ソリトン方程式の内部対称性とその応用

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はしがき

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研究組織

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1. String equations and bispectral problems

Relation between string equations and the bispectral problem has intrigued many. The general form of string equation is $[L,[L,P]]=0$, while a solution to the bispectral problem is a pair (L,Θ), with Θ being a function of x, such that $(\operatorname{ad} L)^{m+1}(\Theta)=0$. Here and in what follows, an upper-case roman letter denotes an ordinary differential operator in a scalar variable x, unless otherwise noted. The map $(L,\Theta)\to(L,P)$, where $P=(\operatorname{ad} L)^{m-1}(\Theta)$, from the space of solutions of the bispectral problem to the space of solutions of the general string equation, illustrates the relation (in the range of this map, there seems no obvious way to see the condition that Θ be a function, so this is rather a general nonsense). This topic is also related to the Krichever theory, through the rank one bispectral (i.e., rational Calogero–Moser) solutions [10, 14], but here we propose another link to the Krichever theory.

Here we consider the equation

$$[L,[L,P]]=0.$$

This equation is only slightly more general than any of the various forms of string equations. E.g., if P and Q satisfy $[P,Q]=1$, then P and any polynomial $L=f(Q)$ of Q satisfy $[L,[L,P]]=0$, and if, moreover, $\operatorname{ord} Q>0$, then so do P and any L such that $[L,Q]=0$. The latter equation suggests that the Krichever theory may come in here:

1. Let $\psi=\psi(x,z)$ be the Baker-Akhiezer function associated to the Krichever data $(C,p,z,\mathcal{F})$, i.e., C is a complete curve of genus g, $p\in C$ a smooth point, z a local coordinate at p ($z(p)=0$), $\mathcal{F}$ a torsion-free rank-1 sheaf (or a line bundle) of degree $g-1$ such that $\mathcal{H}om_{\mathcal{O}}(\mathcal{F},\mathcal{F})\simeq\mathcal{O}$, and ψ a (unique (up to a constant)) section, depending on the scalar parameter x, of $\mathcal{F}$ on $C\setminus\{p\}$ which, with respect to a trivialization of $\mathcal{F}$ near p, looks like $(1/z+O(1))\exp(x/z)$. Let ∇ be a rational vector field on C with no poles away from p, and let $\tilde\nabla$ be a rational lifting of ∇ on $\mathcal{F}$ with no poles away from p, i.e., $\tilde\nabla$ maps any rational section ϕ of $\mathcal{F}$ with no poles away from p to another such section $\tilde\nabla\phi$, such that if f is a rational function on C with no poles away from p, then

$$\tilde\nabla(f\phi)=(\nabla f)\phi+f\tilde\nabla\phi. \tag{1}$$

The set of such $\tilde\nabla$'s, for any fixed ∇, is an affine space, isomorphic to the space of (rational) functions on C with no poles away from p. The Krichever construction tells us that for any rational function f on C with no poles away from p, there is an ordinary differential operator P_f in x such that $f\psi=P_f\psi$. Similarly, there is an ordinary differential operator Q such that $\tilde\nabla\psi=Q\psi$. Substituting ψ for ϕ in (1), and noting

$$\tilde\nabla(f\psi)=\tilde\nabla(P_f\psi)=P_f\tilde\nabla(\psi)=P_fQ\psi,$$

$$(\nabla f)\psi=P_{(\nabla f)}\psi$$

and

$$f\tilde\nabla\psi=fQ\psi=Qf\psi=QP_f\psi,$$

one obtains

$$P_{(\nabla f)} = [P_f, Q].$$

Since ∇f is another (rational) function on C, $P_{(\nabla f)}$ commutes with P_f. Hence letting $L = P_f$ and $P = Q$, we obtain a solution of $[L, [L, P]] = 0$. This construction is more general than the previous ones, in the sense that there may not be any Q such that $[P, Q] = 1$.

2. However, if one considers the equation $[L, [L, P]] = 0$ together with the condition that P is a function (i.e., multiplication operator, having in mind the usual assumption in the bispectral problem), then the situation is different. Here and in what follows, polynomial means a constant-coefficient polynomial.

Claim *If $[L, [L, P]] = 0$ and P is a function, then $L = f(Q)$ for some polynomial f, and $[Q, P] = c$ is a constant (and hence $[L, P] = cf'(Q)$ is a polynomial of Q).*

To avoid the trivial case, we assume $[L, P] \neq 0$. (This implies $[Q, P] \neq 0$, so we may take $c = 1$ in the above claim.)

First note that (i) $[L, P]$ is not necessarily a polynomial of L, but (ii) if it is a polynomial of L, then it is a constant.

(ii) is obvious: since P is a function, the order of the operator $[L, P]$ is less than that of L, so if $[L, P] = f(L)$, then the degree of the polynomial f is 0.

To see (i), let us start with any pair (L, P) such that $[L, [L, P]] = 0$, $[L, P] \neq 0$, and P is a function. If $[L, P]$ is not a polynomial of L, there is nothing to prove. If $[L, P]$ is a polynomial of L, then by (ii) above, it is a constant, say $c \neq 0$. Now let $f(x)$ be any polynomial of degree > 1, and consider the pair $(f(L), P)$ in place of (L, P). Clearly, it also satisfies the conditions, i.e., $[f(L), [f(L), P]] = 0$ and P is a function. But $[f(L), P] = f'(L)[L, P] = cf'(L)$ is not a polynomial of $f(L)$, proving (i).

Note that if P is a function, the order of $[L, P]$ is less than that of L.

Let R be the ring of differential operators which commute with L, and let $Q \in R$ be an element of the lowest positive order, i.e., $q := \operatorname{ord} Q > 0$ and for any $S \in R$, $s := \operatorname{ord} S \geq q$ if $s > 0$. Then

Lemma *If $[L, [L, P]] = 0$ and if the order of $[L, P] \neq 0$ is less than that of L, then R coincides with the ring of polynomials of Q. In particular, L and $[L, P]$ are polynomials of Q.*

Proof: Since Q commutes with L, it becomes a constant-coefficient Laurent series of $L^{-1/l}$,

$$Q = \sum_{k=-\infty}^{q} a_k L^{k/l},$$

where l is the order of L, and $a_q \neq 0$. Using this expansion, and the fact that $[L, P]$ commutes with L, we have

$$[Q, P] = \sum_{k=-\infty}^{q} (k/l) a_k L^{k/l-1} [L, P],$$

which still commutes with L, and has order less than q since the order of $[L, P]$ is less than that of L. Hence, by the minimality of q, $[Q, P]$ is a constant. Multiplying Q by a constant if necessary, we assume $[Q, P] = 1$.

Since R also is the ring of differential operators which commute with Q, every element S of R can be expanded as a Laurent series in $Q^{-1/q}$. Thus

$$S = \sum_{k=-\infty}^{s} a_k Q^{k/q},$$

and hence

$$[S, P] = \sum_{k=-\infty}^{s} (k/q) a_k Q^{k/q-1},$$

which belongs to R (since it commutes with Q and is a differential operator as a commutator of two differential operators), and $\operatorname{ord}[S, P] = \operatorname{ord} S - q$ unless S is a constant. This implies that $s = \operatorname{ord} S$ is a multiple of q. Indeed, if there exists an $S \in R$ such that $\operatorname{ord} S$ is not a multiple of q, then there is an element, say S_0, of minimal order among all such S's. Then $[S_0, P]$ belongs to R, and $\operatorname{ord}[S_0, P]$ is less than $\operatorname{ord} S_0$ and is not a multiple of q. This is a contradiction.

Using this, it follows by induction on the order of operators that R is the ring of polynomials of Q.

This suggests the additional simplicity is offerred when P is a function. It may be interesting to look at equations like $(\operatorname{ad} L)^{m+1} P = 0$ under the assumption that P be a function (relevant set-up for the bispectral problem).

2. Calogero-Moser type KP solutions

KP solutions of rational Calogero-Moser type provide concrete examples of the bispectral problem as well as the Krichever theory. All the information on a solution is contained in a pair of square matrices (X, Y) such that $[X, Y] + 1$ is of rank one. It is obvious that such a Y exists if the eigenvalues of X are all distinct. More generally, if pairs (X_1, Y_1) and (X_2, Y_2) possess this property, and if no eigenvalue of X_1 is the same as any eigenvalue of X_2, then a pair of block matrices

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 & * \\ * & Y_2 \end{pmatrix}$$

possess the same property, after choosing suitable off-diagonal blocks of Y. Subtlety comes in when X has some eigenvalue with multiplicity.

In general, a necessary and sufficient condition for an X to have some Y with this condition satisfied is that for any eigenvalue λ of X, the sizes of Jordan blocks belongging to λ are at least two apart from each other (so in the simplest example of X with two Jordan blocks belonging to the same eigenvalue λ, λ has multiplicity 4, and the two Jordan blocks have sizes 1 and 3, respectively). Needless to say, this fact is related to the cell decomposition of Grassmannian (or the theory of Schur functions), but Wilson's original proof of it, although it may seem a little subtle, is already quite concrete and elementary (cf. [14, lemma 6.9])! This illustrates the degree of concreteness seen in this topic.

Here we look at a different side of this subject, and give a proof, which does not assume bispectrality etc., of the fact that the spectral curve of any rank one ordinary differential operator with rational coefficients is unicursal. As pointed out by E. Horozov [3], after suitable reformulation using the language of Weyl algebra etc., it is also straightforward to work out difference operator analogue of this result.

Let $\mathcal{A}$ be a commutative ring of ordinary differential operators of rank 1, i.e., for any $n \gg 0$ there exists $P \in \mathcal{A}$ of order n. We assume $\mathbb{C} \subset \mathcal{A}$, and that every $P \in \mathcal{A}$ is of the form

$$P = c_0\Big(\frac{d}{dx}\Big)^n + c_1\Big(\frac{d}{dx}\Big)^{n-1} + (\text{terms of order} \le n-2), \tag{1}$$

where $c_0,\ c_1 \in \mathbb{C}$, $c_0 \neq 0$.

The *Krichever data* for $\mathcal{A}$ is a quadruple $(C, p, z, \mathcal{F})$, where C is a complete, reduced irreducible curve over $\mathbb{C}$, $p \in \mathbb{C}$ a smooth point, z a local coordinate at p, i.e., $z \in \mathfrak{m}_p - \mathfrak{m}_p^2$, and $\mathcal{F}$ a torsion-free rank 1 sheaf on C of degree $g-1$, such that $C \setminus \{p\} = \operatorname{Spec}(\mathcal{A})$, and the unique analytic section ψ of $\mathcal{F}$ on $C \setminus \{p\}$ with singularity at p of the form $(1/z)\exp(x/z)(1+O(z))$, for generic $x \in \mathbb{C}$, gives the Baker-Akhiezer function, a common eigenfunction for $\mathcal{A}$.

Lemma 1 (Diximier [2]) *If the coefficients of $P \in \mathcal{A} \setminus \mathbb{C}$ are rational functions in x, then the coefficients have no poles at $x = \infty$.*

The following proof works whenever the coefficients of P have Laurent series expansion with at most poles at infinity.

Proof. Suppose the contrary. Let $p := \operatorname{ord} P > 0$ and write

$$P = c_0\Big(\frac{d}{dx}\Big)^p + c_1\Big(\frac{d}{dx}\Big)^{p-1} + a_2(x)\Big(\frac{d}{dx}\Big)^{p-2} + a_3(x)\Big(\frac{d}{dx}\Big)^{p-3} + \cdots + a_p(x),$$

and let

$$s := \max\{(\deg a_i(x))/i \mid i = 2, 3, \ldots, p\} > 0, \tag{2}$$

where $\deg a_i(x)$ is the order of pole of $a_i(x)$ at $x = \infty$. Expand each $a_i(x)$ in a Laurent series around $x = \infty$, and define the weight of the monomial $x^i(d/dx)^j$ to be $i + sj$. If the maximum in (2) is achieved at $i = i_1, i_2, \ldots, i_k$, where $2 \le i_1 < i_2 < \cdots < i_k \le p$, we have

$$P = c_0 f(y)\Big(\frac{d}{dx}\Big)^p + (\text{terms of weight} < p), \tag{3}$$

where $y := x^s(d/dx)^{-1}$, and $f(y) := 1 + d_1 y^{i_1} + d_2 y^{i_2} + \cdots + d_k y^{i_k}$ for some $d_1, \ldots, d_k \in \mathbb{C}$. Note that s is not necessarily an integer, but $si_1, \ldots, si_k$ are. Note also that monomials of x and d/dx commute modulo terms of lower weight.

Next, let $Q \in \mathcal{A}$ be such that $p = \operatorname{ord} P$ and $q := \operatorname{ord} Q$ are relatively prime. Such a Q exists since $\mathcal{A}$ has rank 1 and $p > 0$. Since P and Q commute, Q is of the form

$$Q = cP^{q/p} + \sum_{i=-\infty}^{q-1} b_i P^{i/p}, \quad c, b_i \in \mathbb{C},\ c \neq 0. \tag{4}$$

Hence, by (3) we have

$$\begin{aligned} Q &= cP^{q/p} + (\text{terms of weight} < q) \\ &= c'(f(y))^{q/p}\Big(\frac{d}{dx}\Big)^q + (\text{terms of weight} < q). \end{aligned}$$

Here $(f(y))^{q/p}$ is computed as a power series in y, but since Q is a differential operator, the series must terminate:

$$(f(y))^{q/p} = g(y) := 1 + d'_1 y^{j_1} + d'_2 y^{j_2} + \cdots + d'_l y^{j_l}$$

or

$$f(y)^q = g(y)^p, \tag{5}$$

for some $d'_1, \ldots, d'_l \in \mathbb{C}$ and $2 \le j_1 < j_2 < \cdots < j_l \le q$. The last inequalities hold since Q is of order q and has the form similar to (1) (i.e., the top two terms have constant coefficients).

Let $\alpha_1, \ldots, \alpha_{i_k} \in \mathbb{C}$ be the roots of polynomial $f(y)$, and $\beta_1, \ldots, \beta_{j_l} \in \mathbb{C}$ the roots of polynomial $g(y)$. Since $(f(y))^q$ has roots $\alpha_1, \ldots, \alpha_{i_k}$, each with multiplicity q, since $(g(y))^p$ has roots $\beta_1, \ldots, \beta_{j_l}$, each with multiplicity p, and since p and q are relatively prime, (5) means the actual multiplicity of each root is a multiple of pq. Since $i_k \le p$ and $j_l \le q$, this is possible only when $i_k = p$, $j_l = q$ and all the α's and β's are equal ($=: \gamma$), so that $f(y) = (1 + y/\gamma)^p$, $g(y) = (1 + y/\gamma)^q$. But this is impossible because i_1 and j_1 are at least 2, so f and g have no term of degree 1 in y. Q.E.D.

Lemma 2 *If some $P \in \mathcal{A} \setminus \mathbb{C}$ has rational coefficients, then the same is true for every $Q \in \mathcal{A}$, and the spectral curve of $\mathcal{A}$ is rational.*

Proof. By lemma 1 the coefficients of $P \in \mathcal{A}$ have no poles at $x = \infty$, and (if $P \notin \mathbb{C}$) by (4) the same is true for every $Q \in \mathcal{A}$. Thus $\mathcal{A} \subset \mathcal{R} := \mathbb{C}[[x^{-1}]][d/dx]$. Note that $I := x^{-1}\mathbb{C}[[x^{-1}]][d/dx]$ is a two-sided ideal of $\mathcal{R}$, and the quotient ring $\mathcal{R}/I$ is commutative and is canonically isomorphic to the ring $\mathcal{R}_0 := \mathbb{C}[d/dx]$ of constant coefficient differential operators. Let $\pi \colon \mathcal{R} \to \mathcal{R}/I \simeq \mathcal{R}_0$ be the canonical projection. One can think of π as 'taking the limit as $x \to \infty$': $\pi(P) = \lim_{y \to \infty} P(x + y, d/dx)$. Since every element of $\mathcal{A}$ is of the form $(d/dx)^n +$ (lower order terms), $\pi|_{\mathcal{A}}$ is injective, $\mathcal{A} \xrightarrow{\sim} \pi(\mathcal{A}) \subset \mathcal{R}_0$. Since $\operatorname{Spec} \mathcal{R}_0$ is an affine line ($\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$), $\operatorname{Spec} \mathcal{A}$ is rational. Q.E.D.

Proposition 3 *Suppose $\mathcal{A}$ satisfies the same conditions as in lemma 2, and suppose $\mathcal{A}$ is also maximal commutative. Then the spectral curve of $\mathcal{A}$ is unicursal.*

Proof. Let $(C, p, z, \mathcal{F})$ be the Krichever data for $\mathcal{A}$. Since C is rational, one can take normalization $\pi \colon \mathbb{P}^1 \to C$ and use the global coordinate z on $\mathbb{P}^1 \setminus \{\pi^{-1}(p)\} \simeq \mathbb{C}$ to represent the Baker-Akhiezer function as a function in z of the form $\psi(x, z) = f(x, z)e^{xz}$ (here we ignore the higher KP times), where $f(x, z)$ is a polynomial in z of degree g (the arighmetic genus of C) with coefficients depending on x. In order for ψ to be a section of $\mathcal{F}$ (via a fixed trivialization) on $C \setminus \{p\}$, ψ has to satisfy a system of g linearly independent linear constraints involving its values and z-derivatives at various points:

$$\begin{aligned}
&\big(P_{11}(d/dz)|_{z=z_1} + \cdots + P_{1N}(d/dz)|_{z=z_N}\big)(f(x,z)e^{xz}) \\
&= \big(e^{xz_1}\tilde{P}_{11}(x, d/dz)|_{z=z_1} + \cdots + e^{xz_N}\tilde{P}_{1N}(x, d/dz)|_{z=z_N}\big) f(x,z) \\
&= 0, \\
&\quad \cdots\cdots \\
&\big(P_{g1}(d/dz)|_{z=z_1} + \cdots + P_{gN}(d/dz)|_{z=z_N}\big)(f(x,z)e^{xz}) \\
&= \big(e^{xz_1}\tilde{P}_{g1}(x, d/dz)|_{z=z_1} + \cdots + e^{xz_N}\tilde{P}_{gN}(x, d/dz)|_{z=z_N}\big) f(x,z) \\
&= 0,
\end{aligned}$$

where $P_{ij}(d/dz)$ are constant coefficient linear differential operators in z, and $\tilde{P}_{ij}(x, d/dz)$ are linear differential operators in z with coefficients in $\mathbb{C}[x]$. Now C is unicursal if and only if for each $i \in \{1, \dots, g\}$ there is only one $j \in \{1, \dots, N\}$ such that the linear differential operator $P_{ij}(d/dz)$ is non-zero (in particular, $N \leq g$). Therefore, only in the unicursal case the exponential functions e^{xz_j} can be factored out from the whole system. If C is *not* unicursal, then at least one of those constraints involves values of $\psi = fe^{xz}$ at two different z_j's, say z_1 and z_2, and the exponentials cannot be factored out. Solving such a system, one sees that some coefficients of f must depend on a nontrivial linear combination of e^{xz_1} and e^{xz_2}, and so the coefficients of any $P \in \mathcal{A} \setminus \{\mathbb{C}\}$, of which f is an eigenfunction, cannot be rational. Q.E.D.

3. Young tableaux and vicious random walks on a line

The following is a small observation which naturally came up in the joint work with Adler and van Moerbeke on matrix integrals and combinatorics. (This was not quite useful at the time, due to the need to code the walk as a Young tableaux before performing permutations of L and R moves.)

The number of $2n$ step walks by $k \geq n$ walkers (one walker moves at a time), in a fully-packed configuration at the beginning and the end of walk, is $(2n-1)!!$ if a wall is put at one end of the chunk of walkers (walkers on a half-line). The idea was to put two such chunks back to back to make a bigger chunk of walkers on the full-line to apply Forrester's trick, and then to count the walks on the left and right halves of the chunk separately by factoring the generating function.

Consider a little more general walks on a half-line: the ones with the end of chunk not facing the wall to be not fully packed at the beginning and the end of walk, but otherwise obey the same rules as before (they are vicious, only one walker walks at a time, to left or right, etc.). I want to use the same idea as above to count the number of such walks, by allowing the chunk of full-line walkers at the beginning and the end of walk to get loose at both ends of the chunk, but keeping enough many (compared to the number of steps) walkers still packed in the middle of the chunk to separate the activities on the left and the right halves of the chunk. The result can be seen as the number of walks from one Young diagram α to another one β in p steps, in the space of Young diagrams. Denote this number by $a_\alpha^\beta(p)$. Thus $a_\emptyset^\emptyset(N) = (2n-1)!!$ if $N = 2n$, and is 0 otherwise. In general, it is easy to see

Claim 0 *In order to have* $a_\alpha^\beta(p) \neq 0$, *we must have* $p \geq |\beta \backslash \alpha| + |\alpha \backslash \beta|$ *and* $p \equiv |\beta| - |\alpha| \mod 2$.

if the wall is on the right of the walkers (half-line walkers, or when we look at the left edge of the chunk of full-line walkers), use standard SYT with shape $\lambda \backslash \mu$ ($\lambda \supset \mu$) as Forrester did, to code the steps of left-walkers or right-walkers (walkers who can move only to the left or right, respectively). For the left-walkers μ and λ give the initial and the final configurations respectively, and for the right-walkers it's the other way around. Pack of walkers with the wall on the left (or when we look at the right edge of the chunk of full-line walkers) should be flipped (left$\leftrightarrow$right) before coding.

Configuration of full-line walkers with two loose ends (we always assume that enough many walkers are packed in the middle), is then coded by a pair of Young diagrams (μ_1, μ_2), μ_1 coding the left edge and μ_2 the right edge. Separating the activities on the left and right edges of the chunk, we see that the number of walks from (μ_1,μ_2) to (λ_1,λ_2) in N steps is

$$A(N) := A^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(N) := \sum_{p=0}^{N} \binom{N}{p} a^{\lambda_1}_{\mu_1}(p) a^{\lambda_2}_{\mu_2}(N-p). \tag{0}$$

Claim 1 *In order to have* $A^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(N) \neq 0$, *there must exist an integer* p, $0 \le p \le N$, *such that* $p \ge |\lambda_1 \setminus \mu_1| + |\mu_1 \setminus \lambda_1|$, $p \equiv |\lambda_1| - |\mu_1| \bmod 2$, $N-p \ge |\lambda_2 \setminus \mu_2| + |\mu_2 \setminus \lambda_2|$, *and* $N-p \equiv |\lambda_2| - |\mu_2| \bmod 2$.

This follows from Claim 0. On the other hand, counting separately the numbers of left- and right-moves, but not distinguishing the activities at the left edge and ones at the right edge, we see

Claim 2 *In order to have* $A^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(N) \neq 0$, *there must exist nonnegative integers* l *and* r *such that* $l+r=N$ *and* $l-r=d:=(|\lambda_1|-|\lambda_2|)-(|\mu_1|-|\mu_2|)$.

Remark 1 *The existence of* l *and* r *as in Claim 2 implies* $N \equiv d \bmod 2$, *as is also seen from the mod 2-part of Claim 1.*

Remark 2 *Since the conditions in Claim 2 determine* l *and* r *uniquely, any walk from* (μ_1,μ_2) *to* (λ_1,λ_2) *must have* l *left moves and* r *right moves.*

Denote the number of all walks from (α_1,α_2) to (β_1,β_2) in s steps, in which walker can move only to the left (resp. right) by $L^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}(s)$ (resp. $R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}(s)$).

Applying Forrester's trick to bring all the left-moves before the right-moves, one gets the formula

$$A(N) = \binom{N}{l} \sum_{\nu_1,\nu_2} L^{\nu_1,\nu_2}_{\mu_1,\mu_2}(l) R^{\lambda_1,\lambda_2}_{\nu_1,\nu_2}(r), \tag{1}$$

where the sum runs over all pairs (ν_1,ν_2) of Young diagrams such that

$$\nu_1 \supset \mu_1, \quad \nu_2 \subset \mu_2, \quad |\nu_1 \setminus \mu_1| + |\mu_2 \setminus \nu_2| = l, \tag{2}$$

and

$$\nu_1 \supset \lambda_1, \quad \nu_2 \subset \lambda_2, \quad |\nu_1 \setminus \lambda_1| + |\lambda_2 \setminus \nu_2| = r \tag{3}$$

hold. Note that the last equalities in (2) and (3) are equivalent to each other since $l-r=d$. Conditions (2) and (3) are needed to have $L^{\nu_1,\nu_2}_{\mu_1,\mu_2}(l) \neq 0$ and $R^{\lambda_1,\lambda_2}_{\nu_1,\nu_2}(r) \neq 0$, respectively.

The right hand side of (1) can be written in terms of $f^{\nu_1\setminus\mu_1}$, $f^{\mu_2\setminus\nu_2}$, $f^{\nu_1\setminus\lambda_1}$ and $f^{\lambda_2\setminus\nu_2}$: using $l_1 := |\nu_1 \setminus \mu_1|$ and $l_2 := |\mu_2 \setminus \nu_2| = l - l_1$ as auxiliary indices, and noting that $r_1 := l_1 + |\mu_1| - |\lambda_1| = |\nu_1 \setminus \lambda_1|$ and $r_2 := l_2 - |\mu_2| + |\lambda_2| = |\lambda_2 \setminus \nu_2|$ add up to r, we have

$$A(N) = \binom{N}{l} \sum_{l=l_1+l2} \sum_{\substack{\nu_1 \supset \mu_1,\lambda_1 \\ |\nu_1\setminus\mu_1|=l_1}} \sum_{\substack{\nu_2 \subset \mu_2,\lambda_2 \\ |\mu_2\setminus\nu_2|=l_2}} \binom{l}{l_1} \binom{r}{l_1+|\mu_1|-|\lambda_1|} f^{\nu_1\setminus\mu_1} f^{\mu_2\setminus\nu_2} f^{\nu_1\setminus\lambda_1} f^{\lambda_2\setminus\nu_2}. \tag{4}$$

It is clear from (0) that the generating function $\hat{A}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(x) := \sum_{N=0}^{\infty} A^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(N) x^N/N!$ factors:

$$\hat{A}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(x) = \hat{a}^{\lambda_1}_{\mu_1}(x) \hat{a}^{\lambda_2}_{\mu_2}(x), \tag{5}$$

where

$$\hat{a}^{\lambda}_{\mu}(x) := \sum_{p=0}^{\infty} a^{\lambda}_{\mu}(p)\frac{x^p}{p!}.$$

After rewriting

$$\binom{N}{l}\binom{l}{l_1}\binom{r}{r_1} = \frac{N!}{l_1!r_1!l_2!r_2!},$$

(4) gives another factorization

$$\hat{A}^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}(x) = g^{\lambda_1}_{\mu_1}(x)h^{\lambda_2}_{\mu_2}(x), \tag{6}$$

where

$$g^{\lambda}_{\mu}(x) := \sum_{l \geq \max\{0,|\lambda|-|\mu|\}} \frac{x^{2l+|\mu|-|\lambda|}}{l!(l+|\mu|-|\lambda|)!} \sum_{\substack{\nu\supset\mu,\lambda \\ |\nu\backslash\mu|=l}} f^{\nu\backslash\mu}f^{\nu\backslash\lambda},$$

$$h^{\lambda}_{\mu}(x) := \sum_{l \geq \max\{0,|\mu|-|\lambda|\}} \frac{x^{2l-|\mu|+|\lambda|}}{l!(l-|\mu|+|\lambda|)!} \sum_{\substack{\nu\subset\mu,\lambda \\ |\mu\backslash\nu|=l}} f^{\mu\backslash\nu}f^{\lambda\backslash\nu}.$$

Note that $h^{\lambda}_{\mu}(x)$ is a polynomial, and in particular $h^{\emptyset}_{\emptyset}(x) = 1$. Both in (5) and (6), the first factor depends only on μ_1 and λ_1, and the second factor on μ_2 and λ_2. Since $\hat{a}^{\emptyset}_{\emptyset}(x) = e^{x/2}$, setting $\mu_2 = \lambda_2 = \emptyset$ gives

$$\hat{a}^{\lambda}_{\mu}(x) = e^{-x/2}g^{\lambda}_{\mu}(x)h^{\emptyset}_{\emptyset}(x) = e^{-x/2}g^{\lambda}_{\mu}(x),$$

and setting $\mu_1 = \lambda_1 = \emptyset$ gives yet another formula

$$\hat{a}^{\lambda}_{\mu}(x) = e^{-x/2}g^{\emptyset}_{\emptyset}(x)h^{\lambda}_{\mu}(x) = e^{-x/2+x^2}h^{\lambda}_{\mu}(x),$$

where we used $g^{\emptyset}_{\emptyset}(x) = e^{x^2}$ which follows easily from the definition and $\sum_{|\nu|=l}(f^{\nu})^2 = l!$ (by RSK).

Comparing the above two formulas also yields $g^{\lambda}_{\mu}(x) = e^{x^2}h^{\lambda}_{\mu}(x)$.

4. Sasha Orlov's BKP solutions of hypergeometric type

This class of solutions, studied extensively by Sasha Orlov, may be useful in the study of combinatorics. In [13], C. Tracy and H. Widom studied the asymptotic behavior of

$$\sum_{\lambda\in\mathcal{D},\lambda_1\leq h} Q_{\lambda}(x)P_{\lambda}(y) \tag{$\dagger$}$$

as $h \to \infty$, where $\mathcal{D}$ is the space of all the strict partitions, $\lambda = (\lambda_1 > \cdots > \lambda_l \geq 0)$, and Q_λ and P_λ are the Schur Q-functions ($P_\lambda(x) = 2^{-l(\lambda)}Q_\lambda(x)$). While the limit as $h \to \infty$, i.e.,

$$\sum_{\lambda\in\mathcal{D}} Q_{\lambda}(x)P_{\lambda}(y) = \prod_{i,j\geq 1}\frac{1+x_iy_j}{1-x_iy_j}$$

is a well-known BKP solution via Miwa change of variables, the nature of ($\dagger$) is not immediately obvious. Sasha [8] pointed out that for any fixed h, ($\dagger$) is indeed a special (rational) case of the BKP solutions of hypergeometric type, thus suggesting a possibility to study it by soliton theory (see [7] for more details):

Using neutral fermions: $[\phi_n, \phi_m]_+ = (-)^m \delta_{m,-n}$, let

$$H_B(t) = (1/2) \sum_{l=1,3,5,\ldots} \sum_{n\in\mathbf{Z}} (-)^{n+1} t_l \phi_n \phi_{-n-l},$$

where $t = (t_1, t_3, t_5, \ldots)$. BKP τ-function is given by

$$\tau_B(t) = \langle 0|e^{H_B(t)} g|0\rangle \tag{1}$$

for some $g \in G^+(W_B)$, where the group $G^+(W_B)$ is defined as in DJKM: If $\mathbf{B}$ is the Clifford algebra on $W_B := \bigoplus_{n\in\mathbf{Z}} \mathbf{C}\phi_n$, and if $\mathbf{B}_+$ is the even part of $\mathbf{B}$ (+1-eigenspace of the involution $\phi_n \mapsto -\phi_n$), then

$$G^+(W_B) = \{g \in \mathbf{B}_+ \mid \exists g^{-1},\ g W_B g^{-1} = W_B\}.$$

Here is Sasha's construction: For any function $r : \mathbf{Z} \to \mathbf{C}$ which satisfies the relation $r(1-n) = r(n)$, let

$$B_k = (1/2) \sum_{n=-\infty}^{\infty} (-)^n \phi_n \phi_{k-n} r(n) r(n-1) \cdots r(n-k+1), \quad k = 1, 3, 5, \ldots,$$

which commute with each other, and for $s = (s_1, s_3, s_5, \ldots)$ let

$$B(s) = \sum_{k=1,3,5,\ldots} s_k B_k, \qquad g = e^{-B(s)}.$$

As the exponential of a quadratic element, g clearly belongs to the group $G^+(W_B)$. Substituting this g into (1), Sasha observes

$$\tau_B(t) = \tau_B(t,s) = 1 + \sum_{\lambda:\ \text{strict partition}} 2^{-l(\lambda)} r_\lambda Q_\lambda(t/2) Q_\lambda(s/2) \tag{2}$$

where $r_\lambda = \prod_{i=1}^k (r(1)r(2)\ldots r(\lambda_i))$, and $Q_\lambda(t/2)$ is the notation of Y.You, a student of V.Kac, and equals to $Q_\lambda(x)$ in the standard notation via Miwa change of variables: $mt_m = \sum_{i=1} x_i^m - (-x_i)^m$.

Taking r to be a suitable step function, one gets Tracy-Widom's restricted sums. Moreover, due to the symmetry between t and s, one sees that $\tau_B(t,s)$ satisfies BKP in both sets of variables.

To prove (2), the following identity (3) of Y.You [15], and Sasha's variation on it (formula (4) below) are the keys:

For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0)$ with k even,

$$\langle 0|e^{H_B(t)} \phi_{\lambda_1} \cdots \phi_{\lambda_k}|0\rangle = 2^{-k/2} Q_\lambda(t/2), \tag{3}$$

where again we use You's notation $Q_\lambda(t/2)$ which is related to the standard Q_λ via Miwa change of variables (see my old message). Sasha makes slight variation of this to obtain

$$\langle 0|\phi_{-\lambda_k} \cdots \phi_{-\lambda_1} e^{-B(s)}|0\rangle = 2^{-k/2} r_\lambda Q_\lambda(t/2). \tag{4}$$

5. Elementary approach to the Schottky problem

So far, we have seen various ways of studying rank one (quasi)rational KP solutions: the Calogero-Moser system, Orlov's hypergeometric solutions, as well as the more standard approach utilizing the theory of Grassmannian or Schur functions, all being quite concrete. Let us proceed one step further and discuss a way to make the main part of my work on S.P. Novikov's conjecture on the Schottky problem [11] more concrete: Since we had a clean-cut answer utilizing only one differential equation (i.e., the KP equation), we did not mention the existence of a less clean-cut but much more elementary answer. This partial solution uses no complex analysis on the abelian variety, and still characterizes the Jacobians using only finitely many (but depending on the dimension of the abelian variety) differential equations. Thus it may be of some theoretical interest. It may also be straightforward to work out analogues of it for the trisecant, Toda and other incarnations of the universal grassmann manifold, giving us some insight.

Note also that in the BKP case, the best possible answer, whose proof goes the same way as in the KP case, already leads to a little less clean-cut answer, due to the existence of counterexamples: The Riemann theta function of an irreducible principally polarized abelian variety X satisfies the first equation in the BKP hierarchy if and only if X, as a polarized variety, contains the Prym variety P of a curve with involution fixing one smooth point and another point, which is either smooth or a space-axial ordinary multiple point at which the involution does not interchange branches. Here $X = P$ if and only if the two fixed points are both smooth (see [12] for details). So the conclusion may be explicit from algebro-geometric point of view, but still a little less clean-cut anyway.

So in what follows I present this partial answer to Novikov's conjecture, with some details added. Our starting point is of course:

Theorem 4 (Krichever) *There is a natural bijection between sets of data as follows: Data A. a) C a complete curve over $\mathbb{C}$ (we always assume C is reduced and irreducible),*
b) $p \in C$ a smooth point, and $z \bmod z^3 \in (\mathfrak{m}_p - \mathfrak{m}_p^2)/\mathfrak{m}_p^3$,
c) $\mathcal{F}$ a torsion-free rank 1 sheaf on C such that

$$h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0.$$

Data B. Commutative subrings $R \subset \mathbb{C}[[x]][d/dx]$,[1] with $\mathbb{C} \subset R$ and for any $n \gg 0$ there is $B \in R$ of the form

$$B = \Big(\frac{d}{dx}\Big)^n + c\Big(\frac{d}{dx}\Big)^{n-1} + (\textit{lower order terms}), \quad c \in \mathbb{C}.$$

Let us recall the basic definitions in the KP theory, intended also to provide quick references to the facts and notions used in this report so far.

Let $\Psi = \mathbb{C}[[x]]((\partial^{-1}))$ be the ring of formal pseudodifferential operators in a single variable x, $\partial = d/dx$, and let $D = \mathbb{C}[[x]][\partial]$ and $\Psi^- = \mathbb{C}[[x]][[\partial^{-1}]]\partial^{-1}$ be the subrings of Ψ of differential operators and of pseudodifferential operators of negative order, respectively. For $P \in \Psi$, we denote $P = P_+ + P_-$ corresponding to the decomposition $\Psi = D + \Psi^-$. The KP hierarchy is defined as

$$\Big(\frac{\partial}{\partial t_n}\Big)L = [B_n, L] = [B_n^c, L], \quad n = 1, 2, \dots, \tag{1}$$

[1] If the above condition on h^i are replaced by $h^0(\mathcal{F}) = h^1(\mathcal{F})$, then one should consider regularizable operators à la Sato.

where

$$L\colon \mathbb{C}^\infty = \lim_{\rightarrow} \mathbb{C}^n \longrightarrow \partial + \Psi^- \quad (\text{or } L \in \partial + \Psi^-[[t_1, t_2, \dots]]), \tag{2}$$

$$B_n = (L^n)_+, \quad B_n^c = -(L^n)_-.$$

A solution L to (1) is called finite dimensional (resp. g-dimensional) if its tangent map at the origin

$$dL_0\colon T_0\mathbb{C}^\infty \longrightarrow T_L(\partial + \Psi^-) \simeq \Psi^-$$

is of finite rank (resp. of rank g). Suppose L is a g-dimensional solution to (1). Let K_L be the kernel of dL_0. Then by the commuting flow property, K_L is also the kernel of the tangent map of L at any $t \in \mathbb{C}^\infty$, and the map L is factored by $\mathbb{C}^\infty/K_L \simeq \mathbb{C}^g$ as $L\colon \mathbb{C}^\infty \to \mathbb{C}^\infty/K_L \to \partial + \Psi^-$. Let

$$R_t = \{B \in D \mid [B, L(t)] = 0\}. \tag{3}$$

Then R_t becomes a commutative ring which satisfies the condition of Data B of Krichever's theorem. K_L and B_t are related with each other by

$$\sum_{n\geq 1} c_n \frac{\partial}{\partial t_n} \in K_L \Longleftrightarrow \sum_{n\geq 1} c_n B_n(t) \in R_t, \tag{3$'$}$$

and every $B \in R_t$ is of the form $c_0 + \sum_{n\geq 1} c_n B_n(t)$. Using (3$'$) and the ring structure of R_t, we can characterize the g-dimensional solutions to (1) by looking at only finitely many equations in (1):

Lemma 5 *If $L\colon \mathbb{C}^n \to \partial + \Psi^-$ is a g-dimensional solution to the first n equations in (1) and $n \geq 2g+1$, then L automatically satisfies the whole hierarchy in the following sense: There exists a linear map $P\colon \mathbb{C}^\infty \to \mathbb{C}^n$ such that $L \circ P$ solves (1), and $L \circ P|\mathbb{C}^n = L$. Moreover, this is a unique extension of L to a solution of the whole hierarchy (1).*

We also define:

$$R_t = \{B \in D \mid [B, L(t)] = 0\}. \tag{3}$$

Since every single equation in (1) actually contains infinitely many differential equations on the coefficients of L, this lemma is not sufficient to characterize finite dimensional solutions to (1) in terms of finitely many differential equations. For this purpose, we consider the Zakharov–Shabat equations obtained from (1):

$$\frac{\partial}{\partial t_m} B_n - \frac{\partial}{\partial t_n} B_m = [B_m, B_n]; \quad m, n = 1, 2, \dots. \tag{4}$$

That (1) implies (4) is just the commuting flow property. Conversely, (4) implies (1) as (4) shows $(\partial/\partial t_n)L^m = [B_n, L^m] + (\text{order} \leq n-2)$ so that $(\partial/\partial t_n)L = [B_n, L] + (\text{order} \leq n-m-1)$ for any m. If we consider only finitely many equations in (4), then we have somewhat weaker characterization of finite dimensional solutions to (1):

Lemma 6 *Let us denote by (ZS^N) the subset of equations in (4) for $m, n = 1, \dots, N$. Then*

a) For any solution $B = (B_1, \dots, B_N)$ to (ZS^N), the kernel K of the tangent map $dB(t)\colon \mathbb{C}^N \to D^N$ is independent of t, so that B is factorized by a fixed linear space $\mathbb{C}^N/K = \mathbb{C}^g$ as $\mathbb{C}^N \longrightarrow \mathbb{C}^N/K \xrightarrow{\overline{B}} D^N$, with $d\overline{B}$ being injective.

b) If, moreover, $N \geq 2g+1$, then there exists $L\colon \mathbb{C}^\infty \longrightarrow \partial + \Psi^-$, such that L solves (1), $(L^m)_+|\mathbb{C}^N = B_m$ for $m = 1, \ldots, N$, and $\ker dL = K$. Such L is unique up to equivalence defined as

$$L \underset{N}{\sim} L' \Leftrightarrow L' = \left(L + \sum_{j=N+1}^{\infty} c_j L^{-j+1}\right) \circ T_c \tag{5}$$

for some $c_j \in \mathbb{C}$, where $T_c\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ is a linear map defined as

$$(T_c t)_i = t_i + \sum_{j=N+1}^{\infty} c_{i+j,j} t_{i+j} \,,$$

where

$$1 + \sum_{j=N+1}^{\infty} c_{n,j} x^j = \left(1 + \sum_{j=N+1}^{\infty} c_j x^j\right)^n .$$

Let L be a solution to (1). A function of the form

$$w = \left(\sum_{-\infty < n \ll \infty} a_n(x) k^n\right) e^{kx}, \quad a_n(x) \in \mathbb{C}[[x]]$$

is called a wave function associated to L if it satisfies

$$\begin{aligned} Lw &= kw \\ (\partial/\partial t_n) w &= B_n w, \quad n = 1, 2, \ldots . \end{aligned} \tag{6}$$

From (6) we observe that $w = Se^{kx+\xi(t,k)}$, $S \in \Psi$, is a wave function associated to L if and only if S satisfies

$$\begin{aligned} LS &= S\partial \\ (\partial/\partial t_n) S &= B_n^c S, \quad n = 1, 2, \ldots . \end{aligned} \tag{6'}$$

If $S \neq 0$ satisfies (6′), then $S = c\partial^n +$ (lower order terms), and $w = Se^{kx+\xi} = ck^n(1 + o(1))e^{kx+\xi}, 0 \neq c \in \mathbb{C}$. Noting that equations still hold when we multiply w by $c^{-1}k^{-n}$ and S from the right by $c^{-1}\partial^{-n}$, we restrict ourselves to $S = 1 +$ (lower order terms). (6) or (6′) imply (1), namely, if $S \in 1 + \Psi^-$ satisfies $(\partial/plt_n)S = B_n^c S$ for $B_n^c = -(S\partial^n S^{-1})_-$, then $L = S\partial S^{-1}$ is a solution of (1), and $w = Se^{kx+\xi}$ is the associated wave function. In (6) or (6′), like in (1), each equation contains infinitely many differential equations for the coefficients of S. However, choosing a finite number of equations from them, we can characterize finite dimensional solutions to (1) via Zakharov–Shabat equations:

Lemma 7 *Let $S\colon \mathbb{C}^N \to 1 + \Psi^-$. Suppose S satisfies*

$$(\partial/\partial t_n) S - B_n^c S \in \Psi(-(N+2-n)), \quad n = 1, \ldots, N, \tag{7}$$

where $B_n^c = -(S\partial^n S^{-1})_-$. Then $B_n = (S\partial^n S^{-1})_+$, $n = 1, \ldots, N$, satisfy (ZS^N). So that if the tangent map of $L = S\partial S^{-1}$ is of rank g, $N \geq 2g+1$, then by Lemma 6 b), there exists a g-dimensional solution $L'\colon \mathbb{C}^\infty \to \partial + \Psi^-$ to (1) such that $L'|\mathbb{C}^N \equiv L \bmod \Psi(-N)$.

The first part of the lemma is shown by using the identity

$$S^{-1}[\partial_m - B_m, \partial_n - B_n]S = [\partial_m - \partial^m, S_n] - [\partial_n - \partial^n, S_m] + [S_m, S_n], \quad m, n = 1, \dots, N,$$

where $\partial_n = \partial/\partial t_n$ and $S_n = S^{-1}(\partial_n(S) - B_n^c S)$.

Noting that equations for $n = 1$ in (1), (6) or (6′) just require that L, w or S be of the form $L(x,t) = L(x+t_1, t_2, t_3, \dots)$ etc., in what follows we confuse x, t_1 and $x+t_1$, and denote $w = Se^{\xi}$ instead of $Se^{kx+\xi(t,k)}$, etc. Let $w = Se^{\xi}$ be a wave function for the KP hierarchy (1), i.e., a solution to (6). Then by integrability condition for (6), there exists a function $\tau\colon \mathbb{C}^\infty \to \mathbb{C}$, called the τ-function associated to L, such that

$$\begin{aligned} e^{-\xi}w &= \text{the full symbol of } S \\ &= \frac{\tau(t_1 - 1/k, t_2 - 1/(2k^2), t_3 - 1/(3k^3), \dots)}{\tau(t_1, t_2, t_3, \dots)}. \end{aligned} \tag{8}$$

See [1] for the proof of (8) and other properties of τ. Given a solution L to (1), w, S and τ are determined up to multiplication by a function of k, multiplication from the right by a constant coefficient pseudodifferential operator, and multiplication by $e^{(\text{linear function of } t)}$, respectively. Each coefficient of L is expressed by a constant coefficient polynomial of second or higher derivatives of $\log\tau$, and every second (or higher) derivative of $\log\tau$ is a constant coefficient differential polynomial of coefficients of L. More precisely, denoting

$$L = \partial + \sum_{j=2}^{\infty} u_j \partial^{-j+1}$$

and denoting the weight of derivatives of u_j as

$$\mathrm{wt}(\partial^\alpha u_j) = \sum_n n\alpha_n + j,$$

we observe that $\partial_i\partial_j \log\tau$ is a weight-homogeneous differential polynomial of u_n of weight $i+j$. In particular, $\partial_i\partial_j \log\tau$ for $i+j \le N$ are determined by $L \bmod \Psi(-N)$, hence by the approximate wave equations (7).

Bilinear identities For $S = 1 + \sum_{n=1}^{\infty} s_n\partial^{-n} \in 1 + \Psi^-[[t]]$, and $w = Se^{\xi}$, we let $S^* = 1 + \sum_{n=1}^{\infty}(-\partial)^{-n} \circ s_n$, formal adjoint of S, and $w^* = S^*e^{-\xi}$. If w is a wave function for the KP hierarchy and satisfies (6), then w^* is called the adjoint wave function and satisfies

$$\begin{aligned} L^*w^* &= kw^* \\ \partial_n w^* &= -B_n^* w^*, \quad n = 1, 2, \dots, \end{aligned}$$

where L^* and B_n^* are the formal adjoints of L and B_n, respectively, defined similarly to S^*. If τ is a τ-function for the KP hierarchy, i.e., if we have (8) for a wave function w, then its adjoint wave function is given by

$$\frac{\tau(t_1 + 1/k, t_2 + 1/(2k^2), t_3 + 1/(3k^3), \dots)}{\tau(t_1, t_2, t_3, \dots)} e^{-\xi},$$

i.e.,

$$\text{the full symbol of } S^* = \frac{\tau(t_1 + 1/(-k), t_2 + 1/(2(-k)^2), t_3 + 1/(3(-k)^3), \dots)}{\tau(t_1, t_2, t_3, \dots)}.$$

See [1]. Now we have the following,

Lemma 8 ([1]) *Formal oscillating functions $w = Pe^{\xi}$ and $w^* = Qe^{-\xi}$, where $P, Q \in 1 + \Psi^-[[t]]$, are a wave function for the KP hierarchy and its adjoint wave function, if and only if they satisfy*

$$\mathrm{Res}_{k=\infty}\, w(t,k)w^*(t',k)dk = 0 \quad \textit{for any} \quad t, t' . \tag{9}$$

A function $0 \neq \tau \in \mathbb{C}[[t]]$ is a τ-function for the KP hierarchy if and only if it satisfies

$$\mathrm{Res}_{k=\infty}\, \tau(t_1 - \frac{1}{k}, t_2 - \frac{1}{2k^2}, \dots)\tau(t_1' + \frac{1}{k}, t_2' + \frac{1}{2k^2}, \dots)e^{\xi(t-t',k)}dk = 0 \quad \textit{for any} \quad t, t' . \tag{9'}$$

This lemma neatly characterizes the wave functions and τ-functions for the KP hierarchy, and (9′) gives infinitely many Hirota bilinear equations for τ-functions. Indeed, if we substitute $t - y$ for t and $t + y$ for t' in (9′) and expand it into Taylor series in y, we get

$$\sum_{l=0}^{\infty} p_l(-2y)p_{l+1}(\tilde{D})e^{\sum_{j=1}^{\infty} y_j D_j}\tau \cdot \tau = 0 , \tag{9''}$$

where D denotes the Hirota bilinear operator, i.e., for $P(X) \in \mathbb{C}[X_1, X_2, \dots]$

$$P(D) \colon \mathbb{C}[[t]] \otimes \mathbb{C}[[t]] \to \mathbb{C}[[t]]$$

is given by

$$P(D)f \cdot f = P(D)f \otimes f = P(\partial_y)f(t+y)f(t-y)|_{y=0} ;$$

and $p_n(X) \in \mathbb{Q}[X_1, X_2, \dots]$ are defined as

$$\sum_{n=1}^{\infty} p_n(X)k^n = e^{\sum_{m=1}^{\infty} X_m k^m} ,$$

so that they are weight-homogeneous polynomials of weight n if we define $\mathrm{wt}(X^\alpha) = \sum n\alpha_n$; and $\tilde{D} = (D_1, (1/2)D_2, (1/3)D_3, \dots)$. For each α the coefficient of y^α in (9″) gives a Hirota bilinear equation for τ homogeneous of weight $\sum n\alpha_n + 1$, counting $\mathrm{wt}(D^\beta) = \sum n\beta_n$. [9] carries a table of those equations for weight up to 11 (complete list up to 9), with interesting observations on the number of Hirota equations for τ of given weight, which are then proved by Date et al in [1]. Equations of the form (9), (9′) or (9″) are called bilinear identities in [1]. Now we pick up finitely many Hirota equations for τ which characterize finite dimensional solutions to the KP hierarchy.

Lemma 9 *a) Formal oscillating functions $w = Pe^{\xi}$ and $w^* = Qe^{-\xi}$, with P and Q as above, are a wave function for the KP hierarchy and its adjoint wave function, if and only if they satisfy*

$$\mathrm{Res}_{k=\infty}(\partial^m w(t,k))w^*(t,k)dk = 0 \quad \textit{for any} \quad t , \tag{10_m}$$

for any $m \geq 0$, and

$$\mathrm{Res}_{k=\infty}(\partial^m \partial_n w(t,k))w^*(t,k)dk = 0 \quad \textit{for any} \quad t , \tag{$10_{m,n}$}$$

for any $m \geq 0$, $n \geq 2$. If we only assume (10_m) for $m = 0, \dots, N$ and ($10_{m,n}$) for $m \geq 0$, $n \geq 2$, $m + n \leq N$, then we have (7) for S given by (8), $B_n^c = -(S\partial^n S^{-1})_-$.

b) A function $0 \neq \tau \in \mathbb{C}[[t]]$ is a τ-function for the KP hierarchy if and only if it satisfies

$$\sum_{l=0}^{m} \frac{(-2)^l}{l!(m-l)!} p_{l+1}(\tilde{D})D_1^{m-l}\tau \cdot \tau = 0 \tag{$10_m'$}$$

for any $m \geq 0$, and

$$\sum_{l=0}^{m} \frac{(-2)^l}{l!(m-l)!}\Big(-2p_{l+n+1}(\tilde{D}) + p_{l+1}(\tilde{D})D_n\Big)D_1^{m-l}\tau \cdot \tau = 0 \qquad (10'_{m,n})$$

for any $m \geq 0$, $n \geq 2$. If we only assume $(10'_m)$ for $m = 0, \dots, N$ and $(10'_{m,n})$ for $m \geq 0$, $n \geq 2$, $m + n \leq N$, then we have (7) for S given by (8), $B_n^c = -(S\partial^n S^{-1})_-$.

Note that (7) characterizes g-dimensional solutions to the KP hierarchy if $N \geq 2g+1$ (Lemma 7). Proof of Lemma 9 a) is the same as one part of Lemma 8, shown in [1]. $(10'_m)$ and $(10'_{m,n})$ are the coefficients of y_1^m and $y_1^m y_n$ in $(9'')$, respectively. So that Lemma 9 b) follows from Lemma 9 a).

Let us rewrite $(10'_m)$ and $(10'_{m,n})$ for $\tau = e^Q\theta$, where $\theta = \theta(t) \in \mathbb{C}[[t]]$ and $Q = Q(t) = \sum_{i,j=2}^{\infty} Q_{i,j}t_it_j$, $Q_{i,j} = Q_{j,i} \in \mathbb{C}$, a quadratic form on $t_2, t_3, \dots$, for later use. The result is the following:

$$\sum_{j=0}^{m+1} C_j F_{j-1,m} = 0\,, \qquad (10''_m)$$

$$\sum_{j=0}^{m+1}\sum_{k=2}^{j} C_{j-k}\frac{4Q_{k,n}}{k}F_{j-1,m} - 2\sum_{j=0}^{m+n+1} C_j F_{j-n-1,m} + \sum_{j=1}^{m+1} C_j F_{j-1,m,n} = 0\,, \qquad (10''_{m,n})$$

where

$$C_j = p_j(\overline{Q}), \quad \overline{Q} = (0,0,0,\overline{Q}_4,\overline{Q}_5,\dots)\,, \quad \overline{Q}_n = 2\sum_{k=2}^{n-2}\frac{Q_{k,n-k}}{k(n-k)}\,,$$

$$F_{k,m} = \sum_{l=k}^{m}\frac{(-2)^l}{l!(m-l)!}p_{l-k}(\tilde{D})D_1^{m-l}\theta \cdot \theta\,,$$

$$F_{k,m,n} = \sum_{l=k}^{m}\frac{(-2)^l}{l!(m-l)!}p_{l-k}(\tilde{D})D_1^{m-l}D_n\theta \cdot \theta\,.$$

Here we set $1/l! = 0$ for $l < 0$.

Baker–Akhiezer functions Let $(C, p, \mathcal{F})$ be a triple of Data A in Krichever's theorem. Let U be a small complex neighborhood of p, and let $z\colon U \to \mathbb{C}$ (holomorphic) be such that $z \bmod z^3 \in \mathfrak{m}_p/\mathfrak{m}_p^3$ is chosen in b) of Data A. We construct a deformation of the sheaf $\mathcal{F}$ to a sheaf $\mathcal{F}^*$ over $C \times \mathbb{C}[[t]]$ similar way to [5]: Let $\mathcal{F}^* = \mathcal{F} \otimes \mathcal{O}_{\mathbb{C}^\infty}$ on $U \times \mathbb{C}^\infty$ and on $(C - p) \times \mathbb{C}^\infty$, but glue $\mathcal{F}^*$ to itself on $(U - p) \times \mathbb{C}^\infty$ by the transition function $e^{\xi(t,1/z)}$. We observe that $H^0(C \times \mathbb{C}[[t]], \mathcal{F}^*(p)) \simeq \mathbb{C}[[t]]$. Let s be a generator of this $\mathbb{C}[[t]]$-module such that on $U \times \mathbb{C}^\infty$, $s = (1/z)(1 + O(z))$ near $p \times \mathbb{C}^\infty$. Looking at s on $(C-p) \times \mathbb{C}^\infty$, we call $s_0^{-1}s\colon (C - p) \times \mathbb{C}^\infty \to \mathbb{C}$ (meromorphic) the Baker-Akhiezer function. Here $s_0 = s_0' \otimes 1$ and s_0' is a nontrivial section of $\mathcal{F}(p)$, e.g., $s_0' = s|_{t=0}$. We have

Lemma 10 (Krichever [4]) *If ψ is a Baker-Akhiezer function and*

$$\psi = \psi(t,z) = (1 + s_1(t)z + s_2(t)z^2 + \cdots)e^{\xi(t,1/z)}$$

near p, then

$$w = \psi(t, 1/k) = (1 + s_1(t)k^{-1} + s_2(t)k^{-2} + \cdots)e^{\xi(t,k)} = Se^{\xi(t,k)}$$

is a wave function for the KP hierarchy, so that $L = S\partial S^{-1}$ is a solution to the KP hierarchy. Moreover, for this S we have

$$R_t = S(t)RS(t)^{-1},$$

where R_t is the ring defined by (3), and

$$R = \{f(\partial^{-1}) \in \mathbb{C}((\partial^{-1})) \mid f(z) \in \Gamma(C - p, \mathcal{O}_C)\}.$$

b) If C is a smooth curve of genus $g \geq 1$, then the Baker-Akhiezer function is given by the formula

$$\psi = \exp\Big(\sum_{n=1}^{\infty} t_n \int^q \omega(n)\Big)\theta(w(q) - At + c)/\theta(w(q) + c), \quad q \in C, \tag{11}$$

where θ is the Riemann theta function on C, $w\colon C \to \mathrm{Jac}(C)$ the Abel map defined as

$$w(q) = \Big(\int^q \omega_1, \ldots, \int^q \omega_g\Big) \text{ modulo periods},$$

$(\omega_1, \ldots, \omega_g)$ normalized basis of abelian differentials of the first kind, $A = (a_{i,j})\colon \mathbb{C}^\infty \to \mathbb{C}^g$ such that

$$\omega_i = (a_{i1} + a_{i2}z + a_{i3}z^2 + \cdots)dz,$$

$\omega(n)$ normalized abelian differentials of the second kind with a unique pole at p of the form $d(z^{-n}) + O(1)$, and c general point of $\mathbb{C}^g$.

Define $Q_{ij} \in \mathbb{C}$, $i, j = 1, 2, \ldots$, as

$$\int^q \omega(i) = z^{-i} - \sum_{j=1}^{\infty} 2q_{i,j}z^j/j + \text{const}.$$

Then $Q_{ij} = Q_{ji}$. Since the wave function for a solution to the KP hierarchy may be changed by multiplication by a function in k and $k = 1/z$, (11) shows

Corollary to Lemma 10 *If C is a smooth curve of genus $g \geq 1$, then the τ-function for the solution to the KP hierarchy corresponding to C is given by*

$$\begin{aligned} \tau(t) &= e^{Q(t)}\theta(At - c) \\ Q(t) &= \sum_{i,j=1}^{\infty} Q_{ij}t_it_j, \end{aligned} \tag{11'}$$

with A and c as above.

By an appropriate change of local coordinate z around p for which z mod z^3 still gives an element of $\mathfrak{m}_p/\mathfrak{m}_p^3$ chosen in Data A, we may always assume

$$Q_{i1} = Q_{1i} = 0.$$

This only changes L to equivalent one defined in (5), in this case $N = 1$. Then by $(10''_{m,n})$, Lemma 7, Lemma 9 etc., we have

Theorem 11 *Given a positive integer g, $(10''_m)$ for $m = 1, \dots, 2g+1$ and $(10''_{m,n})$ for $m \geq 0$, $n \geq 2$, $m+n \leq 2g+1$ — we shall call the set of those equations (A) — characterize the Jacobian varieties of genus g as follows: a genus g Riemann theta function is associated to a Jacobian variety of a complete smooth curve if and only if there exist $Q_{ij} \in \mathbb{C}$, $i, j \geq 2$, $i+j \leq 2g+1$, $Q_{ij} = Q_{ji}$, and $a_{ij} \in \mathbb{C}$, $i = 1, \dots, g$, $j = 1, \dots, 2g+1$, so that (A) is satisfied by these Q's, and θ substituted by $\theta \circ A$, where A is the linear map defined by matrix (a_{ij}).*

Algebraic equations and elimination of parameters Applying $(10''_m)$ and $(10''_{m,n})$ to $\theta(A\cdot)$ and Fourier expanding, we get finite number of algebraic equations on theta constants and derivatives of theta at two division points. Starting with the definition of theta

$$\theta(z) = \sum_{p \in \mathbb{Z}^g} \exp(2\pi i p \cdot z + \pi i p \cdot Tp), \quad T \in \mathfrak{H}_g,$$

and

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, T) = \sum_{q \in \mathbb{Z}^g + a} \exp(2\pi i q \cdot (z+b) + \pi i q \cdot Tq), \quad a, b \in \mathbb{Q}^g, \quad T \in \mathfrak{H}_g,$$

and substituting $\theta(A\cdot)$ for θ in the definition of $F_{k,m}$, we have

$$F_{k,m} = \sum_{\delta \in (\mathbb{Z}/2\mathbb{Z})^g} F^\delta_{k,m} \theta\begin{bmatrix} \delta/2 \\ 0 \end{bmatrix}(2At, 2T),$$

$$F_{k,m,n} = \sum_{\delta \in (\mathbb{Z}/2\mathbb{Z})^g} F^\delta_{k,m,n} \theta\begin{bmatrix} \delta/2 \\ 0 \end{bmatrix}(2At, 2T),$$

where

$$F^\delta_{k,m} = \sum_{l=k}^{m} \frac{(-2)^l}{(m-l)!l!} \sum_{q \in \mathbb{Z}^g + \delta/2} (4\pi i q \cdot a_1)^{m-l} p_{l-k,q} \exp(2\pi i q \cdot Tq),$$

$$F^\delta_{k,m,n} = \sum_{l=k}^{m} \frac{(-2)^l}{(m-l)!l!} \sum_{q \in \mathbb{Z}^g + \delta/2} (4\pi i q \cdot a_1)^{m-l} (4\pi i q \cdot a_n) p_{l-k,q} \exp(2\pi i q \cdot Tq),$$

with

$$p_{k,q} = \frac{1}{k!} \partial^k_\lambda \exp\left(\sum_{j=1}^{\infty} \lambda^j 4\pi i q \cdot a_j / j \right) \bigg|_{\lambda=0} = p_k(a(q)),$$

$$a(q) = (a(q)_1, a(q)_2, \dots), \quad a(q)_n = 4\pi i q \cdot a_n / n.$$

So that $(10''_m)$ and $(10''_{m,n})$ are equivalent to

$$\sum_{j=0}^{m+1} C_j F^\delta_{j-1,m} = 0 \quad \text{for all } \delta \in (\mathbb{Z}/2\mathbb{Z})^g, \tag{12_m}$$

and

$$\sum_{j=0}^{m+1} \sum_{k=2}^{j} C_{j-k} \frac{4Q_{k,n}}{k} F^\delta_{j-1,m} - 2 \sum_{j=0}^{m+n+1} C_j F^\delta_{j-n-1,m} + \sum_{j=1}^{m+1} C_j F^\delta_{j-1,m,n} = 0 \quad \text{for all } \delta \in (\mathbb{Z}/2\mathbb{Z})^g, \tag{$12_{m,n}$}$$

respectively. As before, (12_m) for $m = 0, \dots, N$ and $(12_{m,n})$ for $m \geq 0$, $n \geq 2$, $m+n \leq N$ (we will call the set of those equations (A′) if $N = 2g+1$), together imply (7), so if $N = 2g+1$, they characterize the Jacobian varieties of genus g as in Theorem 11.

$F^\delta_{k,m}$ and $F^\delta_{k,m,n}$ are linear combinations of derivatives of theta at two division points, whose coefficients are weight homogeneous polynomials of a_{ij} of weight $m-k$ and $m+n-k$, respectively, where we define

$$\mathrm{wt}(a_{ij}) = j.$$

Therefore, if we also define

$$\mathrm{wt}(Q_{ij}) = i + j,$$

then the left hand sides of (12_m) and $(12_{m,n})$ are linear combinations of derivatives of theta at two division points, with coefficients being weight homogeneous polynomials of weight $m+1$ and $m+n+1$, respectively. Therefore, at least in principle, we can eliminate the unknown parameters a_{ij} and Q_{ij} from (A′) to get algebraic equations which characterize the Jacobian varieties of genus g, and the resulting equations should give a vector-valued modular form [6, p. 229] since the zero set of those equations should be independent under the action of $Sp(2g, \mathbb{Z})$. As the first step, we eliminate Q_{ij} from (A′).

First we assume

$$\theta\begin{bmatrix}\delta/2\\ 0\end{bmatrix}(0, 2T) \neq 0 \tag{13}$$

for simplicity. This is true for some $\delta \in (\mathbb{Z}/2\mathbb{Z})^g$, since

$$\theta(z)^2 = \sum_{\delta\in(\mathbb{Z}/2\mathbb{Z})^g} \theta\begin{bmatrix}\delta/2\\ 0\end{bmatrix}(0, 2T)\theta\begin{bmatrix}\delta/2\\ 0\end{bmatrix}(2z, 2T) \not\equiv 0.$$

Then by (12_m) and $C_0 = 1$, we have

$$C_j = (-1)^j \left(F^\delta_{0,0}F^\delta_{1,1}\cdots F^\delta_{j-1,j-1}\right)^{-1} \det G^\delta_j, \quad j = 1, 2, \dots, 2g+2, \tag{14}$$

where

$$G^\delta_j = \begin{bmatrix} F^\delta_{-1,0} & F^\delta_{-1,1} & \cdots & F^\delta_{-1,j-2} & F^\delta_{-1,j-1} \\ F^\delta_{0,0} & F^\delta_{0,1} & \cdots & F^\delta_{0,j-2} & F^\delta_{0,j-1} \\ 0 & F^\delta_{1,1} & \cdots & F^\delta_{1,j-2} & F^\delta_{1,j-1} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & F^\delta_{j-2,j-2} & F^\delta_{j-2,j-1} \end{bmatrix}.$$

Note that $F^\delta_{k,k} = \dfrac{(-2)^k}{k!}\theta\begin{bmatrix}\delta/2\\ 0\end{bmatrix}(0, 2T) \neq 0$ by (13). Formula (14) is obtained as follows:

$$\begin{aligned} &(12_m) \text{ for } m = 0, 1, \dots, N, \text{ fixed } \delta \\ &\Leftrightarrow (C_0, \dots, C_{N+1})H^\delta_N = (1, 0, 0, \dots, 0) \\ &\Leftrightarrow (C_0, \dots, C_{N+1}) = (1, 0, 0, \dots, 0)(H^\delta_N)^{-1}, \end{aligned}$$

where

$$H^\delta_N = \begin{bmatrix} 1 & F^\delta_{-1,0} & F^\delta_{-1,1} & \cdots & F^\delta_{-1,N} \\ 0 & F^\delta_{0,0} & F^\delta_{0,1} & \cdots & F^\delta_{0,N} \\ 0 & 0 & F^\delta_{1,1} & \cdots & F^\delta_{1,N} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & F^\delta_{N,N} \end{bmatrix}.$$

Computing $(1,0,0,\dots,0)(H_N^\delta)^{-1}$ (= the first row of $(H_N^\delta)^{-1}$) by using Cramer's rule, we get (14). We observe $F^\delta_{k-1,k}=0$ for $k=0,\,1,\,\dots$; $F^\delta_{-1,1}=F^\delta_{-1,2}=0$. So the condition

$$C_1=C_2=C_3=0$$

is automatically satisfied. Since C_j should be independent of δ, $F^\delta_{k,m}$ must satisfy

$$\sum_{j=0}^{m+1}(-1)^j F^\delta_{j,j}F^\delta_{j+1,j+1}\cdots F^\delta_{m,m}\det G_j^\delta F^\rho_{j-1,m}=0,$$

or equivalently

$$\det\begin{bmatrix} F^\rho_{-1,m} & F^\delta_{-1,0} & F^\delta_{-1,1} & \cdots & F^\delta_{-1,m} \\ F^\rho_{0,m} & F^\delta_{0,0} & F^\delta_{0,1} & \cdots & F^\delta_{0,m} \\ F^\rho_{1,m} & 0 & F^\delta_{1,1} & \cdots & F^\delta_{1,m} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ F^\rho_{m,m} & 0 & \cdots & 0 & F^\delta_{m,m}\end{bmatrix}=0, \tag{21}$$

for any δ, $\rho\in(\mathbb{Z}/2Z)^g$, $m=4,\,\dots\,,\,2g+1$. This is a necessary and sufficient condition that C_j do not depend on δ. Assuming (13) again and substituting (14) for C_j in $(12_{m,n})$, we get systems of linear equations for Q_{ij}. Solving them and equating the answers, or requiring $Q_{ij}=Q_{ji}$, $p_j(\bar{Q})=C_j$, we get algebraic equations for the derivatives of theta at two division points which do not involve Q_{ij}, although they still involve a_{ij}.

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