学位申請 論文

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The Radiation Reaction Effects in the Solutions of the Perturbed Einstein Equations

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Abstract

The gravitational radiation reaction effects in the systems described by the perturbations of given solutions of the Einstein equations are considered. There are two kinds of perturbations to be considered; one is the perturbation induced by no external source and the other is the perturbation due to the presence of a source particle. For the former case, we find that there exists a conserved current constructed from a quadratic combination of the solutions to the linearly perturbed equations, provided that the unperturbed geometry admits a Killing vector. Thus, some effects of radiation reaction are found to be included in the linear approximation. For the latter case, it is found that the usual perturbation expansion scheme fails but there is a possible approach analogous to the one in the Lorentz-Dirac theory of charged particles in order to include the reactive effects. By this approach we find that a naive argument on the energy conservation leads an additonal reactive term which contributes to the energy equation. However this term is found to be negligible if the particle is under a quasi-periodic motion.

§ I Introduction

Much work has been done on the theory of gravitatonal waves since Einstein discovered the existence of wave solutions to his equation. However, mainly because of the non-localizability of the energy of gravitational fields, we know almost nothing about how the generation of gravitational waves affects a system which radiates. One of systems in which the radiation reaction problems can be treated fairly easily is the one which can be described by perturbations of a known solution of the Einstein equations. In this paper, we will present an analysis of the perturbed Einstein equations up to second order in the perturbation amplitude, since the amplitude of gravitational waves induced by the small perturbations is of this order.

There are two kinds of perturbations to be considered. One of them may be called homogeneous in the sense that the perturbation is due to no external source. The other is the one due to external sources, especially by the presence of a particle, and therefore may be called inhomogeneous.

A good example for the homogeneous case is the non-radial pulsation of a spherical star. Thorne made an analysis of this example to first order in the pulsation amplitude.[1)] He found that radiation damping of the pulsation occurs and the associated energy loss rate balances the radiation power of the emitted gravitational waves. One might expect that his conclusion could be generalized; i.e. the (effective) energy-momentum tensor induced by the first order perturbations could be defined and would follow the covariant conservation laws. However, in § II, we will show this is generally not true. None the less, we find that if the unperturbed geometry has a Killing vector, one can construct a conserved current out of the quantities satisfying the equations of linear perturbations.

For inhomogeneous perturbations, the known linear approximation scheme requires the source particle to follow a geodesic of the unperturbed space-time.[2)] As a result no radiation reaction effect can be included contrary to homogeneous perturbations. In § III, we will consider the inhomogeneous perturbation of a vacuum space-time. We find that the simple perturbation expansion does not work in this case. Then, we will present a natural approach in order to include the radiation reaction effects. this approach is analogous to the one in the Lorentz-Dirac theory of charged particles, but in the gravitational theory there are intrinsic difficulties such as non-renormalizability of the self-fields or non-localizability of the gravitational field energy. Then without any explicit calculations, the perturbation of the Schwarzschild geometry will be discussed. We find there that, for a particle under a quasi-periodic motion, the radiation reaction effect on the energy of the particle agrees with a naive argument on the energy conservation.

In § IV, the cases considered in § II and § III will be compared, by restricting them to the perturbations of the Schwarzschild geometry, in order to understand better the differences underlying between these two cases.

In §V, discussion on the nature of inhomogeneous perturbations will be given.

§II The conserved current of the linearized fields

Let us consider a one-parameter family of solutions $(g_{\alpha\beta}(\varepsilon),$ $\phi_A(\varepsilon))$ to the Einstein equations*)

$$G^{\mu\nu}(g_{\alpha\beta}(\varepsilon)) = 8\pi T^{\mu\nu}(\phi_A(\varepsilon); g_{\alpha\beta}(\varepsilon)) \quad , \tag{2-1a}$$

and the equations of matter

$$F^A(\phi_B(\varepsilon); g_{\alpha\beta}(\varepsilon)) = 0 \quad , \tag{2-1b}$$

where ϕ_A is a matter field to be taken into account and the capital Latin indices represent the tensor indices and/or species of the matter field. We denote the $\varepsilon=0$ solution of this family by $(\overset{\circ}{g}_{\alpha\beta}, \overset{\circ}{\phi}_A)$ and assume it to be a globally stable solution, i.e. for any $\varepsilon \ll 1$, $(g_{\alpha\beta}(\varepsilon), \phi_A(\varepsilon))$ is everywhere close to $(\overset{\circ}{g}_{\alpha\beta}, \overset{\circ}{\phi}_A)$. Also the solutions of this family are assumed to vary smoothly with respect to ε. Then, the perturbed Einstein and matter equations are obtained by expanding Eqs.(2-1) in terms of ε. Instead of performing this procedure directly, however, we derive the perturbed equations by appealing to the action principle, since the properties of perturbations can be studied more easily.

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*) In this paper, we use the units c=G=1 and use the metric with signature (− + + +).

Let S be the action which gives the (Einstein and matter) field equations. Then S is the sum of the gravitational action S_G and the matter action S_M;

$$\mathcal{S}[g_{\alpha\beta},\phi_A] = \mathcal{S}_G[g_{\alpha\beta}] + \mathcal{S}_M[\phi_A\,;\,g_{\alpha\beta}] \quad , \tag{2-2}$$

where

$$\mathcal{S}_G[g_{\alpha\beta}] = \frac{1}{16\pi}\int R\sqrt{-g}\,d^4x \qquad \text{and} \qquad \mathcal{S}_M[\phi_A\,;g_{\alpha\beta}] = \int L_M(\phi_A,\nabla_\alpha\phi_A)\sqrt{-g}\,d^4x \ .$$

The Einstein and matter field equations are respectively obtained by requiring that S be stationary under variations of $g_{\mu\nu}$ and ϕ_A;

$$\frac{\delta\mathcal{S}[g_{\alpha\beta},\phi_B]}{\delta g_{\mu\nu}} = -\frac{\sqrt{-g}}{16\pi}\left[G^{\mu\nu}(g_{\alpha\beta}) - 8\pi T^{\mu\nu}(\phi_B\,;g_{\alpha\beta})\right] = 0 \ , \tag{2-3a}$$

$$\frac{\delta\mathcal{S}[g_{\alpha\beta},\phi_B]}{\delta\phi_A} = \frac{\delta\mathcal{S}_M[\phi_B\,;g_{\alpha\beta}]}{\delta\phi_A} = \sqrt{-g}\,F^A(\phi_B\,;g_{\alpha\beta}) = 0 \ . \tag{2-3b}$$

Now, in order to obtain the perturbed equations, we set $(g_{\mu\nu},\ \phi_A) = (\mathring{g}_{\mu\nu}+\varepsilon h_{\mu\nu},\ \mathring{\phi}_A+\varepsilon\varphi_A)$ in the action S and expand it in terms of ε ;

$$\mathcal{S}[g,\phi] = \mathcal{S}[\mathring{g},\mathring{\phi}] + \varepsilon\mathcal{S}_1[h,\varphi\,;\mathring{g},\mathring{\phi}] + \varepsilon^2\mathcal{S}_2[h,\varphi\,;\mathring{g},\mathring{\phi}] + O(\varepsilon^3) \ , \tag{2-4}$$

where indices of the fields are suppressed for simplicity. S_1 and S_2 are defined by

$$\mathcal{S}_1[h,\varphi\,;\mathring{g},\mathring{\phi}] = \int d^4x\left(\frac{\delta\mathcal{S}}{\delta g_{\mu\nu}}\bigg|_{\varepsilon=0}h_{\mu\nu} + \frac{\delta\mathcal{S}_M}{\delta\phi_A}\bigg|_{\varepsilon=0}\varphi_A\right) \quad , \tag{2-5}$$

$$\mathcal{S}_2[h,\varphi\,,\mathring{g},\mathring{\phi}] = \int d^4x\,d^4x'\left(\frac{1}{2}\frac{\delta^2\mathcal{S}}{\delta g_{\alpha\beta}\delta g_{\mu\nu}}\bigg|_{\varepsilon=0}h_{\alpha\beta}h_{\mu\nu} + \frac{\delta^2\mathcal{S}_M}{\delta g_{\mu\nu}\delta\phi_A}\bigg|_{\varepsilon=0}h_{\mu\nu}\varphi_A + \frac{1}{2}\frac{\delta^2\mathcal{S}}{\delta\phi_A\,\delta\phi_B}\bigg|_{\varepsilon=0}\varphi_A\varphi_B\right) . \tag{2-6}$$

Since $(\overset{\circ}{g}, \overset{\circ}{\phi})$ is a solution to Eqs.(2-3), S_1 vanishes and Eq.(2-4) becomes

$$S[g,\phi] = S[\overset{\circ}{g},\overset{\circ}{\phi}] + \varepsilon^2 S_2[h,\varphi;\overset{\circ}{g},\overset{\circ}{\phi}] + O(\varepsilon^3) \quad . \tag{2-7}$$

Then, noting that variations of $g_{\mu\nu}$ is equivalent to those of $\varepsilon h_{\mu\nu}$, we insert the expression (2-7) of $S[g,\phi]$ into Eq.(2-3a) and obtain

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{\delta S}{\varepsilon\delta h_{\mu\nu}} = \varepsilon\frac{\delta S_2}{\delta h_{\mu\nu}} + O(\varepsilon^2) = 0 \quad . \tag{2-8}$$

Also, from the same argument on variations of ϕ_A, we obtain

$$\frac{\delta S}{\delta\phi_A} = \frac{\delta S}{\varepsilon\delta\varphi_A} = \varepsilon\frac{\delta S_2}{\delta\varphi_A} + O(\varepsilon^2) = 0 \quad . \tag{2-9}$$

Thus, the perturbed field equations to the linear order in ε can be derived from the action S_2 and are given by

$$\frac{\delta S_2[h,\varphi;\overset{\circ}{g},\overset{\circ}{\phi}]}{\delta h_{\mu\nu}} = 0 \quad , \quad \frac{\delta S_2[h,\varphi;\overset{\circ}{g},\overset{\circ}{\phi}]}{\delta\varphi_A} = 0 \quad . \tag{2-10}$$

We note that, apart from an irrelevant total divergence, one may introduce the Lagrangian L of the fields $h_{\mu\nu}$ and φ_A, which is integrated to give S_2, i.e.,

$$S_2[h,\varphi;g,\phi] = \int L(h,\nabla h,\varphi,\nabla\varphi;\phi,\nabla\phi,g)\sqrt{-g}\,d^4x \quad , \tag{2-11}$$

where covariant differentiation is denoted by ∇.*) In the follow-

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*) Through out this paper covariant differentiation is denoted by ; or ∇ interchangingly. Similarly, partial differentiation is denoted by , or ∂ .

ing as well as in Eq.(2-11), a naught over an unperterbed quantity will be omitted; thus we regard (g, ϕ) as an unperterbed solution, unless otherwise stated.

Now, as S_2 contains all information about the properties of the linear perturbations, in particular it contains information about the existence or non-existence of conservation equations, which must be a consequence of a certain invariace property of S_2. Therefore let us consider an infinitesimal transformation of the coordinates x^μ in the integral (2-11);

$$x'^\mu = x^\mu - \varepsilon \xi^\mu \quad ; \quad \varepsilon \ll 1 \quad . \tag{2-12}$$

Because of the general covariance of L, S_2 is invariant under this transformation, and we obtain the identity

$$0 = \delta S_2 = \varepsilon \int d^4x \left(\frac{\delta S_2}{\delta g_{\mu\nu}} \pounds_\xi g_{\mu\nu} + \frac{\delta S_2}{\delta \phi_A} \pounds_\xi \phi_A + \frac{\delta S_2}{\delta h_{\mu\nu}} \pounds_\xi h_{\mu\nu} + \frac{\delta S_2}{\delta \varphi_A} \pounds_\xi \varphi_A \right) , \tag{2-13}$$

where $\pounds_\xi$ is the Lie derivative operator with respect to ξ^μ. For a field ϕ_A, the general form of $\pounds_\xi \phi_A$ is written as

$$\pounds_\xi \phi_A = \phi_{A;\alpha} \xi^\alpha + f_A{}^{\alpha\beta} \xi_{\alpha;\beta} \quad , \tag{2-14}$$

where $f_A{}^{\alpha\beta}$ is a function of ϕ_A and $g_{\mu\nu}$. Especially for the metric $g_{\mu\nu}$, one has

$$\pounds_\xi g_{\mu\nu} = \xi_{\mu;\nu} + \xi_{\nu;\mu} \quad . \tag{2-15}$$

If a pair $(h_{\mu\nu}, \varphi_A)$ saitsfies the linearized equations (2-10), the third and fourth terms in the integrand of Eq.(2-13) vanish.

Then, inserting Eqs.(2-14) and (2-15) into Eq.(2-13), performing integration by parts, and noting that ξ^{μ} is arbitrary, we obtain

$$(\overset{\text{eff}}{T}{}^{\mu\nu} - \overset{\text{eff}}{J}{}^{A} f_{A}{}^{\mu\nu})_{;\nu} + \overset{\text{eff}}{J}{}^{A} \phi_{A}{}^{;\mu} = 0 \quad , \tag{2-16}$$

where $\overset{\text{eff}}{T}{}^{\mu\nu}$ and $\overset{\text{eff}}{J}{}^{A}$ are defined by

$$\overset{\text{eff}}{T}{}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_2}{\delta g_{\mu\nu}} \quad , \tag{2-17a}$$

$$\overset{\text{eff}}{J}{}^{A} = -\frac{1}{\sqrt{-g}} \frac{\delta S_2}{\delta \phi_A} \quad . \tag{2-17b}$$

We shall call them the effective energy-momentum tensor and the effective current, respectively, of the linear perturbation. The reason for the presence of the adjective, "effective", will be explained later.

From Eq.(2-16), one finds that no covariant conservation equation may exist in general. However, if the unperturbed fields $g_{\mu\nu}$ and ϕ_A admit a Killing vector K^{μ}, the Lie derivatives of $g_{\mu\nu}$ and ϕ_A with respect to K^{μ} vanish;

$$\pounds_K g_{\mu\nu} = \pounds_K \phi_A = 0 \quad . \tag{2-18}$$

In this case, the contraction of Eq.(2-16) with K^{μ} gives

$$\left[K_{\mu} (\overset{\text{eff}}{T}{}^{\mu\nu} - \overset{\text{eff}}{J}{}^{A} f_{A}{}^{\mu\nu}) \right]_{;\nu} = 0 \quad . \tag{2-19}$$

Thus, the vector P_K^{μ} defined by

$$P_K^{\mu} = K_{\nu} (\overset{\text{eff}}{T}{}^{\nu\mu} - \overset{\text{eff}}{J}{}^{A} f_{A}{}^{\nu\mu}) \tag{2-20}$$

is a conserved current constructed from the solution $(h_{\mu\nu}, \varphi_A)$ of the linearized field equations. Therefore, we conclude that if the unperturbed fields admit a Killing vector there exists a corresponding conservation equation involving the fields of a linearized solution quadratically. This implies that some effects of radiation reaction are included already in the linear approximation. As examples, the application of the above discussion to the electromagnetic field and a perfect fluid cases as matter fields is considered in the appendix.

Now we give the reason for calling $\overset{eff}{T}{}^{\mu\nu}$ and $\overset{eff}{J}{}^{A}$ as "effective"; let us denote an unperturbed solution as $(\overset{\circ}{g}, \overset{\circ}{\phi})$ temporarily as before, and a linearized solution as $(\overset{1}{h}, \overset{1}{\varphi})$. Once $(\overset{\circ}{g}, \overset{\circ}{\phi})$ and $(\overset{1}{h}, \overset{1}{\varphi})$ are fixed, the field variables $g_{\mu\nu}$ and ϕ_A are expressed as

$$g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \varepsilon \overset{1}{h}_{\mu\nu} + \varepsilon^2 h_{\mu\nu} \quad , \tag{2-21a}$$

$$\phi_A = \overset{\circ}{\phi}_A + \varepsilon \overset{1}{\varphi}_A + \varepsilon^2 \varphi_A \quad , \tag{2-21b}$$

and variations of $g_{\mu\nu}$ and ϕ_A are respectively equivalent to those of $\varepsilon^2 h_{\mu\nu}$ and $\varepsilon^2 \varphi_A$. Then, repeating the same argument which led to Eqs.(2-8) and (2-9), and abandoning a naught over an unperturbed field again, we obtain the second order perturbation equations

$$\frac{\delta S_2[h, \varphi; g, \phi]}{\delta h_{\mu\nu}} + \frac{\delta S_2[\overset{1}{h}, \overset{1}{\varphi}; g, \phi]}{\delta g_{\mu\nu}} = 0 \quad , \tag{2-22a}$$

$$\frac{\delta S_2[h,\varphi;g,\phi]}{\delta\varphi_A} + \frac{\delta S_2[\overset{1}{h},\overset{1}{\varphi};g,\phi]}{\delta\phi_A} = 0 \quad , \tag{2-22b}$$

where we have made use of the definition of S_2 (see Eq.(2-6)). One finds that Eqs.(2-22) have the same form as the linearized field equations (2-10) except the presence of external sources, which are infact $\overset{eff}{T}{}^{\mu\nu}$ and $\overset{eff}{J}{}^{A}$. Therefore, if we introduce a pair of fields $(\hat{g}, \hat{\phi})$ defined by

$$(\hat{g}, \hat{\phi}) = (g + \varepsilon^2 h \ , \phi + \varepsilon^2\varphi) \quad , \tag{2-23}$$

Eqs.(2-22a) and (2-22b) are respectively equivalent to

$$G^{\mu\nu}(\hat{g}) = 8\pi\left[T^{\mu\nu}(\hat{\phi};\hat{g}) + \varepsilon^2 \overset{eff}{T}{}^{\mu\nu}(\overset{1}{h},\overset{1}{\varphi};\hat{g})\right] \quad , \tag{2-24a}$$

and

$$F^{A}(\hat{\phi};\hat{g}) = \varepsilon^2 \overset{eff}{J}{}^{A}(\overset{1}{h},\overset{1}{\varphi};\hat{g}) \quad , \tag{2-24b}$$

to the order of ε^2. Thus, the pair $(\hat{g}, \hat{\phi})$ can be considered as the "background", on which the linearized fields propagate and which is due to the energy-momentum (and the current) of both the linearized fields and the unperturbed fields.

The gravitational radiation damping of the pulsation of a static star studied by Thorne[1] is a simple but good example of the above result. In this case the static nature of the unperturbed metric implies the existence of a time like Killing vector t^{μ}, whose direction is parallel to the fluid 4-velocity u^{μ} of the star. Therefore, by using Eq.(A-16) in the appendix, we obtain the integrable energy conservation equation

$$(t_\mu \overset{\text{eff}}{T}{}^{\mu\nu})_{;\nu} = \frac{1}{\sqrt{-g}}\,\partial_\nu(\sqrt{-g}\,t_\mu \overset{\text{eff}}{T}{}^{\mu\nu}) = 0 \quad . \tag{2-25}$$

If one gives initial data such that there are no incoming gravitational waves and perturbations are reasonably confined near and inside the star, the non-radial pulsation modes eventually generate the outgoing gravitational waves. Then, as a consequence of Eq.(2-25), the sum of the properly defined pulsation energy and the gravitational wave energy conserves and the radiation damping of the pulsation follows.

The argument given in this section applies also to the perturbation of a vacuum space-time (e.g. black hole oscillations). Since no matter field is present in this case, $\overset{\text{eff}}{T}{}^{\mu\nu}$ formally coinsides with Isaacson's effective energy-momentum tensor of the gravitational wave,[3] and satisfies the covariant conservation equations. Then, Eq.(2-24a) becomes identical to Eq.(2.3b) of Ref.(3), except that, in our case, the main part of the background curvature is given <u>a priori</u> and the contribution made by $\overset{\text{eff}}{T}{}^{\mu\nu}$ to the curvature is assumed to be always small.

§ III The perturbation of a vacuum metric induced by the presence of a particle

Let the metric $\overset{\circ}{g}_{\alpha\beta}$ represent a solution to the vacuum Einstein equations;

$$G^{\mu\nu}(\overset{\circ}{g}_{\alpha\beta}) = 0 \tag{3-1}$$

We introduce a length scale L over which the characteristic component of the metric $\overset{\circ}{g}_{\alpha\beta}$ changes. Thus the magnitude of the curvature is of order $1/L^2$. We perturb this space-time by putting a particle with small mass m and (almost spherical) radius ℓ, such that the relations $m \ll L$ and $\ell \ll L$ hold. Then the equations governing the perturbation are obtained by considering a family of solutions $(g_{\alpha\beta}(\varepsilon),\ \Phi_A(\varepsilon))$ which satisfy

$$G^{\mu\nu}(g_{\alpha\beta}(\varepsilon)) = 8\pi\,\varepsilon\, T^{\mu\nu}(\Phi_A(\varepsilon)\,;\, g_{\alpha\beta}(\varepsilon)) \quad , \tag{3-2}$$

where $\varepsilon\, T^{\mu\nu}(\Phi_A(\varepsilon);\ g_{\mu\nu}(\varepsilon))$ is the energy-momentum tensor of the particle and $\varepsilon = m/L$. $\Phi_A(\varepsilon)$ represents suitable variables which describe a motion of the particle. Because of the contracted Bianchi identities, Eq.(3-2) implies

$$\nabla_\nu T^{\mu\nu}(\Phi_A(\varepsilon)\,;\, g_{\alpha\beta}(\varepsilon)) = 0 \quad , \tag{3-3}$$

which is the equation of motion of the particle. Therefore no consideration is needed particularly to the equations of matter; the Einstein equations contain all the information that one needs.

The assumptions made in the above are: (a) if the particle is nearly spherical in shape and sufficiently small in size, the perturbation is independent of the particle's structure, and (b) the particle's self-field can be separated out properly so that the equation of motion is independent of it. Although, there exists no rigorous proof (or disproof) of these assumptions at the moment, we adopt them in order that no other parameter except ε would appear in Eqs.(3-2) and (3-3).

Now we apply the perturbation expansion scheme given in § II to the present case, in order to see if it is possible to include radiation reaction effects in this scheme. The action for the present system is given by

$$S[g,\phi;\varepsilon] = S_G[g] + \varepsilon S_P[\phi;g] \quad , \tag{3-4}$$

where εS_p is the action for the particle. Note that S has an explicit ε dependence from the beginning. Although the stationary action principle applied to Eq.(3-4) with a fixed ε gives the correct Einstein equations (3-2), since one considers ε as the expansion parameter, one should set

$$(g_{\mu\nu}, \phi_A) = (\overset{\circ}{g}_{\mu\nu} + \varepsilon h_{\mu\nu},\ \overset{\circ}{\phi}_A + \varepsilon \varphi_A) \tag{3-5}$$

in Eq.(3-4). Then the derivative of S with respect to ε becomes

$$\begin{aligned} \frac{\partial}{\partial\varepsilon} S \Big|_{\varepsilon=0} &= \int \frac{\delta S_G}{\delta g_{\mu\nu}}\Big|_{\varepsilon=0} h_{\mu\nu}\, d^4x + S_P\Big|_{\varepsilon=0} \\ &= S_P\Big|_{\varepsilon=0} \end{aligned} \tag{3-6}$$

which is apparently non-vanishing. This contradicts with the usual stationary action principle. One should be reminded that Eq.(3-3) has a meaningful limit when ε approaches zero;

$$\overset{\circ}{\nabla}_\nu T^{\mu\nu}(\overset{\circ}{\phi}_A ; \overset{\circ}{g}_{\alpha\beta}) = 0 \quad , \tag{3-7}$$

which is known as the test particle equation of motion. Thus we have a motion described by Eq.(3-7) without the presence of a particle. This is precisely due to the term $S_p|_{\varepsilon=0}$ left over in Eq.(3-6).

Still, one might hope that radiation reaction effects could be included in this scheme by proceeding to a higher order. However, one can show it is in vain: Since a test particle's trajectory is fixed through Eq.(3-7), when the deviation from this trajectory becomes large due to the reactive effects, the scheme fails. This happens if the duration period of motion becomes comparable to L/ε. On the other hand, since the gravitational radiation power is of order ε^2, the radiation reaction is non-neglisible if $T\,\varepsilon^2 \sim m$, where T is the duration period of motion. Then any scheme that can include the reactive effects should be able to describe a motion over the time period of order $T \sim m/\varepsilon^2$. But $m/\varepsilon^2 = L/\varepsilon$. Thus one must abandon the scheme given in § II.

It is clear that the failure of the above method owes to the existence of the test particle equation of motion (3-7). Therefore we seek for a method in which the particle variables ϕ_A are not expanded in terms of ε explicitly. Thus, setting $g_{\mu\nu}(\varepsilon) = \overset{\circ}{g}_{\mu\nu} + \varepsilon h_{\mu\nu}(\varepsilon)$ only, Eq.(3-2) becomes

$$\overset{1}{G}{}^{\mu\nu}(h;\overset{\circ}{g}) + \varepsilon \overset{2}{G}{}^{\mu\nu}(h;\overset{\circ}{g}) = 8\pi \left[T^{\mu\nu}(\phi;\overset{\circ}{g}) + \varepsilon \overset{1}{T}{}^{\mu\nu}(h,\phi;g) + O(\varepsilon^2) \right] \quad (3\text{-}8)$$

where the tensors $\overset{1}{G}{}^{\mu\nu}(h;\overset{\circ}{g})$ and $\overset{2}{G}{}^{\mu\nu}(h;\overset{\circ}{g})$ are respectively the linear and quadratic terms with respect to $h_{\mu\nu}$ in the expansion of $G^{\mu\nu}(\overset{\circ}{g} + \varepsilon h)$, and $\overset{1}{T}{}^{\mu\nu}(h,\phi;\overset{\circ}{g})$ is the linear term with respect to $h_{\mu\nu}$ in the expansion of $T^{\mu\nu}(\phi;\overset{\circ}{g} + \varepsilon h)$.

Now, abandoning a naught over the unpurturbed metric, one can express $\overset{1}{G}{}^{\mu\nu}$ as

$$-16\pi\sqrt{-g}\,\overset{1}{G}{}^{\mu\nu}(h;g) = \frac{\delta}{\delta g_{\mu\nu}} S_g[h,g]\Big|_{vac} \tag{3-9}$$

with

$$S_g[h,g] = \int d^4x \left(\frac{\delta S_G}{\delta g_{\mu\nu}} h_{\mu\nu}\right) \quad ,$$

where the subscript "vac" is to remind us the vacuum nature of $g_{\mu\nu}$. Then, repeating the same argument which led to Eq.(2-16) in § II, the invariance of Sg implies

$$\left(\frac{1}{\sqrt{-g}}\frac{\delta S_g}{\delta g_{\mu\nu}}\right)_{;\nu}\Big|_{vac} = -16\pi\,\overset{1}{G}{}^{\mu\nu}(h;g)_{;\nu} = 0 \quad , \tag{3-10}$$

which is actually the perturbed Bianchi identity. By using Eq.(3-10), the divergence of Eq.(3-8) becomes

$$T^{\mu\nu}(\phi;g)_{;\nu} = -\varepsilon\left[\overset{1}{T}{}^{\mu\nu}(h,\phi;g) - \frac{1}{8\pi}\overset{2}{G}{}^{\mu\nu}(h;g)\right]_{;\nu} + O(\varepsilon^2) \quad . \tag{3-11}$$

This can be interpreted as the equation of motion of the particle in the external gravitational field $g_{\mu\nu}$.

From Eqs.(3-8) and (3-11), it is suggested that one may consider the equations,

$$\overset{1}{G}{}^{\mu\nu}(h;g) = 8\pi T^{\mu\nu}(\phi;g) \tag{3-12a}$$

and

$$T^{\mu\nu}(\phi;g)_{;\nu} = -\varepsilon\left[\overset{1}{T}{}^{\mu\nu}(h,\phi,g) - \frac{1}{8\pi}\overset{2}{G}{}^{\mu\nu}(h,g)\right]_{;\nu} \quad , \tag{3-12b}$$

as the set of basic equations to the order of ε . However at first glance they seem to contradict with Eq.(3-10). Thus we must either abandon the idea of considering Eqs.(3-12) as the basic equations or show that the tensor $\overset{1}{G}{}^{\mu\nu}(h;\, g)$ in Eq.(3-12a) can be regarded as something different from the linearized part of the Einstein tensor.

Note that, from Eq.(3-8), one finds $\overset{1}{G}{}^{\mu\nu}(h;\, g)$ is invariant under a transformation of $h_{\mu\nu}$ given by

$$h'_{\mu\nu} = h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu} \quad , \tag{3-13}$$

where ξ^{μ} is an arbitrary vector field. This is the well-known gauge ambiguity that resides in $h_{\mu\nu}$. Thus one may specify a gauge in which arguments can be made easily. Let us choose the so-called Lorentz gauge;

$$\psi^{\mu\nu}{}_{;\nu} = 0 \qquad \text{where} \qquad \psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} h^{\alpha}_{\alpha} \quad . \tag{3-14}$$

In this gauge $\overset{1}{G}{}^{\mu\nu}(h;\, g)$ is written as

$$\begin{aligned} \overset{1}{G}{}^{\mu\nu}(h;g) &= \frac{1}{2}\left[-\psi^{\mu\nu;\alpha}{}_{;\alpha} - 2R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}\psi^{\alpha\beta}\right] \\ &\equiv \frac{1}{2} L^{\mu\nu}{}_{\alpha\beta}\psi^{\alpha\beta} \quad . \end{aligned} \tag{3-15}$$

Since this is explicitly hyperbolic, one can introduce Green functions of the operator $L^{\mu\nu}{}_{\alpha\beta}$, which satisfy

$$L^{\mu\nu}{}_{\alpha\beta}\, G^{\alpha\beta}(x, x')_{\rho'\sigma'} = \delta^{\mu}_{(\rho'}\delta^{\nu}_{\sigma')}\, \delta^{(4)}(x, x') \quad , \tag{3-16}$$

where $\delta^{(4)}(x,x')$ is the invariant 4-dimensional delta function, defined by

$$\int \delta^{(4)}(x, x') \sqrt{-g}\, d^4x = 1 \quad . \tag{3-17}$$

Then, by using a Green function with appropriate boundary conditions, we can rewrite Eq.(3-8) in the integral form

$$\psi^{\mu\nu}(x)=16\pi\int G^{\mu\nu}(x,x')_{\rho'\sigma'}\left[T^{\rho'\sigma'}(\phi;g)+\varepsilon f^{\rho'\sigma'}(\psi,\phi;g)+O(\varepsilon^2)\right]\sqrt{-g}\,d^4x' \quad , \qquad (3\text{-}18)$$

where

$$f^{\mu\nu}(\psi,\phi;g)=\overset{1}{T}{}^{\mu\nu}(h,\phi;g)-\frac{1}{8\pi}\overset{2}{G}{}^{\mu\nu}(h;g) \quad . \qquad (3\text{-}19)$$

Then, to the order ε , $\psi^{\mu\nu}$ is given by

$$\overset{1}{\psi}{}^{\mu\nu}(x)=16\pi\int G^{\mu\nu}(x,x')_{\rho'\sigma'}\left[T^{\rho'\sigma'}(\phi;g)+\varepsilon f^{\rho'\sigma'}(\overset{0}{\psi},\phi;g)\right]\sqrt{-g}\,d^4x' \quad , \qquad (3\text{-}20)$$

where $\overset{0}{\psi}{}^{\mu\nu}$ in the arguments of $f^{\rho'\sigma'}$ is given by

$$\overset{0}{\psi}{}^{\mu\nu}(x)=16\pi\int G^{\mu\nu}(x,x')_{\rho'\sigma'}T^{\rho'\sigma'}(\phi;g)\sqrt{-g}\,d^4x' \quad . \qquad (3\text{-}21)$$

Note that, from Eq.(3-14), $\overset{1}{\psi}{}^{\mu\nu}$ satisfies

$$\overset{1}{\psi}{}^{\mu\nu}{}_{;\nu}=O(\varepsilon^2) \quad . \qquad (3\text{-}22)$$

Now, operating $L^{\mu\nu}{}_{\alpha\beta}$ on the both sides of Eq.(3-20), taking the divergence, and noting Eq.(3-22), we obtain

$$T^{\mu\nu}(\phi;g)_{;\nu}+\varepsilon f^{\mu\nu}(\overset{0}{\psi},\phi;g)_{;\nu}=O(\varepsilon^2) \quad . \qquad (3\text{-}23)$$

By neglecting the orders higher than ε , this equation with $\overset{0}{\psi}{}^{\mu\nu}$ given by (3-21) is identical to the set of Eqs.(3-12). Thus, $\overset{1}{G}{}^{\mu\nu}$ in Eq.(3-12a) should not be regarded as the linearized part of the Einstein tensor but rather as a specific tensor derived by imposing a prescribed gauge condition on $h_{\mu\nu}$.*)

Eq.(3-23) is the equation of motion which includes the lowest order effects of radiation reaction, and one can calculate the amount of emitted gravitational waves from $\mathring{\psi}^{\mu\nu}$. We note that this procedure of deriving the equation of motion is equivalent to the one in classical electromagnetism,[4] but the resulting equation of motion (if one could obtain it) would have a very different structure. In electromagnetism, the notion of a point particle is accepted; the electromagnetic self-energy of a charged particle can be renormalized into the physical mass of the particle, and one obtains the Dirac's (locally defined) radiation reaction term. In gravity, however, the regularization problem of the self-field is a great obstacle. Moreover, even if one ignores it, one can never obtain any local force term for the radiation reaction. This can be made clear from a simple dimensional analysis: Since the 4-acceleration vector $\frac{D}{ds}u^{\mu}$ has dimension of $(\text{length})^{-1}$, the only possible local form the equation of motion may have is

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*) We remark that when the field equations are written in a prescribed gauge, the fulfillment of the gauge condition by the fields implies that of the equations of motion. then, if only approximate equations of motion are needed, the gauge condition can be correspondingly loosened. Conversely, if the gauge condition is slightly violated, the Bianchi identities (i.e. the equations of motion) written in this gauge need not be exactly satisfied.

$$m\frac{D}{ds}u^{\mu} = m^2 f^{\mu}{}_{\alpha}{}^{\beta\gamma\delta} R^{\alpha}{}_{\beta\gamma\delta} \qquad , \tag{3-24}$$

where the tensor $f^{\mu}{}_{\alpha}{}^{\beta\gamma\delta}$ is non-dimensional and must be constructed only from the 4-velocity u^{μ} and the metric $g_{\mu\nu}$. The factor m^2 in the r. h. s. of Eq.(3-24) arises from the fact that the radiation reaction force should be of order ε^2 (see Eq.(3-12b)). Then, it is easy to see that any term of this form vanishes because of the symmetry of Riemann tensor as well as the vacuum nature of the (unperturbed) geometry. Thus, the radiation reaction term should necessarily be non-local.

Even though we are unable to obtain the equation of motion in an explicit form, we may still deduce an implication of Eq.(3-12b); the point is that it is in the form of divergence. For simplicity, let $g_{\mu\nu}$ be a Schwarzschild metric. We denote the Killing vector which is timelike outside the horizon as K^{μ}.[5] Then the contraction of Eq.(312b) with K^{μ} gives

$$(K_{\mu}(T^{\mu\nu} + \varepsilon f^{\mu\nu})\sqrt{-g}\,)_{,\nu} = 0 \qquad . \tag{3-25}$$

Adopting the Schwarzscild coordinates (t, r, θ, φ), we integrate Eq.(3-25) over a 3-dimensional compact region $U(t)$ surrounded by the spheres $r=r_0$ and $r=r_1$ $(r_0 < r_1)$, which includes the spatial volume $V(t)$ occupied by the particle. Then

$$\varepsilon\frac{d}{dt}\int_{U(t)}(-1)(T^0_0 + \varepsilon f^0_0)\, r^2 dr d\Omega = -\varepsilon^2\left[\int_{r=r_0}(f^1_0\, r^2)\, d\Omega + \int_{r=r_1}(-f^1_0\, r^2)\, d\Omega\right] . \tag{3-26}$$

If one places the boundary spheres $r=r_0$ and $r=r_1$ close enough (but not equal) to the horizon $r=2M$ and to the infinity $r=+\infty$, respectively, the r. h. s. of Eq.(3-26) becomes the gravitational energy flux out of the region $U(t)$. This is so because in vacuum the tensor $f^{\mu\nu}$ is equal to the effective energy-momentum tensor of gravitational waves $-\frac{1}{8\pi}\overset{2}{G}{}^{\mu\nu}$,[3] which is actually at high frequency limit near the horizon or the infinity.

Now, we introduce the "energy" of the particle at time t with respect to infinity defined by

$$E(t) = \varepsilon \int_{V(t)} (-T^0_0)\, r^2 dr\, d\Omega \qquad , \qquad (3\text{-}27)$$

which is the conserved energy in the test particle limit. By using the definition of $E(t)$, Eq.(3-26) is rewritten as

$$\frac{d}{dt}E(t) + \varepsilon^2 \frac{d}{dt}\int_{V(t)} (-f^0_0)\, r^2 dr d\Omega = -\varepsilon^2 \left[\frac{d}{dt}\int_{U(t)-V(t)} \overset{GW}{T}{}^0_0\, r^2 dr d\Omega + P\right] \qquad , \qquad (3\text{-}28)$$

where $\overset{GW}{T}{}^{\mu\nu} = -\frac{1}{8\pi}\overset{2}{G}{}^{\mu\nu}$ and $\varepsilon^2 P$ is the energy flux out of the region $U(t)$. Eq.(3-28) shows that the energy change of the particle is not solely due to the radiation. Thus the energy conservation law in the naive sense seems to be violated. However one must be aware that, although the radiation power $\varepsilon^2 P$ is gauge invariant, the coordinates of the particle's trajectory on the unperturbed geometry crucially depends on a gauge chosen. Therefore the "energy" $E(t)$ of the particle is not gauge invariant. In order to give a physical meaning to it, one generally averages Eq.(3-28) over several frequency periods of the generated gravitational wave. But it is possible only if the particle's motion is quasi-periodic. If it is so, the second term in the l. h. s. of Eq.(3-28) becomes negligible, and the definition of $E(t)$ becomes meaningful.

The above argument may well be applied to more general cases. It has been believed that the use of quadrupole formula must yield the energy loss of a weakly gravitating system correctly. But recently, the validity of the formula has been questioned by several authors.[6),7)] One of the problems to be answered is how to define, if possible, the energy of the system. The above result suggests that, only for systems under quasi-periodic motion, this can be solved and the use of quadrupole formula may be justified.

Finally we note that the approximation scheme developed here is analogous to Newtonian expansion, since the parameter $\varepsilon\left(=\frac{m}{L}\right)$ serves as the coupling constant between the particle and the (external) gravitational field.

§IV Comparison between the perturbations considered in §II and §III

We have seen that the homogeneous perturbations considered in §II (hereafter we call it type A) have a nice feature that already in the linear order approximation some effects of the radiation reaction are included, provided the unperturbed geometry admit at least one Killing vector. On the other hand, the inhomogeneous ones considered in §III (type B) require a different approximation scheme if one wants to take into account the effects of radiation reaction.

In this section, we compare these two types in the case of perturbations of the Schwarzschild geometry, in order to understand the difference between them better.

The linearized Einstein equations are known to be expanded in terms of tensor spherical harmonics,[8] because of the spherical symmetry of the Schwarzschild geometry. The metric perturbations are accordingly characterized by the angular indices (ℓ, m) and parity $(-1)^{\ell}$ or $(-1)^{\ell+1}$. It is known that the $\ell=0$ perturbation corresponds to that of the Schwarzschild mass and the $\ell=1$ perturbations to small translation and stationary rotation.[2] Therefore, the $\ell=0$ and $\ell=1$ perturbations are non-radiative.

Noting the facts mentioned in the above paragraph, we discuss about the energy conservation for those two types of perturbations. For this purpose, we tentatively consider the Einstein equations $(G^{\mu\nu}=8\pi T^{\mu\nu})$ and the contracted Bianchi identities $(G^{\mu\nu}{}_{;\nu}=0)$ as though they were independent of each other.

First consider the Einstein equations: In case of type A perturbations, we can exclude the $\ell=0$ and $\ell=1$ components of the linear perturbations, since we are interested only in radiative cases. Then, the stability of the Schwarzschild geometry[9] enables us to proceed to the second order, and we again obtain the linear equations with respect to the second order variation of the metric (see Eqs.(2-21a) and (2-22a)). The $\ell=0$ components then appear in this order, which are due to the effective energy-momentum tensor of the linearized field $\overset{1}{h}_{\mu\nu}$. Thus, the lowest order change in the energy (mass) of the system is determined by $\overset{1}{h}_{\mu\nu}$, and one expects that there may be a certain conservation law associated with $\overset{1}{h}_{\mu\nu}$. In case of type B perturbations, however, there exist the $\ell=0$ components already in the linear order, which account for the mass of the particle. Then, according to the general rules of perturbation theories, we expect the energy conservation to hold only to this order and the radiation reaction not to be included.

Next consider the contracted Bianchi identities: For type A, the $\ell=0$ components of the identities give trivial $0=0$ relations in the linear order, but the law of the (spherically averaged) evergy conservation in the second order. This conservation law might be regarded as an additional restriction on the behaviour of the solution $\overset{1}{h}_{\mu\nu}$. However, as seen in §II, it holds if and only if $\overset{1}{h}_{\mu\nu}$ satisfies the linearized Einstein equations. Thus, this restriction is just a restatement of the trivial consistency between the Einstein equations and the contracted Bianchi identities. For type B, the situation is very different; the $\ell=0$ components of the contracted Bianchi identities give non-trivial restrictions on the motion of the source particle in the first order, and if one tries to proceed to the second order, one fails to obtain any meaningful equations because of the reason given in § III.

Thus, the consistency between the Einstein equations and the contracted Bianchi identities forbids us to use the usual perturbation expansion scheme for the type B case, while it leads to the coservation of the perturbed energy and consequently gives radiation reaction effects for the type A case.

In order to include radiation reaction effects for the type B case, one therefore has to abandon this consistency up to a certain degree. The method developed in § III is a natural way to do so and one eventually obtains a set of equations as Eqs.(3-12).

To conclude this section, we also mention the difference in the character of the initial value problem between each type of perturbations. For type A, we can set up initial data in the usual manner. However for type B, we can only give asymptotic (past) initial data, since what we are interested in is the perturbation caused by the presence of a particle. This fact leads to the well-known difficulty when one desires to consider a bounded orbital motion of the particle.

§ V Discussion

We have intentionally avoided to discuss on several difficulties associated with the inhomogeneous perturbations caused by a source particle. First of all, there is the self-energy problem of the particle. This cannot be left unsolved if one wants to perform actual calculations. There seems to be two ways to approach this problem: One is to consider a particle of finite size with a certain structure. Then one takes a suitable limit to separate out the self-field and possibly to get rid of the structure dependence at the same time. The other is to consider a point particle from the beginning. Then, one gives a reasonable regularization procedure to extract out the infinity associated with the self-field.

However, even if one succeeds to solve the above problem, one may not be able to obtain the equation of motion in a closed form; the particle's motion may depend on the global structure of space-time as well as on the history of the particle. For example, consider the perturbation of a black hole induced by the presence of a particle. It is known that the quasi-normal modes are enhanced and their contribution to the gravitational radiation is non-negligible in the case of the test particle approximation.[10),11)] Since the existence of quasi-normal modes and their (complex) frequency values are intrinsic to the black hole geometry,[12)] it is plausible that the radiation reaction depends very much on the global structure of space-time.

Apparently, much more work should be done on the inhomogeneous perturbations.

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Appendix

In this appendix, we take the electromagnetic field and a perfect fluid as example fields, and give corresponding expressions of Eq.(2-16) and the conserved current P_k^{μ} (see Eq.(2-20)).

(a) The electromagnetic field

The Lagrangian is taken to be

$$L_M = -\frac{1}{16\pi} F_{\mu\nu} F_{\alpha\beta} g^{\mu\alpha} g^{\nu\beta} \qquad , \tag{A-1}$$

where $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$ is the field strength. The field to be varied is the (vector) potential A_ν. Then, the action S_2 is expressed as

$$S_2 = \int L(f, h, \nabla h; F, g) \sqrt{-g}\, d^4x \qquad , \tag{A-2}$$

where indices are supressed and $f_{\mu\nu} = a_{\nu;\mu} - a_{\mu;\nu}$ is the perturbed field strength. Note that S_2 preserves gauges invariance of the unperturbed electromagnetic field by itself.

Now, the Lie derivative of A_μ with respect to a vector ξ^μ is given by

$$\pounds_\xi A_\mu = A_{\mu;\alpha} \xi^\alpha + \xi_{\alpha;\mu} A^\alpha \qquad , \tag{A-3}$$

and Eq.(2-16) is written as

$$(\overset{\text{eff}}{T}{}^{\mu\nu} - \overset{\text{eff}}{J}{}^{\nu} A^\mu)_{;\nu} + \overset{\text{eff}}{J}{}^{\nu} A_\nu{}^{;\mu} = 0 \qquad , \tag{A-4}$$

where

$$\overset{\text{eff}}{J}{}^{\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_2}{\delta A_\nu}$$

Because of the invariance of S_2 under gauge transformation of A_μ, the conservation of $\overset{\text{eff}}{J}{}^{\nu}$ is identically satisfied;

$$\overset{\text{eff}}{J}{}^{\nu}{}_{;\nu} = 0 \quad . \tag{A-5}$$

Thus, Eq.(A-4) can be rewritten in a more comprehensive form

$$\overset{\text{eff}}{T}{}^{\mu\nu}{}_{;\nu} + F^{\mu\nu}\overset{\text{eff}}{J}{}_{\nu} = 0 \quad . \tag{A-6}$$

The conserved current associated with a Killing vector K^{μ} is

$$P_K^{\mu} = K_{\nu}\left(\overset{\text{eff}}{T}{}^{\nu\mu} - A^{\nu}\overset{\text{eff}}{J}{}^{\mu}\right) \quad . \tag{A-7}$$

(b) A perfect fluid

The Lagrangian of a perfect fluid is given by

$$L_M = -\rho\,(1 + e) \tag{A-8}$$

where ρ and e are the density and the internal energy per unit mass, respectively. It is assumed that e is a function of ρ alone, and the particle number conservation holds;

$$e = e\,(\rho) \quad , \tag{A-9}$$

$$(\rho\, u^{\mu})_{;\mu} = 0 \quad , \tag{A-10}$$

where u^{μ} is the fluid 4-velocity which satisfies the condition

$$g_{\mu\nu}u^{\mu}u^{\nu} = -1 \quad . \tag{A-11}$$

The action is a functional of the flow lines of fluid elements which are to be varied. The variations of ρ and u^{μ} are determined by the conditions that Eqs.(A-10) and (A-11) are kept satisfied.

We represent the flow lines by a set of 4 functions $\{z^{\nu}(s,y^{i})\}$ whose values are the coordinates of the trajectory of a fluid element. The argument s and y^{i} are, resepctively, the proper time interval from a given spacelike hypersurface and the labels attached to a fluid element on this surface. If one denotes the varied flow lines by $\{z^{\nu}(s,y^{i};\lambda)\}$, they are related to the unvaried flow lines by

$$z^{\nu}(s,y^{i};\lambda)=\int_{0}^{\lambda}\frac{\partial}{\partial\lambda'}z^{\nu}(s,y^{i};\lambda')\,d\lambda' + z^{\nu}(s,y^{i};0) \quad . \tag{A-12}$$

Thus, the vector $\xi=\left[\frac{\partial}{\partial\lambda}\right]_{(s,y^{i})}$ is the variable which plays the fundamental role in the perturbed equations. Specifically, the expression corresponding to Eq.(2-21b) is given by

$$\xi^{\mu}=\overset{1}{\xi}{}^{\mu}+\varepsilon\,\eta^{\mu} \quad , \tag{A-13}$$

where $\overset{1}{\xi}{}^{\mu}$, together with $\overset{1}{h}_{\mu\nu}$, compose a solution to the linearized field equations, and the action S_2 is expressed as

$$S_2=\int L(\xi,\nabla\xi,h,\nabla h\ ;\ \rho,\nabla\rho,u,\nabla u,g)\sqrt{-g}\,d^4x \quad . \tag{A-14}$$

Then, Eq.(2-16) becomes

$$\left[\overset{\mathrm{eff}}{T}{}^{\mu\nu}-\rho\,\gamma^{\mu\alpha}\left(\overset{\mathrm{eff}}{J}{}_{\alpha}{}^{\nu}-\overset{\mathrm{eff}}{J}_{\alpha}u^{\nu}\right)\right]_{;\nu}+\overset{\mathrm{eff}}{J}\rho^{;\mu}-\rho\overset{\mathrm{eff}}{J}_{\nu}u^{\nu;\mu}=0 \quad , \tag{A-15}$$

where $\overset{\mathrm{eff}}{J}=-\frac{1}{\sqrt{-g}}\frac{\delta S_2}{\delta\rho}$, $\overset{\mathrm{eff}}{J}_{\alpha}=-\frac{1}{\rho\sqrt{-g}}\frac{\delta S_2}{\delta u^{\alpha}}$ and $\gamma^{\mu\nu}=g^{\mu\nu}+u^{\mu}u^{\nu}$.

If the unperturbed fields admit a killing vector K^{ν}, the conserved current is given by

$$P_K^{\mu}=K_{\nu}\left[\overset{\mathrm{eff}}{T}{}^{\nu\mu}-\rho\,\gamma^{\nu\alpha}\left(\overset{\mathrm{eff}}{J}\delta_{\alpha}^{\mu}-\overset{\mathrm{eff}}{J}_{\alpha}u^{\mu}\right)\right] \quad . \tag{A-16}$$

References

1) K.S. Thorne, Phys. Rev. Lett. 21 (1968), 320.

2) F.J. Zerilli, Phys. Rev. D2 (1970), 2141.

3) R.A. Isaacson, Phys. Rev. 166 (1968), 1272.

4) B.S. DeWitt and R.W. Breme, Ann. Phys. 9 (1960), 220. See also, e.g., F. Rohrlich, Classical Charged Particles (Addison-Wesley, Mass., 1965).

5) See, e.g., S.W. Hawking and G.F.R. Ellis, The large scale structure of space-time (Cambridge Univ. Press, Cambridge, 1973), p.149.

6) A. Rosenblum, Phys. Rev. Lett. 41 (1978), 1003.

7) J. Ehlers, A. Rosenblum, J.N. Goldberg, and P. Havas, Astrophys. J. 208 (1976), L77.

8) F.J. Zerilli, J. Math. Phys. 11 (1970), 2203.

9) V.C. Vishveshwara, Phys. Rev. D1 (1970), 2870.

10) M. Davis, R. Ruffini, W.H. Press, and R.H. Price, Phys. Rev. Lett. 27 (1971), 1466.

11) M. Davis, R. Ruffini and J. Tiomno, Phys. Rev. D5 (1972), 2932.

12) S. Chandrasekhar and S. Detweiler, Proc. Roy. Soc. A344 (1975), 441.