Linear Periodically Time-Varying Scaling and Its Properties

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Abstract

Stimulated by a general necessary and sufficient condition for robust stability of sampled-data systems stated in an operator-theoretic framework, we introduce a novel technique called linear periodically time-varying (LPTV) scaling. We then give a simple example of sampled-data systems in which this new scaling allows exact robust stability analysis for static uncertainties while the conventional linear time-invariant (LTI) scaling fails to do so. This leads us to an interesting question whether LPTV scaling can be effective also in other situations, e.g., in the robust stability analysis of continuous-time feedback systems regarded as a special class of sampled-data systems, or for other classes of uncertainties. We thus study some basic properties of LPTV scaling by confining ourselves to the so-called D-scaling, and show that it provides no advantage over LTI scaling when it is applied to continuous-time LTI feedback systems, regardless of the class of uncertainties we take into consideration. This demonstrates that the LPTV scaling of the type we deal with in this paper is in some sense a special technique for sampled-data systems but is indeed an effective and more natural technique than the conventional LTI scaling as far as such systems are concerned. The technique can be further extended to include what is called noncausal LPTV scaling, and the implication of the present study in such a larger framework of LPTV scaling is also described.

Keywords: robust stability, structured uncertainty, linear periodically time-varying scaling, frequency response operator, quadratic separator.

1 Introduction

The widespread use of digital controllers has stimulated a lot of studies about robust stability of sampled-data systems against uncertainties in the continuous-time plant, which is typically considered under the setting of Fig. 1, where solid lines and dashed lines denote continuous-time and discrete-time signals, respectively. The system in this figure, which we denote by Σ_Δ, can be regarded as the feedback connection of the system Σ_0 shown in Fig. 2 and the uncertainty Δ, for which reason we refer to Σ_0 and

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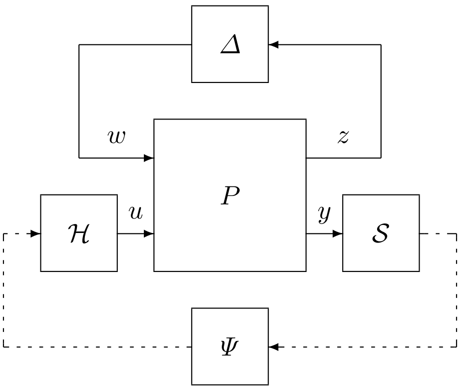

Fig. 1: Uncertain closed-loop sampled-data system Σ_Δ.

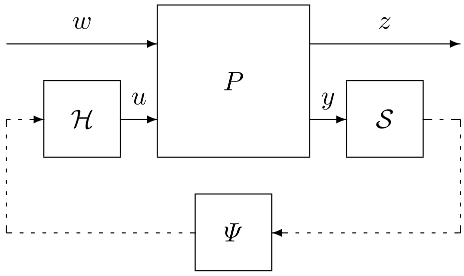

Fig. 2: Open-loop sampled-data system Σ_0.

Σ_Δ respectively as the open-loop sampled-data system and the closed-loop sampled-data system, for lack of better terminologies. For some classes of uncertainties Δ such as the set $\boldsymbol{\Delta}_h$ of norm-bounded linear h-periodic unstructured dynamical systems (where h denotes the underlying sampling period) [11] as well as the set $\boldsymbol{\Delta}_{\mathrm{LTI}}$ of norm-bounded linear time-invariant (LTI) unstructured dynamical systems [3], [5], corresponding necessary and sufficient conditions have been characterized by some sort of norms of the (unscaled) open-loop sampled-data system Σ_0. By applying scaling on Σ_0 with some constant matrices or LTI systems, sufficient conditions for robust stability could readily be obtained for some kind of structured uncertainties. Such scaling, however, is not specific to sampled-data systems since it is a standard technique in the continuous-time setting. Moreover, in view of the fact that, unlike in the LTI case, a gap exists [11], [3],[5] between the necessary and sufficient conditions for robust stability of sampled-data systems under the (unstructured) uncertainties $\boldsymbol{\Delta}_h$ and $\boldsymbol{\Delta}_{\mathrm{LTI}}$, it might be rational to anticipate that such standard scaling may not be as natural a framework in the sampled-data setting as in the continuous-time case, and that another class of effective scaling could exist that is more suited to the sampled-data setting.

On the other hand, by exploiting a Nyquist stability criterion for sampled-data systems together with the idea of separators [9], a general necessary and sufficient condition was derived in [7] for robust stability of sampled-data systems. Even though the condition is not necessarily readily confirmable (i.e., it is not straightforward in general to find a suitable separator satisfying the condition and thus ensuring robust stability even when such one does exist), the arguments of [7] are significant in introducing a class of operators from which we may take a separator so as to ensure robust stability. Stimulated by this observation and the feature of that particular class, we introduce a novel technique called linear periodically time-varying (LPTV) scaling in this paper as opposed to the widely employed LTI scaling with constant matrices and LTI systems. We first give a simple example in which this new scaling leads to exact robust stability analysis against some kind of (even unstructured) static uncertainties, while the conventional LTI scaling fails to do so. This leads us to an interesting question whether LPTV scaling can be effective also in other situations, e.g., in the robust stability analysis of continuous-time feedback systems regarded as a special class of sampled-data systems, or for other classes of uncertainties. We hence proceed to studying some basic properties of LPTV scaling, and show that it provides no advantage over LTI scaling when it is applied to continuous-time systems as far as (causal) LPTV scaling of the so-called D-scaling type is concerned. This is true regardless of the (spatial) structure of uncertainties as well as that of the scaling elements to be employed, and whether or not scaling elements are taken to be static or dynamic.

Through these discussions, this paper, together with our recent studies, opens a new promising horizon for robust stability analysis via the novel idea of LPTV scaling, which is actually not limited to sampled-data systems. Indeed, an extended idea of noncausal LPTV scaling has been suggested recently [4],[6], [8] to get around the limit of the (causal) LPTV scaling (of the D-scaling type) studied in this paper, and also to reduce the conservativeness of the analysis further. Relationships to such directions, together with the implication of the present study viewed in the context of such a larger framework of LPTV scaling, are also described.

We use the following notation in this paper: $\lambda(\cdot)$ denotes the set of the eigenvalues of a finite-dimensional matrix. The complex conjugate transpose of a matrix and the adjoint of an operator are denoted by $(\cdot)^*$, and $\|\cdot\|$ denotes the induced norm of a matrix or an operator, unless otherwise stated. The complex plane is denoted by $\mathbf{C}$, and the unit circle on the complex plane is denoted by $\partial\mathbf{D}$ where $\mathbf{D} := \{z : |z| > 1\}$.

2 Preliminaries

Frequency-domain treatment of sampled-data systems and linear continuous-time periodic systems plays an important role in the following arguments of this paper. Some preliminary results about such treatment are thus reviewed in this section.

2.1 Lifting-Based Transfer Operators of Sampled-Data Systems and Linear Continuous-Time Periodic Systems

Let us consider the open-loop sampled-data system shown in Fig. 2, where P denotes the continuous-time generalized plant, Ψ the discrete-time controller, $\mathcal{H}$ the zero-order hold, and $\mathcal{S}$ the ideal sampler. The underlying sampling period will be denoted by h. For simplicity we assume throughout the paper that w and z are vectors with the same number of entries so that Δ in Fig. 1 is square.

Suppose that P and Ψ are described respectively by

$$\frac{dx}{dt} = Ax + B_1 w + B_2 u, \quad z = C_1 x + D_{11} w + D_{12} u, \quad y = C_2 x \tag{1}$$

$$\xi_{k+1} = A_\Psi \xi_k + B_\Psi y_k, \quad u_k = C_\Psi \xi_k + D_\Psi y_k \tag{2}$$

where $y_k = y(kh)$ and $u(t) = u_k$ $(kh \leq t < (k+1)h)$. The Hilbert space of square integrable vector functions $f(\cdot)$ over the time interval $[0, h)$ with the standard inner product will be denoted by $\mathcal{K}$, regardless of the dimension of the underlying vector space. The underlying Euclidean space for $x(t)$ will be denoted by $\mathcal{F}_x$. We define $\mathcal{F}_\xi$ in a similar way, and further define $\mathcal{F} := \mathcal{F}_x \oplus \mathcal{F}_\xi$. Then, the lifting-based transfer operator [14],[2],[15] characterizing the transfer characteristics of the sampled-data system Σ_0 is defined by $\widehat{G}(z) := \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D} : \mathcal{K} \to \mathcal{K}$ as a function in $z \in \mathbf{C} \setminus \lambda(\mathcal{A})$ with a suitably defined matrix $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ and operators $\mathcal{B} : \mathcal{K} \to \mathcal{F}$, $\mathcal{C} : \mathcal{F} \to \mathcal{K}$, $\mathcal{D} : \mathcal{K} \to \mathcal{K}$.

In a similar fashion, a finite-dimensional linear continuous-time h-periodic system W has its lifting-based transfer operator $\widehat{W}(z)$ [2]. Thus, if the uncertainty Δ is in the class of such h-periodic systems or in its subset, then we can consider its transfer operator $\widehat{\Delta}(z)$. If the h-periodic system W is static, then it can be identified with an h-periodic matrix function $D_W(t)$ corresponding to the direct feedthrough matrix of the state-space representation of W, and its transfer operator $\widehat{W}(z) = \widehat{D}_W$ is nothing but the operator of multiplication by $D_W(t)$ which maps $f(\cdot) \in \mathcal{K}$ to $D_W(\cdot)f(\cdot) \in \mathcal{K}$. For this reason, with a slight abuse of notation, we denote $\widehat{D}_W$ simply by D_W or even by W.

2.2 Harmonic Frequency Response Operator of Linear Continuous-Time Periodic Systems

The lifting-based approach to sampled-data systems and linear continuous-time periodic systems was reviewed in the preceding subsection. Another useful approach to such systems is a harmonic analysis type of method [1],[13],[16], which we review in this subsection. More precisely, we review here only the harmonic frequency response operators of linear continuous-time periodic systems, since they suffice in the following discussions.

To this end, we introduce the (vector) signal of the form $f(t) = \sum_{k=-\infty}^{\infty} f_k e^{j\varphi_k t}$, where $\varphi_k := \varphi + k\omega_s$ ($k = 0, \pm1, \pm2, \cdots$) with $\omega_s := 2\pi/h$. We also assume that $\varphi \in \mathcal{I}_0 := [-\omega_s/2, \omega_s/2)$. Let us define $\underline{f} := [\cdots, f_{-1}^T, f_0^T, f_1^T, \cdots]^T$, denote the condition $\sum_{k=-\infty}^{\infty} \|f_k\|^2 < \infty$ simply by $\underline{f} \in l_2$ and denote $\left(\sum_{k=-\infty}^{\infty} \|f_k\|^2\right)^{1/2}$ by $\|\underline{f}\|$ for simplicity. If $\underline{f} \in l_2$, then we say that $f(t)$ is an SD-sinusoid (where SD stands for sampled-data) with generalized amplitude/phase $\underline{f}$ and fundamental angular frequency φ. If the monodromy matrix associated with the h-periodic system W has no eigenvalues on $\partial\mathbf{D}$, then W maps the input SD-sinusoid f to an output SD-sinusoid with the same fundamental angular frequency under some appropriate initial state, where the generalized amplitude/phase $\underline{g}$ of the output SD-sinusoid is given by

$$\underline{g} = \underline{W}(j\varphi)\underline{f} \tag{3}$$

with an appropriately defined infinite-dimensional matrix $\underline{W}(j\varphi)$, which is called the harmonic frequency response of the h-periodic system W. The frequency response gain of such W at the (fundamental) angular frequency φ is defined by

$$\|\underline{W}(j\varphi)\| = \sup_{\underline{f} \in l_2} \frac{\|\underline{W}(j\varphi)\underline{f}\|}{\|\underline{f}\|} \tag{4}$$

and it is known that

$$\|\underline{W}(j\varphi)\| = \|\widehat{W}(e^{j\varphi h})\| \tag{5}$$

where $\widehat{W}(z)$ denotes the transfer operator of W reviewed in the preceding subsection and the right hand side denotes the norm of $\widehat{W}(e^{j\varphi h})$ induced on $\mathcal{K}$. Furthermore, the L_∞ norm of W (or in fact the H_∞ norm if W is internally stable) is given by $\|W\|_\infty = \max_{\varphi \in \mathcal{I}_0} \|\underline{W}(j\varphi)\| = \max_{\varphi \in \mathcal{I}_0} \|\widehat{W}(e^{j\varphi h})\|$. If W is linear time-invariant as a special case, then the above quantity coincides with the L_∞ norm of the transfer matrix $W(s)$ of W (or in fact the H_∞ norm of $W(s)$ if W is internally stable).

Finally, for later arguments we introduce some notations relevant to $\underline{W}(j\varphi)$:
conformably to the partitioning of $\underline{f} := [\cdots, f_{-1}^T, f_0^T, f_1^T, \cdots]^T$ in (3), we partition $\underline{W}(j\varphi) =:$
$[\cdots, \underline{W}_{-1}(j\varphi), \underline{W}_0(j\varphi), \underline{W}_1(j\varphi), \cdots]$ where $\underline{W}_k(j\varphi)$ corresponds to f_k, and further partition $\underline{W}_k(j\varphi) =:$

$[\cdots, W_{-1,k}(j\varphi)^T, W_{0k}(j\varphi)^T, W_{1k}(j\varphi)^T, \cdots]^T$ conformably to the partitioning of $\underline{g} := [\cdots, g_{-1}^T, g_0^T, g_1^T, \cdots]^T$, where $W_{ik}(j\varphi)$ corresponds to g_i.

3 Robust Stability Condition and Periodically Time-Varying Scaling

A general necessary and sufficient robust stability condition of a separator-type for sampled-data systems was derived in [7]. To review this robust stability condition, we first need to review the following definition.

Definition 1 The set of measurable, essentially bounded, symmetric matrix functions $\varPhi : [0, h] \to \mathbf{R}^{m\times m}$ is denoted by $\boldsymbol{\varPhi}^{m\times m}$. The set of operators of multiplication by $\varPhi \in \boldsymbol{\varPhi}^{m\times m}$, i.e., $\varLambda : f(\cdot) \in \mathcal{K} \mapsto \varPhi(\cdot)f(\cdot) \in \mathcal{K}$, is denoted by $\boldsymbol{\varLambda}^{m\times m}$. The set of (linear bounded) operators given by the sum of $\varLambda \in \boldsymbol{\varLambda}^{m\times m}$ and a linear self-adjoint compact operator is denoted by $\boldsymbol{\varTheta}^{m\times m}$. When m is understood from the context or when it is not of particular importance, we simply denote these sets by $\boldsymbol{\varPhi}$, $\boldsymbol{\varLambda}$ and $\boldsymbol{\varTheta}$, respectively.

With the above-defined set $\boldsymbol{\varTheta}$, the following separator type of theorem was derived in [7].

Theorem 1 Suppose that the open-loop sampled-data system $\varSigma_0$ is internally stable, and that $\boldsymbol{\varDelta}$ is a set such that (i) every $\varDelta \in \boldsymbol{\varDelta}$ is finite-dimensional linear h-periodic and internally stable, and (ii) $\kappa\varDelta \in \boldsymbol{\varDelta}$ whenever $\varDelta \in \boldsymbol{\varDelta}$ and $0 < \kappa < 1$. Then, $\varSigma_\varDelta$ is well-posed and internally stable for every $\varDelta \in \boldsymbol{\varDelta}$ if and only if there exists $\varTheta \in \boldsymbol{\varTheta}$ possibly dependent on $z \in \partial\mathbf{D}$ and $\varepsilon > 0$ possibly dependent on $\varDelta$ such that

$$\begin{bmatrix} I & \widehat{G}(z)^* \end{bmatrix} \varTheta \begin{bmatrix} I \\ \widehat{G}(z) \end{bmatrix} \le 0 \ (\forall z \in \partial\mathbf{D}), \quad \begin{bmatrix} \widehat{\varDelta}(z)^* & I \end{bmatrix} \varTheta \begin{bmatrix} \widehat{\varDelta}(z) \\ I \end{bmatrix} \ge \varepsilon I \ (\forall \varDelta \in \boldsymbol{\varDelta},\ \forall z \in \partial\mathbf{D}) \tag{6}$$

Now, we are in a position to introduce linear periodically time-varying (LPTV) scaling based on the above theorem. Let us first consider the causal finite-dimensional linear continuous-time h-periodic system W, and denote by $D_W(t)$ its direct feedthrough matrix. Let us assume that the singularities of $\widehat{W}(z)$ do not lie on $\partial\mathbf{D}$ and thus $\widehat{W}(z)$ is well-defined for each $z \in \partial\mathbf{D}$, which is indeed the case if and only if the monodromy matrix associated with W has no eigenvalues on $\partial\mathbf{D}$. Then, it is not hard to see that $\widehat{W}(z)^*\widehat{W}(z)$ belongs to $\boldsymbol{\varTheta}$ whenever $z \in \partial\mathbf{D}$ (see, e.g., [16]). Hence, we can consider taking the separator

$$\varTheta = \begin{bmatrix} -\gamma(z)^2\widehat{W}(z)^*\widehat{W}(z) & 0 \\ 0 & \widehat{W}(z)^*\widehat{W}(z) \end{bmatrix} \tag{7}$$

with $\gamma(z) > 0$ in (6). This separator induces LPTV scaling under mild assumptions, which can be seen as follows.

We first study invertibility of $\widehat{W}(z)$. The spectrum of the operator of multiplication by $D_W(\cdot)$ defined on $\mathcal{K}$ is given by

$$\begin{aligned} \lambda_{[0,h]}(D_W) := \{\lambda \,|\, &\text{the set of } t \in [0,h] \text{ such that } |\det(\lambda I - D_W(t))| < \gamma \text{ has} \\ &\text{nonzero measure whenever } \gamma > 0\} \end{aligned} \tag{8}$$

and hence if

$$\exists \gamma > 0 \text{ such that } |\det D_W(t)| \geq \gamma \text{ for almost every } t \in [0, h] \tag{9}$$

then D_W is invertible as an operator on $\mathcal{K}$, and thus $\widehat{W}(z)^{-1}$ is well-defined as a function in z except at some singularities. If the singularities of $\widehat{W}(z)^{-1}$ do not exist on $\partial\mathbf{D}$ (this condition, together with the invertibility condition of D_W mentioned just above, is our standing assumptions in this paper), the condition (6) reduces to checking

$$\|\widehat{W}(z)\widehat{G}(z)\widehat{W}(z)^{-1}\| \leq \gamma(z), \ \|\widehat{W}(z)\widehat{\Delta}(z)\widehat{W}(z)^{-1}\| \leq (1-\varepsilon)/\gamma(z), \quad \forall \Delta \in \boldsymbol{\Delta}, \ \forall z \in \partial\mathbf{D} \tag{10}$$

with $\varepsilon > 0$ possibly dependent on Δ. This is equivalent to scaling $\widehat{G}(z)$ and $\widehat{\Delta}(z)$ with the causal h-periodic system W on the output and W^{-1} on the input and then applying the small-gain theorem to the scaled systems (note that we do not necessarily assume that W and $\Delta \in \boldsymbol{\Delta}$ commute in this paper). This clearly suggests the use of LPTV scaling for robust stability analysis of sampled-data systems.

Remark 1 We can introduce more general separators other than (7) based on Definition 1 (see, e.g., [4]), but in this paper we confine ourselves to the class given by (7) for simplicity; it corresponds to the so-called D-scaling. From the form of (7), we can confirm that one may confine the scaling element W to such a class that $\widehat{W}(z)$ is self-adjoint and positive definite for each $z \in \partial\mathbf{D}$ (in addition to the standing assumptions stated earlier) as in the continuous-time case. In particular, the condition (9) implies that $\widehat{W}(z)$ is noncompact.

Remark 2 It is often the case that $\boldsymbol{\Delta}$ is a set of (diagonally) structured norm-bounded uncertainties. In such a case, we can consider a restricted class of W that commutes with every $\Delta \in \boldsymbol{\Delta}$ as in the conventional μ-analysis [10]. Then, possible dependency of $\gamma(z)$ on z together with possible dependency of ε on Δ can virtually be ignored.

4 Example of Exact Robust Stability Analysis via Periodically Time-Varying Scaling

Let us consider the sampled-data system Σ_Δ with

$$P = \begin{bmatrix} 0 & 1 \\ G & 0 \end{bmatrix}, \quad \Psi = 1 \tag{11}$$

where we assume that w, u, z and y are all scalar signals. We also assume that the transfer function of G is given by $G(s) = 1/(1+s)$, and that the uncertainty Δ is a static h-periodic system. In other words, it is an h-periodic gain $\Delta(t)$. What we analyze in this section is the robust stability radius $\rho := \inf_{\Delta \in \boldsymbol{\Delta}_d} \|\Delta\|$, where $\boldsymbol{\Delta}_d$ denotes the set of h-periodic gains $\Delta(t)$ for which Σ_Δ is not internally stable, while $\|\Delta\|$ is defined as the L_2-induced norm, which can be equivalently represented as $\|\Delta\| = \text{ess sup}_{t\in[0,h)} |\Delta(t)|$.

To this end, we apply the LPTV scaling suggested in the preceding section. For simplicity, we confine ourselves to the case when W is static and thus $W = D_W(t)$. To be more precise, to conform to our standing assumptions, we assume $W = D_W \in \boldsymbol{W}$, where $\boldsymbol{W}$ is defined as the set of h-periodic functions $D_W(t)$ satisfying the condition (9). Obviously, such W and Δ commute and thus $\|W\Delta W^{-1}\| = \|\Delta\|$. Hence, the essential issue will be the scaling of $\widehat{G}(z)$ with W pertinent to (10). To study this issue, we note that $\widehat{G}(z)$ here is described by

$$\widehat{G}(z) = \mathcal{C}(z - e^{-h})^{-1}\mathcal{B} \tag{12}$$

$$\mathcal{B} : w(\cdot) \in \mathcal{K} \mapsto y \in \mathbf{C},\ y = \int_0^h e^{-(h-t)} w(t)dt, \qquad \mathcal{C} : u \in \mathbf{C} \mapsto z(\cdot) \in \mathcal{K},\ z(t) \equiv u \tag{13}$$

Hence, the H_∞ norm of the scaled transfer operator described by $\widehat{W}\widehat{G}(z)\widehat{W}^{-1}$ is given by

$$\|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty = (\mathcal{C}_W^*\mathcal{C}_W)^{1/2}(\mathcal{B}_W\mathcal{B}_W^*)^{1/2}\|(z - e^{-h})^{-1}\|_\infty \tag{14}$$

where $\mathcal{B}_W := \mathcal{B}D_W^{-1}$ and $\mathcal{C}_W := D_W\mathcal{C}$. Noting that

$$\mathcal{B}_W\mathcal{B}_W^* = \int_0^h e^{-2(h-t)} D_W(t)^{-2}dt, \quad \mathcal{C}_W^*\mathcal{C}_W = \int_0^h D_W(t)^2 dt \tag{15}$$

it follows from the Cauchy-Schwarz inequality that

$$(\mathcal{C}_W^*\mathcal{C}_W)^{1/2}(\mathcal{B}_W\mathcal{B}_W^*)^{1/2} \geq \int_0^h e^{-(h-t)}dt \tag{16}$$

Note that the right hand side is independent of $W(t) = D_W(t)$ and is equal to $1 - e^{-h}$, and that the equality is attained if and only if $D_W(t)^2 = \kappa e^{-(h-t)}$ for some $\kappa > 0$. Since $\|(z - e^{-h})^{-1}\|_\infty = (1 - e^{-h})^{-1}$, it follows that $M(h) := \min_{W \in \boldsymbol{W}} \|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty = 1$ regardless of the sampling period h.

Combining the above discussions, we are immediately led to the conclusion that $\rho \geq M(h)^{-1} = 1$ ($\forall h$), by taking $\gamma(z) = M(h)$ in (10). We can indeed confirm that $\rho = 1$ is the exact robust stability radius for any $h > 0$. To see this, let us consider the case when Δ is a real constant, in which case testing stability of the closed-loop sampled-data system Σ_Δ reduces, due to the special form of Σ_Δ here, to that of closed-loop *discrete-time* system with Δ being the output feedback gain, and thus we are immediately led to the allowable range $-1 < \Delta < (1 + e^{-h})/(1 - e^{-h})$ for which Σ_Δ is internally stable. Noting that $(1 + e^{-h})/(1 - e^{-h}) > 1$ and that a real constant Δ can be regarded as a special case of an h-periodic function $\Delta(t)$, it follows that $\rho \leq 1$. Since $\rho \geq 1$ as shown above, we have established that $\rho = 1$ regardless of h. Note that the above arguments in particular implies that the stability radius remains unchanged in this example if the class of uncertainties is restricted to include only constant gains and also if it is relaxed to include (not necessarily h-periodic) general gains $\Delta(t)$.

To see the effectiveness of the above optimal LPTV scaling, it would be suggestive to note that the (unscaled) H_∞ norm of $\widehat{G}(z)$ is given by $N(h) := [h(1 + e^{-h})/2(1 - e^{-h})]^{1/2}$ (> 1), and as h becomes large, the lower bound of the exact stability margin given by $N(h)^{-1}$ becomes arbitrarily small and thus arbitrarily conservative compared with $\rho = 1$, even though $N(h)^{-1}$ is indeed the exact stability margin if we are to deal with h-periodic *dynamical* uncertainties [11],[5]. This situation is illustrated in Fig. 3 for different sampling periods h, where the solid line denotes our optimally LPTV scaled norm $M(h) = 1$ ($\forall h$), while the dashed line denotes the unscaled H_∞ norm $N(h)$ mentioned above. The effectiveness of LPTV scaling is evident.

The third line, the dash-dot line, in this figure, on the other hand, denotes the 'structured singular value' μ in the context of sampled-data systems whose reciprocal gives the exact robust stability radius if we are to deal with the uncertainties that are linear time-invariant and dynamical [3],[5]. It is a general fact that this μ is a lower bound of the H_∞ norm of $\widehat{G}(z)$ scaled with linear time-invariant systems, but in this particular example, we can see from the discussions in Section V of [5] that μ in fact gives the exact[†] infimum of the

[†]As shown in [5], μ is given by the maximum over a finite frequency range of some frequency-dependent infinite series with nonnegative terms. Due to truncations of the infinite series as well as frequency gridding in the computation, the plot of μ in Fig. 3 is, strictly speaking, a lower bound (or an 'optimistic' estimate) of μ.

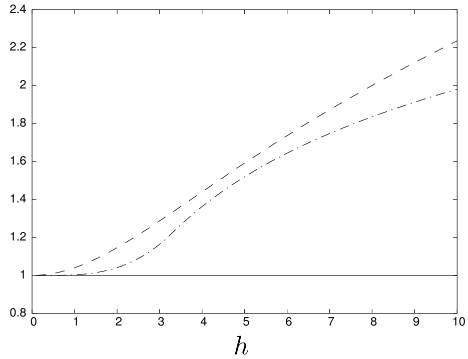

Fig. 3: The optimally LPTV-scaled H_∞ norm, the unscaled H_∞ norm and the optimally LTI-scaled H_∞ norm of Σ_0

scaled H_∞ norm of $\widehat{G}(z)$ under all possible linear time-invariant frequency-dependent scaling. Hence, μ^{-1} gives the best available lower bound of the robust stability radius when Δ is assumed to be a constant gain and the conventional frequency-dependent scaling is applied for the radius estimation. The figure, however, clearly indicates that this lower bound via the standard scaling is also quite conservative compared with the exact radius obtained by the optimal LPTV scaling.

5 Properties of Linear Periodically Time-Varying Scaling

We have seen in the example in the preceding section that LPTV scaling could be quite useful for less conservative robust stability analysis of sampled-data systems with static uncertainties. This leads us to an interesting question whether LPTV scaling can be effective also in other situations, especially in the robust stability analysis of continuous-time feedback systems regarded as a special class of sampled-data systems. Even though we believe that the idea of LPTV scaling is quite promising particularly when it is extended to noncausal LPTV scaling [4],[6],[8], what we show in this section is that the effectiveness of (causal) LPTV scaling of the D-scaling type studied in this paper is limited to sampled-data systems (and h-periodic systems). More precisely, we show that this class of LPTV scaling does not provide any advantage over the conventional LTI scaling in whatever sense when it is applied to continuous-time LTI feedback systems. Thus, this class of LPTV scaling could be interpreted as a class that is in some sense specific to sampled-data systems, but can indeed be a quite effective tool in dealing with such sampled-data systems. Some words will be added as to this observation in this section, and also in the Concluding Remarks.

5.1 Fundamental Relation in Frequency Response Gain

We first consider the system shown in Fig. 4, corresponding to the D-scaling with (7), where we assume that G is linear time-invariant and stable (which is possibly static). Let us consider an arbitrary angular frequency $\varphi \in \mathcal{I}_0$ and an arbitrary integer L, and define $\omega := \varphi + L\omega_s$. Furthermore, we consider the sinusoid $v(t) = v_0 e^{j\omega t}$ with an arbitrary complex vector v_0. Then, the signal f in Fig. 4 in the steady-state consistent with this v is the sinusoid $f(t) = f_0 e^{j\omega t}$ with $f_0 = G(j\omega)v_0$, where $G(s)$ denotes the transfer matrix of G. Since $\widetilde{f} = Wf$ and $\widetilde{v} = Wv$, it follows from the harmonic frequency response representation of the h-periodic

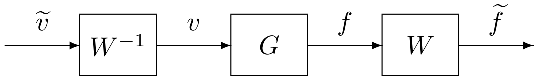

Fig. 4: Linear time-invariant system with linear periodically time-varying scaling.

W that in the steady-state, $\widetilde{f}(t)$ and $\widetilde{v}(t)$ are SD-sinusoids with fundamental angular frequency φ and generalized amplitudes/phases $\underline{\widetilde{f}} := \underline{W}_L(j\varphi)f_0$ and $\underline{\widetilde{v}} := \underline{W}_L(j\varphi)v_0$, respectively. Hence, by the definition (4) together with the relation (5), it follows that the frequency response gain $\|\widehat{W}(e^{j\varphi h})\widehat{G}(e^{j\varphi h})\widehat{W}(e^{j\varphi h})^{-1}\|$ of the scaled system at the fundamental angular frequency φ, which we denote by $\gamma_\varphi(G_W)$ for notational simplicity, satisfies

$$\gamma_\varphi(G_W) \geq \frac{\|\underline{\widetilde{f}}\|}{\|\underline{\widetilde{v}}\|} = \frac{\|\underline{W}_L(j\varphi)f_0\|}{\|\underline{W}_L(j\varphi)v_0\|} \tag{17}$$

By the invertibility assumption[‡] of $\underline{W}(j\varphi)$ ($\forall\varphi \in \mathcal{I}_0$), it follows that $\underline{W}_L(j\varphi)$ has full column rank for any integer L and $\varphi \in \mathcal{I}_0$. Thus, the finite-dimensional matrix given by the infinite series

$$S_L(j\varphi) := \underline{W}_L(j\varphi)^*\underline{W}_L(j\varphi) = \sum_{i=-\infty}^{\infty} W_{iL}(j\varphi)^* W_{iL}(j\varphi) \tag{18}$$

(which is convergent by the boundedness [16] of $\underline{W}(j\varphi)$) is positive definite for any L and φ. Hence, $S_L(j\varphi)^{-1/2}$ is well-defined and $v_0 = S_L(j\varphi)^{-1/2}v_0'$ induces a one-to-one correspondence between v_0 and v_0', by which it follows from (17) that

$$\gamma_\varphi(G_W) \geq \sup_{v_0} \frac{\|\underline{\widetilde{f}}\|}{\|\underline{\widetilde{v}}\|} = \sup_{v_0'} \frac{\|\underline{W}_L(j\varphi)\cdot G(j(\varphi + L\omega_s))S_L(j\varphi)^{-1/2}v_0'\|}{\|\underline{W}_L(j\varphi)\cdot S_L(j\varphi)^{-1/2}v_0'\|} \tag{19}$$

A direct computation of the denominator on the right hand side leads to $\|v_0'\|$ while that of the numerator leads to $\|S_L(j\varphi)^{1/2}G(j(\varphi + L\omega_s))S_L(j\varphi)^{-1/2}v_0'\|$, and hence

$$\gamma_\varphi(G_W) \geq \|S_L(j\varphi)^{1/2}G(j(\varphi + L\omega_s))S_L(j\varphi)^{-1/2}\|, \quad \forall L = 0, \pm 1, \pm 2, \cdots, \ \forall\varphi \in \mathcal{I}_0 \tag{20}$$

Now, let us define

$$W_{\mathrm{LTI}}(j\omega) := S_L(j\varphi)^{1/2} \tag{21}$$

where ω, φ and L are related by the one-to-one correspondence $\omega = \varphi + L\omega_s$ ($\varphi \in \mathcal{I}_0$) between ω and the pair (φ, L). Then, from (20), we readily have

$$\gamma_\varphi(G_W) \geq \|W_{\mathrm{LTI}}(j\omega)G(j\omega)W_{\mathrm{LTI}}(j\omega)^{-1}\|, \quad \forall\omega = \varphi + L\omega_s \ (L = 0, \pm 1, \pm 2, \cdots), \ \forall\varphi \in \mathcal{I}_0 \tag{22}$$

The frequency response gain of the LTI system H viewed as an h-periodic system is given by

$$\gamma_\varphi(H) = \sup_{\substack{\omega = \varphi + L\omega_s \\ L = 0, \pm 1, \pm 2, \cdots}} \|H(j\omega)\| \tag{23}$$

from (4). Hence, we readily have the following result from the inequality (22).

Theorem 2 With respect to the reduction of the frequency response gain of the scaled system $G_W = WGW^{-1}$, LPTV scaling with a general dynamic h-periodic W provides no advantage over LTI frequency-dependent scaling with $W_{\mathrm{LTI}}(j\omega)$.

[‡]This is our standing assumption (recall Section 3); note that $\widehat{W}(e^{j\varphi h})$ is invertible if and only if $\underline{W}(j\varphi)$ is.

5.2 Limit of the Ability of the LPTV Separator (7)

We are now in a position to state the main result of this section.

Theorem 3 Suppose that the discrete-time controller Ψ is in fact absent and thus Σ_Δ is a continuous-time system, and that P and Δ are both LTI. Then, the LPTV separator of the D-scaling type given by (7) provides no advantage over the conventional D-scaling in the continuous-time systems.

Proof. Since Δ is also LTI, we also have (22) with G replaced by Δ. Hence, it follows that if the LPTV separator (7) satisfies (6), and thus $\gamma_\varphi(G_W)\gamma_\varphi(\Delta_W) \leq 1-\varepsilon,\ \forall\varphi \in \mathcal{I}_0,\ \forall\Delta \in \boldsymbol{\Delta}$ for some $\varepsilon > 0$ possibly dependent on Δ (see (10)), then we also have

$$\|W_{\mathrm{LTI}}(j\omega)G(j\omega)W_{\mathrm{LTI}}(j\omega)^{-1}\| \cdot \|W_{\mathrm{LTI}}(j\omega)\Delta(j\omega)W_{\mathrm{LTI}}(j\omega)^{-1}\| \leq 1-\varepsilon, \quad \forall\omega,\ \forall\Delta \in \boldsymbol{\Delta} \tag{24}$$

This in particular implies that robust stability of Σ_Δ can also be concluded by the small-gain theorem via the conventional frequency-dependent D-scaling with $W_{\mathrm{LTI}}(j\omega)$ given by (21). Q.E.D.

We give some remarks that make clearer the implication of this theorem.

First of all, it would be worth noting that nothing is assumed about the uncertainty set $\boldsymbol{\Delta}$ other than the basic assumptions stated in Theorem 1. In particular, each $\Delta \in \boldsymbol{\Delta}$ can be static or dynamic, and also (spatially) structured or unstructured. Regarding the latter viewpoint, suppose that the h-periodic system W has some structure such as block-diagonal forms; we often consider such structure in D-scaling so that the scaling element and the structured uncertainties commute. In such a case, the structure of W is inherited to W_{LTI}, as seen from the construction of W_{LTI}. Hence, LPTV scaling (7) cannot lead to such kind of "dreamy" contribution as, e.g., equivalently introducing more general D-scaling for which commutativity between the scaling element and the uncertainties may not be obvious or even fails (note that the proof of the theorem does not rely on such commutativity at all).

Second, suppose that W is static, i.e., W is a periodic gain matrix $D_W(t)$. Then, it is not hard to see that the resulting $W_{\mathrm{LTI}}(j\omega)$ is actually independent of ω. That is, W_{LTI} is in fact scaling with a constant matrix, whose spatial structure is simply inherited from that of $D_W(t)$. Hence, LPTV scaling (7) cannot lead to any contribution from the viewpoint of the dynamics of the scaling elements, not only theoretically but also from any practical point of view such as ease in the associated computations.

The above arguments might sound trivial at a glance, but once we extend the notion of LPTV scaling to include what we call noncausal LPTV scaling and then observe the associated discussions [4],[6],[8], we believe that the importance of Theorem 3 and these arguments becomes evident as the clarification about the ability of (causal) LPTV scaling studied in this paper. This is because in the case of noncausal LPTV scaling, even static LPTV scaling turns out to have an ability of inducing frequency-dependent scaling if it is interpreted in the context of the conventional scaling in the continuous-time systems, and thus it is a meaningful and promising idea to apply *noncausal* LPTV scaling also to LTI continuous-time feedback systems.

From these arguments and the example in the preceding section, we can conclude that the LPTV scaling (7), which is a subclass of general noncausal LPTV scaling, can be an effective tool for robust stability analysis (and synthesis) of sampled-data systems with at least one static uncertainty block, even though its effectiveness is a feature that is specific to the setting of sampled-data (and LPTV) systems. This subclass can be applied also to static sector nonlinearities while noncausal LPTV scaling is hard to apply, although we do not deal with this topic in this paper due to limited space.

Remark 3 We only remark here that with some additional arguments, the example in Section 4 can also be interpreted as showing the effectiveness of (causal) LPTV scaling to static sector nonlinearities. To this end, let us consider the case when $\varDelta$ is in fact a time-invariant/time-varying static nonlinearity $\varUpsilon$ belonging to the sector $[-\alpha, \alpha]$ $(\alpha > 0)$. By definition, this implies that $|w(t)| \le \alpha |z(t)|$. If we apply the LPTV scaling $W = D_W(t)$ as in Section 4, we are led to the scaled systems $W\varSigma_0 W^{-1}$ and $W\varUpsilon W^{-1}$. Let us denote the input and output of $W\varUpsilon W^{-1}$ by z_W and w_W, respectively. Then, it is easy to see that

$$|w_W(t)| = |D_W(t)| \cdot |w(t)| \le \alpha |D_W(t)| \cdot |z(t)| = \alpha |z_W(t)| \tag{25}$$

regardless of $W = D_W(t)$. That is, $W\varUpsilon W^{-1}$ is a sector nonlinearity belonging to the same sector $[-\alpha, \alpha]$. Since we have seen in Section 4 that the L_2-induced norm of $W\varSigma_0 W^{-1}$ (which equals the H_∞ norm $\|\widehat{W}\widehat{G}(z)\widehat{W}^{-1}\|_\infty$) can be minimized to 1 by a suitable choice of $W = D_W(t)$, it follows from the small-gain theorem that L_2-stability is assured if $\alpha < 1$. Considering the case when $\varUpsilon$ is in fact linear as a special case, we see, together with our preceding observations, that the above condition is not only sufficient but also necessary in the sense of absolute stability analysis [12]. This demonstrates that (causal) LPTV scaling is useful also for static sector nonlinearities. Combining this with the preceding observations, we are led to the following consequence as far as the example in Section 4 is concerned: (causal) LPTV scaling can assure stability of the system $\varSigma_\varDelta$ for every sampling period whenever $\varDelta$ is a static time-invariant/time-varying linear/nonlinear system whose L_2 gain is less than 1, while time-invariant scaling fails to do so. It is not hard to see that we cannot arrive at such a relation like (25) and thus deal with nonlinearities if noncausal LPTV scaling [4], [6],[8] is employed. This is because noncausal LPTV scaling induces superposition of data at different time instants that is hard to reconstruct without a linearity assumption.

6 Concluding Remarks

We introduced a novel technique called linear periodically time-varying (LPTV) scaling to robust stability analysis of sampled-data systems, and showed that it is generally more effective than the conventional LTI scaling in such a context. In particular, we gave a simple example in which applying LPTV scaling does lead to exact robust stability analysis without any conservatism while the conventional LTI scaling fails to do so. This suggests that LPTV scaling is indeed a more natural technique than the conventional LTI scaling as far as sampled-data systems (and periodic systems) with at least one static uncertainty block are concerned. At the same time, however, we established that the effectiveness of the (causal) LPTV scaling of the form (7) studied in this paper is limited to sampled-data systems (and periodic systems) in the sense that it provides no advantage over the conventional LTI scaling when the feedback loop consists of only LTI systems (Theorem 3). We would like to close the paper by giving some further remarks on this observation together with some related important issues pertinent to our current studies.

As stated in Section 5, we first remark that the idea of LPTV scaling has been extended further in our recent study [4]; according to the extended discussions therein, the technique we introduced in this paper is classified into the subclass of *causal* LPTV scaling, while a more general technique called noncausal LPTV scaling can also be introduced and is proved more effective, in general. In this respect, the above observation (Theorem 3) in the context of LTI feedback systems is quite important as the clarification about the limit of the ability of *causal* LPTV scaling (7) studied in this paper. Even though we confined ourselves to linear uncertainties in this paper, however, the technique of causal LPTV scaling can be applied also to static sector nonlinearities while noncausal LPTV scaling is hard to apply. Thus, the idea of causal LPTV scaling developed in this paper is of independent significance even though it forms a subclass of a more general class of LPTV scaling and its effectiveness *in continuous-time LTI feedback systems* is limited.

Detailed discussions on the further extended notion of noncausal LPTV scaling, its relationship to causal LPTV scaling, promising features of noncausal LPTV scaling beyond Theorem 3, optimization of causal/ noncausal LPTV scaling via some sort of equivalent discretization, controller synthesis via causal/noncausal LPTV scaling, and the application of causal LPTV scaling to static sector nonlinearities, and so on, however, are far beyond the scope of this paper especially because the full discussions take a huge space (see the preliminary discussions in [4],[8]), and thus will be reported in details independently.

References

[1] M. Araki, Y. Ito and T. Hagiwara, Frequency-response of sampled-data systems, *Automatica*, Vol. 32, No. 4, 483–497 (1996).

[2] B. A. Bamieh and J. B. Pearson, A general framework for linear periodic systems with applications to H_∞ sampled-data control, *IEEE Trans. Automat. Contr.*, Vol. AC-37, No. 4, pp. 418–435 (1992).

[3] G. Dullerud and K. Glover, Robust stabilization of sampled-data systems to structured LTI perturbations, *IEEE Trans. Automat. Contr.*, Vol. 38, No. 10, pp. 1497–1508 (1993).

[4] T. Hagiwara, Causal/noncausal linear periodically time-varying scaling for robust stability analysis and their properties, Proc. the 17th International Symposium on Mathematical Theory of Networks and Systems, pp. 742–752 (2006).

[5] T. Hagiwara and M. Araki, Robust stability of sampled-data systems under possibly unstable additive/ multiplicative perturbations, *IEEE Trans. Automat. Contr.*, Vol. 43, No. 9, pp. 1340–1346 (1998).

[6] T. Hagiwara and Y. Ohara, Noncausal linear periodically time-varying scaling for discrete-time systems, 3rd IFAC Workshop PSYCO '07 (2007).

[7] T. Hagiwara and Y. Tsuruguchi, Nyquist stability criterion of sampled-data systems with the 2-regularized determinant and its applications to robust stability analysis, Proc. the 16th International Symposium on Mathematical Theory of Networks and Systems, THA3-4 (2004).

[8] T. Hagiwara and H. Umeda, Robust stability analysis and robust stabilization of sampled-data systems via noncausal periodically time-varying scaling, to be presented in the 46th IEEE Conference on Decision and Control.

[9] T. Iwasaki and S. Hara, Well-posedness of feedback systems: insights into exact robustness analysis and approximate computations, *IEEE Transactions on Automatic Control*, Vol. 43, No. 5, pp. 619–630 (1998).

[10] A. Packard and J. Doyle, The complex structured singular value, *Automatica*, Vol. 29, No. 1, pp. 71–109 (1993).

[11] N. Sivashankar and P. P. Khargonekar, Robust stability and performance analysis of sampled-data systems, *IEEE Trans. Automat. Contr.*, Vol. 38, No. 1, pp. 58–69 (1993).

[12] M. Vidyasagar, *Nonlinear Systems Analysis*, Second ed., Prentice-Hall, NJ (1993).

[13] N. M. Wereley, *Analysis and Control of Linear Periodically Time Varying Systems*, Ph. D. Thesis, Dept. of Aeronautics and Astronautics, M.I.T., Cambridge, MA (1990).

[14] Y. Yamamoto, A function space approach to sampled-data systems and tracking problems, *IEEE Trans. Automat. Contr.*, Vol. AC-39, No. 4, pp. 703–713 (1994).

[15] Y. Yamamoto and P. P. Khargonekar, Frequency response of sampled-data systems, *IEEE Trans. Automat. Contr.*, Vol. AC-41, No. 2, pp. 166–176 (1996).

[16] J. Zhou and T. Hagiwara, Existence conditions and properties of the frequency response operators of continuous-time periodic systems, *SIAM Journal on Control and Optimization*, Vol. 40, No. 6, pp. 1867–1887 (2002).