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<th>Title</th>
<th>On the homotopy types of the groups of equivariant diffeomorphisms</th>
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<td>Citation</td>
<td>Kyoto University (京都大学)</td>
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<td>Issue Date</td>
<td>1979-11-24</td>
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<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.r3994">https://doi.org/10.14989/doctor.r3994</a></td>
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<td>author</td>
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Kyoto University
学位申請論文

阿印孝順
On the homotopy types of the groups of equivariant diffeomorphisms

By Kojun Abe

§0. Introduction

The purpose of this paper is to study the homotopy type of the group of the equivariant diffeomorphisms of a closed connected smooth G-manifold M, when G is a compact Lie group and the orbit space M/G is homeomorphic to a unit interval [0,1].

Let $\text{Diff}_G^\infty(M)_0$ denote the group of equivariant $C^\infty$ diffeomorphisms of the G-manifold M which are G-isotopic to the identity, endowed with $C^\infty$ topology. If M/G is homeomorphic to [0,1], then M has two or three orbit types $G/H$, $G/K_0$ and $G/K_1$. We can choose the isotropy subgroups $H$, $K_0$, $K_1$ satisfying $H \subset K_0 \cap K_1$. Moreover the G-manifold structure of M is determined by an element $\eta$ of a factor group $N(H)/H$, where $N(H)$ is the normalizer of $H$ in G (see §1). Let $\Omega(N(H)/H; (N(H) \cap N(K_0))/H, (N(H) \cap N(\eta K_1 \eta^{-1}))/H)_0$ denote the connected component of the identity of the space of paths $a: [0,1] \to N(H)/H$ satisfying $a(0) \in (N(H) \cap N(K_0))/H$ and $a(1) \in (N(H) \cap N(\eta K_1 \eta^{-1}))/H$.

Theorem. $\text{Diff}_G^\infty(M)_0$ has the same homotopy type as the path space $\Omega(N(H)/H; (N(H) \cap N(K_0))/H, (N(H) \cap N(\eta K_1 \eta^{-1}))/H)_0$.

The paper is organized as follows. In §1, we study the G-manifold structure of M and give a differentiable structure of M/G such that the functional structure of M/G is induced from that of M. This differentiable structure is important to study the structure of $\text{Diff}_G^\infty(M)_0$. In §2, we define a group homomorphism $\Phi: \text{Diff}_G^\infty(M)_0 \to \text{Diff}_C^\infty([0,1])_0$ and prove that $\Phi$ is a continuous homomorphism between topological groups. In §3, we prove that there exists a global
continuous section of \( P \) and \( \text{Ker} P \) is a deformation retract of \( \text{Diff}^\omega_G(M)_0 \).

In §4, we study the group structure of \( \text{Ker} P \). In §5 and §6, we prove our Theorem.

In this paper we assume that all manifolds and all actions are differentiable of class $C^\infty$.

In this section we study the G-manifold structure of $M$. First we see that it is sufficient for us to consider $\eta=1$ (see Lemma 1.1). Next we give a differentiable structure of $M/G$ such that the functional structure of $M/G$ is induced from that of $M$ (see Lemma 1.2).

Let $M$ be a closed connected smooth G-manifold such that $M/G$ is homeomorphic to $[0,1]$. We denote this homeomorphism by $f$. Let $\pi: M \to M/G$ be the natural projection. Put $M_0 = (f \circ \pi)^{-1}([0,1/2])$ and $M_1 = (f \circ \pi)^{-1}([1/2,1])$. Let $x_i$ be a point of $M$ with $f(\pi(x_i)) = i$ for $i=0,1$. Then $M_i$ is a G-invariant closed tubular neighborhood of the orbit $G(x_i)$ (cf. G. Bredon [3, Chapter VI, §6]). Moreover $M$ is equivariantly diffeomorphic to a union of the G-manifolds $M_0$ and $M_1$ such that their boundaries are identified under a G-diffeomorphism $\eta': \partial M_0 \to \partial M_1$. Let $V_i$ be a normal vector space of $G(x_i)$ at $x_i$ and $K_i$ be the isotropy subgroup of $x_i$ for $i=0,1$. Then $V_i$ is a representation space of $K_i$. From the differentiable slice theorem, $M_i$ is equivariantly diffeomorphic to a smooth G-bundle $G\times_{K_i} D(V_i)$ which is associated to the principal $K_i$ bundle $\eta_i: G \to G/K_i$, where $D(V_i)$ is a unit disc in $V_i$.

Let $H$ be a principal isotropy subgroup of the G-manifold $M$. We can assume that $H$ is a subgroup of $K_0\cap K_1$. Let $e_i \in S(V_i)$ be a point such that the isotropy subgroup of $e_i$ is $H$ for $i=0,1$, where $S(V_i)$ is a unit sphere in $V_i$. There exists a G-diffeomorphism $h_i: G/H \to G\times_{K_i} S(V_i)$ given by $h_i(gH) = [g,e_i]$, $i=0,1$. Then we have a G-diffeomorphism

$$h_0': G/H \to G\times_{K_0} S(V_0) = \partial M_0 \xrightarrow{\eta'} \partial M_1 = G\times_{K_1} S(V_1) \xrightarrow{h_1^{-1}} G/H.$$

Since any G-map $G/H \to G/H$ is given by a right translation of an element
of $N(H)/H$, $\eta''$ defines an element $\eta \in N(H)/H$.

Put $x'_i = \eta \cdot x_i$. Then the isotropy subgroup $K'_i$ of $x'_i$ is $\eta K_i \eta^{-1}$.

Let $V'_i$ be a normal vector space of the orbit $G(x'_i) = G(x_i)$ at $x'_i$. Put $e'_i = (dn)_{x'_i} (e_i) \in S(V'_i)$. There exists a $G$-diffeomorphism $u: G_{x'_i} D(V'_i) \to G_{K'_i} D(V'_i)$ given by $u([g,v]) = [g^{-1}, \eta \cdot v]$. Then $(u \circ \eta')(\eta, e'_0) = u([g \eta, e'_0]) = [g, e'_1]$ for $[g, v] \in G_{K'_0} S(V'_0)$. Therefore $M$ is equivariantly diffeomorphic to a union of the $G$-bundles $G_{K'_0} D(V'_0)$ and $G_{K'_1} D(V'_1)$ such that their boundaries are identified under a $G$-diffeomorphism $u \circ \eta'$. Now we have:

Lemma 1.1. Let $M$ be a closed connected smooth $G$-manifold such that $M/G$ is homeomorphic to $[0,1]$. Then $M$ has two or three orbit types $G/H$, $G/K_0$ and $G/K_1$ with $H \subseteq K_0 \cap K_1$, and there exist representation spaces $V_i$, $i = 0,1$, of $K_i$ such that $M$ is equivariantly diffeomorphic to a union of $G$-bundles $G_{K'_0} D(V'_0)$ and $G_{K'_1} D(V'_1)$ with their boundaries identified under a $G$-diffeomorphism $\eta: G_{K'_0} S(V'_0) \to G_{K'_1} S(V'_1)$.

Moreover we may assume that $\eta([g, e_0]) = [g, e'_1]$, where $e'_1$ is a point of $S(V'_1)$ such that the isotropy subgroup of $e'_1$ is $H$ for $i = 0,1$.

Hereafter we shall assume that $M$ is a $G$-manifold as in Lemma 1.1.

Let $\xi: [0,1] \to \mathbb{R}$ be a smooth function such that

\[ \xi(r) = r^2 \quad \text{for} \quad 0 \leq r \leq 1/2, \]
\[ \xi'(r) > 0 \quad \text{for} \quad 0 < r < 1 \quad \text{and} \]
\[ \xi(r) = r - 1/2 \quad \text{for} \quad 7/8 < r \leq 1. \]

Let $\theta: M = G_{K'_0} D(V'_0) \cup G_{K'_1} D(V'_1) \to [0,1]$ be a map given by

\[ \theta([g,v]) = \xi(||v||) \quad \text{for} \quad [g,v] \in G_{K'_0} D(V'_0), \]
\[ \theta([g,v]) = 1 - \xi(||v||) \quad \text{for} \quad [g,v] \in G_{K'_1} D(V'_1). \]

Since $\theta$ is a $G$-map, there exists a map $\phi: M/G \to [0,1]$ such that $\phi \circ \eta = \theta$. It is easy to see that $\phi$ is a homeomorphism. We give a differentiable structure of $M/G$ by $\phi$.

Lemma 1.2. In the above situation, we have
(1) \( \theta \) is a smooth map,

(2) there exists a G-diffeomorphism \( \alpha: \Theta^{-1}((0,1)) \to G/H \times (0,1) \) such that \( \theta \circ \alpha^{-1} \) is the projection on the second factor, and

(3) \( f: M/G \to R \) is smooth if and only if \( f \circ \pi: M \to R \) is smooth.

Proof. (1) Let \( \alpha_i: G \times D(V_i) \to G/H \times (0,1) \) be a map given by \( \alpha_i([g, x_i]) = (gH, r) \) for \( g \in G \) and \( r \in (0,1) \) (\( i = 0, 1 \)). Then it is easy to see that \( \alpha_i \) is a G-diffeomorphism. Since \( \alpha_0 \circ \alpha_1 = \alpha_0 \) on \( G \times D(V_0) \), the composition \( \beta: \Theta^{-1}((0,1)) = G \times D(V_0) \cup G \times D(V_1) \) \( \xrightarrow{\alpha_0 \circ \alpha_1} G/H \times (0,1) \cup G/H \times (0,1) = G/H \times (0,2) \) is a G-diffeomorphism. Note that

\( (\theta \circ \beta^{-1})(gH, r) = \begin{cases} \xi(r) & \text{for } 0 < r \leq 1, \\ 1 - \xi(2-r) & \text{for } 1 < r < 2. \end{cases} \)

Thus \( \theta \circ \beta^{-1} \) is a smooth map, and \( \theta \) is a map on \( \Theta^{-1}((0,1)) \). From the definition, \( \theta \) is a smooth map on \( \Theta^{-1}(r) \) for \( r \neq 1/2 \). Therefore \( \theta \) is a smooth map.

(2) Let \( \bar{\theta}: (0,2) \to (0,1) \) be a smooth map given by

\( \bar{\theta}(r) = \begin{cases} \xi(r) & \text{for } 0 < r \leq 1, \\ 1 - \xi(2-r) & \text{for } 1 < r < 2. \end{cases} \)

Since \( \bar{\theta}'(r) > 0 \) for \( 0 < r < 2 \), \( \bar{\theta} \) is a diffeomorphism. Let \( \alpha: \Theta^{-1}((0,1)) \to G/H \times (0,1) \) be a G-diffeomorphism given by \( \alpha = (1, \bar{\theta}) \circ \beta \). Then \( (\theta \circ \alpha^{-1})(gH, r) = (\theta \circ \beta^{-1})(gH, \bar{\theta}^{-1}(r)) = r \), and \( \theta \circ \alpha^{-1} \) is the projection on the second factor.

(3) Let \( f: M/G \to R \) be a function such that \( f \circ \pi: M \to R \) is smooth. We shall prove that \( f \circ \Phi^{-1}: [0,1] \to R \) is smooth. Since

\( (f \circ \pi \circ \alpha^{-1})(gH, r) = (f \circ \Phi^{-1} \circ \theta \circ \alpha^{-1})(gH, r) = (f \circ \Phi^{-1})(r) \) for \( 0 < r < 1 \), \( f \circ \Phi^{-1} \) is smooth on \( (0,1) \). Let \( i_0: D_{1/2}(V_0) = \{ v \in D(V_0); ||v|| \leq 1/2 \} \) \( \xrightarrow{i_0} G \times D_{1/2}(V_0) \) be an inclusion given by \( i_0(v) = [1, v] \). Note that \( (\theta \circ i_0)(v) = ||v||^2 \) for \( v \in D_{1/2}(V_0) \). By Corollary 5.3 of G. Bredon [3, Chapter VI, §5], \( f \circ \Phi^{-1} \) is smooth if and only if \( (f \circ \Phi^{-1}) \circ (\theta \circ i_0) \) is
smooth. Since \((f \circ \phi^{-1}) \circ (\theta \circ i_0) = f \circ i_0\), which is smooth, then \(f \circ \phi^{-1}\) is smooth on \([0, 1/4]\). Similarly we can prove that \(f \circ \phi^{-1}\) is smooth on \([3/4, 1]\). Since \((f \circ \phi^{-1})(r) = (f \circ \phi^{-1} \circ \theta \circ \alpha^{-1})(1H, r) = (f \circ \pi \circ \alpha^{-1})(1H, r)\) for \(0 < r < 1\), \(f \circ \phi^{-1}\) is smooth on \((0, 1)\). This completes the proof of Lemma 1.2.

Remark 1.3. Lemma 1.2 is essentially proved by G. Bredon [3, Chapter VI, §5], and (3) implies that the functional structure of \(M/G\) is induced from that of \(M\).
§2. On the group homomorphism $P$.

In this section we shall define a group homomorphism $P : \text{Diff}_{G}^{\infty}(M)_{0} \to \text{Diff}_{[0,1]}^{\infty}$, and we shall prove $P$ is continuous.

We shall identify the orbit space $M/G$ with $[0,1]$ by the homeomorphism $\phi$ in §1, therefore the projection $\pi : M \to M/G$ is identified with the smooth map $\theta : M \to [0,1]$. Let $h : M \to M$ be a $G$-diffeomorphism of $M$ which is $G$-isotopic to the identity $1_{M}$, and let $f : [0,1] \to [0,1]$ be the orbit map of $h$. Since $f \circ \pi = \pi \circ h$ is a smooth map, $f$ is a smooth map by Lemma 1.2 (3). Similarly the inverse map $f^{-1}$ of $f$ is smooth, and $f$ is a diffeomorphism. Then there exists an abstract group homomorphism $P : \text{Diff}_{G}^{\infty}(M)_{0} \to \text{Diff}_{[0,1]}^{\infty}$ which is given by $P(h) = f$, where $\text{Diff}_{[0,1]}^{\infty}$ is the group of $C^{\infty}$ diffeomorphism of $[0,1]$, endowed with $C^{\infty}$ topology.

**Proposition 2.1.** $P : \text{Diff}_{G}^{\infty}(M)_{0} \to \text{Diff}_{[0,1]}^{\infty}$ is a continuous homomorphism of topological groups.

Let $C^{\infty}(M_{1}, M_{2})$ denote the set of all smooth maps from a compact smooth manifold $M_{1}$ to a smooth manifold $M_{2}$, endowed with $C^{\infty}$ topology. Before the proof of Proposition 2.1, we begin with some lemmas.

**Lemma 2.2.** Let $M_{i}$ be a compact smooth manifold and $N_{i}$ be a smooth manifold for $i = 1, 2$. Then we have

1. Let $\phi : N_{1} \to N_{2}$ be a smooth map, and let $\phi_{*} : C^{\infty}(M_{1}, N_{1}) \to C^{\infty}(M_{1}, N_{2})$ be a map which is given by $\phi_{*}(f) = \phi \circ f$. Then $\phi_{*}$ is continuous.
2. Let $\phi : M_{1} \to M_{2}$ be a smooth map, and let $\phi_{*} : C^{\infty}(N_{2}, N_{1}) \to C^{\infty}(M_{1}, N_{1})$ be a map which is given by $\phi_{*}(f) = f \circ \phi$. Then $\phi_{*}$ is continuous.
3. Let $\phi : M_{1} \to N_{2}$ be a smooth map and let $\phi_{#} : C^{\infty}(M_{1}, N_{1}) \to C^{\infty}(M_{1}, N_{1} \times N_{2})$ be a map which is given by $\phi_{#}(f) = (f, \phi)$. Then $\phi_{#}$ is continuous.
(4) Let \( \phi: M_2 \to N_2 \) be a smooth map and let \( \phi_1: C^\infty(M_1,N_1) \to C^\infty(M_1 \times M_2, N_1 \times N_2) \) be a map given by \( \phi_1(f) = f \times \phi \). Then \( \phi_1 \) is continuous.

(5) Let \( \kappa: C^\infty(M_1,N_1) \times C^\infty(M_2,N_2) \to C^\infty(M_1,N_1 \times N_2) \) be a map given by \( \kappa(f,g)(x) = (f(x), g(x)) \) for \( x \in M_1 \). Then \( \kappa \) is continuous.

(6) Let \( L \) be a smooth manifold. Let \( \text{comp}: C^\infty(M_1,N_1) \times C^\infty(N_1,L) \to C^\infty(M_1,L) \) be a map given by \( \text{comp}(f,g) = g \circ f \). Then \( \text{comp} \) is continuous.

Proof. (1) and (2) are proved by R. Abraham [2, Theorem 11.2 and 11.3]. It is easy to see (3), (4) and (5). From J. Cerf [4, Chapter I, §4, Proposition 5], (6) follows.

Lemma 2.3. Let \( X \) be a topological space. Let \( M \) be a compact smooth manifold and \( N \) be a smooth manifold. Choose an open covering \( \{U_i\} \) of \( M \) such that each closure \( \bar{U}_i \) of \( U_i \) is a regular submanifold of \( M \) which is contained in a coordinate neighborhood of \( M \). Then a map \( \Psi: X \to C^\infty(M,N) \) is continuous if and only if each composition \( \Psi_i: X \xrightarrow{\psi} C^\infty(M,N) \xrightarrow{j_i^*} C^\infty(\bar{U}_i,N) \) is continuous for each \( i \), where \( j_i: U_i \hookrightarrow M \) is an inclusion.

Proof. From Lemma 2.2 (2), if \( \Psi \) is continuous, then \( \Psi_i \) is continuous for each \( i \). We can choose \( \{U_i\} \) as a coordinate covering of \( M \). Let \( \{V_\lambda\} \) be a coordinate covering of \( N \). Let \( f \in C^\infty(M,N) \) and \( K \subseteq U_i \) be a compact subset such that \( f(K) \subseteq V_\lambda \) for some \( \lambda \). \( N_r(f, U_i, V_\lambda, K, \varepsilon) \) \( (r = 0, 1, 2, \ldots, 0 < \varepsilon \leq \infty) \) denote the set of \( C^r \) maps \( g: M \to N \) such that \( g(K) \subseteq V_\lambda \) and \( \|D^k f(x) - D^k g(x)\| < \varepsilon \) for any \( x \in K \), \( k = 0, 1, 2, \ldots, r \).

Then the \( C^\infty \) topology on \( C^\infty(M,N) \) is generated by these sets \( N_r(f, U_i, V_\lambda, K, \varepsilon) \) (see M. Hirsch [6, Chapter 2, §1]).

Let \( x \in X \) and let \( f = \Psi(x) \). For any open neighborhood \( W \) of \( f \), there exist above sets \( N_k = N_{r_k}^k \) \( (f, U_i_k, V_{\lambda_k}, K_k, \varepsilon_k) \), \( k = 1, 2, \ldots, n \), such that \( \bigcap_{k=1}^n N_k \subseteq W \). Note that \( j_{i_k}^*: C^\infty(M,N) \to C^\infty(\bar{U}_{i_k}, N) \) is an open map and \( (j_{i_k}^*)^{-1}(j_{i_k}^*(N_k)) = N_k \). Since \( \Psi_i \) is continuous, \( \Psi_i^{-1}(N_k) = j_{i_k}^{-1}(j_{i_k}^*(N_k)) \) is an open neighborhood of \( x \) in \( X \), for each \( k \). Then \( \bigcap_{k=1}^n \)
Y^{-1}(N_k) is an open neighborhood of x in X. Since \( \psi(\bigcap_{k=1}^{n} \psi^{-1}(N_k)) \subseteq \bigcap_{k=1}^{n} N_k \subseteq W \), \( \psi \) is continuous at x. This completes the proof of Lemma 2.3.

Remark. Lemma 2.2 and Lemma 2.3 are hold in the cases of manifolds with corners.

Let \( C^\infty([-1/2,1/2], \mathbb{R}) \) denote the set of all smooth functions \( f: [-1/2,1/2] \to \mathbb{R} \) satisfying \( f(-x) = f(x) \), endowed with \( C^\infty \) topology. Let \( T: C^\infty([-1/2,1/2], \mathbb{R}) \to C^\infty([0,1/4], \mathbb{R}) \) denote a map defined by \( T(f)(x) = f(\sqrt{x}) \). Then we have

Lemma 2.4. The above map \( T \) is well defined and continuous.

Proof. Put \( f = T(\hat{f}) \) for each \( \hat{f} \in C^\infty([-1/2,1/2], \mathbb{R}) \). Since \( \hat{f} \) is a \( C^\infty \) even function, we have the Taylor expansion

\[
\hat{f}(x) = \hat{f}(0) + (\hat{f}'(0)/2)x^2 + \ldots + (\hat{f}^{(2n-2)}(0)/(2n-2)!x^{2n-2}
+ (\int_0^1 ((1-t)^{2n-1}/(2n-1)!) \hat{f}^{(2n)}(tx) \, dt)x^{2n}
\]

for \(-1/2 \leq x \leq 1/2, n=1,2,\ldots \). Thus we have

\[
f(x) = \hat{f}(0) + (\hat{f}'(0)/2)x + \ldots + (\hat{f}^{(2n-2)}(0)/(2n-2)!x^{n-1}
+ (\int_0^1 ((1-t)^{2n-1}/(2n-1)!) \hat{f}^{(2n)}(tx) \, dt)x^n
\]

for \(0 \leq x \leq 1/4\). By the composite mapping formula, we can compute

the n-th derivative

\[
D^n(f^{(2n)}(t\sqrt{x})x^n)
= \sum_{p=0}^{n} E_{q=0}^{p} \sum_{i_1+\ldots+i_q=p} B(p,i_1,\ldots,i_q) \hat{f}^{(2n+q)}(t\sqrt{x})x^{q/2}t^q,
\]

where \( B(p,i_1,\ldots,i_q) \) is a real number. Put \( f_i = T(\hat{f}_i) \) for \( \hat{f}_i \in C^\infty([-1/2,1/2], \mathbb{R}) \) \( (i=1,2) \). Then there exists a positive number \( A_n \) such that

\[
\sup_{0 \leq x \leq 1/4} |D^n f_1(x) - D^n f_2(x)|
\leq A_n \cdot \max_{0 \leq q \leq 3n} (\sup_{-1/2 \leq x \leq 1/2} |D^q f_1(x) - D^q f_2(x)|)
\]

- 9 -
for each \( n = 1, 2, \ldots \). Note that

\[
\sup_{0 \leq x \leq 1/4} |f_1(x) - f_2(x)| = \sup_{-1/2 \leq x \leq 1/2} |\hat{f}_1(x) - \hat{f}_2(x)|.
\]

Therefore \( T \) is a continuous map, and this completes the proof of Lemma 2.4.

**Proof of Proposition 2.1.** Let \( J \) denote a closed interval \([0, 1/4]\), \([1/5, 4/5]\) or \([3/4, 1]\). By Lemma 2.3, it is sufficient to show that the composition

\[
P_J : \text{Diff}^\infty_G(M) \xrightarrow{p} \text{Diff}^\infty(0, 1, \mathbb{C}) \xrightarrow{j^*} C^\infty(J, [0, 1])
\]

is continuous, where \( j : J \hookrightarrow [0, 1] \) is an inclusion map.

We shall first consider the case \( J = [0, 1/4] \). Let \( \iota : [-1/2, 1/2] \to [0, 1/4] \) be a map given by \( \iota(x) = x^2 \). Let \( \hat{\iota} : [-1/2, 1/2] \to G \times_{K_0} D(V_0) \subseteq M \) be a map given by \( \hat{\iota}(r) = [r, e_0] \), where \( e_0 \) is a point of \( S(V_0) \) as in §1. Then \( \iota \circ \hat{\iota} = \iota \). Let \( \hat{P}_J \) denote the composition

\[
\text{Diff}^\infty_G(M) \xrightarrow{\pi^*} C^\infty([-1/2, 1/2], M) \xrightarrow{\pi^*} C^\infty([-1/2, 1/2], [0, 1]).
\]

Then \( \hat{P}_J(h) = \pi \circ h \circ \hat{\iota} = \pi \circ P(h) \circ \iota \) for \( h \in \text{Diff}^\infty_G(M) \), and the image of \( \hat{P}_J \) is contained in \( C^\infty([-1/2, 1/2], R) \). Note that \( \hat{P}_J = T \circ P_J \). Combining Lemma 2.2 and Lemma 2.4, \( P_J \) is continuous.

Next consider the case \( J = [1/5, 4/5] \). By Lemma 1.2, there is a \( G \)-diffeomorphism \( \alpha : \pi^{-1}([1/5, 4/5]) \to G/H \times [1/5, 4/5] \). Let \( i : \pi^{-1}([1/5, 4/5]) \subseteq M \) be the inclusion map and let \( k : [1/5, 4/5] \to G/H \times [1/5, 4/5] \) be a map given by \( k(r) = (1H, r) \). Then \( P_J \) is the composition

\[
\text{Diff}^\infty_G(M) \xrightarrow{(i \circ \alpha^{-1} \circ k)^*} C^\infty([1/5, 4/5], M) \xrightarrow{\pi} C^\infty([1/5, 4/5], [0, 1])
\]

which is continuous by Lemma 2.2.

We can prove that \( P_J \) is continuous in the case \( J = [3/4, 1] \) similarly as in the case \( J = [0, 1/4] \), and this completes the proof of Proposition 2.1.
§3. A continuous global section of $P$.

In §2 we have proved that $P: \text{Diff}_G^\infty(M)_0 \to \text{Diff}_G^\infty[0,1]$ is continuous. Thus the image of $P$ is contained in the connected component $\text{Diff}_G^\infty[0,1]_0$ of the identity. In this section we shall construct a continuous global section of $P: \text{Diff}_G^\infty(M)_0 \to \text{Diff}_G^\infty[0,1]_0$.

Let $f$ be an element of $\text{Diff}_G^\infty[0,1]_0$. We shall define a map $\Psi(f): M \to M$ as follows: $\Psi(f)$ is defined on $\pi^{-1}((0,1))$ by the composition $\pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1) \xrightarrow{1,f} G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1))$, and $\Psi(f) = 1$ on $\pi^{-1}(0) \cup \pi^{-1}(1)$.

Proposition 3.1. $\Psi(f)$ is a $G$-diffeomorphism of $M$.

In order to prove Proposition 3.1, we need the following lemma and notations.

Lemma 3.2. Let $\Psi_1: \text{Diff}_G^\infty[0,1]_0 \to \text{Diff}_G^\infty(D^n)$ be a map defined by

$$\Psi_1(f)(v) = \begin{cases} \sqrt{f\left(\frac{v}{\|v\|^2}\right)} & \text{for } v \neq 0, \\ 1 & \text{for } v = 0, \end{cases}$$

where $D^n$ denotes an $n$-dimensional unit disc. Then $\Psi_1$ is a well-defined and continuous map.

Notations 3.3. For $i = 0,1$, we shall use the following notations:

- $\pi_i: G \to G/K_i$ the natural projection,
- $U_i$ an open disc neighborhood of $1K_i$ in $G/K_i$,
- $\sigma_i: U_i \to G$ a smooth local cross section of $\pi_i$,
- $\sigma_{i,a}: aU_i \to G (a \in G)$ a smooth local cross section of $\pi_i$ defined by $\sigma_{i,a}(x) = a \cdot \sigma_i(a^{-1}x)$.

Put $M_i = G \times D(V_i)$ and $M_i(r) = G \times_{K_i} D(V_i)$, where $D_r(V_i)$ is a closed $r$-disc in $V_i$ ($0 < r \leq 1$).

- $p_i: M_i \to G/K_i$, $p_i,r: M_i(r) \to G/K_i$ the bundle projections,
- $\phi_{i,a}: p_i^{-1}(aU_i), \to U_i \times D(V_i)$ $(a \in G)$ a chart of $p_i$ over $aU_i$ defined
by $\phi(a, (g, v)) = (a^{-1} \pi_1(g), ((a, a^{-1} \pi_1(g))^{-1} g, v))$.

$\pi_2: G \to G/H$ the natural projection,

$U_2$ an open disc neighborhood of $1H$ in $G/H$,

$\sigma_2: U_2 \to G$ a smooth local cross section of $\pi_2$.

Proof of Proposition 3.1. Put $h = \psi(f)$. We shall first prove that $h$ is smooth on $\pi^{-1}(0)$. Since $f(0) = 0$, there exists a real number $c$ such that $0 < c < 1/2$ and $f(c^2) < 1/4$. Then $h(\pi^{-1}([0, c^2])) \subseteq \pi^{-1}([0, 1/4])$, and $h(M_0(\varepsilon)) \subseteq M_0(1/2)$. For $(g, r, 0) \in G \times D_0(0) \subseteq (V_0 - 0)$, $0 < r < \varepsilon$, $h((g, r, 0)) = (a^{-1} \circ (1, f) \circ a)((g, r, 0)) = (a^{-1} \circ (1, f))(gH, r^2) = a^{-1}(gH, f(r^2)) = (g, \sqrt{f(r^2)} e_0)$. Then, for $(g, v) \in G \times D_0(0)$, $h((g, v)) = (g, \sqrt{|v|^2} / |v|) = (g, \sqrt{f(v^2) / |v|}) = (g, \psi_1(f)(v))$. Since $h((g, 0)) = (g, 0)$, $h((g, v)) = (g, \psi_1(f)(v))$ for any $(g, v) \in M_0(\varepsilon)$. Then the composition

$\tilde{h}: U_0 \times D_0(V_0) \xrightarrow{(\phi_0, a)^{-1}} p_0, 1/2(aU_0)$

$\xrightarrow{h} p_0, 1/2(aU_0)$

$\xrightarrow{\phi_0, a} U_0 \times D_1/2(V_0)$

is given by $\tilde{h}(x, v) = (x, \psi_1(f)(v))$ for $a \in G$. Since $\psi_1(f)$ is a smooth map by Lemma 3.2, $h$ is smooth on $\pi^{-1}(0)$. Similarly we can prove that $h$ is smooth on $\pi^{-1}(1)$. Since $h$ is smooth on $\pi^{-1}((0, 1))$ by the definition, $h$ is a smooth map. Since $h^{-1} = \psi(f^{-1})$, $h^{-1}$ is also a smooth map. Thus $h$ is a $G$-diffeomorphism of $M$, and this completes the proof of Proposition 3.1.

In order to prove Lemma 3.2, we need the following assertion.

Assertion 3.4. Let $\phi: \text{Diff}^\infty([0, 1]) \to C^\infty([0, 1], \mathbb{R})$ be a map given by

$\phi(f)(x) = \begin{cases} \sqrt{f(x)} / x & \text{for } x \neq 0, \\ f'(0) & \text{for } x = 0. \end{cases}$

Then $\phi$ is a well defined continuous map.

Proof. For $f \in \text{Diff}^\infty([0, 1])$, we have the Taylor expansion
\[ f(x) = f'(0)x + x^2 \int_0^1 (1-t)f''(tx) \, dt \quad \text{for } 0 \leq x \leq 1. \]

Put \( F(x) = f'(0) + x^2 \int_0^1 (1-t)f''(tx) \, dt \) for \( 0 \leq x \leq 1 \). Then \( \phi(f) = \sqrt{F} \).

Note that \( F(x) > 0 \) for \( 0 \leq x \leq 1 \). It is easy to see that \( \phi \) is continuous.

**Proof of Lemma 3.2.** Let \( N: D^n \to [0,1] \) be a map given by \( N(v) = ||v||^2 \). Let \( i: D^n \to \mathbb{R}^n \) be the inclusion and let \( \psi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be the scalar multiplication. Since \( \psi_1(f) = \phi(f)(||v||^2)v \), \( \psi_1(f) \) is a smooth map by Assertion 3.4. Since \( \psi_1(f^{-1}) = \psi_1(f)^{-1} \), \( \psi_1(f)^{-1} \) is also a smooth map. Thus \( \psi_1(f) \) is a diffeomorphism of \( D^n \). Note that \( \psi_1 \) is the composition \( \psi_1: \text{Diff}^{\infty}([0,1])_0 \xrightarrow{\phi} C^{\infty}([0,1],\mathbb{R}) \xrightarrow{N^*} C^{\infty}(D^n,\mathbb{R}) \xrightarrow{i\#} C^{\infty}(D^n,\mathbb{R}^n) \). Combining Assertion 3.4 and Lemma 2.2, \( \psi \) is continuous. This completes the proof of Lemma 3.2.

**Proposition 3.5.** \( \psi: \text{Diff}^{\infty}([0,1])_0 \to \text{Diff}^{\infty}_G(M) \) is continuous.

**Proof.** Let \( B_i \subset U_i \) be a closed disc neighborhood of \( 1K_i \) in \( G/K_i \) for \( i = 0,1 \). Let \( B_2 \subset U_2 \) be a closed disc neighborhood of \( 1H \) in \( G/H \). We can choose \( \{ \text{int}(p_{0i}^{-1}aB_0), \text{int}(p_{1i}^{-1}aB_1), \text{int}(a^{-1}(aB_2 \times [\epsilon/2,1-\epsilon/2]) \mid a \in G \} \) as an open covering of \( M \) for \( 0 < \epsilon < 1/2 \). Let \( W = \{ f \in \text{Diff}^{\infty}([0,1])_0 \mid f([0,\epsilon^2]) \subset [0,1/4], f([1-\epsilon^2,1]) \subset [3/4,1] \} \). Then \( W \) is an open neighborhood of the identity in \( \text{Diff}^{\infty}([0,1])_0 \). Since \( \psi \) is a homomorphism as an abstract group, it is enough to show that \( \psi \) is continuous on \( W \). Let \( C \) denote one of the sets \( p_{0i}^{-1}aB_0, p_{1i}^{-1}aB_1 \) or \( a^{-1}(aB_2 \times [\epsilon/2,1-\epsilon/2]) \) for \( a \in G \). If we can prove that the composition \( \psi_C: W \xrightarrow{\psi} \text{Diff}^{\infty}_G(M)_0 \xrightarrow{i^*} C^{\infty}(C,M) \) is continuous for each \( C \), then \( \psi \) is continuous on \( W \) by Lemma 2.3, where \( i: C \subset M \) is an inclusion map.

First consider the case \( C = p_{0i}^{-1}aB_0 \). \( \psi(f)(C) \) is contained in \( p_{0i}^{-1}aU_0 \) for each \( f \in W \). Note that \( \psi(f)([g,v]) = [g,\psi_1(f)(v)] \) for \([g,v] \in C \) and \((\phi_0,a)^*\psi(f)\phi_0^{-1}(x,v) = (x,\psi_1(f)(v))\) for \((x,v) \in B_0 \times D_\epsilon(V_0) \). Thus \( \psi_C \) is given by the composition...
where $j: B_0 \hookrightarrow U_0$ and $k: \pi_0^{-1}(aU_0) \hookrightarrow M$ are inclusions. Combining Lemma 3.2 and Lemma 2.2, $\psi_C$ is continuous.

Now consider the case $C = a^{-1}(B_0 \times [\varepsilon/2, 1-\varepsilon/2])$. $\psi_C$ is given by the composition

where $\iota: [\varepsilon/2, 1-\varepsilon/2] \hookrightarrow [0,1]$, $j: B_0 \hookrightarrow C/H$ and $k: \pi^{-1}((0,1)) \hookrightarrow M$ are inclusion maps. By Lemma 2.2, $\psi_C$ is continuous.

We can prove that $\psi_C$ is continuous in the case $C = \pi_1^{-1}(aB_1)$ similarly as in the case $C = \pi_0^{-1}(aB_0)$, and this completes the proof of Proposition 3.5.

By Proposition 3.5, $P: \text{Diff}^\infty_G(M)_0 \to \text{Diff}^\infty[0,1]_0$ is a globally trivial fibration. Then we have

Corollary 3.6. $\text{Diff}^\infty_G(M)_0$ is homeomorphic to $\text{Diff}^\infty[0,1]_0 \times \text{Ker} P$. 

- 14 -
§4. On the group $\text{Ker } P$.

In this section we shall define a group homomorphism $L: \text{Ker } P \rightarrow Q$, where $Q$ is a subgroup of $C^\infty([0,1], \mathbb{C}/\mathbb{H})$, and we shall prove that $L$ is a group monomorphism between topological groups (see Lemma 4.5 and Proposition 4.6).

Let $h$ be an element of $\text{Ker } P$. Let $h$ be the composition

$$
G/H \times (0,1) \xrightarrow{\alpha^{-1}} \pi^{-1}((0,1)) \xrightarrow{h} \pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1).
$$

Then $h$ is a level preserving $G$-diffeomorphism. Let $a: (0,1) \rightarrow N(H)/H$ be a smooth map satisfying $h(gH,r) = (ga(r),r)$ for $(gH,r) \in G/H \times (0,1)$.

Proposition 4.1. With the above notations, there exists a smooth map $\tilde{a}: [0,1] \rightarrow N(H)/H$ such that

1. $\tilde{a} = a$ on $(0,1)$,
2. $\tilde{a}(i) \in (N(H) \cap N(K_i))/H$ for $i = 0,1$.

To prove Proposition 3.1, we need the following lemma.

Lemma 4.2. Let $G$ be a compact Lie group. Let $K$ and $N$ be closed subgroups of $G$. Let $\pi: G \rightarrow G/K$ be the natural projection. Then there exists a smooth local section $\sigma$ of $\pi$, which is defined on an open neighborhood $U$ of $1K$, such that $\sigma(1K) = 1$ and $\sigma(x) \in N$ for $x \in \pi(N) \cap U$.

Proof. Let $\pi_1: N \rightarrow N/(N \cap K)$ be a natural projection. Let $i: N \subseteq G$ be the inclusion and let $I: N/(N \cap K) \rightarrow G/K$ be a map satisfying $\pi \circ i = I \circ \pi_1$. Since $I(N/(N \cap K)) = \pi(N)$ is an orbit of the natural action $N \times G/K \rightarrow G/K$, $I$ is an imbedding. Let $U$ be a disc neighborhood around $\pi(1)$ in $G/K$ and let $U_1$ be a disc neighborhood around $\pi_1(1)$ in $N/(N \cap K)$. Since $I$ is an imbedding, we can assume $I(U_1) = U \cap I(N/(N \cap K)) = U \cap \pi(N)$. Let $\sigma_1: U_1 \rightarrow N$ be a smooth local section of $\pi_1$ satisfying $\sigma_1(\pi_1(1)) = 1$. Then $\sigma_1 \circ I^{-1}$ is a smooth section.
defined on $I(U_1)$. We can extend $o(I^{-1})$ to a smooth local section defined on $U$. Then $o(\tau(1)) = 1$ and $o(U \cap \tau(N)) \subseteq N$. This completes the proof of Lemma 4.2.

**Lemma 4.3.** Let $G$ be a compact connected Lie group. Let $V$ be a representation of $G$ such that $G$ acts transitively and effectively on a unit sphere $S(V)$ of $V$. Let $H$ be the isotropy subgroup of a point of $S(V)$. Then we have the following list:

<table>
<thead>
<tr>
<th>$G$</th>
<th>$SO(n)$ $(n \geq 3)$</th>
<th>$SU(n)$ $(n \geq 2)$</th>
<th>$U(n)$ $(n \geq 1)$</th>
<th>$Sp(n)$ $(n \geq 1)$</th>
<th>$Sp(n) \times \mathbb{Z}_2 S^3(n \geq 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>$SO(n-1)$</td>
<td>$SU(n-1)$</td>
<td>$U(n-1)$</td>
<td>$Sp(n-1)$</td>
<td>$H_1$</td>
</tr>
<tr>
<td>$N(H)/H$</td>
<td>$\mathbb{Z}_2$</td>
<td>$U(1)$</td>
<td>$U(1)$</td>
<td>$Sp(1)$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

Proof. It is known that $G$ and $H$ are the above Lie groups (c.f. W. C. Hsiang and W. Y. Hsiang [7, §1]). We can determine the Lie group $N(H)/H$ by an immediate calculation except for $G = G_2$, $Spin(7)$, $Spin(9)$. For the cases $G = G_2$, $Spin(7)$, $Spin(9)$, we can determine $N(H)/H$ by using I. Yokota's definitions of these Lie groups in [9, Chapter 5].

**Lemma 4.4.** (1) Let $F: [-1, 1] \to \mathbb{R}$ be a smooth function such that $F(0) = 0$. Put $f(x) = F(x)/x$ for $x \neq 0$ and $f(0) = F'(0)$ for $x = 0$. Then $f: [-1, 1] \to \mathbb{R}$ is a well defined smooth function.

(2) Put $C^\infty_0([-1, 1], \mathbb{R}) = \{ F \in C^\infty([-1, 1], \mathbb{R}); F(0) = 0 \}$, endowed with

- 16 -
endowed with $C^\infty$ topology. Let $\Phi: C^\infty_0([-1,1],\mathbb{R}) \to C^\infty([-1,1],\mathbb{R})$
be a map given by $\Phi(F)(x) = f(x)$. Then $\Phi$ is continuous.

Proof. For $F \in C^\infty_0([-1,1],\mathbb{R})$, we have $\Phi(F)(x) = f(x) = F'(0) +$
$x\int_0^1 (1-t)F''(tx) \, dt$. Then the $n$-th derivative $f^{(n)}(x) = x\int_0^1 (1-t)$
$\tau^n_{\tau^0}(n+2)(tx) \, dt + n\int_0^1 (1-t)\tau^n-1\tau^{(n+1)}(tx) \, dt$. Thus there exists
a positive number $A$ such that $\|\Phi(F)\|_n \leq A\|F\|_{n+2}$, and Lemma 4.4 follows.

Proof of Proposition 4.1. Let $\epsilon (0 < \epsilon \leq 1/2)$ be a real number.
Let $W_i$ and $U_i$ be open neighborhoods of $1K_i$, satisfying $W_i \subset U_i$
for $i = 0,1$. Put $O = \{h \in \text{Ker } P : h(p^{-1}(W_i)) \subset p^{-1}(U_i) \text{ for } i = 0,1\}$.
Then $O$ is an open neighborhood of the identity in $\text{Ker } P$. By Corollary
3.6, $\text{Ker } P$ is connected, and $O$ generates the topological group $\text{Ker } P$.
Thus we can assume $h \in O$.

Let $\tilde{h}$ be the composition
$\tilde{h} = (\phi_0^{-1})(1K_0, r_0) \in \text{Hom}(\text{Ker } P) \subset G/H$.

Then
$$(a \circ h \circ \phi_0^{-1})(1K_0, r_0) = (a \circ h)([1, r_0])$$
$$(a \circ h)([1, r_0]) = h(1H, r^2)$$
$$(a \circ h)([1, r_0]) = (a(r^2), r^2) \text{ for } |r| \leq \epsilon, r \neq 0.$$
\[ \tilde{h}_1(r) = \bar{\pi}_0(a(r^2)), \]
\[ \tilde{h}_2(r) = (\sigma_0 \circ \bar{\pi}_0)(a(r^2))^{-1} \cdot a(r^2) \cdot r e_0, \]
for \( |r| \leq \varepsilon, \ r \neq 0. \)

Here we can assume that \( \sigma_0(1 K_0) = 1 \) and \( \sigma_0(\pi_0(N(H)) \cap U_0) \subset N(H) \) by Lemma 4.2. Let \( b: [-\varepsilon, \varepsilon] \rightarrow G \) be a smooth map given by \( b(r) = \sigma_0(\tilde{h}_1(r)) \). Then \( b(r) = \sigma_0(\bar{\pi}_0(a(r^2)) \in \sigma_0(\pi_0(N(H)) \cap U_0), \) and \( b(r) \in N(H) \) for \( r \neq 0. \) Since \( b \) is a smooth map, \( b(r) \in N(H) \) for \( r = 0. \)

For \( [1, 0] \in \pi^{-1}(0) \), we have \( h([1, 0]) = (h \circ \phi_{0, 1}^{-1})(1 K_0, 0) = \phi_{0, 1}^{-1}(i(0)) \]
\[ = \phi_{0, 1}^{-1}(\tilde{h}_1(0), 0) = [b(0), 0]. \] Note that \( p_0 \) is a G-diffeomorphism on the zero section of \( p_0 \) and \( h(\pi^{-1}(0)) = \pi^{-1}(0) \). Then the composition \( p_0 \circ h \circ \phi_{0, 1}^{-1} G/K_0 \rightarrow G/K_0 \) is a G-diffeomorphism, and \( (p_0 \circ h) \pi^{-1}(0)) = p_0([b(0), 0]) = b(0) K_0. \) Thus \( b(0) \in N(K_0), \) and \( b(0) \in N(H) \cap N(K_0). \)

Put \( J = [-\varepsilon, 0) \cup (0, \varepsilon] \). Let \( c: J \rightarrow N(H)/H \) be a smooth map given by \( c(r) = b(r)^{-1} \cdot a(r^2). \) Since \( \bar{\pi}_0(c(r)) = \bar{\pi}_0(\sigma_0(\bar{\pi}_0(a(r^2)))^{-1}. \)
\( a(r^2)) = 1 K_0, \) then \( c(r) \in K_0/H. \) Thus \( c(r) \in N(H, K_0)/H \) for \( r \in J. \) Since \( \text{Ker} \ P \) is connected, the maps \( a, b \) and \( c \) are homotopic to the constant maps. Note that the identity component \( (N(H, K_0)/H)^0 \) of \( N(H, K_0)/H \) is contained in \( (N(H, K_0) \cap K_0^0)/H \), and there exists an isomorphism \( (N(H, K_0) \cap K_0^0)/H \cong (N(H, K_0) \cap K_0^0)/(H \cap K_0^0) \) as a Lie group, where \( K_0^0 \) is the identity component of \( K_0. \) Then there exists a smooth map \( \hat{c}: J \rightarrow (N(H, K_0)/H)^0 \) such that \( \text{Ker} \ p_0 \cong (N(H, K_0) \cap K_0^0)/(H \cap K_0^0) \) \( \cong N(H \cap K_0^0, K_0^0)/(H \cap K_0^0). \) Now we shall prove that \( \hat{c} \) can be extended to a smooth map on \( [-\varepsilon, \varepsilon], \) and so is \( c. \)

Note that \( K_0 \) acts transitively on the unit sphere \( S(V_0) \) of \( V_0. \)
If \( \dim S(V_0) = 0, \) then \( K_0/H = Z_2 \) and \( N(H, K_0)/H = Z_2. \) In this case \( \hat{c} \) is a trivial map, and it is clear that \( \hat{c} \) can be extended to a smooth map on \( [-\varepsilon, \varepsilon]. \) Now we assume \( \dim S(V_0) > 0. \) Since \( S(V_0) \) is connected, \( K_0^0 \) acts transitively on \( S(V_0) \) and \( K_0^0/(K_0^0 \cap H) \) is diffeomorphic to \( S(V_0). \)
Put $D = \bigcap_{g \in K^0} g(K_0^0 \cdot H) g^{-1}$ which is the kernel of the action $K_0^0 \times S(V_0) \rightarrow S(V_0)$. Put $\tilde{K}_0 = K_0^0 / D$ and $\tilde{H} = (H \cap K_0^0) / D$. Then $\tilde{K}_0$ acts transitively and effectively on $S(V_0)$ and $\tilde{K}_0 / \tilde{H}$ is diffeomorphic to $S(V_0)$. Put $\tilde{N}_0 = N(\tilde{H}, \tilde{K}_0) / \tilde{H}$ which is isomorphic to $N(H \cap K_0^0, K_0^0) / (H \cap K_0^0)$ as a Lie group. The pair $(\tilde{K}_0, \tilde{N}_0)$ is one of pairs $(G, N(H)/H)$ in the list of Lemma 4.3. Now we shall prove that $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$. If $\tilde{N}_0 = Z_2$, this is clear since $\hat{c}$ is a trivial map.

Consider the case $\tilde{K}_0 = SU(n)$ ($n \geq 1$) and $\tilde{N}_0 = U(1)$. In this case $V_0$ is an $n$-dimensional complex vector space and $\tilde{N}_0 = U(1)$ acts on $V_0$ as a scalar multiplication. We can regard $\mathbb{C}^n$ as a 2n-dimensional real vector space $\mathbb{R}^{2n}$ and $N_0$ as $SO(2)$. Then there exist smooth functions $c_i : J \rightarrow \mathbb{R}$, $i = 1, 2$, such that

$$\hat{c}(r) = \begin{bmatrix} c_1(r) & -c_2(r) \\ c_2(r) & c_1(r) \end{bmatrix} \in SO(2) \text{ for } r \in J.$$ 

Note that $\tilde{h}_2 : [-\varepsilon, \varepsilon] \rightarrow D_\varepsilon(V_0)$ is a smooth map and $\tilde{h}_2(r) = c(r) \cdot r e_0 = \hat{c}(r) \cdot r e_0$ for $r \neq 0$. In this case $e_0 = (1, 0, \ldots, 0) \in S^{2n-1}$ and $\tilde{h}_2(r) = (c_1(r) r, c_2(r) r, 0, \ldots, 0)$ for $r \in J$. Put $c_i(0) = \lim_{r \rightarrow 0} c_i(r)$ for $i = 1, 2$. From Lemma 4.4, $c_i : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}$, $i = 1, 2$, are smooth functions and $\hat{c}$ can be extended to a smooth map on $[-\varepsilon, \varepsilon]$.

Now consider the case $\tilde{K}_0 = Sp(n)$ ($n \geq 1$) and $\tilde{N} = Sp(1)$. In this case $V_0$ is an $n$-dimensional quaternionic vector space $\mathbb{H}^n$ and $N_0 = Sp(1)$ acts on $V_0$ as a scalar multiplication on the right. We can regard $\mathbb{H}^n$ as $\mathbb{R}^{4n}$ and $Sp(1)$ as a subgroup of $SO(4)$ naturally.

By the similar way as in the case $K_0 = SU(n)$, there exist smooth functions $c_i : J \rightarrow \mathbb{R}$, $i = 1, 2, 3, 4$, such that $h_2(r) = (c_1(r) r, c_2(r) r, c_3(r) r, c_4(r) r, 0, \ldots, 0)$ for $r \in J$, and we can extend $\hat{c}$ to a smooth map on $[-\varepsilon, \varepsilon]$.

The proof of the other cases are similar to those of the above cases. Thus we can extend $c$ to a smooth map on $[-\varepsilon, \varepsilon]$. Since $c(r) \in N(H, K_0^0) / H$
for \( r \neq 0 \), we see \( c(0) \in N(H,K_0)/H \). Put \( \tilde{a}(0) = b(0) \cdot c(0) \). Since \( b(0) \in N(H) \cap N(K_0) \) and \( c(0) \in N(H,K_0)/H \), we have \( \tilde{a}(0) \in (N(H) \cap N(K_0))/H \).

Let \( \hat{a} : [-1/2,1/2] \to N(H)/H \) be a map given by \( \hat{a}(r) = \tilde{a}(r^2) \). Since \( \hat{a}(r) = b(r) \cdot c(r) \) for \(-1 \leq r \leq 1\), \( \hat{a} \) is a smooth map. Since \( \hat{a} \) is an even map and \( \hat{a}(x) = \hat{a}(\sqrt{r}) \) for \( 0 \leq r \leq 1/4 \), \( \hat{a} \) is a smooth map on \([0,1/4]\) by Lemma 2.4. Thus we can extend the map \( \hat{a} \) to a smooth map \( \tilde{a} \) on \([0,1]\) satisfying \( \tilde{a}(0) \in (N(H) \cap N(K_0))/H \). Similarly we can extend \( \hat{a} \) to a smooth map \( \tilde{a} \) on \([0,1]\) satisfying \( \tilde{a}(1) \in (N(H) \cap N(K_1))/H \). This completes the proof of Proposition 4.1.

Let \( Q \) denote the set of smooth maps \( f : [0,1] \to N(H)/H \) satisfying \( f(i) \in (N(H) \cap N(K_i))/H \) for \( i = 0,1 \), endowed with \( C^\infty \) topology. Using Proposition 4.1, we define a map \( L : \text{Ker}P \to Q \) by \( L(h) = a^{-1} \).

Lemma 4.5. \( L : \text{Ker}P \to Q \) is a group monomorphism.

Proof. Let \( h_i \in \text{Ker}P \) for \( i = 1,2 \). For \( 0 < r < 1 \) and \( g \in G \), we have

\[
(g \cdot L(h_2 \circ h_1)(r^{-1})) = (a \cdot h_2 \circ h_1 \circ a^{-1})(gH,r)
\]

\[
= ((a \cdot h_2 \circ a^{-1}) \circ (a \cdot h_1 \circ a^{-1}))(gH,r)
\]

\[
= (a \cdot h_2 \circ a^{-1})(g \cdot L(h_1)(r^{-1}),r)
\]

\[
= (g \cdot L(h_1)(r^{-1}) \cdot L(h_2)(r^{-1}),r).
\]

Thus \( L(h_2 \circ h_1) = L(h_2) \cdot L(h_1) \) on \((0,1)\). Since \( L(h_1) \), \( L(h_2) \) and \( L(h_1 \circ h_2) \) are smooth maps on \([0,1]\), \( L(h_2 \circ h_1) = L(h_2) \cdot L(h_1) \) on \([0,1]\). Thus \( L \) is a group homomorphism. Suppose \( L(h) = 1 \) for \( h \in \text{Ker}P \). Then \( (h \circ a^{-1})(gH,r) = a^{-1}(gH,r) \) for \( g \in G \) and \( 0 < r < 1 \), and \( h = 1 \) on \( \pi^{-1}((0,1)) \). Thus \( h = 1 \) on \( M \), and \( L \) is a monomorphism.

Proposition 4.6. \( L \) is a continuous map.

Proof. We shall use the notations in the proof of Proposition 4.1. Since \( L \) is a group homomorphism, it is sufficient to show Proposition 4.6 that \( L : O \to Q \) is continuous. Let \( O \) denote a closed
interval \([0, \varepsilon^2], [\varepsilon^2/2, 1-\varepsilon^2/2] \) or \([1-\varepsilon^2, 1] \). By Lemma 2.3, it is sufficient to prove that \(L_1: O \xrightarrow{L} Q \xrightarrow{j^*} C^\infty(I,N(H)/H) \) is continuous, where \(j: I \hookrightarrow [0,1] \) is an inclusion map.

First we shall consider the case \(I = [0, \varepsilon^2] \). Let \(L_1 \) be the composition

\[
\begin{align*}
0 \xrightarrow{(k \circ \phi_{0,1}^{-1} \circ i)^*} & C^\infty([-\varepsilon, \varepsilon], p_{0,1}^{-1}(U_0)) \\
& \xrightarrow{(\sigma \circ \rho_1 \circ \phi_{0,1}^{-1})^*} C^\infty([-\varepsilon, \varepsilon], G),
\end{align*}
\]

where \(k: p_{0,1}^{-1}(W_0) \hookrightarrow M \) is an inclusion map. Then \(L_1 \) is continuous by Lemma 2.2. Note that \(L_1(h) = b \).

Let \(L_2: O \xrightarrow{} C([-\varepsilon, \varepsilon], (N(H,K_0)/H)^0) \) be a map given by \(L_2(h) = c \). We shall prove that \(L_2 \) is continuous. This is trivial in the case \(N(H,K_0)/H = Z_2 \). Consider the case \(K_0 = SU(n) \) \((n \geq 2) \). In this case \(V_0 = C^n = R^{2n} \) and \(N_0 = U(1) = S^0(2) \). Put \(C_0^\infty([-\varepsilon, \varepsilon]), V_0) = \{ F \in C^\infty([-\varepsilon, \varepsilon]), V_0) : F(0) = 0 \} \), endowed with \(C^\infty \) topology. Let \(\phi: C_0^\infty([-\varepsilon, \varepsilon]), V_0) \to C^\infty([-\varepsilon, \varepsilon], R^2) \) be a map defined by \(\phi(F) = (\phi(F^1), \phi(F^2)) \), where \(F = (F^1, \ldots, F^{2n}) \) and \(\phi(F^i) \) is a map defined in Lemma 4.4. Then \(\phi \) is continuous by Lemma 4.4. Let \(m: R^2 \to M_2(R) \) denote a smooth map defined by

\[
m(x,y) = \left[ \begin{array}{cc} x & -y \\ y & x \end{array} \right],
\]

where \(M_2(R) \) denote the set of all \(2 \times 2 \) matrices over \(R \). Let \(L'_2 \) denote the composition

\[
\begin{align*}
0 \xrightarrow{(k \circ \phi_{0,1}^{-1} \circ i)^*} & C^\infty([-\varepsilon, \varepsilon], p_{0,1}^{-1}(U_0)) \\
& \xrightarrow{(\rho_1 \circ \phi_{0,1}^{-1})^*} C^\infty([-\varepsilon, \varepsilon], D_{\varepsilon}(V_0)).
\end{align*}
\]

From Lemma 2.2, \(L'_2 \) is continuous. Note that \(L'_2(h) = \tilde{h}_2 \) and \(L'_2(0) \) is contained in \(C_0^\infty([-\varepsilon, \varepsilon], V_0) \). Let \(\hat{L}_2 \) denote the composition

\[
\begin{align*}
0 \xrightarrow{L'_2} & C_0^\infty([-\varepsilon, \varepsilon], V_0) \\
& \xrightarrow{\phi} C^\infty([-\varepsilon, \varepsilon], R^2) \\
& \xrightarrow{m^*} C^\infty([-\varepsilon, \varepsilon], M_2(R)).
\end{align*}
\]

Then \(\hat{L}_2(h) = c \) and \(\hat{L}_2 \) is continuous. This implies that \(L_2 \) is
continuous by using Lemma 2.2. Similarly we can see that $L_2$ is continuous in the other cases.

Let $\mu: G \times G/H \to G/H$ be a map defined by the left translation and let $\iota: (N(H,K_0)/H)^0 \hookrightarrow G/H$ be an inclusion map. Then the composition

$$
L: 0 \xrightarrow{(L_1, \iota^*L_2)} C^\infty([-\epsilon,\epsilon], G) \times C^\infty([-\epsilon,\epsilon], G/H)
$$

$$
\xrightarrow{\kappa} C^\infty([-\epsilon,\epsilon], G \times G/H)
$$

$$
\xrightarrow{L_{-*}} C^\infty([-\epsilon,\epsilon], G/H)
$$

is continuous by Lemma 2.2, where $\kappa$ is defined by $\kappa(f_1,f_2)(r) = (f_1(r),f_2(r))$. Note that $L(h) = b \cdot c = a$ and $L(0)$ is contained in $C^\infty([-\epsilon,\epsilon], N(H)/H)$. Here $C^\infty([-\epsilon,\epsilon], N(H)/H)$ denote the set of all smooth even maps $f: [-\epsilon,\epsilon] \to N(H)/H$, endowed with $C^\infty$ topology.

Let $T: C^\infty([-\epsilon,\epsilon], N(H)/H) \to C^\infty([-\epsilon,\epsilon], N(H)/H)$ be a map defined by $T(f)(r) = f(\sqrt{r})$. By the same argument as in the proof in Lemma 2.4, we can prove that $T$ is continuous. Thus $L_I = T \circ L$ is continuous.

Now consider the case $I = [\epsilon^2/2, 1-\epsilon^2/2]$. $L_I$ is given by the composition

$$
0 \xrightarrow{k^*} C^\infty(\pi^{-1}(I), \pi^{-1}(I))
$$

$$
\xrightarrow{(\alpha^{-1} \circ \iota)^*} C^\infty(I, \pi^{-1}(I))
$$

$$
\xrightarrow{(q_2 \circ \alpha)^*} C^\infty(I, G/H),
$$

where $k: \pi^{-1}(I) \hookrightarrow M$ is an inclusion, $\iota: I \to G/H \times I$ is a map given by $\iota(r) = (1H, r)$ and $q_2: G/H \times I \to G/H$ is the projection on the first factor. Thus $L_I$ is continuous. We can see that $L_I$ is continuous in the case $I = [1-\epsilon^2, 1]$ similarly as in the case $I = [0, \epsilon^2]$, and this completes the proof of Proposition 4.6.
§5. Subgroups of the topological groups \( Q \) and \( \text{Ker} \ P \).

In this section we shall consider subgroups \( Q_1 \) and \( S \) of the topological groups \( Q \) and \( \text{Ker} \ P \), respectively, such that \( L(S) = Q_1 \), and we shall prove that the inclusions \( Q_1 \rightarrow Q_0 \) and \( S \hookrightarrow \text{Ker} \ P \) are homotopy equivalence, where \( Q_0 \) is the identity component of \( Q \).

Put \( Q_1 = \{ a \in Q_0 ; \ a(r) = a(0) \ \text{for} \ 0 \leq r \leq 1/4 \ \text{and} \ a(r) = a(1) \ \text{for} \ 3/4 \leq r \leq 1 \} \). Then \( Q_1 \) is a topological subgroup of \( Q_0 \). Let \( i : Q_1 \hookrightarrow Q_0 \) be an inclusion.

**Lemma 5.1.** \( i : Q_1 \hookrightarrow Q_0 \) is a homotopy equivalence.

**Proof.** Let \( \sigma : [0,1] \rightarrow [0,1] \) be a smooth map such that
\[
\sigma(r) = 0 \ \text{for} \ 0 \leq r \leq 1/4,
\]
\[
\sigma(r) = 1 \ \text{for} \ 3/4 \leq r \leq 1.
\]
Let \( \mu_t : [0,1] \rightarrow [0,1] (0 \leq t \leq 1) \) be a smooth homotopy given by
\[
\mu_t(r) = t \sigma(r) + (1-t)r.
\]
Since \( (a \circ \mu_t)(i) \in (N(H) \cap N(K_i))/H \) for \( i = 0, 1 \), \( a \circ \mu_t \) is an element of \( Q \). Define \( q : Q_0 \times [0,1] \rightarrow Q \) by \( q(a,t) = a \circ \mu_t \).

Let \( \nu : [0,1] \rightarrow C^\infty([0,1],[0,1]) \) be a map given by \( \nu(t) = \mu_t \). Then it is easy to see that \( \nu \) is continuous. Note that \( q \) is given by the composition
\[
Q_0 \times [0,1] \xrightarrow{(1,\nu)} Q_0 \times C^\infty([0,1],[0,1]) \xrightarrow{\text{comp}} C^\infty([0,1],N(H)/H),
\]
where \( \text{comp} \) is given by \( \text{comp}(a,f) = a \circ f \). By Lemma 2.2 (6), \( q \) is continuous. Then \( q(Q_0 \times [0,1]) \) is contained in \( Q_0 \). Let \( q_t : Q_0 \rightarrow Q_0 \) be a map given by \( q_t(a) = q(a,t) \). Since \( \nu_t = \sigma \), \( q_t(Q_0) \) is contained in \( Q_1 \). Thus \( q \) is a homotopy between \( q_0 = 1_{Q_0} \) and \( q_1 = i \circ q_1 \). Note that \( q_t(Q_1) \) is contained in \( Q_1 \) for any \( t \). Then \( q_1 : Q_1 \times [0,1] \rightarrow Q_1 \) is a homotopy between \( 1_{Q_1} \) and \( q_1 \circ i \). Therefore Lemma 5.1 follows.

Put \( S = L^{-1}(Q_1) \subseteq \text{Ker} \ P \). Let \( i : S \hookrightarrow \text{Ker} \ P \) be an inclusion.
Lemma 5.2. \( \mathcal{S} \rightarrow \text{Ker } P \) is a homotopy equivalence.

Proof. Put \( a = L(h^{-1}) \) for \( h \in \text{Ker } P \). Let \( h_t : M \rightarrow M \) (\( 0 \leq t \leq 1 \)) be a map as follows: \( h_t \) is given on \( \pi^{-1}((0,1)) \) by the composition

\[
\pi^{-1}((0,1)) \xrightarrow{\alpha} G/H \times (0,1) \xrightarrow{h_t} G/H \times (0,1) \xrightarrow{\pi^{-1}} \pi^{-1}((0,1)),
\]

where \( h_t \) is defined by \( h_t(gH, r) = (g \cdot q_t(a(r)), r) \). \( h_t(gk_0) = ga(i) \cdot k_0(i = 0, 1) \) for \( g \in G \). Here we need the following

Assertion 5.3. \( h_t \) is a smooth map for any \( t \).

Proof. By the definition, \( h_t \) is smooth on \( \pi^{-1}((0,1)) \). We shall prove that \( h_t \) is smooth on \( \pi^{-1}(0) \). Let \( a_0 \) be an element of \( G \) such that \( a_0 \cdot H = a(0) \) and \( a_0 \in N(H) \cap N(K_0) \). For \( [g, 0] \in p_{0,1/2}^{-1}(1K_0) \), \( (p_{0,1/2} \circ h)([g, 0]) = \pi_0(ga_0) = \pi_0(a_0) \in a_0U_0 \). Then there exists a neighborhood \( \mathcal{W}_0 \) of \( 1K_0 \) in \( G/K_0 \) such that \( (p_{0,1/2} \circ h)(p_{0,1/2}^{-1}(\mathcal{W}_0)) \) is contained in \( a_0U_0 \). For \( [g, re_0] \in p_{0,1/2}^{-1}(\mathcal{W}_0) \) and \( 0 \leq t \leq 1 \),

\[
(p_{0,1/2} \circ h_t)([g, re_0]) = \pi_0(ga_t(a(r^2))) = \pi_0(ga(1-t) r^2)) = (p_{0,1/2} \circ h)([g, \sqrt{1-t} re_0])
\]

which is contained in \( (p_{0,1/2} \circ h)(p_{0,1/2}^{-1}(\mathcal{W}_0)) \). Then \( h_t(p_{0,1/2}^{-1}(g\mathcal{W}_0)) \) is contained in \( p_{0,1/2}^{-1}(ga_0U_0) \) for \( g \in G \) and \( 0 \leq t \leq 1 \).

Let \( \tilde{h} : \mathcal{W}_0 \times D_{1/2}((V_0) \rightarrow U_0 \times D_{1/2}(V_0) \) be a map given by \( \tilde{h} = \phi_0, ga_0 h^{-1} \) for \( g \in G \). Let \( \rho_1 : U_0 \times D_{1/2}(V_0) \rightarrow U_0 \) and \( \rho_2 : U_0 \times D_{1/2}(V_0) \rightarrow D_{1/2}(V_0) \) be the projections on the first factor and the second factor respectively. Put \( g' = ga_0 \) and put \( \tilde{h}_i = \rho_1 \circ \tilde{h} \) for \( i = 0, 1 \). Then \( \tilde{h}_i \) is a smooth map and

\[
\tilde{h}_1(x, rke_0) = g^{-1}(g_0(x)k \cdot \pi_0(a(r^2)) \)
\]

\[
\tilde{h}_2(x, rke_0) = g_0(x)(g_0(x)k \cdot \pi_0(a(r^2))^{-1} g_0(x)ka(r^2) \cdot re_0
\]

for \( x \in \mathcal{W}_0 \) and \( k \in K_0 \), where \( \pi_0 : G/H \rightarrow G/K_0 \) is the natural projection.

Put \( \tilde{h}_i = \rho_1 \circ \phi_0 g, \circ h_t \circ \phi_0^{-1} \) for \( i = 0, 1 \). Then

\[
\tilde{h}_1(x, rke_0) = g^{-1}(g_0(x)k \cdot \pi_0(a(\mu_t(r^2))\)
\]

- 24 -
\[ h_t(x, r e_0) = \sigma_0, g, (g_0(x) k \cdot \tau_0 (a(u_t(r^2)))^{-1} g_0(x) k a(u_t(r^2)) \cdot re_0 \]
for \( x \in W_0 \) and \( k \in K_0 \).

Since \( \sigma(r^2) = 0 \) for \( r \leq 1/2 \), \( u(r^2, t) = (1-t)r^2 \) for \( 0 \leq r \leq 1/2 \). Then
\[ h_t^1(x, v) = h_t^1(x, \sqrt{1-t} v) \quad \text{for} \quad 0 \leq t \leq 1 \]
and \( h_t^2(x, v) = 1/\sqrt{1-t} h_t^2(x, \sqrt{1-t} v) \)
for \( 0 \leq t < 1 \). Thus \( h_t^1 (0 \leq t \leq 1) \) and \( h_t^2 (0 < t < 1) \) are smooth maps.

By the Taylor formula (c.f. J. Dieudonné, [5, Chapter VIII, (8, 14, 3)]), we have
\[ h_t(x, v) = h_2(x, 0) + \left( \frac{\partial h_t^2}{\partial v} (x, 0, t) \right) dt v, \]
where \( \partial h_t^2 \) is the derivative of \( h_t^2 \). Since \( h_2(x, 0) = 0 \),
\[ h_t^2(x, v) = \left( \frac{\partial h_t^2}{\partial v} (x, \sqrt{1-t} v) \right) dt v \quad \text{for} \quad 0 \leq t < 1. \]
Then \( h_t^2(x, v) = \lim_{t \to 1} h_t^2(x, v) = \left( \frac{\partial h_t^2}{\partial v} (x, 0) \right) v, \) and \( h_t^2 \) is a smooth map.
Therefore \( h_t \) is smooth on \( \pi^{-1}(0) \) for any \( 0 \leq t \leq 1 \). Similarly we can prove that \( h_t \) is smooth on \( \pi^{-1}(1) \), and Assertion 5.3 follows.

Proof of Lemma 5.2 continued. Let \( \bar{q}: \text{Ker} P \times [0,1] \to \text{Ker} P \) be a map defined by \( \bar{q}(h, t) = h_t \). By Assertion 5.3, \( h_t \) and \( h_t^{-1} \) are smooth maps, and \( \bar{q} \) is a well defined map. Next we shall prove that \( \bar{q} \) is continuous. Let \( W_i \) be a neighborhood of \( 1K_i \) in \( G/K_i \) satisfying \( \bar{W}_i \subset U_i \) for \( i = 0, 1 \). Put \( O = \{ h \in \text{Ker} P \mid h(p_i, 1/2^{-1}(\bar{W}_i)) \subset p_i, 1/2^{-1}(U_i) \}
for \( i = 0, 1 \} \). Then \( O \) is an open neighborhood of \( 1_M \) in \( \text{Ker} P \). For \( h \in O \), \( g \in G \), \( 0 \leq t \leq 1 \), \( h_t(p_i, 1/2^{-1}(g \bar{W}_i)) \) is contained in \( p_i, 1/2^{-1}(g U_i) \) (\( i = 0, 1 \)). Let \( W_2 \) be an open neighborhood of \( 1H \) in \( G/H \) satisfying \( \bar{W}_2 \subset U_2 \).
Let \( C \) be one of the sets \( \{ p_i, 1/2^{-1}(g \bar{W}_i) \} \) (\( i = 0, 1 \), \( g \in G \), \( a^{-1}(g \bar{W}_2 \times [1/5, 4/5]) \) (\( g \in G \} \). By Lemma 2.3, it is sufficient to show that the composition \( \bar{q}_C: O \times [0,1] \to \text{Ker} P \)
\[ j_C: C \to C \to C \to (C, M) \]
continuous for any \( C \), where \( j_C: C \to M \) is an inclusion map.

First consider the case \( C = p_0, 1/2^{-1}(g \bar{W}_o) \). Let \( v_1: C \to C \to C \to C \to (\bar{W}_0 \times D_{1/2}(V_0), U_0) \) be a map given by \( v_1(f, t)(x, v) = f(x, \sqrt{1-t} v) \). Let \( v_2: C \to C \to C \to C \to C \to (\bar{W}_0 \times D_{1/2}(V_0), U_0) \)
be a map given by \( v_2(f, t)(x, v) = (\int_0^1 (Df)(x, \sqrt{1-t} v) d\zeta) (V) \). It is easy
to see that \(v_1\) and \(v_2\) are continuous. Note that \(\tilde{q}_C\) is the composition

\[
0 \times [0,1] \xrightarrow{(j_C^*,1)} C^\infty(C, P_0,1/2 (gU_0)\times [0,1])
\]

\[
(\phi_0, g)^* \circ (\phi_0, g)^* \circ \tilde{q}_C \xrightarrow{\phi_0, g)^*} C^\infty(\tilde{W}_0^0 \times D_{1/2} (V_0), U_0 \times D_{1/2} (V_0))\times [0,1]
\]

\[
(\rho_1^*, (\rho_2^*)^* \xrightarrow{\rho_1^*, (\rho_2^*)^*} C^\infty(\tilde{W}_0^0 \times D_{1/2} (V_0), U_0) \times C^\infty(\tilde{W}_0^0 \times D_{1/2} (V_0), D_{1/2} (V_0))
\]

\[
\kappa \xrightarrow{\kappa} C^\infty(\tilde{W}_0^0 \times D_{1/2} (V_0), U_0) \times C^\infty(\tilde{W}_0^0 \times D_{1/2} (V_0), U_0)
\]

Here \(v\) is given by \(v(f_1, f_2, t) = (v_1(f_1, t), v_2(f_2, t))\) and \(\kappa\) is the map defined in Lemma 2.2 (5). Then \(\tilde{q}_C\) is continuous by Lemma 2.2.

Next consider the case \(C = a_{-1}(gW_2 \times [1/5, 4/5])\). Let \(m: N(H)/H \times G/H \to G/H\) be a map defined by \(m(nH, gH) = (gn)H\) and \(p_2: G/H \times [1/5, 4/5] \to [0,1]\) be a map given by \(p_2(gH, r) = r\). Let \(\delta: Q_0 \to Q_0\) be a map given by \(\delta(a) = a_{-1}\). Then the map \(\tilde{q}_C\) is the composition

\[
0 \times [0,1] \xrightarrow{(1,1, L)} Q_0 \times [0,1] \xrightarrow{\delta \circ \tilde{q}} Q_0 \xrightarrow{\delta \circ \tilde{q}} C^\infty(G/H \times [1/5, 4/5], N(H)/H)
\]

\[
(1, G/H \times [1/5, 4/5]) \xrightarrow{\delta, G/H \times [1/5, 4/5]} C^\infty(G/H \times [1/5, 4/5], N(H)/H \times G/H \times [1/5, 4/5])
\]

\[
\xrightarrow{m} C^\infty(G/H \times [1/5, 4/5], G/H \times [1/5, 4/5])
\]

\[
(\alpha \circ j_C)^* \circ (\alpha_{-1})^* \xrightarrow{\alpha \circ j_C, (\alpha_{-1})^*} C^\infty(C, a_{-1}(G/H \times [1/5, 4/5])) \to C^\infty(C, a_{-1}(G/H \times [1/5, 4/5]))
\]

which is continuous because \(L\) and \(q\) are continuous.

Similarly as in the case \(C = P_0, 1/2(\tilde{g}W_0)\), we can see that \(\tilde{q}_C\) is continuous in the case \(C = P_0, 1/2(\tilde{g}W_1)\). Thus \(\tilde{q}\) is continuous.

Since \(q_1(Q_0) \subset Q_1\), \(\tilde{q}_1(\text{Ker } P) \subset S\). Therefore \(\tilde{q}\) is a homotopy between \(\tilde{q}_0 = 1_{\text{Ker } P}\) and \(\tilde{q}_1 = 1_{\text{Ker } \tilde{q}_1}\). Since \(q(Q_1 \times [0,1]) \subset Q_1\), \(\tilde{q}(S \times [0,1]) \subset S\).

Then \(\tilde{q}: S \times [0,1] \to S\) is a homotopy between \(1_S\) and \(\tilde{q}_1 \circ 1\). Thus \(\tilde{q}\) is a homotopy equivalence, and this completes the proof of Lemma 5.2.


§ 6. Proof of Theorem.

In this section, we shall see that $L: S \to Q_1$ is an isomorphism between topological groups, and we shall prove our Theorem.

Proposition 6.1. $L: S \to Q_1$ is an isomorphism between topological groups.

Before the proof of Proposition 6.1, we begin with some lemmas. For any topological subgroup $K$ of $G$, $K^0$ denotes the identity component of $K$.

Lemma 6.2. For any $a \in N(K_0)^0 \cap N(H)$, there exist $a' \in N(H^0) \cap K_0^0$ and $n \in \text{Cent}(K_0^0)$ such that $a = n \cdot a'$, where $\text{Cent}(K_0^0)$ is the centralizer of $K_0^0$ in $G$.

Proof. Since $N(K_0)^0$ is a compact connected Lie group, there exist a torus group $T$ and a simply connected semi-simple compact Lie group $G'$ such that $\hat{N}_0 = T \times G'$ is a finite covering group of $N(K_0)^0$ (c.f. L. Pontrjagin [8, §64]). Let $q_0: \hat{N}_0 \to N(K_0)^0$ be the covering projection. Put $\hat{K}_0 = q_0^{-1}(K_0^0)$. Since $K_0^0$ is a normal subgroup of $N(K_0)^0$, $\hat{K}_0$ is a normal subgroup of $\hat{N}_0$. Then $\hat{K}_0^0$ is also a normal subgroup of $\hat{N}_0$. Here we need the following:

Assertion 6.3. There exists a closed normal subgroup $K'_0$ of $\hat{N}_0$ such that $\hat{N}_0$ is isomorphic to the product group $\hat{K}_0^0 \times K'_0$ as a Lie group.

Proof. There exist simple Lie groups $G_i (1 \leq i \leq r)$ such that $G' = G_1 \times \ldots \times G_r$. Since $\hat{K}_0^0$ is a compact connected Lie group, there exist simply connected simple Lie groups $K_j (1 \leq j \leq s)$ and a torus group $T'$ such that $\tilde{K}_0 = T' \times K_1 \times \ldots \times K_s$ is a finite covering of $\hat{K}_0^0$. Let $\rho_i: \hat{N}_0 \to T \times G_1 \times \ldots \times G_r \to G_i$ be a projection on the direct factor $G_i (1 \leq i \leq r)$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$, $\rho_i(\hat{K}_0^0)$ is a normal subgroup of $G_i$. 

- 27 -
Since $G_i$ is a simple Lie group, $\rho_i(\hat{K}_0^0) = G_i$ or $\{1\}$. If $\rho_i(\hat{K}_0^0) = G_i$, $\rho_i(\hat{P}_0(\hat{K}_j))$ is a normal subgroup of $G_i$. Thus $\rho_i(\hat{P}_0(\hat{K}_j)) = G_i$ or $\{1\}$, for $\mathbf{lsisr}$, $\mathbf{lsjss}$.

Put $\rho'_i = \rho_i \circ P_0$. If $\rho'_i(\hat{K}_j) = \rho'_i(\hat{K}_j)$ ($j_1 \neq j_2$), then $\rho'_i(\hat{g}_1) \cdot \rho'_i(\hat{g}_2) = \rho'_i(\hat{g}_1 \cdot \hat{g}_2) = \rho'_i(\hat{g}_2 \cdot \hat{g}_1) = \rho'_i(\hat{g}_2) \cdot \rho'_i(\hat{g}_1)$ for $\hat{g}_1 \in \hat{K}_j_1$, $\hat{g}_2 \in \hat{K}_j_2$. Then $\rho'_i(\hat{K}_j)$ is a commutative normal subgroup of $G_i$, and $\rho'_i(\hat{K}_j) = \{1\}$.

If $\rho'_i(\hat{K}_j) = G_i$, then $\rho'_i(T')$ is a normal subgroup of $G_i$, hence $\rho'_i(T') = \{1\}$. Therefore, if $\rho'_i(\hat{K}_j) = G_i$, then $\rho'_i(T') = \{1\}$ and $\rho'_i(\hat{K}_n) = \{1\}$ for $n \neq j$.

Assume $\rho'_i(\hat{K}_j) = G_i$ and $\rho'_i(\hat{K}_j) = G_i$ for $i \neq j$. Let $\rho'_i : \hat{K}_0^0 \rightarrow G_i \times G_i$ be a map defined by $\rho'_i(\hat{k}) = (\rho'_i(\hat{k}), \rho'_i(\hat{k}))$. Since $\hat{K}_0^0$ is a normal subgroup of $\hat{N}_0$ and $\rho'(\hat{K}_0^0) = \rho'(\hat{K}_j)$, $\rho'(\hat{K}_j)$ is a normal subgroup of $G_i \times G_i$. Then, for $x, y \in K_j$, there exists $k \in K_j$ such that $(\rho'_{i_1}(x), 1) \rho'(y) (\rho'_{i_1}(x)^{-1}, 1) = \rho'(k)$. Then $\rho'_i(\hat{x}xy^{-1}) = \rho'_i(x)\rho'_i(y)\rho'_i(x)^{-1} = \rho'_i(k)$ and $\rho'_i(y) = \rho'_i(x)$. Since $K_j, G_i$ are simply connected simple Lie groups, $\rho'_i : K_j \rightarrow G_i$ is an isomorphism between the Lie groups. Thus $\hat{x}xy^{-1} = k = y$ for any $x, y \in K_j$, and $K_j$ must be a commutative Lie group, which is a contradiction since $K_j$ is a simple Lie group.

Thus we may assume that $\rho'_i(\hat{K}_j) = G_j$ and $\rho'_i(\hat{K}_j) = \{1\}$ ($i \neq j$) for $\mathbf{lsjss}$, $\mathbf{lsisr}$. For $i > s$, $\rho'_i(\hat{K}_0^0) = \rho'_i(\hat{K}_0^0) = \rho'_i(T')$ which is a commutative normal subgroup of $G_i$, hence $\rho'_i(T') = \{1\}$. Then $P_0(T')$ is a subgroup of $T$, and there exists a torus subgroup $S$ of $T$ such that $T = P_0(T') \times S$.

Put $K' = S \times G_{s+1} \times \ldots \times G_r$. Then $\hat{N}_0 = \hat{K}_0^0 \times K'$, and Assertion 6.3 follows.

Proof of Lemma 6.2 continued. By Assertion 6.3, there exists a closed normal subgroup $K'_0$ of $\hat{N}_0$ such that $\hat{N}_0 = \hat{K}_0^0 \times K'_0$. Since $\hat{K}_0^0$ is a connected group, $q_0(\hat{K}_0^0) = K_0^0$. Then $N(K_0^0) = q_0(\hat{N}_0) = q_0(\hat{K}_0^0) \cdot q_0(K'_0)$.
= K_0^0 \cdot q_0(K_0'). Note that q_0(K_0') is contained in Cent(K_0^0). Thus, for a \in N(K_0)^0 \cap N(H), there exists a' \in K_0^0 and n \in Cent(K_0^0) such that a = a' \cdot n. Since N(H) \subseteq N(H^0) and H^0 \subseteq K_0^0, H^0 = aH^0 a^{-1} = a' \cdot nH^0 n^{-1} a', a' = a_0^0 a_1^{-1}. Thus a' \in N(H^0) and Lemma 6.2 follows.

For a \in Q_1, we define a map h: \mathbb{M} \rightarrow \mathbb{M} as follows:

$$
h(a^{-1}(g, r)) = a^{-1}(ga(r)^{-1}, r) \quad \text{for} \quad (g, r) \in G/H \times (0, 1),
$$

$$
h([g, 0]) = [ga(i)^{-1}, 0] \quad \text{for} \quad [g, 0] \in \pi^{-1}(i)(i = 0, 1).
$$

Lemma 6.4. h is a smooth map.

Proof. Choose a_0 \in (N(H) \cap N(K_0))^0 \cap N(H^0) \cap N(K_0)^0 such that a_0^{-1} = a_0 H. There exists a neighborhood W_0 of 1K_0 in G/K_0 such that \pi_0^{-1}(W_0) \cdot a_0 is contained in a_0 \cdot \pi_0^{-1}(U_0). Since a(r) = a(0) for 0 \leq r \leq 1/4, h(p_0, 1/2(g, w_0)) is contained in p_0, 1/2(ga_0 U_0). Let \widetilde{h}_1: W_0 \times D_{1/2}(V_0) \rightarrow U_0 be a map given by the composition p_1 \circ \phi_0, ga_0 \circ h_0^{-1}, g. And let \widetilde{h}_2: W_0 \times D_{1/2}(V_0) \rightarrow D_{1/2}(V_0) be a map given by the composition \rho_2 \circ \phi_0, ga_0 \circ h_0^{-1}, g. Note that

$$
(h \circ \phi_0, g)^{-1}((x, r, k, e_0)) = h([g_0(x), k, e_0]) = h(a^{-1}(g_0(x) k H, r^2)) = a^{-1}(g_0(x) k a_0 H, r^2) = [g_0(x) k a_0, e_0]
$$

for x \in W_0, k \in K_0, 0 \leq r \leq 1/2. Since a_0 \in N(K_0), ka_0 K_0 = a_0 K_0. Then

$$
\widetilde{h}_1(x, v) = a^{-1}_0(x) a_0 K_0 \quad \text{for} \quad (x, v) \in W_0 \times D_{1/2}(V_0), \quad \text{and}
$$

$$
\widetilde{h}_2(x, r, k, e_0) = g_0(x, a_0 K_0) g_0^{-1}(x) a_0 K_0 \quad \text{for} \quad (x, v) \in W_0, k \in K_0, 0 \leq r \leq 1/2. \quad \text{Thus} \quad \tilde{h}_1 \quad \text{is a smooth map and} \quad \tilde{h}_2 \quad \text{is smooth on}
$$

W_0 \times (D_{1/2}(V_0) - 0). We shall prove that \tilde{h}_2 is smooth on W_0 \times 0, hence h is smooth on \pi^{-1}(0). This is trivial in the case dim S(V_0) = 0.

Let \xi_{a_0, g}: W_0 \rightarrow G be a map given by \xi_{a_0, g} = g_0, a_0 (g_0(x) a_0 K_0)^{-1} g_0^{-1}(x). Then \xi_{a_0, g} is a smooth map. By Lemma 6.2, there

- 29 -
exist $a'_0 \in N(H^0_0) \cap K_0^0$ and $n \in \text{Cent}(K_0^0)$ such that $a_0 = na'_0$. Then
$$\tilde{h}_2(x, rke_0) = \xi a_0 \cdot g(x) nka'_0 \cdot rke_0 = \xi a_0 \cdot g(x) nka'_0 \cdot re_0$$
for $x \in W_0$, $k \in K_0^0$ and $0 < r < 1/2$. Note that $N(H^0_0) \cap K_0^0 = N(H^0_0, K_0^0)$.

Assertion 6.5. For $a \in N(H^0_0, K_0^0)$, let $\psi_a: D(V_0) \to D(V_0)$ be a map defined by $\psi_a(rke_0) = rkae_0$ for $0 < r < 1$, $k \in K$. Then $\psi_a$ is a diffeomorphism. Moreover, let $\psi: N(H^0_0, K_0^0) \to \text{Diff}^c(D(V_0))$ be a map given by $\psi(a) = \psi_a$, then $\psi$ is continuous.

Proof. If $\dim S(V_0) = 0$, then $K_0^0 \subset H$ and $\psi_a = 1_{D(V_0)}$. In this case, the proof is trivial. We assume $\dim S(V_0) > 0$. Since $S(V_0) = K_0^0/H$ is connected, $K_0^0$ acts transitively on $S(V_0)$. Let $L$ be the ineffective kernel of the action $K_0^0 \times S(V_0) \to S(V_0)$. Put $\bar{K} = K_0^0/L$ and $\bar{H} = (H \cap K_0^0)/L$. Then $\bar{K}$ acts transitively and effectively on $S(V_0)$ and $\bar{H}$ is an isotropy subgroup of this action. By Lemma 4.3, $\bar{K}$, $\bar{H}$ and $N(\bar{H}, \bar{K})/\bar{H}$ are $G$, $H$ and $N(H)/H$ in Lemma 4.3, respectively. Hence $\bar{H}$ is connected. Since the identity component of $H \cap K_0^0$ is $H^0$, $\bar{H} = H^0 \cdot L/L$. For $a \in N(H^0_0, K_0^0)$, the left coset $aL$ is an element of $N(\bar{H}, \bar{K})$. Then $a$ defines an element $\bar{a} \in N(\bar{H}, \bar{K})/\bar{H}$. Note that $\psi_a(rke_0) = rkae_0$ for $0 < r < 1$, $k \in K_0^0$.

Consider the case $\bar{K} = \text{SU}(n)$ ($n \geq 2$), $\bar{H} = \text{SU}(n-1)$ and $N(\bar{H}, \bar{K})/\bar{H} = U(1)$. In this case, $V_0 = \mathbb{C}^n$ and $U(1)$ acts on $V_0$ as a scalar multiplication. Thus $\psi_a(rke_0) = \bar{a} \cdot rke_0$ for $rke_0 \in D(V_0)$. Hence $\psi_a$ is a diffeomorphism. It is easy to see that $\psi$ is continuous.

Next consider the case $\bar{K} = \text{Sp}(n)$ ($n \geq 2$), $\bar{H} = \text{Sp}(n-1)$ and $N(\bar{H}, \bar{K})/\bar{H} = \text{Sp}(1)$. In this case, $V_0 = \mathbb{H}^n$ and $\text{Sp}(1)$ acts on $V_0$ as a scalar multiplication on the right. Then $\psi_a(v) = v \cdot \bar{a}$ for $v \in D(V_0)$, hence $\psi_a$ is a diffeomorphism and $\psi$ is continuous. Similarly we can see that $\psi_a$ is a diffeomorphism and $\psi$ is continuous in the other cases, and Assertion 6.5 follows.

Proof of Lemma 6.4 continued. Since $\tilde{h}_2(x, v) = \xi a_0 \cdot g(x) n \cdot \psi_{a_0}(v)$,
by Assertion 6.5, \( \tilde{h}_2 \) is a smooth map. Thus \( \tilde{h}_1 \) and \( \tilde{h}_2 \) are smooth maps, hence \( h \) is smooth on \( \pi^{-1}(0) \). Similarly we can see that \( h \) is smooth on \( \pi^{-1}(1) \). By the definition, \( h \) is smooth on \( \pi^{-1}((0, 1)) \), and this completes the proof of Lemma 6.4.

Let \( \hat{L}(a) \) be a smooth map \( h: M \to M \) in Lemma 6.4, for \( a \in Q_1 \).

Since \( \hat{L}(a^{-1}) = \hat{L}(a)^{-1} \), \( h \) is a diffeomorphism of \( M \). By the definition, \( h \) is an equivariant map. Thus we have a map \( \hat{L}: Q_1 \to \text{Diff}_G^\infty(M) \).

Note that \( \hat{L} \) is an abstract group homomorphism.

Lemma 6.6. \( \hat{L}: Q_1 \to \text{Diff}_G^\infty(M) \) is continuous:

Proof. Let \( W_i \) be a neighborhood of \( 1K_i \) in \( G/K_i \) such that \( \bar{W}_i \subset U_i \) (\( i = 0, 1 \)), and let \( W_2 \) be a neighborhood of \( 1H \) in \( G/H \) such that \( \bar{W}_2 \subset U_2 \).

Put \( A_i = \{ n \in N(K_i)^0; n^{-1}\bar{W}_i n \subset U_i \} \). Then \( A_i \) is an open neighborhood of the identity in \( N(K_i)^0 \). Let \( q_i^\alpha: \hat{N}_i \to N(K_i)^0 \) be a finite covering such that \( \hat{N}_i \) is a direct product \( T_i \times G'_i \). Here \( T_i \) is a torus group and \( G'_i \) is a simply connected semi-simple compact Lie group. Put \( \hat{K}_i = q_i^{-1}(K_i^0) \). By Assertion 6.3, there exists a closed normal subgroup \( K'_i \) of \( \hat{N}_i \) such that \( \hat{N}_i = \hat{K}_i^0 \times K'_i \). Let \( s_i \) be a smooth local cross section of \( q_i^\alpha \) defined on an open neighborhood \( B_i \) of the identity in \( N(K_i)^0 \). Since \( \pi_2^\alpha: (N(H) \cap N(K_i))^0 \to ((N(H) \cap N(K_i))/H)^0 \) is a fibration, there exists a smooth local cross section \( t_i \) of \( \pi_2^\alpha \) defined on an open neighborhood \( E_i \) of \( 1H \) such that \( t_i(E_i) \subset A_i \cap B_i \).

Put \( O = \{ a \in Q_1; a(i)^{-1} \in E_i \ (i = 0, 1) \} \). Then \( O \) is an open neighborhood of the identity. Since \( \hat{L} \) is a group homomorphism, it is enough to show Lemma 6.6 that \( \hat{L} \) is continuous on \( O \). Let \( C \) denote one of the sets \( \{ p_i^{-1}(g\bar{W}_1) \ (i = 0, 1, g \in G), \ a^{-1}(g\bar{W}_2 \times [1/5, 4/5]) \ (g \in G) \} \).

By Lemma 2.3, if \( \hat{L}_C: O \to \text{Diff}_G^\infty(M)^0 \overset{j^*}{\to} C^\infty(C, M) \) is continuous for any \( C \), then \( \hat{L} \) is continuous, where \( j_C: C \subset M \) is an inclusion map.
First consider the case $C = p_{0,1/2}^{-1}(g\bar{w}_i)$. Let $\beta_1: \hat{N}_0 = \hat{k}_0 \times k'_0 \rightarrow \hat{k}_0$ and $\beta_2: \hat{N}_0 \rightarrow k'_0$ be the projection on the first factor and the second factor respectively. Let $L_1$ be the composition

$$0 \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{(\xi', q_0^* \beta_2 s_0)} C^\infty(\bar{w}_0, G) \times \text{Cent}(k_0^0) \xrightarrow{m} C^\infty(\bar{w}_0, G).$$

Here $r$, $\xi'$ and $m$ are given by $r(a) = a(0)^{-1}$, $\xi'_q(a_0)(x) = \xi'_{a_0}g(x)$ and $m(f, n)(x) = f(x) \cdot n$, respectively. Put $a_0 = (t_0 \circ r)(a)$ for $a \in 0$.

Then $\pi_0(\xi'_q, a_0(x)) = \pi_0(a_0^{-1})$ for $w \in \bar{W}_0$ and $\pi_0((q_0 \circ \beta_2 \circ s_0)(a_0)) = \pi_0(a_0)$. Therefore $L_1(a) \in k_0$ for any $a \in 0$, and $L_1(0) \subseteq C^\infty(\bar{W}_0, k_0)$. Let $L_2$ be the composition

$$0 \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \cap B_0 \xrightarrow{q_0^* \beta s_0} N(H^0, k_0^0) \xrightarrow{\psi} \text{Diff}(D_{1/2}(V_0)).$$

By Assertion 6.5, $L_2$ is continuous. Let $L_3$ be the composition

$$0 \xrightarrow{(L_1, L_2)} C^\infty(\bar{w}_0, k_0) \times \text{Diff}(D_{1/2}(V_0)) \xrightarrow{(\rho_1^*, \rho_2^*)} C^\infty(\bar{w}_0 \times D_{1/2}(V_0), k_0) \times C^\infty(\bar{w}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)) \xrightarrow{\kappa} C^\infty(\bar{w}_0 \times D_{1/2}(V_0), k_0 \times D_{1/2}(V_0)) \xrightarrow{\mu} C^\infty(\bar{w}_0 \times D_{1/2}(V_0), D_{1/2}(V_0)),$$

where $\mu$ is given by $\mu(k, v) = k \cdot v$, and $\kappa$ is the map in Lemma 2.2. Then $L_3$ is continuous, and $L_3(a) = \tilde{h}_2$. Let $\gamma: A_0 \rightarrow C^\infty(\bar{w}_0, U_0)$ be a map defined by $\gamma(a_0)(x) = a_0^{-1}g_0(x)a_0k_0$. $\gamma$ is a restriction map to $A_0$ of a map $\bar{\gamma}: N(k_0) \rightarrow C^\infty(G/k_0, G/k_0)$ given by $\bar{\gamma}(n)(gk_0) = n^{-1}gnk_0$.

Since $\bar{\gamma}$ is a continuous map, $\gamma$ is continuous. Let $L_4$ be the composition

$$0 \xrightarrow{r} E_0 \xrightarrow{t_0} A_0 \xrightarrow{\gamma} C^\infty(\bar{w}_0, U_0) \xrightarrow{\rho_1^*} C^\infty(\bar{w}_0 \times D_{1/2}(V_0), U_0 \times D_{1/2}(V_0)).$$

Then $L_4$ is continuous and $L_4(h) = \tilde{h}_1$. $L_C$ is the composition

$$0 \xrightarrow{(L_4, L_3)} C^\infty(\bar{w}_0 \times D_{1/2}(V_0), U_0 \times D_{1/2}(V_0)) \xrightarrow{\kappa} C^\infty(\bar{w}_0 \times D_{1/2}(V_0), U_0 \times D_{1/2}(V_0)) \xrightarrow{(\phi_0, g)} C^\infty(C, p_0^{-1}(gU_0)) \subseteq C^\infty(C, M).$$

Thus $L_C$ is continuous.

Now consider the case $C = a^{-1}(g\bar{w}_2 \times [1/5, 4/5])$. Let $m: g\bar{w}_2 \times N(H)/H$
be a map defined by $m(gH, nH) = gnH$, and let $\rho: G/H \times [1/5, 4/5] \to [1/5, 4/5]$ be the projection on the second factor. Then $LC$ is given by the composition

$$0 \xrightarrow{i \ast \delta} C^\infty([1/5, 4/5], N(H)/H)$$

$$\xrightarrow{(1,g\tilde{W}_2)} C^\infty(g\tilde{W}_2 \times [1/5, 4/5], g\tilde{W}_2 \times N(H)/H)$$

$$\xrightarrow{m \ast} C^\infty(g\tilde{W}_2 \times [1/5, 4/5], G/H)$$

$$\xrightarrow{P_\#} C^\infty(g\tilde{W}_2 \times [1/5, 4/5], G/H \times [1/5, 4/5])$$

$$\xrightarrow{a \ast \circ (a^{-1})} C^\infty(C, a^{-1} (G/H \times [1/5, 4/5]) \hookrightarrow C^\infty(C, M),$$

where $i: [1/5, 4/5] \subseteq [0,1]$ is the inclusion map and $\delta: N(H)/H \to N(H)/H$ is a map given by $\delta(a) = a^{-1}$. By Lemma 2.2, $LC$ is continuous.

We can see that $LC$ is continuous in the case $C = p_{1,1/2}^{-1}(g\tilde{W}_1)$ similarly as in the case $C = p_{0,1/2}^{-1}(g\tilde{W}_0)$, and this completes the proof of Lemma 6.6.

Proof of Proposition 6.1. From Lemma 6.6, $L(Q_1)$ is contained in $Diff_G^\infty(M)$. Then, by the definition, $L(Q_1)$ is contained in $S$, and $L = L^{-1}$. Combining Lemma 4.5, Proposition 4.6 and Lemma 6.6, $L: S \to Q_1$ is an isomorphism between topological groups, and this completes the proof of Proposition 6.1.

Proof of Theorem. By Corollary 3.6, $Diff_G^\infty(M)$ has the same homotopy type as $Ker P$. Combining Lemma 5.1, Lemma 5.2 and Proposition 6.1, $Ker P$ has the same homotopy type as $Q_0$. Note that $Q_0$ has the same homotopy type as the path space $\mathcal{N}(N(H)/H; (N(H) \cap N(K_0))/H, (N(H) \cap N(K_1))/H)_0$. This completes the proof of our Theorem.
§7. Concluding remarks.

From our Theorem, we have the following:

Corollary 7.1. (1) If \( K_0 = K_1 = G \), then \( \text{Diff}_G^\infty(M)_0 \) has the same homotopy type as \( (N(H)/H)_0 \).

(2) If \( N(H)/H \) is a finite group, then \( \text{Diff}_G^\infty(M)_0 \) is contractible.

Remark 7.2. In K. Abe and K. Fukui [1], we have proved that \( \text{Diff}_G^\infty(M)_0 \) is perfect if \( M \) is a \( G \)-manifold with one orbit type and \( \dim M/G \geq 1 \). But, by using Proposition 3.1, we can see that \( \text{Diff}_G^\infty(M)_0 \) is not perfect in the case \( M/G = [0,1] \).

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References

