$\boldsymbol{x} = \hat{\boldsymbol{x}} + \Delta \boldsymbol{x}, \tag{18}$$

where, $\hat{\boldsymbol{x}} \equiv (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_m)^{\mathrm{T}}$ is a vector of the solution of the HB equation and $\Delta \boldsymbol{x} \equiv (\Delta x_1, \Delta x_2, \dots, \Delta x_m)^{\mathrm{T}}$ is a vector of its variations.

Using $\Delta \boldsymbol{x}$ and Taylor expansion, we obtain the quadratic approximation $g_{\Delta\mathrm{EB}}(\Delta \boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{e})$ of the error bound as follows;

$$\begin{aligned} g_{\mathrm{EB}}(\boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{e}) &\approx g_{\Delta\mathrm{EB}}(\Delta \boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{e}) \\ &= \sum_{i=1}^{m} a_{ii} \Delta x_i^2 + 2 \sum_{i=1}^{m} \sum_{j=i+1}^{m} a_{ij} \Delta x_i \Delta x_j + 2 \sum_{i=1}^{m} a_{0i} \Delta x_i + a_{00}. \end{aligned} \tag{19}$$

Then, the approximated error bound is rewritten by

$$g_{\Delta\mathrm{EB}}(\Delta \boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{e}) = (1, \Delta \boldsymbol{x}^{\mathrm{T}}) \boldsymbol{A} \begin{bmatrix} 1 \\ \Delta \boldsymbol{x} \end{bmatrix} = 0, \tag{20}$$

where

$$\boldsymbol{A} \equiv \boldsymbol{A}(\hat{\boldsymbol{x}}, \boldsymbol{\mu}, \boldsymbol{e}) \equiv \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{01} & a_{11} & \cdots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{0m} & a_{1m} & \cdots & a_{mm} \end{bmatrix} \in \mathbb{R}^{(m+1)\times(m+1)}.$$

In order to obtain the projection of the approximated error bound $g_{\Delta\mathrm{EB}}(\Delta \boldsymbol{x}; \boldsymbol{\mu}, \boldsymbol{e})$, we decompose $\boldsymbol{A}(\hat{\boldsymbol{x}}, \boldsymbol{\mu}, \boldsymbol{e})$ and $\Delta \boldsymbol{x}$ into

$$\boldsymbol{A} \equiv \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_2^{\mathrm{T}} & \boldsymbol{A}_3 \end{bmatrix}, \begin{bmatrix} 1 \\ \Delta \boldsymbol{x} \end{bmatrix} \equiv \begin{bmatrix} \Delta \boldsymbol{x}_1 \\ \Delta \boldsymbol{x}_2 \end{bmatrix}, \Delta \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ \Delta x_1 \\ \Delta x_2 \end{bmatrix}, \Delta \boldsymbol{x}_2 = \begin{bmatrix} \Delta x_3 \\ \vdots \\ \Delta x_m \end{bmatrix}, \tag{21}$$

where partial matrices of $\boldsymbol{A}$ denote $\boldsymbol{A}_1 \in \mathbb{R}^{3\times3}$, $\boldsymbol{A}_2 \in$

$\mathbb{R}^{(m-2)\times 3}$ and $\boldsymbol{A}_3 \in \mathbb{R}^{(m-2)\times(m-2)}$, respectively. Now let

$$\nabla g_{\Delta\mathrm{EB}} = \left(\frac{\partial g_{\Delta\mathrm{EB}}}{\partial \Delta x_1}, \dots \frac{\partial g_{\Delta\mathrm{EB}}}{\partial \Delta x_m}\right)^{\mathrm{T}} \in \mathbb{R}^m, \tag{22}$$

be a gradient vector of $g_{\Delta\mathrm{EB}}$. Then, the boundary of the projected error bound to (x_1, x_2) plane satisfies that $\nabla g_{\Delta\mathrm{EB}}$ is orthogonal to the following unit vectors which are parallel to the $x_3, \dots, x_m$ axes.

$$\begin{aligned}&(0,0,0,1,0,\dots,0)^{\mathrm{T}},\\ &(0,0,0,0,1,\dots,0)^{\mathrm{T}},\\ &\qquad\vdots\\ &(0,0,0,0,0,\dots,1)^{\mathrm{T}}.\end{aligned} \tag{23}$$

Namely, the projection of $g_{\Delta\mathrm{EB}}(\Delta \boldsymbol{x})$ satisfies

$$\frac{\partial g_{\Delta\mathrm{EB}}(\Delta \boldsymbol{x})}{\partial \Delta x_k} = 0, \quad k = 3, \dots, m. \tag{24}$$

Thus, applying this relation to Eq. (20), we obtain a constraint for the projection;

$$\boldsymbol{A}_2^{\mathrm{T}} \Delta \boldsymbol{x}_1 + \boldsymbol{A}_3 \Delta \boldsymbol{x}_2 = \boldsymbol{0}. \tag{25}$$

As a result, the projection of the approximated error bound is represented by

$$\begin{aligned}(1, \Delta \boldsymbol{x}^{\mathrm{T}})\boldsymbol{A}\begin{bmatrix}1\\ \Delta \boldsymbol{x}\end{bmatrix} &= \left(\Delta \boldsymbol{x}_1^{\mathrm{T}}, \Delta \boldsymbol{x}_2^{\mathrm{T}}\right)\begin{bmatrix}\boldsymbol{A}_1\Delta \boldsymbol{x}_1 + \boldsymbol{A}_2 \Delta \boldsymbol{x}_2\\ \boldsymbol{0}\end{bmatrix}\\ &= \Delta \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{A}_1\Delta \boldsymbol{x}_1 + \Delta \boldsymbol{x}_1^{\mathrm{T}}\boldsymbol{A}_2\Delta \boldsymbol{x}_2\\ &= \Delta \boldsymbol{x}_1^{\mathrm{T}}\left(\boldsymbol{A}_1 - \boldsymbol{A}_2\boldsymbol{A}_3^{-1}\boldsymbol{A}_2^{\mathrm{T}}\right)\Delta \boldsymbol{x}_1 = 0.\end{aligned} \tag{26}$$

Finally, the substitution of $\Delta x_1 = x_1 - \hat{x}_1$, $\Delta x_2 = x_2 - \hat{x}_2$ into Eq. (26) gives the approximated projection of the error bound.

The quadratic approximation algorithm is written by

S1. We calculate the algebraic representation (6) of the error bound using Gröbner base.
S2. We set the target complex plane (x_1, x_2) and other variables $x_3, \dots, x_m$.
S3. We obtain algebraic representations of the elements $a_{ij}(\hat{\boldsymbol{x}}, \boldsymbol{\mu}, \boldsymbol{e})$, $(i, j = 0, \dots, m, i \le j)$ of the matrix $\boldsymbol{A}$ with the solution $\hat{\boldsymbol{x}}$ of the HB equation and the system parameters $\boldsymbol{\mu}$, $\boldsymbol{e}$.
S4. We determine a_{ij} by the substitution of the given solution $\hat{\boldsymbol{x}}$ and parameters $\boldsymbol{\mu}$, $\boldsymbol{e}$ into $a_{ij}(\hat{\boldsymbol{x}}, \boldsymbol{\mu}, \boldsymbol{e})$, $(i, j = 0, \dots, m, i \le j)$.
S5. We obtain the projection of the approximated error bound by $\boldsymbol{A}_1 - \boldsymbol{A}_2\boldsymbol{A}_3^{-1}\boldsymbol{A}_2^{\mathrm{T}}$ and the substitution of $\Delta x_1 = x_1 - \hat{x}_1$, $\Delta x_2 = x_2 - \hat{x}_2$ into Eq. (26).

Although the proposed error bound approximately guarantees solutions of the HB equation, the projection of the error bound can be plotted easily on a two-dimensional space by this algorithm.

4.2 Example

We apply the quadratic approximation of the error bound to Duffing Eq. (12) where we assume that zero frequency components are zero for simplicity. We consider the approximated projection to fundamental frequency component, namely, $x_1 = x_{1\mathrm{r}}, x_2 = x_{1\mathrm{s}}, x_3 = x_{2\mathrm{r}}, \dots, x_{m-1} = x_{n\mathrm{r}}, x_m = x_{n\mathrm{s}}$, and $f_1 = f_{1\mathrm{r}}, f_2 = f_{1\mathrm{s}}, f_3 = f_{2\mathrm{r}}, \dots, f_{m-1} = f_{n\mathrm{r}}, f_m = f_{n\mathrm{s}}$, where $m = 2n$. Let the solution of the HB equation $\boldsymbol{f}(\boldsymbol{x}; \mu, E) = \boldsymbol{0}$ be $\hat{\boldsymbol{x}} = (\hat{x}_1, \dots, \hat{x}_m)^{\mathrm{T}}$. Then the matrix $\boldsymbol{A}$ of the approximated error bound is represented by the elements

$$a_{00} = -81H^2\hat{\alpha}_0^8\hat{\beta}_0, \tag{27}$$

$$a_{0i} = -81H^2\hat{\alpha}_0^8\hat{x}_i + 4\hat{\alpha}_0^7\hat{\alpha}_{0i}\hat{\beta}_0, \tag{28}$$

$$\begin{aligned}a_{ii} = &-81H^2\hat{\alpha}_0^6(16\hat{\alpha}_0\hat{\alpha}_{0i}\hat{x}_i + 8\hat{\alpha}_0\hat{\alpha}_{ii}\hat{\beta}_0 + 28\hat{\alpha}_{0i}^2\hat{\beta}_0 + \hat{\alpha}_0^2)\\ &(225\hat{\alpha}_0^4H^2 - 30\hat{\alpha}_0^4H - 270\hat{\alpha}_0^6H^3 + 1)\sum_{k=1}^{m}\left(\frac{\partial f_k(\hat{\boldsymbol{x}}; \mu, E)}{\partial \hat{x}_i}\right)^2\end{aligned} \tag{29}$$

$$\begin{aligned}a_{ij} = &-81H^2\hat{\alpha}_0^6(8\hat{\alpha}_0\hat{\alpha}_{0i}\hat{x}_j + 8\hat{\alpha}_0\hat{\alpha}_{0j}\hat{x}_i + 28\hat{\alpha}_{0i}\hat{\alpha}_{0j}\hat{\beta}_0\\ &+4\hat{\alpha}_0\hat{\alpha}_{ij}\hat{\beta}_0) + (225\hat{\alpha}_0^4H^2 - 30\hat{\alpha}_0^4H - 270\hat{\alpha}_0^6H^3 + 1)\\ &\sum_{k=1}^{m}\left(\frac{\partial f_k(\hat{\boldsymbol{x}}; \mu, E)}{\partial \hat{x}_j}\frac{\partial f_k(\hat{\boldsymbol{x}}; \mu, E)}{\partial \hat{x}_i}\right)\end{aligned} \tag{30}$$

where

$$\hat{\alpha}_0 = \sum_{k=1}^{n}\sqrt{\hat{x}_{2k}^2 + \hat{x}_{2k+1}^2}, \hat{\alpha}_{0i} = \frac{\hat{x}_i}{\hat{\alpha}_i}, \hat{\alpha}_{ii} = \frac{\hat{x}_i}{2\hat{\alpha}_i} - \frac{\hat{x}_i^2}{2\hat{\alpha}_i^3},$$

$$\hat{\alpha}_{ij} = \begin{cases}-\frac{\hat{x}_i\hat{x}_j}{2\hat{\alpha}_i} & |i-j| = 1\\ 0 & |i-j| \ne 1\end{cases}, \hat{\alpha}_i = \begin{cases}\sqrt{\hat{x}_i^2 + \hat{x}_{i+1}^2} & i = 1 \bmod 2\\ \sqrt{\hat{x}_{i-1}^2 + \hat{x}_i^2} & i = 0 \bmod 2\end{cases},$$

$$\hat{\beta}_0 = \sum_{k=1}^{m}\hat{x}_k^2, H = \frac{1}{(n+1)\sqrt{(n+1)^2 + \mu^2}},$$

$i = 1, \dots, m, \ j = 2, \dots, m, \ i < j.$

In order to confirm the validity of the approximation, the projections by the proposed method and the method in [9] are shown in Fig. 2 where $\mu = 0.1$, $E = 0.35$, $n = 4, 6, 8$. We can see that the projection of the approximated error bound is very close to the projection in [9].

Moreover, the projection of the approximated error bound with the parameter E varying from 0.1 to 0.4 is shown in Fig. 3 where $\mu = 0.1$ and $n = 4$. Because the elements (27), (28), (29) and (30) contain the system parameters symbolically, the approximated error bound can be easily obtained even if we change the system parameters as shown in Fig. 3.

Further, the computational time of the proposed method and the method in [9] for $n = 4, 6, 8, 20$ is shown in Table 2 when we vary the parameters μ from 0.1 to 1.0, E from 0.1 to 0.4. Additionally, we also show the solving time of the HB equation in Table 2. Although the proposed method in Table 2 does not contain the computational cost of $g_{\mathrm{EB}}(\boldsymbol{x}; \mu, E)$, $g_{\mathrm{EB}}(\boldsymbol{x}; \mu, E)$ is calculated only once and the

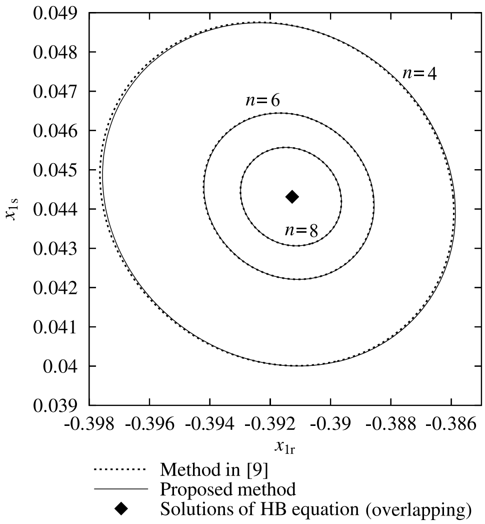

Fig. 2 Projections of error bound for HB method on the (x_{1r}, x_{1s}) plane ($\mu = 0.1$, $E = 0.35$ and $n = 4, 6, 8$).

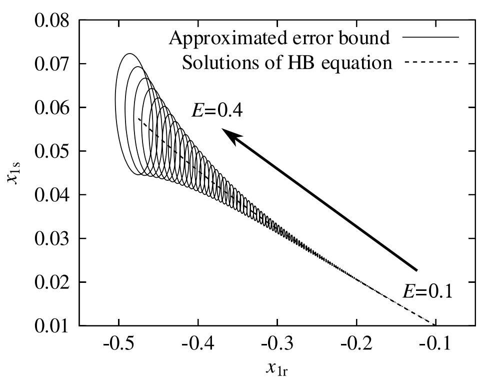

Fig. 3 Projection of approximated error bound with parameter E varying from 0.1 to 0.4 ($\mu = 0.1$ and $n = 4$).

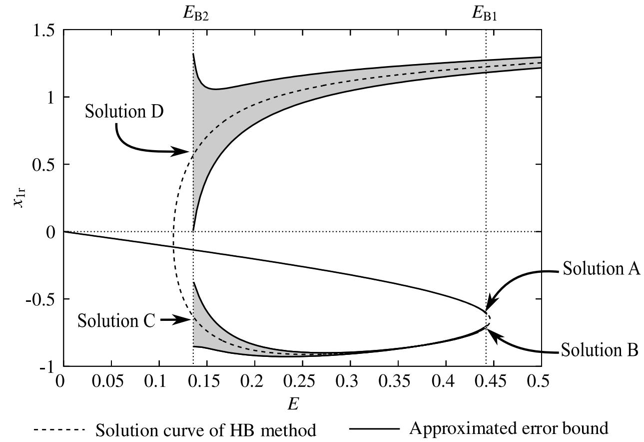

Fig. 4 Bifurcation diagram of HB method and approximated error bound ($\mu = 0.1$ and $n = 30$).

Table 2 Computational time of projection of error bound [s] (μ varied from 0.1 to 1.0 and E varied from 0.1 to 0.4, using Newton method with 90 $\times$ 300 $\times$ 32 points).

Method	$n = 4$	$n = 6$	$n = 8$	$n = 20$
HB method	1.50	2.23	3.47	16.63
Method in [9]	237.02	303.73	390.45	1172.47
Proposed method	7.52	8.70	10.29	26.22

Calculated by a PC with Xeon 3.06 GHz CPU.

computational cost is very low as shown in Table 1. Thus, we can confirm that the proposed method reduces the computational cost of the error bound dramatically. Although the conventional method is very time-consuming compared with solving the HB equation, the proposed method approximately guarantees the solutions as fast as solving the HB equation.

5. Break Point of Error Bound

5.1 Break Point and Singular Point of Error Bound

When we set the system parameters close to the bifurcation parameters, there exist two solutions of the HB equation in a neighborhood. In such cases, two error bounds containing these solutions are broken by a collision of each other, and we can not hence guarantee the solutions near the bifurcation point. Let us call this collision point a break point of the error bound. We propose a method to obtain accurate break points using the algebraic representation of the error bound.

Because a gradient vector of singular points equals zero in general [16], [17], the break point of the error bound satisfies the following relations based on the algebraic representation (6)

$$\begin{cases} g_{\mathrm{EB}}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) = 0 \\ \nabla g_{\mathrm{EB}}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) = \boldsymbol{0} \end{cases}, \tag{31}$$

where a gradient vector $\nabla g_{\mathrm{EB}}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e})$ is written by

$$\nabla g_{\mathrm{EB}} = \left(\frac{\partial g_{\mathrm{EB}}}{\partial x_{0r}}, \frac{\partial g_{\mathrm{EB}}}{\partial x_{1r}}, \frac{\partial g_{\mathrm{EB}}}{\partial x_{1s}}, \dots \frac{\partial g_{\mathrm{EB}}}{\partial x_{nr}}, \frac{\partial g_{\mathrm{EB}}}{\partial x_{ns}}\right)^{\mathrm{T}} \in \mathbb{R}^m. \tag{32}$$

Thus, if we view one system parameter $\varepsilon \in \{\boldsymbol{\mu}, \boldsymbol{e}\}$ as the variable, the simultaneous Eq. (31) gives the break point $(\boldsymbol{x}, \varepsilon)$. We obtain the break point of the error bound by numerical method using an initial point $((\hat{\boldsymbol{x}}_1 + \hat{\boldsymbol{x}}_2)/2, \hat{\varepsilon})$, where $\hat{\varepsilon}$ is the parameter value close to the bifurcation parameters, and $\hat{\boldsymbol{x}}_1, \hat{\boldsymbol{x}}_2$ denote two close numerical solutions of the HB equation $\boldsymbol{f}(\boldsymbol{x}; \hat{\varepsilon}) = \boldsymbol{0}$.

The algorithm to obtain the break point of the error bound is described by

S1. We calculate the algebraic representation (6) of the error bound using Gröbner base.
S2. We select the parameter ε in the system parameters $\boldsymbol{\mu}, \boldsymbol{e}$.
S3. Using the initial value $((\hat{\boldsymbol{x}}_1 + \hat{\boldsymbol{x}}_2)/2, \hat{\varepsilon})$, we obtain the break point of the error bound by solving Eq. (31).

5.2 Example

We obtain the break points of the error bound for Duffing Eq. (12), where we assume that zero frequency component is neglected for simplicity. Let us select the parameter $\varepsilon = E$

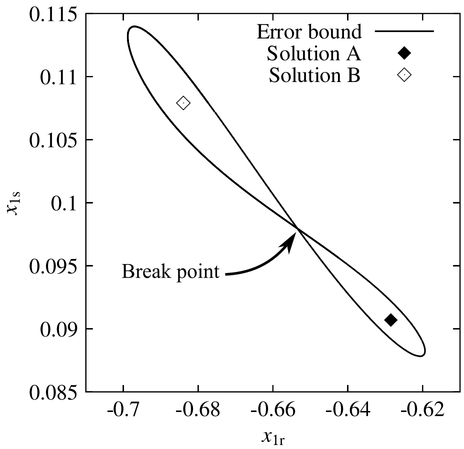

Fig. 5 Projection (x_{1r}, x_{1s}) of error bound at parameter E_{B1} of break point (E_{B1} = 0.445168, μ = 0.1 and n = 30).

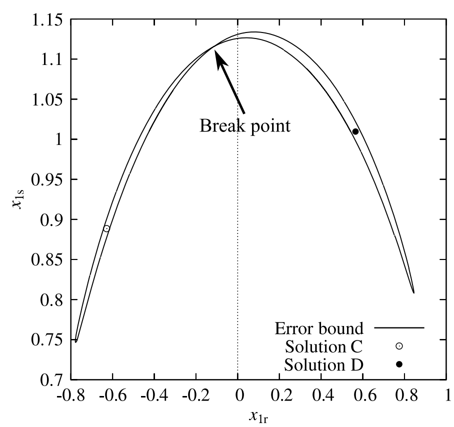

Fig. 6 Projection (x_{1r}, x_{1s}) of error bound at parameter E_{B2} of break point (E_{B2} = 0.135366, μ = 0.1 and n = 30).

and let $\mu = 0.1$, $n = 30$. Then the bifurcation diagram E-x_{1r} is shown in Fig. 4. Additionally, we also show the approximated error bound in Fig. 4. Namely, we can guarantee the solutions of the HB equation in a gray region. Thus, the bifurcation points in Fig. 4 lie close to $E = 0.45$ and $E = 0.12$.

Using the proposed method, we can calculate the parameters $E_{B1} = 0.445168$ and $E_{B2} = 0.135366$ of the break points. We show the projection of the error bound at E_{B1} and E_{B2} in Fig. 5 and Fig. 6, respectively. The Solution A, B, C and D in Fig. 5 and Fig. 6 correspond to the Solution A, B, C and D in Fig. 4. Thus, we can confirm that the proposed method enables to obtain the accurate break points of the error bound by the collisions.

6. Conclusion

We proposed an algebraic representation of an error bound for HB method using Gröbner base. Further, we proposed an efficient method to calculate the algebraic representation using transformations of variables. The proposed method does not depend on linear elements of the system and the number of specific frequency components. Next, we proposed a fast computational method of an approximated error bound by a quadratic approximation using the algebraic representation. We confirmed that the quadratic approximation guarantees approximately the solutions as fast as solving the HB equation. Moreover, we proposed a method to obtain accurate break points of the error bound near bifurcation points. In this way, algebraic approach is very powerful for high dimensional varieties such as error bounds.

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Appendix A: Derivation of HB Equation

Let us apply the HB method to the system equation (1). We assume that Eq. (1) has a periodic solution with the period 2π. Thus, the solution $u(\tau)$ is given by

$$u(\tau) \equiv \sum_{k=0}^{\infty} \Re\left[x_k^* e^{jk\tau}\right] = \sum_{k=0}^{\infty} \Re\left[(x_{kr} + jx_{ks})e^{jk\tau}\right], \quad \text{(A· 1)}$$

where $x_k^* \in \mathbb{C}$, $x_{kr}, x_{ks} \in \mathbb{R}$ for $k = 0, \ldots, n$, $\mathbb{C}$ is a set of complex numbers, $\mathbb{R}$ is a set of real numbers, $x_{0s} = 0$ and $\Re[\cdot]$ denotes the real part. Now, assuming that a projection operator K_L expresses the truncation of Fourier series, we approximate the above solution by

$$u_L(\tau) \equiv K_L u(\tau) \equiv \sum_{k=0}^{n} \Re\left[x_k^* e^{jk\tau}\right] = \sum_{k=0}^{n} \Re\left[(x_{kr} + jx_{ks})e^{jk\tau}\right], \quad \text{(A· 2)}$$

Using the operator K_L and approximated solution (A· 2), we rewrite Eq. (1) to

$$\sum_{k=0}^{n} \Re\left[\left(x_k^* - G_1(jk;\boldsymbol{\mu})\left\{(e_k^* - G_2(jk;\boldsymbol{\mu})y_k^*\right\}\right)e^{jk\tau}\right] = 0, \quad \text{(A· 3)}$$

due to $s^n e^{j\tau} = (jk)^n e^{j\tau}$, where

$$K_L v(\tau) = \sum_{k=0}^{n} e_k^* e^{jk\tau}, \quad K_L N[u_L(\tau)] = \sum_{k=0}^{n} y_k^* e^{jk\tau}. \quad \text{(A· 4)}$$

By this relation, HB equation is written by

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{e}) \equiv (f_{0r}, f_{1r}, f_{1s}, \ldots, f_{nr}, f_{ns})^T = \boldsymbol{0} \in \mathbb{R}^m, \quad \text{(A· 5)}$$
$$f_{0r} \equiv \Re\left[x_0^* - G_1(0;\boldsymbol{\mu})\left\{e_0^* - G_2(0;\boldsymbol{\mu})y_0^*\right\}\right],$$
$$f_{kr} \equiv \Re\left[x_k^* - G_1(jk;\boldsymbol{\mu})\left\{e_k^* - G_2(jk;\boldsymbol{\mu})y_k^*\right\}\right],$$
$$f_{ks} \equiv \Im\left[x_k^* - G_1(jk;\boldsymbol{\mu})\left\{e_k^* - G_2(jk;\boldsymbol{\mu})y_k^*\right\}\right],$$
$$\boldsymbol{x} \equiv (x_{0r}, x_{1r}, x_{1s}, \ldots, x_{nr}, x_{ns})^T \in \mathbb{R}^m,$$
$$\boldsymbol{e} \equiv (e_{0r}, e_{1r}, e_{1s}, \ldots, e_{nr}, e_{ns})^T \in \mathbb{R}^m,$$
$$e_k^* = e_{kr} + je_{ks}, \quad e_0^* = e_{0r},$$
$$k = 1, \ldots, n,$$

where $\Im[\cdot]$ denotes the imaginary.

Appendix B: Error Bound of HB Method for Periodically Forced System

B.1 Definition for Error Bound

Let a projection operator K_H be

$$u_H(\tau) = K_H u(\tau) = \sum_{k=n+1}^{\infty} \Re\left[(x_{kr} + jx_{ks})e^{jk\tau}\right], \quad \text{(A· 6)}$$
$$u(\tau) = u_L(\tau) + u_H(\tau), \quad K_L + K_H = I,$$

where I is an identity operator. Then, we define norms by

$$l^i\text{norm} : \|u(\tau)\|_i \equiv \left\{\sum_{k=0}^{\infty}\left(\sqrt{x_{kr}^2 + x_{ks}^2}\right)^i\right\}^{1/i} \text{for } i = 1, 2, \quad \text{(A· 7)}$$
$$L^\infty\text{norm} : \|u(\tau)\|_\infty \equiv \sup_{\tau\in[0,2\pi)} |u(\tau)|. \quad \text{(A· 8)}$$

where

$$\|u_L(\tau)\|_i \equiv \left\{\sum_{k=0}^{n}\left(\sqrt{x_{kr}^2 + x_{ks}^2}\right)^i\right\}^{1/i}, \quad \text{(A· 9)}$$
$$\|u_H(\tau)\|_i \equiv \left\{\sum_{k=n+1}^{\infty}\left(\sqrt{x_{kr}^2 + x_{ks}^2}\right)^i\right\}^{1/i} \text{for } i = 1, 2. \quad \text{(A· 10)}$$

The l^1 norm satisfy $\|u(t)\|_\infty \le \|u(t)\|_1$.

B.2 Estimation of High Frequency Components

We estimate the high frequency components. Let λ be any positive number satisfying

$$\lambda \ge \left\|\frac{dN[u]}{du}\right\|_\infty. \quad \text{(A· 11)}$$

Applying the operator K_H to Eq. (1), we obtain the relation;

$$u_H(\tau) = -K_H G(s;\boldsymbol{\mu})N[u_L(\tau) + u_H(\tau)] \quad \text{(A· 12)}$$

If

$$\lambda H \equiv \lambda \sup_{k>n}\left|G(jk;\boldsymbol{\mu})\right| < 1, \quad \text{(A· 13)}$$

is satisfied where $H(\boldsymbol{\mu}) = \sup_{k>n}\left|G(jk;\boldsymbol{\mu})\right| \in \mathbb{R}$ is a constant, then there exists a unique u_H [9].

Using the contraction mapping theorem and the mean value theorem, we obtain the following relations;

$$\|u_H\|_i \le \frac{\lambda H}{1 - \lambda H}\|u_L\|_i \ \text{ for } \ i = 1, 2. \quad \text{(A· 14)}$$

The inequality estimates the high frequency components by the low frequency components for l^1 and l^2 norms.

B.3 Determination of λ

The estimation of the high frequency components (A· 14) gives the following relation;

$$\|u\|_1 \le \|u_L\|_1 + \|u_H\|_1 \le \left(1 + \frac{\lambda H}{1 - \lambda H}\right)\|u_L\|_1. \quad \text{(A· 15)}$$

Thus, if we determine the variable λ as

$$\lambda = \sum_{i=0}^{p}(2i + 1)c_{2i+1}\left(1 + \frac{\lambda H}{1 - \lambda H}\right)^{2i}\|u_L\|_1^{2i} \quad \text{(A· 16)}$$
$$\ge \sum_{i=0}^{p}(2i+1)c_{2i+1}\|u\|_1^{2i} = \left\|\frac{dN[u]}{du}\right\|_1$$
$$\ge \left\|\frac{dN[u]}{du}\right\|_\infty, \quad \text{(A· 17)}$$

then λ satisfies Eq. (A· 11).

B.4 Error Bound by Homotopy Invariance

Applying the operator $K_{\rm L}$ to Eq. (1), we obtain

$$\begin{aligned}\mathrm{FT}(u_{\rm L}) &\equiv G^{-1}(s;\boldsymbol{\mu})u_{\rm L} - \{G_2^{-1}(s;\boldsymbol{\mu})v(\tau) - K_{\rm L}N[u_{\rm L}+u_{\rm H}]\}\\ &= 0. \end{aligned} \tag{A· 18}$$

Equation (A· 18) corresponds to the low frequency components of Eq. (1), and $u_{\rm L}$ in Eq. (A· 18) is the exact solution of Eq. (1). Now, if we set $u_{\rm H} = 0$ in Eq. (A· 18), then we obtain

$$\begin{aligned}\mathrm{FH}(u_{\rm L}) &\equiv G^{-1}(s;\boldsymbol{\mu})u_{\rm L} - \{G_2^{-1}(s;\boldsymbol{\mu})v(\tau) - K_{\rm L}N[u_{\rm L}]\}\\ &= 0 \end{aligned} \tag{A· 19}$$

which corresponds to the HB Eq. (A· 5). That is, $u_{\rm L}$ in Eq. (A· 19) is an approximated solution by the HB Eq. (A· 5).

In order to obtain the error bound from Eqs. (A· 18) and (A· 19), we use the following lemma of the homotopy invariance theorem [18]–[20].

Lemma 1. *Let Ω be an open bounded set in $\mathbb{R}^m$ and let $\boldsymbol{f},\boldsymbol{g}{:}\overline{\Omega} \to \mathbb{R}^m$ be two continuous maps where $\overline{\Omega}$ denotes the closure of the set Ω. Let $\boldsymbol{y} \in \mathbb{R}^m$ be a certain vector. Suppose further that ε satisfies $0 < \varepsilon = \min\{\|\boldsymbol{f}(z) - \boldsymbol{y}\|_2 \mid z \in \partial\Omega\}$ where $\partial\Omega$ is the boundary of the set Ω. If*

$$\|\boldsymbol{f}(z) - \boldsymbol{g}(z)\|_2 < \varepsilon \quad \forall z \in \partial\Omega, \tag{A· 20}$$

then $\deg(\boldsymbol{f},\Omega,\boldsymbol{y}) = \deg(\boldsymbol{g},\Omega,\boldsymbol{y})$ where we denote by $\deg(\boldsymbol{f},\Omega,\boldsymbol{y})$ the degree of $\boldsymbol{f}$ with respect to Ω at $\boldsymbol{y}$.

If we set $\boldsymbol{f}(z) = \mathrm{FH}$, $\boldsymbol{g}(z) = \mathrm{FT}$, $z = \boldsymbol{x}$, $\boldsymbol{y} = 0$ in Eq. (A· 20), then $\deg(\mathrm{FH},\Omega,0) = \deg(\mathrm{FT},\Omega,0)$ is satisfied. Namely, if there exists a region Ω containing a single solution of Eq. (A· 19) on whose boundary

$$\|\mathrm{FT}(u_{\rm L}) - \mathrm{FH}(u_{\rm L})\|_2 < \|\mathrm{FH}(u_{\rm L})\|_2 \tag{A· 21}$$

holds, then a solution of Eq. (A· 18) exists belonging to Ω.

Using the mean value theorem and the estimation of the high frequency components (A· 14), we obtain the following relation from Eq. (A· 21);

$$\begin{aligned}\|\mathrm{FT}(u_{\rm L}) - \mathrm{FH}(u_{\rm L})\|_2 &= \|K_{\rm L}N[u_{\rm L}] - K_{\rm L}N[u_{\rm L}+u_{\rm H}]\|_2\\ &\le \lambda\|u_{\rm H}\|_2 \le \frac{\lambda^2 H}{1-\lambda H}\|u_{\rm L}\|_2. \end{aligned} \tag{A· 22}$$

Because inequality

$$\frac{\lambda^2 H}{1-\lambda H}\|u_{\rm L}\|_2 < \|\mathrm{FH}(u_{\rm L})\|_2 \tag{A· 23}$$

satisfies Eq. (A· 21), we can define the error bound for the HB method by

$$\frac{\lambda^2 H}{1-\lambda H}\|u_{\rm L}\|_2 = \|\mathrm{FH}(u_{\rm L})\|_2. \tag{A· 24}$$

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